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Cash flow matching with risks controlled by buffered probability of exceedance and conditional value-at-risk

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Abstract Bond immunization is an important topic in portfolio management. This paper presents a scenario based optimization framework for solving a cash flow matching problem. In this problem, the time horizon of the cash flow generated by the liability is longer than the maturities of the available bonds, and the interest rates are uncertain. Bond purchase decisions are made each period to generate cash flows to cover the obligations due in the future. We use buffered probability of exceedance (bPOE) and conditional value-at-risk (CVaR) to control for the risk of shortfalls. The initial cost of the hedging portfolio of bonds is minimized and optimal positions in bonds are calculated at all time periods. We also study the methodology when solving the optimization problem to minimize bPOE instead of CVaR, which has important practical relevance. The methodology we present in this paper is quite general and can be extended to other financial optimization problems. We use portfolio safeguard optimization package to solve the optimization problems.

Keywords Risk management · Bond immunization · Buffered probability of exceedance (bPOE) · Conditional value-at-risk (CVaR)

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1 Introduction

Bond immunization, including duration matching and cash flow matching, is an important portfolio optimization problem. The objective of cash flow matching problem is to match the liabilities with some asset cash flows. By liabilities we mean the negative cash flows resulting from the contractual obligations.

Hiller and Eckstein (1993), Zenios (1995), and Consigli and Dempster (1998) proposed stochastic programming approaches for cash flow matching problems. Iyengar and Ma (2009) used conditional value-at-risk (CVaR) (Rockafellar and Uryasev 2002) to constrain risks in the bond cash flow matching problem. In their paper, considered bonds of various maturities make coupon payments each period and the principals are paid when the bonds mature. But unlike a classic cash flow matching problem, the liability has a longer time horizon than the maturities of the currently available bonds. Hence some bond purchases have to be made in future periods. Moreover, the future prices of the bonds are uncertain. Therefore, the resulting portfolio cannot be entirely immunized to the changes in interest rates. The model considered in that article is intended to design a portfolio providing needed cash flows with a high probability and to minimize the initial portfolio cost. Our paper follows the general setting suggested by Iyengar and Ma (2009).

We consider new variants of the cash flow matching problem with downfall risks controlled by buffered probability of exceedance (bPOE) (Mafusalov and Uryasev 2014; Norton and Uryasev 2014; Uryasev 2014). bPOE is a function closely related to the probability of exceedance (POE), which is the chance that the liability amount is higher than the cash flow generated by the bond portfolio at least at one time period. bPOE is an extension of the so-called buffered probability of failure considered by Rockafellar and Royset (2010). The value of bPOE is about two times larger than POE (Mafusalov and Uryasev 2014). We compare optimization problem statements in which risks are controlled by bPOE and by CVaR, respectively, and explore the important implication and the practical relevance of bPOE. Although the paper describes a specific cash flow matching problem, the suggested approach is quite general and it can be used in many other finance problems. The bPOE concept is introduced in the basic engineering context (reliability of component design), and this is the first exposure of this concept to the finance optimization community.

It is important to note that constraints on bPOE and CVaR are equivalent in the sense explained later on in the paper. However, the problems of minimizing bPOE and CVaR are quite different. Minimizing bPOE is intended to reduce the probability of an undesirable event. In this paper, the undesirable event is when assets are below or only slightly above the liabilities at some time moment. This event includes tail outcomes such that the average of the tail equals to the threshold. Compared to bPOE minimization, the CVaR minimization problem fixes the probability of an event and minimizes the average of the tail outcomes (in $ units). We want to emphasize that controlling for the probabilities of the upper tail of the loss distribution is in the basis of risk management for fixed income securities. So-called "ratings" are assigned by the rating agencies for fixed income securities based on their default probabilities. The optimization objective in probability format corresponds to the problem of maximizing the rating of the portfolio. This type of problems are not considered in financial literature because of the high computational complexity. To the best of our knowledge, this is the first paper suggesting the constructive approach for solving realistic rating maximization problems with a large number of scenarios.

We conduct a case study demonstrating that the cash flow matching problems with bPOE functions can be efficiently solved with convex and linear programming. The optimization was done with the portfolio safeguard (PSG) package (American Optimal Decisions 2009). PSG provides compact and intuitive problem formulations and codes for solving risk management problems.

The rest of the paper is organized as follows. Section 2 defines CVaR and bPOE functions. Section 3 describes the mathematical formulations of the cash flow matching problems with risks controlled by CVaR and bPOE functions. Section 4 presents an approach for minimizing bPOE. Section 5 describes a case study demonstrating a cash flow matching problem with the shortfall risk controlled by CVaR and bPOE. Section 6 concludes.

