LIFETIME PORTFOLIO SELECTION UNDER UNCERTAINTY: THE CONTINUOUS-TIME CASE

Robert C. Merton *

I Introduction

MOST models of portfolio selection have been one-period models. I examine the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous-time model where his income is generated by returns on assets and these returns or instantaneous "growth rates" are stochastic. P. A. Samuelson has developed a similar model in discrete-time for more general probability distributions in a companion paper [8].

I derive the optimality equations for a multi-asset problem when the rate of returns are generated by a Wiener Brownian-motion process. A particular case examined in detail is the two-asset model with constant relative risk-aversion or iso-elastic marginal utility. An explicit solution is also found for the case of constant absolute risk-aversion. The general technique employed can be used to examine a wide class of intertemporal economic problems under uncertainty.

In addition to the Samuelson paper [8], there is the multi-period analysis of Tobin [9]. Phelps [6] has a model used to determine the optimal consumption rule for a multi-period example where income is partly generated by an asset with an uncertain return. Mirrless [5] has developed a continuous-time optimal consumption model of the neoclassical type with technical progress a random variable.

II Dynamics of the Model: The Budget Equation

In the usual continuous-time model under certainty, the budget equation is a differential equation. However, when uncertainty is introduced by a random variable, the budget equation must be generalized to become a stochastic differential equation. To see the meaning of such an equation, it is easiest to work out the discrete-time version and then pass to the limit of continuous time.

Define

$W(t)$ = total wealth at time t

$X_i(t)$ = price of the i^{th} asset at time t, ($i = 1, \ldots, m$)

$C(t)$ = consumption per unit time at time t

$w_i(t)$ = proportion of total wealth in the i^{th} asset at time t, ($i = 1, \ldots, m$)

Note
$$\left(\sum_{i=1}^{m} w_i(t) \underset{t}{\equiv} 1 \right)$$

The budget equation can be written as
$$W(t) = \left[\sum_{i=1}^{m} w_i(t_0) \frac{X_i(t)}{X_i(t_0)} \right] \cdot \left[W(t_0) - C(t_0)h \right] \tag{1}$$

where $t \equiv t_0 + h$ and the time interval between periods is h. By subtracting $W(t_0)$ from both sides and using $\sum_{i=1}^{m} w_i(t_0) = 1$, we can rewrite (1) as,
$$\begin{aligned} W(t) - W(t_0) &= \left[\sum_{i=1}^{m} w_i(t_0) \left(\frac{X_i(t) - X_i(t_0)}{X_i(t_0)} \right) \right] \cdot \left[W(t_0) - C(t_0)h \right] - C(t_0)h \\ &= \left[\sum_{i=1}^{m} w_i(t_0) \left(e^{g_i(t_0)h} - 1 \right) \right] \cdot \left[W(t_0) - C(t_0)h \right] - C(t_0)h \end{aligned} \tag{2}$$

where
$$g_i(t_0)h \equiv \log \left[X_i(t)/X_i(t_0) \right],$$

the rate of return per unit time on the i^{th} asset. The $g_i(t)$ are assumed to be generated by a stochastic process.

In discrete time, I make the further assumption that $g_i(t)$ is determined as follows,
$$g_i(t)h = (\alpha_i - \sigma_i^2/2)h + \triangle Y_i \tag{3}$$

where α_i, the "expected" rate of return, is con-

* This work was done during the tenure of a National Defense Education Act Fellowship. Aid from the National Science Foundation is gratefully acknowledged. I am indebted to Paul A. Samuelson for many discussions and his helpful suggestions. I wish to thank Stanley Fischer, Massachusetts Institute of Technology, for his comments on section 7 and John S. Flemming for his criticism of an earlier version.

stant; and $Y_i(t)$ is generated by a Gaussian random-walk as expressed by the stochastic difference equation,

$$Y_i(t) - Y_i(t_0) \equiv \triangle Y_i = \sigma_i Z_i(t)\sqrt{h} \qquad (4)$$

where each $Z_i(t)$ is an independent variate with a standard normal distribution for every t, σ_i^2 is the variance per unit time of the process Y_i, and the mean of the increment $\triangle Y_i$ is zero.

Substituting for $g_i(t)$ from (3), we can rewrite (2) as,

$$W(t) - W(t_0) = \sum_1^m w_i(t_0)(e^{(a_i - \sigma_i^2/2(h + \triangle Y)} - 1)(W(t_0) - C(t_0)h) - C(t_0)h. \qquad (5)$$

Before passing in the limit to continuous time, there are two implications of (5) which will be useful later in the paper.

$$E(t_0)\ [W(t) - W(t_0)] = \left\{ \sum_1^m w_i(t_0)a_i W(t_0) - C(t_0) \right\} h + 0(h^2) \qquad (6)$$

and

$$E(t_0)[(W(t) - W(t_0))^2] = \sum_{i=1}^m \sum_{j=1}^m w_i(t_0)w_j(t_0) \cdot E(t_0)\ (\triangle Y_i \triangle Y_j) \cdot W^2(t_0) + 0(h^2) \qquad (7)$$

where $E(t_0)$ is the conditional expectation operator (conditional on the knowledge of $W(t_0)$), and $0(\cdot)$ is the usual asymptotic order symbol meaning "the same order as."

The limit of the process described in (4) as $h \to 0$ (continuous time) can be expressed by the formalism of the stochastic differential equation,[1]

$$dY_i = \sigma_i Z_i(t)\sqrt{dt} \qquad (4')$$

and $Y_i(t)$ is said to be generated by a Wiener process.

