Homework Set 1

Ming Yang, SY2021120

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Section B

Problem 1

(a)

Proof.

$$a \leq b \Rightarrow a^2 \leq ab \Rightarrow a \leq (ab)^{1/2}$$

$\square$

(b)

Proof. Let R_1 be the distribution area of class ω_1, and R_2 be the area of class ω_2

$$p(error) = \int_{R_1} p(x, \omega_2)dx + \int_{R_2} p(x, \omega_1)dx$$

In R_1 we always have $p(\omega_2|x) \leq p(\omega_1|x)$.

$$p(\omega_2|x) \leq \{p(\omega_1 \mid x)p(\omega_2 \mid x)\}^{1/2}$$

$$\begin{aligned}
\int_{R_1} p(x, \omega_2)dx &= \int_{R_1} p(\omega_2|x) \\
&\leq \int_{R_1} \{p(\omega_1 \mid x)p(\omega_2 \mid x)\}^{1/2} p(x)dx \\
&= \int_{R_1} \{p(x, \omega_1)p(x, \omega_2)\}^{1/2} dx
\end{aligned}$$

and similar situations apply for R_2:

$$\int_{R_2} p(x, \omega_1)dx \leq \int_{R_2} \{p(x, \omega_1)p(x, \omega_2)\}^{1/2} dx$$

$$
\begin{aligned}
p(error) &= \int_{R_1} p(x,\omega_2)dx + \int_{R_2} p(x,\omega_1)dx \\
&\leq \int_{R_1} \{p(x,\omega_1)p(x,\omega_2)\}^{1/2}\,dx + \int_{R_2} \{p(x,\omega_1)p(x,\omega_2)\}^{1/2}\,dx \\
&= \int \{p(x,\omega_1)p(x,\omega_2)\}^{1/2}\,dx
\end{aligned}
$$

□

Problem 2

(a)

Proof.

$$
\begin{aligned}
R(a_i \mid \boldsymbol{x}) &= \sum_{j=1}^{c} \lambda(a_i \mid \omega_j) p(\omega_j \mid \boldsymbol{x}) \\
&= \sum_{j=1, j\neq i}^{c} \lambda_s p(\omega_j \mid \boldsymbol{x}) \\
&= \lambda_s \sum_{j=1, j\neq i} p(\omega_j \mid \boldsymbol{x}) \\
&= \lambda_s (1 - p(\omega_j \mid \boldsymbol{x}))
\end{aligned}
$$

For taking the rejection decision we have:

$$
\begin{aligned}
R(a_{c+1} \mid \boldsymbol{x}) &= \sum_{j=1}^{c} \lambda(a_i \mid \omega_j) p(\omega_j \mid \boldsymbol{x}) \\
&= \sum_{j=1}^{c} \lambda_r p(\omega_j \mid \boldsymbol{x}) \\
&= \lambda_r \sum_{j=1} p(\omega_j \mid \boldsymbol{x}) \\
&= \lambda_r
\end{aligned}
$$

Action a_i is taken if its risk is smaller than the risk of taking another action a_j, $i \neq j$:

$$
\begin{aligned}
\forall j = 1, ..., c, j \neq i \\
R(a_i \mid \boldsymbol{x}) \leq R(a_j \mid \boldsymbol{x}) \Rightarrow \\
\lambda_s (1 - p(\omega_i \mid \boldsymbol{x})) \leq \lambda_s (1 - p(\omega_j \mid \boldsymbol{x})) \Rightarrow \\
p(\omega_i \mid \boldsymbol{x}) \geq p(w_j \mid \boldsymbol{x})
\end{aligned}
$$

Also, the risk of action a_i has to be less than the risk of rejection, That is:

$$
\begin{aligned}
R(a_i \mid \boldsymbol{x}) \leq R(a_{c+1} \mid \boldsymbol{x}) \Rightarrow \\
\lambda_s (1 - p(\omega_i \mid \boldsymbol{x})) \leq \lambda_r \Rightarrow \\
p(\omega_i \mid \boldsymbol{x}) \geq 1 - \frac{\lambda_r}{\lambda_s}
\end{aligned}
$$

So, the minimum risk is obtained:

- if we decide ω_i if $P(\omega_i \mid \boldsymbol{x}) \geq P(\omega_j \mid \boldsymbol{x})$ for all j and if $P(\omega_i \mid \boldsymbol{x}) \geq 1 - \frac{\lambda_r}{\lambda_s}$,
- reject otherwise.

$\square$

(b)

- If $\lambda_r = 0$, then for all values of $\boldsymbol{x}$, the system always rejects the decision, since this action has a cost of zero.
- If $\lambda_r > \lambda_s$, then the rejection decision will never be taken.