2 Definition of CVaR and bPOE

Suppose a random variable L is the future loss (or the return with a minus sign) of some investment. By definition, Value-at-Risk at level α is the α-quantile of L,

$$\mathrm{VaR}_\alpha(L) = \inf\{z | F_L(z) > \alpha\},$$

where F_L denotes cumulative distributions function (CDF) of the random variable L. Conditional value-at-risk (CVaR) for a continuous distribution equals the expected loss exceeding VaR (see, e.g., Rockafellar and Uryasev 2000),

$$\mathrm{CVaR}_\alpha(L) = E[L | L \geq VaR_\alpha(L)].$$

This formula justifies the name of CVaR as a conditional expectation. For general distributions, the definition is more complicated, and can be found in Rockafellar and Uryasev (2002).

There are two probabilistic characteristics associated with VaR and CVaR (Mafusalov and Uryasev 2014; Norton and Uryasev 2014; Uryasev 2014). The first characteristic is the *Probability of Exceedance (POE)*, which equals 1− CDF,

$$p_z(L) = P(L > z) = 1 - F_L(z).$$

By definition, CDF is the inverse function of VaR. The second probabilistic characteristic is called *Buffered Probability of Exceedance (bPOE)*. There are two slightly different variants of bPOE, so-called upper bPOE and lower bPOE. We use only *Upper bPOE* in this paper which is defined as follows,

$$\bar{p}_z(L) = \min_{\lambda \geq 0} E[\lambda(L - z) + 1]^+, \tag{1}$$

where $[A]^+ = \max\{0, A\}$. Formula (1) is considered in (Mafusalov and Uryasev 2014; Norton and Uryasev 2014) as a property of bPOE, but it is convenient to use it as a definition, as in this paper. Further on, upper bPOE will be called bPOE (without mentioning that it is upper bPOE). It has been proved in (Mafusalov and Uryasev 2014) that bPOE equals $1 - \alpha$ on the interval $EL < z < \sup L$, where α is an inverse function of CVaR, i.e., a unique solution of the equation

$$CVaR_\alpha(L) = z, \tag{2}$$

where $\sup L$ is the essential supremum[1] of the random value L.

[1] The essential supremum of the random value L is the smallest number a such that probability of the set $\{L > a\}$ equals zero.

Therefore, *bPOE equals the probability,* $1-\alpha$, *of the tail such that CVaR for this tail is equal to* z. The formula (1), to some extent, could be surprising; the expression does not immediately come across as a probability of some event and it is not obvious that the value belongs to the interval [0, 1] for any real value z.

At the point $z = \sup L$, the solution of Eq. (2) may not be unique. For $z = \sup L$, the smallest solution equals $1 - P(L = z)$. The formula (1) corresponds to this smallest solution

$$\bar{p}_{z=\sup L}(L) = P(L = z) = \max\{1-\alpha \mid CVaR_\alpha(L) = z\}.$$

So bPOE (which is upper bPOE) corresponds to the *largest* value for $1-\alpha$.

The largest solution of Eq. (2) at the point $z = \sup L$ equals $\alpha = 1$, which corresponds to the *smallest* value of $1-\alpha = 0$, i.e.,

$$0 = \min\{1-\alpha \mid CVaR_\alpha(L) = z\}.$$

Lower bPOE equals zero at point $z = \sup L$ and is defined as follows,

$$\bar{p}_z^{Lower}(L) = \begin{cases} 0, & \text{for } z = \sup L; \\ \bar{p}_z(L) = \min_{\lambda \geq 0} E[\lambda(L-z)+1]^+, & \text{otherwise.} \end{cases} \tag{3}$$

For continuous distributions of L, we have $P(L = \sup L) = 0$, and subsequently, $\bar{p}_z^{Lower}(L) = \bar{p}_z(L)$. Upper and lower bPOE differ only at the supremum point $z = \sup L$. We opt for Upper bPOE defined by formula (1) thanks to its simplicity.

