CO342 Graph Theory

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1 Math239 Review

- adjacent
  two vertex are adjacent if they are joined by edge
- incident
  vertex v is incident to edge e if e contains v
- neighbor
  vertex u is neighbor of vertex v if there is edge uv
- neighborhood
  neighborhood of u is all vertex that are neighbor to u
- degree
  size of neighborhood
- complete graph K_n:
  a graph has n vertex and all possible arcs
- bipartite graph
  very edge join two vertex in different set
- k-regular
  every vertex has degree k
- subgraph
  h contains some of vertex and some of the edges of G
- path
  a set of adjacent vertex
- cycle
- connected graph
- component
  a maximal connected subgraph
- tree
- planar graph
  can be drawn in a plane without crossing
- subdivision
  subdivision of edge e(uv)yields a graph containing new vertex w and with an edge set replacing uw by uw and vw
- face of a planar graph
  connected region of plane bounded by vertex and edge

2 Connectivity

2.1 Vertex Cut

Definition 2.1.1 (Cut Vertex)
Let G be a connected graph and let $x \in V(G)$. We say x is cutvertex of G if graph G-x obtained from G by deleting the vertex x is disconnected

Definition 2.1.2 (Vertex Cut)
We say $w \subseteq V(G)$ is a vertex cut of connected graph G if G-w is disconnected

Theorem 2.1.1
If G is a connected graph that is not complete, then G has a vertex cut

2.2 Connectivity

Definition 2.2.1 (k-connectivity)
Let G be a connected graph, and let $k \geq 1$ be an integer, we say G is k-connected if

- $|V(G)| \geq k + 1$
- a has no vertex cut of size $\leq k - 1$

Definition 2.2.2 (Minimum Degree)

$$\delta(G) = min\{d(v); v \in V(G)\}$$

Lemma 2.2.1
If G is k-connected, then $\delta(G) \geq k$

> **Proof**
> Let $x \in V(G)$ be a vertex of degree $\delta(G)$, by definition $|V(G)| \geq k + 1$
> If $|V(G)| \geq \delta(G) + 2$ then N(x) is a vertex cut of G, So $|N(x)| = \delta(G) \geq k$
> If $|V(G)| \leq \delta(G) + 1$ Then $k + 1 \leq |V(G)| \leq \delta(G) + 1$
> In both case $k \leq \delta(G)$

Note: converse of this lemma is NOT true

Lemma 2.2.2
Let G be a graph with n vertex, let $1 \leq k \leq n-1$. If $\delta(G) \geq \frac{n+k-2}{2}$, then G is k-connected

> **Proof**
> we have $|V(G)| \geq k+1$. Suppose on the contrary that G has a vertex cut W with $|w| \leq k-1$. Let H be the smallest component of G-w. Then $|H| \leq \frac{n-|W|}{2}$. For $v \in V(H)$, v can have edge in H or W only. we see $d(v) \leq |W| + (|H|-1)$. So
>
> $$\delta(G) \leq d_G(v) \leq |W| + |H| - 1 \leq |W| + \frac{n-|W|}{2} - 1 \leq \frac{n}{2} + \frac{|W|}{2} - 1 \leq \frac{n}{2} + \frac{k-1}{2} - 1 = \frac{n+k-3}{2}$$
>
> this is contradiction to our assumption, so G has no vertex cut with size $\leq k-1$. G is k-connected

Note: this lemma is NOT if and only if