By applying the same limit process to the discrete-time budget equation, we write (5) as

$$dW = \left[\sum_1^m w_i(t)a_i W(t) - C(t) \right] dt + \sum_1^m w_i(t)\sigma_i Z_i(t) W(t)\sqrt{dt}. \qquad (5')$$

The stochastic differential equation (5′) is the generalization of the continuous-time budget equation under uncertainty.

[1] See K. Itô [4], for a rigorous discussion of stochastic differential equations.

A more familiar equation would be the *averaged* budget equation derived as follows: From (5), we have

$$E(t_0)\left[\frac{W(t) - W(t_0)}{h}\right] = \sum_1^m w_i(t_0)a_i[W(t_0) - C(t_0)h] - C(t_0) + 0(h). \qquad (8)$$

Now, take the limit as $h \to 0$, so that (8) becomes the following expression for the defined "mean rate of change of wealth":

$$\overset{\circ}{W}(t_0) \underset{\text{def.}}{\equiv} \lim_{h \to 0} E(t_0)\left[\frac{W(t) - W(t_0)}{h}\right] = \sum_1^m w_i(t_0)a_i W(t_0) - C(t_0). \qquad (8')$$

III The Two-Asset Model

For simplicity, I first derive the optimal equations and properties for the two-asset model and then, in section 8, display the general equations and results for the m-asset case.

Define

$w_1(t) \equiv w(t)$ = proportion invested in the risky asset

$w_2(t) = 1 - w(t)$ = proportion invested in the sure asset

$g_1(t) = g(t)$ = return on the risky asset (Var $g_1 > 0$)

$g_2(t) = r$ = return on the sure asset (Var $g_2 = 0$)

Then, for $g(t)h = (a - \sigma^2/2)\ h + \triangle Y$, equations (5), (6), (7), and (8′) can be written as,

$$W(t) - W(t_0) = [w(t_0)(e^{(a - \sigma^2/2(h + \triangle Y)} - 1) + (1 - w(t_0))(e^{rh} - 1)] \cdot (W(t_0) - C(t_0)h - C(t_0)h. \qquad (9)$$

$$E(t_0)\ [W(t) - W(t_0)] = \left\{ [w(t_0)(a - r) + r]W(t_0) - C(t_0) \right\} h + 0(h^2). \qquad (10)$$

$$E(t_0)\ [(W(t) - W(t_0))^2] = w^2(t_0)\ W^2(t_0)\ E(t_0)[(\triangle Y)^2] + 0(h^2) = w^2(t_0)W^2(t_0)\sigma^2 h + 0(h^2). \qquad (11)$$

$$dW = [(w(t)(a - r) + r)W(t) - C(t)]dt + w(t)\sigma Z(t)W(t)\sqrt{dt}. \qquad (12)$$

$$\overset{\circ}{W}(t) = [w(t)(a - r) + r]W(t) - C(t). \qquad (13)$$

The problem of choosing optimal portfolio selection and consumption rules is formulated as follows,

$$\operatorname{Max} E \left\{ \int_0^t e^{-\rho t} U[C(t)]\, dt + B[W(T),T] \right\} \tag{14}$$

subject to: the budget constraint (12),

$C(t) \geqq 0;\ W(t) > 0;\ W(0) = W_0 > 0$

and where $U(C)$ is assumed to be a strictly concave utility function (i.e., $U'(C) > 0$; $U''(C) < 0$); where $g(t)$ is a random variable generated by the previously described Wiener process. $B[W(T),T]$ is to be a specified "bequest valuation function" (also referred to in production growth models as the "scrap function," and usually assumed to be concave in $W(T)$). "E" in (14) is short for $E(0)$, the conditional expectation operator, given $W(0) = W_0$ as known.

To derive the optimality equations, I restate (14) in a dynamic programming form so that the Bellman principle of optimality [2] can be applied. To do this, define,

$$I[W(t),t] \equiv \operatorname*{Max}_{\{C(s),w(s)\}} E(t) \left[\int_t^T e^{-\rho s} U[C(s)]ds + B[W(T),T] \right] \tag{15}$$

where (15) is subject to the same constraints as (14). Therefore,

$$I[W(T),T] = B[W(T),T]. \tag{15'}$$

In general, from definition (15),

$$I[W(t_0),t_0] = \operatorname*{Max}_{\{C(s),w(s)\}} E(t_0) \left[\int_{t_0}^t e^{-\rho s} U[C(s)]ds + I[W(t),t] \right] \tag{16}$$

and, in particular, (14) can be rewritten as

$$I(W_0,0) = \operatorname*{Max}_{\{C(s),w(s)\}} E[\int_0^t e^{-\rho s} U[C(s)]ds + I[W(t),t]]. \tag{14'}$$

If $t \equiv t_0 + h$ and the third partial derivatives of $I[W(t_0),t_0]$ are bounded, then by Taylor's theorem and the mean value theorem for integrals, (16) can be rewritten as

$$\begin{aligned} I[W(t_0,t_0] = \operatorname*{Max}_{\{C,w\}} E(t_0) \Big\{ & e^{-\rho \bar{t}} U[C(t)] + I[W(t_0),t_0] + \frac{\partial I[W(t_0),t_0]}{\partial t} \\ & + \frac{\partial I[W(t_0),t_0]}{\partial W} [W(t) - W(t_0)] + \frac{1}{2} \frac{\partial^2 I[W(t_0),t_0]}{\partial W^2} \cdot \\ & [W(t) - W(t_0)]^2 + 0(h^2) \Big\} \end{aligned}$$

where $\bar{t} \in [t_0,t]$. (17)