3 Controlling risks by CVaR and bPOE

We start the description of the bond immunization problem with the deterministic variant. Let p_0 be the vector of the initial prices of the bonds available in the market, x_0 be the vector of the initial positions in these bonds, and l_t and c_t be the liability and the payment vectors at period t, respectively. The classic cash flow matching problem minimizes the initial value of the portfolio, under the condition that the portfolio of bonds can generate cash flows large enough to cover the amount of liability at each time period,

$$\begin{aligned} &\min_{x_0} \; p_0^T x_0 \\ &\text{subject to} \\ &l_t - c_t^T x_0 \leq 0, \quad t = 1, \dots, N, \\ &\quad -x_0 \leq 0. \end{aligned} \tag{4}$$

In this cash flow matching problem (4), the liability has a shorter (or equal) time horizon compared to the maturities of the available bonds. Therefore, the resulting portfolio can be truly immunized.

The considered cash flow matching problem further on has a longer horizon than the maturities of bonds available at the initial time period. Hence, the bonds purchased at the initial period do not generate a cash flow with a horizon long enough to cover the stream of liabilities. Bonds have to be purchased in later periods and therefore prices of them are uncertain. We simulate future prices of bonds using interest rate scenarios generated by Prof. Ken Kortanek with the Hull and White (1990) interest rate model. Rebonato (1998) provides a comprehensive introduction to the calibration of interest rate models.

We consider the setting similar to the bond-matching problem in Iyengar and Ma (2009). At every period t, $t = \{1, \dots, N\}$, the amount of liability to pay off is l_t. We assume that the same set of bonds are available for investment for each t, $t \in \{0, \dots, N\}$. We denote by p_0 a deterministic price vector of bonds at time 0, and denote by p_t a random price vector of bonds at time $t \in \{1, \dots, N\}$. We denote by $c^j_{t,n}$ the cash flow at time moment $n \in \{1, \dots, N\}$ from bond $j \in \{1, \dots, M\}$ purchased at time period $t \in \{0, \dots, n-1\}$, and $c_{t,n} = (c^1_{t,n}, \dots, c^M_{t,n})^T$ is the corresponding column vector. Suppose x^j_t is the number of shares of bond $j \in \{1, \dots, M\}$ purchased at time step $t \in \{0, \dots, N\}$, and $x_t = (x^1_t, \dots, x^M_t)^T$ is the corresponding column vector, then x_t is a vector of real non-negative numbers. An optimal value of this vector is determined by solving an optimization problem every period.

With these notations, the underperformance L_t of the replicating portfolio of bonds relative to the liability amount at the end of time period t equals

$$L_t = l_t + p_t^T x_t - \sum_{s=0}^{t-1} c^T_{s,t} x_s, \quad t = 1, \dots, N.$$

Let us denote by $L(x_0, \dots, x_N)$ the random maximum loss over all time periods

$$L(x_0, \dots, x_N) = \max_{0 \le t \le N} L_t.$$

Further on, for simplicity, we will skip the argument in this maximum loss function. We consider the following cash flow matching problem with the constraint on CVaR of the maximum loss,

$$\begin{aligned} &\min_{x_0, \dots, x_N} \; p_0^T x_0 \\ &\text{subject to} \\ &CVaR_\alpha(L) \le z, \\ &\qquad x_t \ge 0, \quad t = 0, \dots, N. \end{aligned} \tag{5}$$

Iyengar and Ma (2009) suggested problem (5) with parameter $z = 0$. The problem finds an optimal investment strategy by minimizing the cost of the initial bond portfolio. The maximum reinvestment risk over time related to the uncertainties in the bond prices is controlled by CVaR. The level of the protection from the reinvestment risk monotonically increases as a function of the confidence level α in CVaR.

Alternatively, we can replace CVaR with bPOE in the constraint,

$$\begin{aligned} &\min_{x_0, \dots, x_N} \; p_0^T x_0 \\ &\text{subject to} \\ &\bar{p}_z(L) \le 1 - \alpha, \\ &\qquad x_t \ge 0, \quad t = 0, \dots, N. \end{aligned} \tag{6}$$

We will show later on in this section that the problems (5) and (6) are nearly equivalent.

Continuing the discussion of problem (5), we can get the objective function which minimizes the reinvestment risk subject to the constraint on the initial budget d by switching the objective and constraint,

$$\begin{aligned} &\min_{x_0, \dots, x_N} CVaR_\alpha(L) \\ &\text{subject to} \\ &p_0^T x_0 \le d, \\ &\qquad x_t \ge 0, \quad t = 0, \dots, N. \end{aligned} \tag{7}$$

