STAT333 Applied Probability

Austin Xia

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1 Course Information

1.1 Contact

Instructor: Steve Drekic
Email: sdrekic@uwaterloo.ca

1.2 Grading Scheme

4 assignments 100%...

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2.1 Review of Elementary Probablity

Definition 2.1.1 (Probability Function)
For each event A of a sample space S, P(A) is defined as the probability of event A, satisfying 3 conditions:

- $0 \leq P(A) \leq 1$
- p(S)=1
- $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ if the sequence of events A_i are mutually exclusive

Definition 2.1.2 (independent)
X and Y are independent rvs if $f(x, y) = f(x)f(y)$

Definition 2.1.3 (multivariate mgf)
$\phi_{x,y}(a, b) = E(e^{ax+by})$

Theorem 2.1.1
if $X_1, X_2 ... X_n$ are independent rvs where $\phi_{X_i}(t)$ is the mgf of $X_i, i = 1, 2, ...n.$ then $T = \sum_{i=1}^{n} X_i$ has mgf $\phi_x(t) = \prod_{i=1}^{n} \phi_{X_i}(t)$

Theorem 2.1.2 (Strong Law of Large Numbers)
If $X_1, ..., X_n$ is an iid sequence of rvs, each having mean $\mu < \infty$, then with probability 1, as $n \to \infty$

$$\bar{X}_n = \frac{X_1 + X_2 ... + X_n}{n} \to \mu$$

2.2 Conditional Distributions and Conditional Expectation

Theorem 2.2.1 (Law of total Expectation)
For rvs X and Y,
$$E[g(X)] = E[E[g(x)|Y]]$$

Theorem 2.2.2
For rvs X and Y,
$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

2.3 Computing Probability by conditioning

recall
$$E[X] = E[E[X|Y]] = \sum_y E[X|Y = y]p_Y(y)$$
similarly
$$P(A) = \sum_y P(A|Y = y)p_Y(y)$$
$P(X < Y)$ is good example of application

3 Discrete-time Markov Chain

Definition 3.0.1 (Stochastic Process)
$\{X(t), t \in T\}$ is called a stochastic process if $X(t)$ is a rv for any $t \in T$

Definition 3.0.2 (DTMC)
A stochastic process $\{X_n, n \in N\}$ is said to be a discrete-time Markov chain (DTMC) if:

- X_n is a discrete rv for all $n \in N$
- the Markov property hold:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, X_{n-1} = x_{n-1}...X_0 = x_0) = P(X_{n+1} = x_{n+1}|X_n = x_n)$$

Definition 3.0.3 (transition probability Matrix)
the transition probability from state i at time n to state j at time n+1 is:
$$P_{nij} = P(X_{n+1} = j|X_n = i),\ n \in \mathbb{N}$$
the transition probability matrix is all number of $P_{i,j}^{(n)}$
we can show by linear algebra that:
$$P^{(n)} = P^n$$

We will only be dealing with stationary or homogeneous situation we at every step we have the same P

Definition 3.0.4 (α_n)
The marginal pmf of X_n is $\alpha_n = (\alpha_{n0}...\alpha_{nk})$
where $\alpha_{nk} = P(X_n = k)\forall k \in \mathbb{N}$

3.1 transiant and recurrance

Definition 3.1.1
The probability that starting from state i, FIRST visit to j occurs at time n is

$$f_{i,j}^{(n)} = P(X_n = j, X_{X-1} \neq j...x_1 \neq j|X_0 = i)$$

note $f_{i,j}^{(1)} = P_{i,j}$

Definition 3.1.2 (transiant)
State i is transient if $f_{i,i} < 1$ otherwise, state i is recurrent

Definition 3.1.3 (M_i)
M_i is a rv which counts the number of times the DTMC visits state i

$$P(M_i = k|X_0 = i) = f_{i,i}^k(1 - f_{i,i})$$

visit k times, then never visit it again. We also notice this is geometric Distribution so

$$E[M_i|X_0 = i] = E[Y - 1] = \frac{1}{1 - f_{i,i}} - 1$$

so if $f_{i,i} = 1$, $E[M_i|X_0 = i] = \infty$, the state is recurrent we can derive that

$$E[M_i|X_0 = i] = \sum P_{i,i}^{(n)}$$

Theorem 3.1.1
if $i \leftrightarrow j$ and state i is recurrent, than state j is recurrent