In (17), take the $E(t_0)$ operator onto each term and, noting that $I[W(t_0),t_0] = E(t_0) I[W(t_0),t_0]$, subtract $I[W(t_0)t_0]$ from both sides. Substitute from equations (10) and (11) for $E(t_0)[W(t) - W(t_0)]$ and $E(t_0)[(W(t) - W(t_0))^2]$, and then divide the equation by h. Take the limit of the resultant equation as $h \to 0$ and (17) becomes a continuous-time version of the Bellman-Dreyfus fundamental equation of optimality, (17′).

$$\begin{aligned} 0 = \operatorname*{Max}_{\{C(t),w(t)\}} \Big[& e^{-\rho t} U[C(t)] + \frac{\partial I_t}{\partial t} \\ & + \frac{\partial I_t}{\partial W} [(w(t)(\alpha - r) + r)W(t) - C(t)] \\ & + 1/2 \frac{\partial^2 I_t}{\partial W^2} \sigma^2 w^2(t) W^2(t)] \Big] \end{aligned} \tag{17'}$$

where I_t is short for $I[W(t),t]$ and the subscript on t_0 has been dropped to reflect that (17′) holds for any $t \in [0,T]$.

If we define $\phi(w,C;W;t) \equiv \left\{ e^{-\rho t} U(C) + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} [(w(t)(\alpha - r) + r)W(t) - C(t)] + 1/2 \frac{\partial^2 I_t}{\partial W^2} \sigma^2 w^2(t) W^2(t) \right\}$, [3] then (17′) can be written in the more compact form,

$$\operatorname*{Max}_{\{C,w\}} \phi(w,C;W,t) = 0. \tag{17''}$$

The first-order conditions for a regular interior maximum to (17″) are,

$$\phi_C[w^*,C^*;W,:t] = 0 = e^{-\rho t} U'(C) - \partial I_t/\partial W \tag{18}$$

and

$$\phi_w[w^*;C^*;W;t] = 0 = (\alpha - r) \frac{\partial I_t}{\partial W} + \frac{\partial^2 I_t}{\partial W^2} wW\sigma^2. \tag{19}$$

A set of sufficient conditions for a regular interior maximum is

$$\phi_{ww} < 0;\ \phi_{cc} < 0;\ \det \begin{bmatrix} \phi_{ww} & \phi_{wC} \\ \phi_{Cw} & \phi_{CC} \end{bmatrix} > 0.$$

$\phi_{wC} = \phi_{Cw} = 0$, and if $I[W(t),t]$ were strictly concave in W, then

$$\phi_{CC} = U''(c) < 0, \text{ by the strict concavity of } U \tag{20}$$

and

[2] The basic derivation of the optimality equations in this section follows that of S. E. Dreyfus [2], Chapter VII.

[3] $\phi(w,C:W:t)$ is short for the rigorous $\phi[w,C;\ \partial I_t/\partial t;\ \partial I_t/\partial W;\ \partial^2 I_t/\partial W^2;\ I_t:W;t]$.

$$\phi_{ww} = W(t)\sigma^2 \frac{\partial^2 I_t}{\partial W^2} < 0^4, \text{ by the strict concavity of } I_t, \quad (21)$$

and the sufficient conditions would be satisfied. Thus a candidate for an optimal solution which causes $I[W(t),t]$ to be strictly concave will be any solution of the conditions (17′)–(21).

The optimality conditions can be re-written as a set of two algebraic and one partial differential equation to be solved for $w^*(t)$, $C^*(t)$, and $I[W(t),t]$.

$$(*)\left\{\begin{array}{lr} \phi[w^*,C^*;W;t] = 0 & (17'') \\ \phi_C[w^*;C^*;W;t] = 0 & (18) \\ \phi_w[w^*,C^*;W;t] = 0 & (19) \\ \text{subject to the boundary condition} & \\ I[W(T),T] = B[W(T),T] \text{ and} & \\ \text{the solution being a feasible solu-} & \\ \text{tion to (14).} & \end{array}\right\}$$

IV Constant Relative Risk Aversion

The system (*) of a nonlinear partial differential equation coupled with two algebraic equations is difficult to solve in general. However, if the utility function is assumed to be of the form yielding constant relative risk-aversion (i.e., iso-elastic marginal utility), then (*) can be solved explicitly. Therefore, let $U(C) = C^\gamma/\gamma$, $\gamma < 1$ and $\gamma \neq 0$ or $U(C) = \log C$ (the limiting form for $\gamma = 0$) where $-U''(C)\,C/U'(C) = 1 - \gamma \equiv \delta$ is Pratt's [7] measure of relative risk aversion. Then, system (*) can be written in this particular case as

$$(*')\left\{\begin{array}{lr} 0 = \left[\dfrac{(1-\gamma)}{\gamma}\dfrac{\partial I_t}{\partial W}\right]^{\gamma/\gamma-1} e^{-\rho t/1-\gamma} & \\ \quad + \dfrac{\partial I_t}{\partial t} + \dfrac{\partial I_t}{\partial W} rW & \\ \quad - \dfrac{(a-r)^2}{2\sigma^2}\dfrac{[\partial I_t/\partial W]^2}{\partial^2 I_t/\partial W^2} & (17'') \\ C^*(t) = \left[e^{\rho t}\dfrac{\partial I_t}{\partial W}\right]^{1/\gamma-1} & (18) \\ w^*(t) = \dfrac{-(a-r)\,\partial I_t/\partial W}{\sigma^2 W\,\partial^2 I_t/\partial W^2} & (19) \\ \text{subject to } I[W(T),T] = \epsilon^{1-\gamma} e^{-\rho T} & \\ [W(T)]^\gamma/\gamma, \text{ for } 0 < \epsilon << 1 & \end{array}\right\}$$

where a strategically-simplifying assumption has been made as to the particular form of the bequest valuation function, $B[W(T),T]$.[5]