Here we want to emphasize that the problem (7) minimizes CVaR risk, instead of directly minimizing the probability that the loss, L, exceeds some threshold z. In the following problem, we place bPOE in the objective function for controlling for the reinvestment risk. bPOE manages the probability of upper tail, similar to POE. bPOE accounts for the losses in the tail with the average equal to the threshold. Since bPOE is an upper bound for POE, minimizing bPOE also lowers POE. Minimizing the probability of upper tail is important for fixed income portfolios, which are designed, usually, for safety concerned investors. Such portfolios should maintain high rating (e.g., AA) to be eligible for institutional investors. Note that ratings are specified in probability terms (target default probability is defined for obtaining a specified rating). Thus setting up the objective as the probability of the upper tail is an important practical consideration. Here is the problem formulation with bPOE in objective,

$$\begin{aligned} &\min_{x_0,\ldots,x_N} \bar{p}_z(L) \\ &\qquad \text{subject to} \\ &p_0^T x_0 \le d, \\ &\quad x_t \ge 0, \quad t = 0,\ldots,N. \end{aligned} \tag{8}$$

Mafusalov and Uryasev (2014) showed the equivalence of constraints on CVaR and on lower bPOE for general distributions (including discrete distributions considered in this paper). Therefore, optimization problems (5) and (6) are equivalent with lower bPOE in (6) for general distributions of L. This equivalence is a generalization of equivalence in a special case with $z = 0$, which was originally stated by Rockafellar and Royset (2010) for continuous distributions. For upper bPOE, with discrete distribution considered in this paper, constraints on CVaR and on upper bPOE are not equivalent. However, the constraint on CVaR and upper bPOE are "nearly" equivalent as we show below.

Since this equivalence is an important fact for understanding the relation between CVaR and bPOE, we prove this "near" equivalence with the following two statements.

Statement 1. Constraint on bPOE implies constraint on CVaR.

Let $0 < \alpha < 1$, then $\bar{p}_z(L) \le 1 - \alpha$ implies $CVaR_\alpha(L) \le z$.

Proof To begin with, we give a quick schematic proof (without details) using known facts from paper (Mafusalov and Uryasev 2014). Since, $\bar{p}_z(L) \ge \bar{p}_z^{Lower}(L)$, we have

$$\bar{p}_z(L) \le 1 - \alpha \;\; => \;\; \bar{p}_z^{Lower}(L) \le 1 - \alpha \;\; <=> \;\; CVaR_\alpha(L) \le z.$$

The paper (Mafusalov and Uryasev 2014) is quite long and it presents statements in a formal mathematical language, which could make it inaccessible to a considerable number of readers with less advanced mathematical backgrounds. Here we present an alternative proof without heavy mathematics.

Inequality $\bar{p}_z(L) < 1$ implies that an optimal λ^* in minimization formula (1) is strictly positive, i.e., $\lambda^* > 0$. Therefore,

$$\bar{p}_z(L) = E[\lambda^*(L - z) + 1]^+ \le 1 - \alpha\,,$$

which can be rearranged as follows

$$-\frac{1}{\lambda^*} + \frac{1}{1-\alpha} E\left[(L - z) + \frac{1}{\lambda^*}\right]^+ \le 0.$$

By substituting the variable $1/\lambda^* = z - \mu^*$ in the last inequality we get

$$\mu^* + \frac{1}{1-\alpha} E[L - \mu^*]^+ \leq z. \tag{9}$$

Therefore, using CVaR minimal representation (see, Rockafellar and Uryasev 2002), we get

$$CVaR_\alpha(L) = \min_\mu \left\{ \mu + \frac{1}{1-\alpha} E[L - \mu]^+ \right\} \leq z. \tag{10}$$

□

Statement 2. Constraint on CVaR implies constraint on bPOE.
Let $0 < \alpha < 1$ and $y > z$, then inequality $CVaR_\alpha(L) \leq z$ implies $\bar{p}_y(L) < 1 - \alpha$.

Proof To begin with, we give a quick schematic proof (without details) using known facts from paper (Mafusalov and Uryasev 2014). Since

$$\bar{p}_z^{Lower}(L) \geq \bar{p}_y \,,$$

we have

$$CVaR_\alpha(L) \leq z \;\; <=> \;\; \bar{p}_z^{Lower}(L) \leq 1 - \alpha \;\; => \;\; \bar{p}_y(L) < 1 - \alpha.$$