Theorem 3.1.2
if $i \leftrightarrow j$ and state i is recurrent, than $f_{i,j} = 1$

Theorem 3.1.3
inconclusion, if sates are within the same communication class, then

- the states communicate with each other
- these states all have the same period
- these sates are either recurrent or all transiant

Theorem 3.1.4
A finite state DTMC has at least one recurrent state

Theorem 3.1.5
for any i and transient state j of DTMC,

$$lim_{n\to\infty} P_{i,j}^{(n)} = 0$$

Definition 3.1.4
$N_i = min\{n \in Z : X_n = i\}$ mean recurrent time

$$m_i = E[N_i|X_0 = i] = \sum n f_{i,i}^{(n)}$$

Definition 3.1.5
Suppose i is recurrent. i is positively recurrent iff $m_i < \infty$, i is nullrecurrent iff $m_i = \infty$

Theorem 3.1.6 (Positive and null recurrence)

- if $i \leftrightarrow j$ and i is positively recurrent then j is also
- in finite state DTMC, there can never be null recurrent states

Definition 3.1.6 (stationary distribution)
if $\sum_{i=0}^{\infty} p_i = 1$ and $p_j = \sum_{i=0}^{\infty} p_i P_{i,j}$
or in matrix form

$$p(1, 1, 1...) = 1 \; p = pP$$

Theorem 3.1.7
if irreducible DTMC is positive reccurent iff stationary distribution exisist
Stationary distributions are not nessarily unique

Theorem 3.1.8 (Basic Limit thm)
For an irredicible, recurrent, aperioadic DTMC, $lim_{n\to\infty}$ exisist and independent of state i satisfying

$$lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j = \frac{1}{m_j} \forall i, j \in N$$

If it is also positive recurrent, then π_j is the unique positive solution to

$$\pi_j = \sum \pi_i P_{i,j} \; \sum \pi_j = 1$$

3.2 Galton Waston Branching Process

$Z_i^{(j)}$ be the number of offspringt produced from individual i in jth generation

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i^{n-1}$$

since X_n is a random sum, we get

$$E[X_n] = \mu E[X_{n-1}]$$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j \alpha_j$$

this is summing x_0 has 0 to infty offsprings

4 The Exponential Distribution and the Poisson Process

$$Y = min\{X_1, ...X_n\} \vdash EXP(\sum_{i=1}^{n} \lambda_i)$$

$$P(X_1 < X_2 ... < X_n) = \prod_{i=1}^{n-1} P(X_i = min\{X_i, X_{i+1} ...\})$$

Theorem 4.0.1
A rv X is memoryless iff

$$P(x > y + z | X > y) = P(X > z) \forall y, z \geq 0$$

Theorem 4.0.2
A rv X is memoryless iff

$$P(x > y + z) = P(X > y) * P(X > z) \forall y, z \geq 0$$

Erlang Distribution:

$$\phi_x(t) = (\frac{\lambda}{\lambda - t})^n$$

which is product of n mgf of $exp(\lambda)$

4.1 poisson Process

Definition 4.1.1 (counting process)
A counting process is a stochastic process in which N(t) represents number of event by time t

Definition 4.1.2 (independent increments)
A counting process has independent increments if $N(t_1) - N(S_1)$ is independent of $N(t_2) - N(s_2)$ whenever the intervels do not overlap

Definition 4.1.3 (stationary increments)
A counting process has independent increments if $N(s+t) - N(s)$ dependes only on t

Definition 4.1.4 (o(h))
A function y=f(x) is said to be "o(h)" (of order h) if

$$lim_{h\to 0}\frac{f(h)}{h} = 0$$

Definition 4.1.5 (Poisson process)
a counting process is a Poisson process at rate λ if

- the process has independent and stationary increments
- $P(N(h) = 1) = \lambda h + o(h)$
- $P(N(h) \geq 2) = o(h)$

Theorem 4.1.1
if $\{N(t), t \geq 0\}$ is a Poisson process at rate λ, $N(t) \sim POI(\lambda t)$

$$\neg\exists x A(x) \vdash \forall x \neg A(x)$$

Theorem 4.1.2
if $\{N(t), t \geq 0\}$ is a Poisson process, then $\{T_i\}$ is a suquence of iid $EXP(\lambda)$