To solve (17″) of (*′), take as a trial solution,

$$\bar{I}_t[W(t),t] = \frac{b(t)}{\gamma} e^{-\rho t}\,[W(t)]^\gamma. \quad (22)$$

By substitution of the trial solution into (17″), a necessary condition that $\bar{I}_t[W(t),t]$ be a solution to (17″) is found to be that $b(t)$ must satisfy the following ordinary differential equation,

$$\dot{b}(t) = \mu b(t) - (1-\gamma)\,[b(t)]^{-\gamma/1-\gamma} \quad (23)$$

subject to $b(T) = \epsilon^{1-\gamma}$, and where $\mu \equiv \rho - \gamma\,[(a-r)^2/2\sigma^2(1-\gamma) + r]$. The resulting decision rules for consumption and portfolio selection, $C^*(t)$ and $w^*(t)$, are from equations (18) and (19) of (*′), then

$$C^*(t) = [b(t)]^{1/\gamma-1}\,W(t) \quad (24)$$

and

$$w^*(t) = \frac{(a-r)}{\sigma^2(1-\gamma)}. \quad (25)$$

The solution to (23) is

$$b(t) = \{[1 + (\nu\epsilon - 1)\,e^{\nu(t-T)}]/\nu\}^{1-\gamma} \quad (26)$$

where $\nu \equiv \mu/(1-\gamma)$.

A sufficient condition for $I[W(t),t]$ to be a solution to (*′) is that $I[W(t),t]$ satisfy

A. $\bar{I}[W(t),t]$ be real (feasibility)

B. $\dfrac{\partial^2 \bar{I}_t}{\partial W^2} < 0$ (concavity for a maximum)

C. $C^*(t) \geqq 0$ (feasibility)

The condition that A, B, and C are satisfied in the iso-elastic case is that

$$[1 + (\nu\epsilon - 1)\,e^{\nu(t-T)}]/\nu > 0, \; 0 \leqq t \leqq T \quad (27)$$

which is satisfied for all values of ν when $T < \infty$.

Because (27) holds, the optimal consumption and portfolio selection rules are,[6]

[4] By the substitution of the results of (18) into (19) at (C^*,w^*), we have the condition $w^*(t)(a-r) > 0$ if and only if $\dfrac{\partial^2 I_t}{\partial W^2} < 0$.

The paper considers only interior optimal solutions. The problem could have been formulated in the more general Kuhn-Tucker form in which case the equalities of (18) and (19) would be replaced with inequalities.

[5] The form of the bequest valuation function (the boundary condition), as is usual for partial differential equations, can cause major changes in the solution to (*). The particular form of the function chosen in (*′) is used as a proxy for the "no-bequest" condition ($\epsilon = 0$). A slightly more general form which can be used without altering the resulting solution substantively is $B[W(T),T] = e^{-\rho t}G(T)\,[W(T)]^\gamma/\gamma$ for arbitrary $G(T)$. If B is not of the iso-elastic family, systematic effects of age will appear in the optimal decision-making.

[6] Although not derived explicitly here, the special case ($\gamma = 0$) of Bernoulli logarithmic utility has (29) with $\gamma = 0$ as a solution, and the limiting form of (28), namely

$$C^*(t) = \left[\frac{\rho}{1 + (\rho\epsilon - 1)e^{\rho(t-T)}}\right] W(t).$$

$$C^*(t) = [\nu/(1 + (\nu\epsilon - 1)\, e^{\nu(t-T)})]\, W(t), \quad \text{for } \nu \neq 0$$
$$= [1/(T - t + \epsilon)]\, W(t), \quad \text{for } \nu = 0 \tag{28}$$

and

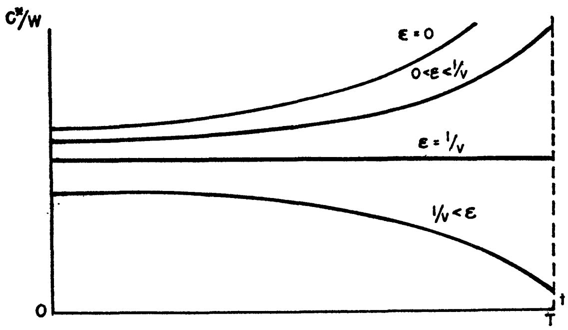
$$w^*(t) = \frac{(\alpha - r)}{\sigma^2(1-\gamma)} \equiv w^*, \text{ a constant independent of } W \text{ or } t. \tag{29}$$

V Dynamic Behavior and the Bequest Valuation Function

The purpose behind the choice of the particular bequest valuation function in (*′) was primarily mathematical. The economic motive is that the "true" function for no bequests is $B[W(T),T] = 0$ (i.e., $\epsilon = 0$). From (28), $C^*(t)$ will have a pole at $t = T$ when $\epsilon = 0$. So, to examine the dynamic behavior of $C^*(t)$ and to determine whether the pole is a mathematical "error" or an implicit part of the economic requirements of the problem, the parameter ϵ was introduced.