Now, we give an alternative proof without using paper (Mafusalov and Uryasev 2014). According to the CVaR minimization representation (Rockafellar and Uryasev 2002),

$$CVaR_\alpha(L) = \mu^* + \frac{1}{1-\alpha} E[L - \mu^*]^+ \,, \tag{11}$$

where $\mu^* = VaR_\alpha(L)$. Consequently,

$$\mu^* \leq \mu^* + \frac{1}{1-\alpha} E[L - \mu^*]^+ \leq z < y. \tag{12}$$

Since $\mu^* < y$, we can define $\mu^* = y - 1/\lambda^*$, where $\lambda^* > 0$. By replacing the variable μ^* in inequality

$$\mu^* + \frac{1}{1-\alpha} E[L - \mu^*]^+ < y$$

we get

$$E[\lambda^*(L - y) + 1]^+ < 1 - \alpha.$$

Consequently,

$$\bar{p}_y(L) = \min_{\lambda \geq 0} E[\lambda(L - y) + 1]^+ \leq E[\lambda^*(L - y) + 1]^+ < 1 - \alpha.$$

□

Mafusalov and Uryasev (2014) demostrated the equivalence of CVaR minimization problem (7) and lower bPOE minimization problem (8) with a convex set of constraints in the following sense:

- for every parameter value α in problem (7) and an optimal objective value z of this problem, the optimization problem (8) with parameter z has the optimal objective value $1 - \alpha$;

- for every parameter value z in problem (8) and an optimal objective value $1-\alpha$ of this problem, the optimization problem (7) with parameter α has the optimal objective value z.

Also, it can be shown that optimization problems (5), (6), (7), (8) generate coinciding parts of the efficient frontiers. Similar results were presented in Norton and Uryasev (2014). By definition, an efficient frontier is a set of Pareto-optimal solutions in a two criteria optimization problem. In this case, the first criterion is the initial investment, $p_0^T x_0$, and the second criterion is CVaR or bPOE.

Firstly, let us fix the parameter α in the considered optimization problems. Some coinciding parts of the efficient frontiers can be generated with: (a) problem (5) by variation of parameter z; (b) problem (6) by variation of parameter z; (c) problem (7) by variation of parameter d.

Now, let us fix the parameter z in the considered optimization problems. Some coinciding parts of the efficient frontiers can be generated with: (a) problem (5) by variation of parameter α; (b) problem (6) by variation of parameter α; (c) problem (8) by variation of parameter d.

The efficient frontiers statements are quite standard in Pareto optimization literature. These statements can be proved, for instance, similar to Proposition 4.4 in Mafusalov and Uryasev (2014). One may ask why bPOE minimization is needed, provided the same efficient frontiers can be generated by using CVaR and bPOE. The truth is, generating an efficient frontier can be computationally demanding, especially, if some non-convex constraints are involved, such as "cardinality" and "buying constraints".[2] By solving a bPOE minimization problem, we can minimize the probability and find an optimal solution without generating the whole efficient frontier.

Informally, the idea can be summarized as follows. Constraints on bPOE and CVaR are nearly equivalent (may slightly differ on one atom for discrete distributions). Therefore, it is hard to say that there is a significant benefit in using bPOE in the constraints compared to CVaR. However, bPOE minimization and CVaR minimization have different real-world implications. The former minimizes the probability of undesirable events in which the loss is above or slightly below the threshold, which is an important practical consideration. Note that bPOE is an upper bound for POE, so bPOE minimization also reduces POE. CVaR minimization leads to the reduction of the average value of the tail with the fixed probability. In other words, CVaR reduces the monetary value of the tail provided the fixed probability.

A reasonable follow-up question would be which is better between the minimizations of bPOE or of CVaR. The simple answer is that it depends upon the preference and the objective of the decision maker. For instance, in Netherland pension system there is a law that a pension fund should cover the obligations under any conditions (i.e., mathematically with probability one). For this case, bPOE should be in the objective, rather than CVaR with some fixed probability of the tail.

bPOE may become a quite important characteristic in fixed income securities as an alternative to POE. Nowadays, POE is used in defining ratings. Provided the edge bPOE has, it can be appreciated in this industry as well because: (a) bPOE rating takes the magnitude of losses into account and will incorporate the information on the length of the tail of the loss distribution; (b) with bPOE it is possble to find a portfolio assuring the highest rating with specified constraints.

[2] See examples of such problems at this case study: http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/case-study-portfolio-replication-with-cardinality-and-buyin-constraints/.

4 bPOE minimization

The maximum loss function, $L = \max_{0\le t\le N} L_t$, depends upon a set of decision vectors, $x_1,\ldots,x_N$. Let us combine these vectors in one decision vector $\mathbf{x} = (x_1,\ldots,x_N)$.

Let us consider a general linear loss function $L_t(\mathbf{x}) = (\mathbf{a}_t)^T\mathbf{x}+b_t$ with random coefficients $\mathbf{a}_t$. Then, (1) implies

$$\bar{p}_z(L) = \min_{\lambda\ge 0} E\left[\lambda\left(\max_{0\le t\le N} L_t - z\right)+1\right]^+ = \min_{\lambda\ge 0} E\left[\lambda \max_{0\le t\le N}\left\{(\mathbf{a}_t)^T\mathbf{x}+b_t - z\right\}+1\right]^+.$$

The minimization problem for bPOE w.r.t. $\mathbf{x}$ can be written as follows,

$$\begin{aligned}\min_{\mathbf{x}} \bar{p}_z(L) &= \min_{\mathbf{x},\lambda\ge 0} E\left[\lambda \max_{0\le t\le N}\{(\mathbf{a}_t)^T\mathbf{x}+b_t - z\}+1\right]^+\\ &= \min_{\mathbf{x},\lambda\ge 0} E\left[\max_{0\le t\le N}\{(\mathbf{a}_t)^T\lambda\mathbf{x}+\lambda(b_t - z)\}+1\right]^+.\end{aligned}$$

By replacing the term $\lambda\mathbf{x}$ with $\mathbf{y}$ in the last equation we get the optimization problem with respect to variables $\mathbf{y}$, λ,

$$\min_{\mathbf{y},\lambda\ge 0} E\left[\max_{0\le t\le N}\left\{(\mathbf{a}_t)^T\mathbf{y}+\lambda(b_t - z)\right\}+1\right]^+ .$$

Further, we suppose that coefficients, $\mathbf{a}_t$, are random vectors with finite discrete distribution, and the random loss function $L_t^k(\mathbf{x}) = (\mathbf{a}_t^k)^T\mathbf{x}+b_t^k$ has scenarios $k = 1,\ldots K$ with probabilities $p_k = 1/K$. By using the new variables $\mathbf{y}$ and λ, we rewrite problem (8) as follows,

$$\begin{aligned}&\min_{\mathbf{y},\lambda}\sum_{k=1}^{K} p_k\Big[\max_{0\le t\le N}\left\{\left(\mathbf{a}_t^k\right)^T\mathbf{y}+\lambda\left(b_t^k - z\right)\right\}+1\Big]^+\\ &\text{subject to}\\ &p_0^T y_0 - d\lambda \le 0,\\ &\lambda \ge 0,\quad y_t\ge 0,\ t=0,\ldots,N.\end{aligned}\tag{13}$$

The objective in (13) is a so-called *partial moment* with threshold -1 of the random function $L = \max_{0\le t\le N}\{(\mathbf{a}_t^k)^T\mathbf{y}+\lambda(b_t^k - z)\}$. This objective function is a piecewise linear convex function w.r.t. the variables $\mathbf{y}$ and λ. The problem (13) can be reformulated as a linear programming (LP) problem. Let us introduce an additional vector of decision variables $\mathbf{u} = (u_1,\ldots,u_K)$. Problem (13) is equivalent to the following LP,

$$\min_{\mathbf{y},\lambda,\mathbf{u}} \sum_{k=1}^{K} p_k u_k \tag{14}$$

subject to

$$u_k \ge \left(\mathbf{a}_t^k\right)^T\mathbf{y}+\lambda\left(b_t^k - z\right)+1,\ t=0,\ldots,N,\ k=1,\ldots,K,$$

$$p_0^T y_0 - d\lambda \le 0,$$

$$\lambda\ge 0,\quad y_t\ge 0,\ t=0,\ldots,N,\ u_k\ge 0,\ k=1,\ldots,K\,.$$

Table 1 Details of treasury bonds

Bond index	Name	Maturity	Coupon rate (%)	Current price
1	T-bill	0.5	0	95.8561
2	T-note	1	4.5	96.1385
3	T-note	2	4.5	92.6873
4	T-note	3	4.5	89.5784
5	T-note	4	4.5	86.7610
6	T-note	5	4.5	84.1959
7	T-bond	10	5.0	77.5948
8	T-bond	15	5.0	71.9232
9	T-bond	20	5.0	68.1357
10	T-bond	25	5.0	65.5990
11	T-bond	30	5.0	63.8989

5 Case study

In this section, we provide a case study to demonstrate the solution of the optimization problems discussed previously. Data, codes, and solutions for this case study are posted at this link.[3] The future bond prices are generated with the Hull and White (1990) interest rate model.[4]

We consider 120 time steps with 0.5 year of step length. Table 1 shows the maturities, coupon rates, and initial prices of the 11 available bonds. We run $K = 200$ simulations of bond prices. Every simulation provides prices of all bonds for all time periods. $p_t^{k,j}$ is the price of bond $j \in \{1, \ldots, M\}$ at time step $t \in \{1, \ldots, N\}$ for simulation $k \in \{1, \ldots, K\}$.