From figure 1, $(C^*/W)_{t=T} \to \infty$ as $\epsilon \to 0$. However, one must not interpret this as an infinite

FIGURE 1. —



rate of consumption. Because there is zero utility associated with positive wealth for $t > T$, the mathematics reflects this by requiring the optimal solution to drive $W(t) \to 0$ as $t \to T$. Because C^* is a flow and $W(t)$ is a stock and, from (28), C^* is proportional to $W(t)$, (C^*/W) must become larger and larger as $t \to T$ to make $W(T) = 0$.[7] In fact, if $W(T-) > 0$, an "impulse" of consumption would be required to make $W(T) = 0$. Thus, equation (28) is valid for $\epsilon = 0$.

[7] The problem described is essentially one of exponential decay. If $W(t) = W_0 e^{-f(t)}$, $f(t) > 0$, finite for all t, and $W_0 > 0$, then it will take an infinite length of time for $W(t) = 0$. However, if $f(t) \to \infty$ as $t \to T$, then $W(t) \to 0$ as $t \to T$.

To examine some of the dynamic properties of $C^*(t)$, let $\epsilon = 0$, and define $V(t) \equiv [C^*(t)/W(t)]$, the instantaneous marginal (in this case, also average) propensity to consume out of wealth. Then, from (28),

$$\dot{V}(t) = [V(t)]^2\, e^{\nu(t-T)} \tag{30}$$

and, as observed in figure 1 (for $\epsilon = 0$), $V(t)$ is an increasing function of time. In a generalization of the half-life calculation of radioactive decay, define τ as that $t \epsilon\, [0,T]$ such that $V(\tau) = nV(0)$ (i.e., τ is the length of time required for $V(t)$ to grow to n times its initial size). Then, from (28),

$$\tau = \log\left[e^{\nu T}\left(1 - \frac{1}{n}\right) + \frac{1}{n} \right]\Big/\nu; \quad \text{for } \nu \neq 0$$
$$= \frac{(n-1)}{(n)}\, T \quad , \text{ for } \nu = 0. \tag{31}$$

To examine the dynamic behavior of $W(t)$ under the optimal decision rules, it only makes sense to discuss the expected or "averaged" behavior because $W(t)$ is a function of a random variable. To do this, we consider equation (13), the averaged budget equation, and evaluate it at the optimal (w^*,C^*) to form

$$\frac{\overset{\circ}{W}(t)}{W(t)} = \alpha_* - V(t) \tag{13′}$$

where $\alpha_* = \left[\dfrac{(\alpha - r)^2}{\sigma^2(1-\gamma)} + r \right]$, and, in section VII, α_* will be shown to be the expected return on the optimal portfolio.

By differentiating (13′) and using (30), we get

$$\frac{d}{dt}\left[\frac{\overset{\circ}{W}}{W}\right] = -\dot{V}(t) < 0 \tag{32}$$

which implies that for all finite-horizon optimal paths, the expected rate of growth of wealth is a diminishing function of time. Therefore, if $\alpha_* < V(0)$, the individual will dis-invest (i.e., he will *plan* to consume more than his expected income, $\alpha_* W(t)$). If $\alpha_* > V(0)$, he will plan to increase his wealth for $0 < t < \bar{t}$, and then, dis-invest at an expected rate $\alpha_* < V(t)$ for $\bar{t} < t < T$ where $\bar{t}$ is defined as the solution to

$$\bar{t} = T + \frac{1}{\nu}\log\left[\frac{\alpha_* - \nu}{\alpha_*}\right]. \tag{33}$$

Further, $\partial\bar{t}/\partial\alpha_* > 0$ which implies that the length of time for which the individual is a net saver increases with increasing expected returns on the portfolio. Thus, in the case $\alpha_* > V(0)$, we find the familiar result of "hump saving." [8]

VI Infinite Time Horizon

Although the infinite time horizon case ($T = \infty$) yields essentially the same substantive results as in the finite time horizon case, it is worth examining separately because the optimality equations are easier to solve than for finite time. Therefore, for solving more complicated problems of this type, the infinite time horizon problem should be examined first.

The equation of optimality is, from section III,

$$0 = \underset{\{C,w\}}{\text{Max}} \left[e^{-\rho t} U(C) + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} [(w(t)(\alpha - r) + r)W(t) - C(t)] + 1/2 \frac{\partial^2 I_t}{\partial W^2} \sigma^2 w^2(t) W^2(t) \right]. \qquad (17')$$

However (17′) can be greatly simplified by eliminating its explicit time-dependence. Define

$$\begin{aligned} J[W(t),t] &\equiv e^{\rho t} I[W(t),t] \\ &= \underset{\{C,w\}}{\text{Max}}\ E(t) \int_t^\infty e^{-\rho(s-t)} U[C] ds \\ &= \underset{\{C,w\}}{\text{Max}}\ E \int_0^\infty e^{-\rho v} U[C] dv, \end{aligned}$$

independent of explicit time. (34)

Thus, write $J[W(t),t] = J[W]$ to reflect this independence. Substituting $J[W]$, dividing by $e^{-\rho t}$, and dropping all t subscripts, we can rewrite (17′) as,

$$0 = \underset{\{C,w\}}{\text{Max}} \ [U(C) - \rho J + J'(W) \cdot \{(w(t)(\alpha - r) + r)W - C\} + 1/2\, J''(W) \sigma^2 w^2 W^2]. \qquad (35)$$

Note: when (35) is evaluated at the optimum (C^*, w^*), it becomes an *ordinary* differential equation instead of the usual partial differential equation of (17′). For the iso-elastic case, (35) can be written as

$$0 = \frac{(1-\gamma)}{\gamma} [J'(W)]^{-\gamma/1-\gamma} - \rho J(W) - \frac{(\alpha - r)^2}{2\sigma^2} \frac{[J'(W)]^2}{J''(W)} + rWJ'(W) \qquad (36)$$

where the functional equations for C^* and w^* have been substituted in equation (36).