Similar to (Iyengar and Ma 2009) we consider the following stream of liabilities:

$$l_t = \begin{cases} 100 & \text{if} \quad t/2 = 0, \ldots, 10, \\ 110 - 2.2 \times \left(\frac{t}{2} - 10\right) & \text{if} \quad t/2 = 11, \ldots, 60, \\ 0 & \text{otherwise.} \end{cases}$$

In order to solve the optimization problems, we use the PSG package containing preprogrammed CVaR, partial moment, and bPOE functions. PSG codes are quite simple and transparent. “Appendix” section contains codes which are used in the case study.

Table 2 contains the results of the optimization runs. This table illustrates several theoretical statements.

Firstly, we show that with the fixed parameter α, coinciding parts of the efficient frontiers can be generated with:

- problem (5) by variation parameter z;
- problem (7) by variation parameter d.

The first line of Table 2 contains the calculation results for problem (5) with $\alpha = 0.9$ and $z = 0$. The optimal objective value equals 1172.368. The second line corresponds to problem (7) with $\alpha = 0.9$ and the right-hand side $d = 1172.368$ which equals the optimal objective value of problem (5). As expected, the optimal objective value of problem (7) equals zero,

[3] http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/case-study-cash-matching-with-bpoe-and-cvar-functions/.

[4] The modeling of interest rate in real world may involve a more sophisticated mathematical design and concern other variables, such as credit default risk in a long run. The purpose of this case study is to demonstrate the approach, which can be extended to other models.

Table 2 Calculation results for three optimization problems

	Constraint setting	Optimal objective value
Problem (5)	$CVaR_{\alpha=0.9}(L) \leq 0$	$p_0^T x_0 = 1172.368$
Problem (7)	$p_0^T x_0 \leq d = 1172.368$	$CVaR_{\alpha=0.9} = 0$
Problem (8)	$p_0^T x_0 \leq d = 1172.368$	$bPOE = 0.1$
Problem (13)	$p_0^T y_0 - d\lambda \leq 0, d = 1172.368$	$bPOE = 0.1$

because CVaR is bounded by zero in problem (5). Therefore, solving of problems (5) and (7) results in the same point $z = 0$, $d = 1172.368$ on the two efficient frontiers. The first efficient frontier is generated by variation of parameter z in problem (5) and plotting z versus objective. The second efficient frontier is generated by variation parameter d in problem (7) and plotting d versus objective.

Secondly, we show that with the fixed parameter z, coinciding parts of the efficient frontiers can be generated with:

- problem (5) by variation parameter α;
- problem (8) by variation parameter d.

The third line in Table 2 contains solution of the bPOE minimization problem (8) with $d = 1172.368$ and $z = 0$. The optimal bPOE with threshold 0 equals 0.1, because $\alpha = 0.9$ and $z = 0$ in the problem (5) and $1 - \alpha = 1 - 0.9 = 0.1$. The calculated point $\alpha = 0.9$, $d = 1172.368$ is on the efficient frontier generated by variation of parameter α in problem (5) and plotting α versus objective. This point is also on the efficient frontier generated by variation of parameter d in problem (8) and plotting d versus objective.

Now, let us demonstrate the numerical equivalence of the problems (8) and (13). The third line in Table 2 contains the solution of the bPOE minimization problem (8) with $d = 1172.368$ and $z = 0$; the optimal bPOE $= 0.1$. The fourth line in Table 2 contains the solution of the bPOE minimization problem (13) with the same parameter values $d = 1172.368$ and $z = 0$. The optimal bPOE $= 0.1$ coincides with the optimal objective in the problem (8).

Finally, we build an efficient frontier by solving problem (13) for a series of values of parameter d. Table 3 contains the calculated values for this frontier (first column = budget d, second column = optimal bPOE), as seen in Fig. 1.

6 Conclusion

This paper introduces a new risk quantity called buffered probability of exceedance (bPOE) to finance optimization literature. The idea is to demonstrate how this new concept is applied to a dynamic stochastic system in a finance context. In particular, we consider a cash flow matching problem in which the time horizon of the cash flow generated by the liability is longer than the maturity of any available bond, and the interest rates are uncertain. Therefore, a series of bond purchasing decisions have to be made in the later periods in order to fully cover the liability amounts. The considered optimization methodology is quite general and the suggested approaches can be applied to other financial systems involving uncertainties.