The first-order conditions corresponding to (18) and (19) are

$$0 = U'(C) - J'(W) \qquad (37)$$

and

$$0 = (\alpha - r) J'(W) + J'' w W \sigma^2 \qquad (38)$$

and assuming that $\lim_{T\to\infty} B[W(T),T] = 0$, the boundary condition becomes the transversality condition,

$$\lim_{t\to\infty} E[I[W(t),t]] = 0 \qquad (39)$$

or

$$\lim_{t\to\infty} E[e^{-\rho t} J[W(t)]] = 0$$

which is a condition for convergence of the integral in (14). A solution to (14) must satisfy (39) plus conditions A, B, and C of section IV. Conditions A, B, and C will be satisfied in the iso-elastic case if

$$V^* \equiv \nu = \frac{\rho}{1-\gamma} - \gamma \left[\frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma} \right] > 0 \qquad (40)$$

holds where (40) is the limit of condition (27) in section IV, as $T \to \infty$ and $V^* = C^*(t)/W(t)$ when $T = \infty$. Condition (39) will hold if $\rho > \gamma \overset{\circ}{W}/W$ where, as defined in (13), $\overset{\circ}{W}(t)$ is the stochastic time derivative of $W(t)$ and $\overset{\circ}{W}(t)/W(t)$ is the "expected" net growth of wealth after allowing for consumption. That (39) is satisfied can be rewritten as a condition on the subjective rate of time preference, ρ, as follows:

for $\gamma < 0$	(bounded utility),	$\rho > 0$
$\gamma = 0$	(Bernoulli log case),	$\rho > 0$
$0 < \gamma < 1$	(unbounded utility),	$\rho > \gamma \left[\frac{(\alpha - r)^2 (2-\gamma)}{2\sigma^2(1-\gamma)} + r \right].$

(41)

Condition (41) is a generalization of the usual assumption required in deterministic optimal consumption growth models when the production function is linear: namely, that $\rho > \text{Max}\,[0, \gamma\beta]$ where β = yield on capital.[9] If a "di-

[8] "Hump saving" has been widely discussed in the literature. (See J. De V. Graaff [3] for such a discussion.) Usually "hump saving" is discussed in the context of work and retirement periods. Clearly, such a phenomenon can occur without these assumptions as the example in this paper shows.

[9] If one takes the limit as $\sigma_*^2 \to 0$ (where σ_*^2 is the variance of the composite portfolio) of condition (41), then

minishing-returns," strictly-concave "production" function for wealth were introduced, then a positive ρ would suffice.

If condition (41) is satisfied, then condition (40) is satisfied. Therefore, if it is assumed that ρ satisfies (41), then the rest of the derivation is the same as for the finite horizon case and the optimal decision rules are,

$$C_\infty^*(t) = \left\{ \frac{\rho}{1-\gamma} - \gamma \left[\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma} \right] \right\} W(t) \tag{42}$$

and

$$w_\infty^*(t) = \frac{(\alpha-r)}{\sigma^2(1-\gamma)}. \tag{43}$$

The ordinary differential equation (35), $J'' = f(J,J')$, has "extraneous" solutions other than the one that generates (42) and (43). However, these solutions are ruled out by the transversality condition, (39), and conditions A, B, and C of section IV. As was expected, $\lim_{T\to\infty} C^*(t) = C_\infty^*(t)$ and $\lim_{T\to\infty} w^*(t) = w_\infty^*(t)$.

The main purpose of this section was to show that the partial differential equation (17′) can be reduced in the case of infinite time horizon to an ordinary differential equation.

VII Economic Interpretation of the Optimal Decision Rules for Portfolio Selection and Consumption

An important result is the confirmation of the theorem proved by Samuelson [8], for the discrete-time case, stating that, for iso-elastic marginal utility, the portfolio-selection decision is independent of the consumption decision. Further, for the special case of Bernoulli logarithmic utility ($\gamma = 0$), the separation goes both ways, i.e., the consumption decision is independent of the financial parameters and is only dependent upon the level of wealth. This is a result of two assumptions: (1) constant relative risk-aversion (iso-elastic marginal utility) which implies that one's attitude toward financial risk is independent of one's wealth level, and (2) the stochastic process which generates the price changes (independent increments assumption of the Wiener process). With these two assumptions, the only feedbacks of the system, the price change and the resulting level of wealth, have zero relevance for the portfolio decision and hence, it is constant.