We control for the reinvestment risks with CVaR and bPOE functions. This paper only considers upper bPOE which is defined by a simple optimization formula. Another paper

Table 3 Efficient frontier

d (Bound on budget)	Minimal bPOE
1162	1
1163	0.9703236
1164	0.8577470
1165	0.7469944
1166	0.6404300
1167	0.5377273
1168	0.4457100
1169	0.3517119
1170	0.2727167
1171	0.2028206
1172	0.1286488
1172.368	0.1000000
1172.5	0.0905712
1172.75	1.091354E−10
1173	2.182642E−11
1174	1.359135E−12
1175	0.000000

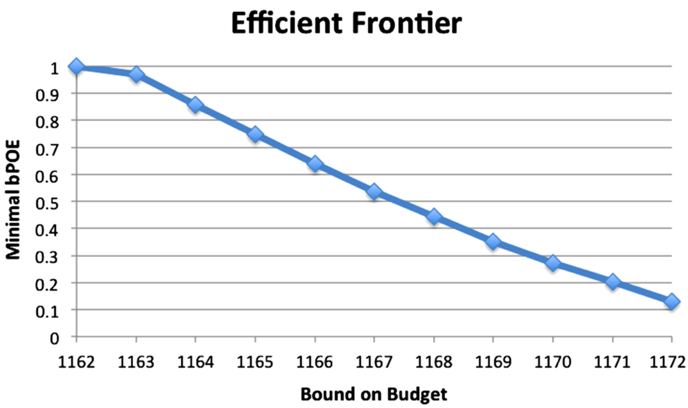

Fig. 1 Efficient frontier: bound on budget d versus minimal bPOE

(Mafusalov and Uryasev 2014) is concentrated on lower bPOE. The upper and lower bPOE may differ only at one maximal atom. An advantage of lower bPOE is that, there is a complete equivalence of CVaR and bPOE constraints. A disadvantage of lower bPOE is that the optimization formula (1) is not valid at the maximum atom. The value at the maximum atom should be assigned to zero, which creates some complications in numerical analysis. This is especially inconvenient when the maximum atom depends upon the control variables. Therefore, this paper reserves the simplicity of numerical calculations and uses upper bPOE. Although, formally speaking, CVaR and upper bPOE constraints are not equivalent, we prove the "near-equivalence" of these constraints.

We demonstrate how to minimize bPOE through the reduction to convex programming. Here we present a simple derivation intended for a broader audience, including researchers who may have less advanced backgrounds in stochastic optimization. Unlike in Mafusalov and Uryasev (2014), which studied the topic in a strict mathematical language with formal mathematical statements, this paper emphasizes the big picture, the overall rationale and the practical relevance of bPOE optimization problems in the finance context.

This paper presents several results about the equivalence of different efficient frontiers. Theoretically, the minimization problem for bPOE can be solved by building the efficient frontier with CVaR constraints and finding the inverse solution corresponding to the threshold. However, we show that it is possible to minimize bPOE in one shot, which is the advantage of using this new technology.

We conduct a case study to numerically demonstrate some theoretical statements. The study suggests that bPOE minimization problems can be solved very efficiently. The considered bPOE minimization problem with 1331 decision variables and 200 scenarios can be solved on a PC in 15 s. Moreover, the code for solving the problem contains only several lines and can be conveniently applied to similar problems. This is another practical advantage of this paper: conveniently verifiable and extendible. Other researchers can download the datasets from the website and quickly modify it, if of interest.

Appendix: Portfolio safeguard (PSG) codes

PSG code for minimizing CVaR with problem (7):

```
minimize
  CVaR(0.9,lmax(matrix_1L,…,matrix_120L))
Constraint: <= 1172.368
  linear(matrix_0)
Box: >= 0
```

PSG code for minimizing bPOE with problem (8):

```
minimize
  bPOE(0,lmax(matrix 1,…,matrix 120))
Constraint: <= 1172.368
  linear(matrix_0)
Box: >= 0
```

PSG code for minimizing bPOE with problem (13):

```
minimize
  pm_pen(-1,lmax(matrix_1L0,…,matrix_120L0))
Constraint: <= 0
  linear(matrix_0)
  -1172.368*variable(lambda)
Box: >= 0
```

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