The optimal proportion in the risky asset,[10] w^*, can be rewritten in terms of Pratt's relative risk-aversion measure, δ, as

$$w^* = \frac{(\alpha-r)}{\sigma^2\delta}. \tag{29′}$$

The qualitative results that $\partial w^*/\partial\alpha > 0$, $\partial w^*/\partial r < 0$, $\partial w^*/\partial\sigma^2 < 0$, and $\partial w^*/\partial\delta < 0$ are intuitively clear and need no discussion. However, because the optimal portfolio selection rule is constant, one can define the optimum composite portfolio and it will have a constant mean and variance. Namely,

$$\alpha_* = E[w^*(\alpha+\triangle Y) + (1-w^*)r] = w^*\alpha + (1-w^*)r = \frac{(\alpha-r)^2}{\sigma^2\delta} + r \tag{44}$$

$$\sigma_*^2 = \mathrm{Var}\,[w^*(\alpha+\triangle Y) + (1-w^*)r] = w^{*2}\sigma^2 = \frac{(\alpha-r)^2}{\sigma^2\delta^2}. \tag{45}$$

After having determined the optimal w^*, one can now think of the original problem as having been reduced to a simple Phelps-Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of an (composite) asset.

Thus, the problem becomes a continuous-time analog of the one examined by Phelps [6] in discrete time. Therefore, for consistency, $C_\infty^*(t)$ should be expressible in terms of α_*, σ_*^2, δ, ρ, and $W(t)$ only. To show that this is, in fact, the result, (42) can be rewritten as,[11]

(41) becomes the condition that $\rho > \max[0,\gamma\alpha^*]$ where α^* is the yield on the composite portfolio. Thus, the deterministic case is the limiting form of (41).

[10] Note: no restriction on borrowing or going short was imposed on the problem, and therefore, w^* can be greater than one or less than zero. Thus, if $\alpha < r$, the risk-averter will short some of the risky asset, and if $\alpha > r + \sigma^2\delta$, he will borrow funds to invest in the risky asset. If one wished to restrict $w^*\epsilon[0,1]$, then such a constraint could be introduced and handled by the usual Kuhn-Tucker methods with resulting inequalities.

[11] Because this section is concerned with the qualitative changes in the solution with respect to shifts in the parameters, the more-simple form of the infinite-time horizon case is examined. The essential difference between $C_\infty^*(t)$ and $C^*(t)$ is the explicit time dependence of $C^*(t)$ which was discussed in section V. For simplicity, the "∞" on subscript $C_\infty^*(t)$ will be deleted for the rest of this section.

$$C^*(t) = \left[\frac{\rho}{\delta} + (\delta-1)\left(\frac{\alpha_*}{\delta} - \frac{\sigma_*^2}{2}\right)\right]W(t) = V\,W(t) \tag{46}$$

where V = the marginal propensity to consume out of wealth.

The tools of comparative statics are used to examine the effect of shifts in the mean and variance on consumption behavior in this model. The comparison is between two economies with different investment opportunities, but with the individuals in both economies having the same utility function.

If θ is a financial parameter, then define $\left[\frac{\partial C^*}{\partial \theta}\right]_{\bar{I}_0}$, the partial derivative of consumption with respect to θ, $I_0[W_0]$ being held fixed, as the intertemporal generalization of the Hicks-Slutsky "substitution" effect, $\left[\frac{\partial C^*}{\partial \theta}\right]_{\bar{U}}$ for static models. $[\partial C^*/\partial\theta - (\partial C^*/\partial\theta)_{\bar{I}_0}]$ will be defined as the intertemporal "income" or "wealth" effect. Then, from equation (22) with I_0 held fixed, one derives by total differentiation,

$$0 = -\frac{1}{\delta-1}\frac{\partial b(0)}{\partial\theta}W_0 + b(0)\left(\frac{\partial W_0}{\partial\theta}\right)_{\bar{I}_0}. \tag{47}$$

From equations (24) and (46), $b(0) = V^{-\delta}$, and so solving for $(\partial W_0/\partial\theta)_{\bar{I}_0}$ in (47), we can write it as

$$\left(\frac{\partial W_0}{\partial\theta}\right)_{\bar{I}_0} = \frac{-\delta W_0}{(\delta-1)V}\frac{\partial V}{\partial\theta}. \tag{48}$$

Consider the case where $\theta = \alpha_*$, then from (46),

$$\frac{\partial V}{\partial\alpha_*} = \frac{(\delta-1)}{\delta} \tag{49}$$

and from (48),

$$\left(\frac{\partial C^*}{\partial\alpha_*}\right)_{\bar{I}_0} = -\frac{W_0}{V}. \tag{50}$$

Thus, we can derive the substitution effect of an increase in the mean of the composite portfolio as follows,

$$\left(\frac{\partial C^*}{\partial\alpha_*}\right)_{\bar{I}_0} = \left[\frac{\partial V}{\partial\alpha_*}W_0 + V\frac{\partial W_0}{\partial\alpha_*}\right]_{\bar{I}_0} = -\frac{W_0}{\delta} < 0. \tag{51}$$

Because $\partial C^*/\partial\alpha_* = (\partial V/\partial\alpha_*)W_0 = [(\delta-1)/\delta]W_0$, then the income or wealth effect is

$$\left[\frac{\partial C^*}{\partial\alpha_*} - \left(\frac{\partial C^*}{\partial\alpha_*}\right)_{\bar{I}_0}\right] = W_0 > 0. \tag{52}$$

Therefore, by combining the effects of (51) and (52), one can see that individuals with low relative risk-aversion ($0 < \delta < 1$) will choose to consume less now and save more to take advantage of the higher yield available (i.e., the substitution effect dominates the income effect). For high risk-averters ($\delta > 1$), the reverse is true and the income effect dominates the substitution effect. In the borderline case of Bernoulli logarithmic utility ($\delta = 1$), the income and substitution effect just offset one another.[12]

In a similar fashion, consider the case of $\theta = -\sigma_*^2$,[13] then from, (46) and (48), we derive

$$\left(\frac{\partial W_0}{\partial(-\sigma_*^2)}\right)_{\bar{I}_0} = \frac{-\delta W_0}{2V} \tag{53}$$

and

$$\left(\frac{\partial C^*}{\partial(-\sigma_*^2)}\right)_{\bar{I}_0} = \frac{-W_0}{2} < 0, \text{ the substitution effect.} \tag{54}$$

Further, $\partial C^*/\partial(-\sigma_*^2) = (\delta-1)W_0/2$, and so

$$\left[\frac{\partial C^*}{\partial(-\sigma_*^2)} - \left(\frac{\partial C^*}{\partial(-\sigma_*^2)}\right)_{\bar{I}_0}\right] = \frac{\delta}{2}W_0 > 0, \text{ the income effect.} \tag{55}$$

To compare the relative effect on consumption behavior of an upward shift in the mean versus a downward shift in variance, we examine the elasticities. Define the elasticity of consumption with respect to the mean as

$$E_1 \equiv \alpha_*\frac{\partial C^*}{\partial\alpha_*}/C^* = \alpha_*(\delta-1)/\delta V \tag{56}$$

and similarly, the elasticity of consumption with respect to the variance as,

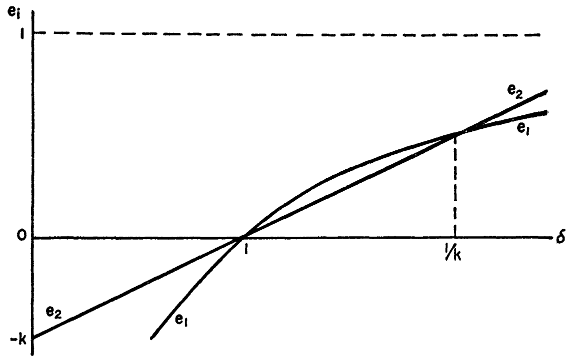
$$E_2 \equiv \sigma_*^2\frac{\partial C^*}{\partial\sigma_*^2}/C^* = -\sigma_*^2(\delta-1)/2V \tag{57}$$

For graphical simplicity, we plot $e_1 \equiv [VE_1/\alpha_*]$ and $e_2 \equiv -[VE_2/\alpha_*]$ and define $k \equiv$

[12] Many writers have independently discovered that Bernoulli utility is a borderline case in various comparative-static situations. See, for example, Phelps [6] and Arrow [1].

[13] Because increased variance for a fixed mean usually (always for normal variates) decreases the desirability of investment for the risk-averter, it provides a more symmetric discussion to consider the effect of a decrease in variance.

FIGURE 2. —



$\sigma_*^2/2\alpha_*$. e_1 and e_2 are equal at $\delta = 1, 1/k$. The particular case drawn is for $k < 1$.

For relatively high variance ($k > 1$), the high risk averter ($\delta > 1$) will always increase present consumption more with a decrease in variance than for the same percentage increase in mean. Because a high risk-averter prefers a steadier flow of consumption at a lower level than a more erratic flow at a higher level, it makes sense that a decrease in variance would have a greater effect than an increase in mean. On the other hand, for relatively low variance ($k < 1$), a low risk averter ($0 < \delta < 1$) will always decrease his present consumption more with an increase in the mean than for the same percentage decrease in variance because such an individual (although a risk-averter) will prefer to accept a more erratic flow of consumption in return for a higher level of consumption. Of course, these qualitative results will vary depending upon the size of k. If the riskiness of the returns is very small (i.e., $k << 1$), then the high risk-averter will increase his present consumption more with an upward shift in mean. Similarly, if the risk-level is very high (i.e., $k >> 1$) the low risk averter will change his consumption more with decreases in variance.

The results of this analysis can be summed up as follows: Because all individuals in this model are risk-averters, when risk is a dominant factor ($k >> 1$), a decrease in risk will have the larger effect on their consumption decisions. When risk is unimportant (i.e., $k << 1$), they all react stronger to an increase in the mean yield. For all degrees of relative riskiness, the low risk-averter will give up some present consumption to attain an expected higher future consumption while the high risk averter will always choose to increase the amount of present consumption.

VIII Extension to Many Assets

The model presented in section IV, can be extended to the m-asset case with little difficulty. For simplicity, the solution is derived in the infinite time horizon case, but the result is similar for finite time. Assume the m^{th} asset to be the only certain asset with an instantaneous rate of return $\alpha_m = r$.[14] Using the general equations derived in section II, and substituting for $w_m(t) = 1 - \sum_{i=1}^{n} w_i(t)$ where $n \equiv m - 1$, equations (6) and (7) can be written as,

$$E(t_0)\ [W(t) - W(t_0)] = [w'(t_0)(\alpha - \hat{r}) + r]\, W(t_0)h - C(t_0)h + 0(h^2) \tag{6}$$

and

$$E(t_0)\ [(W(t) - W(t_0))^2] = w'(t_0)\,\Omega\, w(t_0)\, W^2(t_0)h + 0(h^2) \tag{7}$$

where

$w'(t_0) \equiv [w_1(t_0), \ldots, w_n(t_0)]$, a n-vector
$\alpha' \equiv [\alpha_1, \ldots, \alpha_n]$
$\hat{r}' \equiv [r, \ldots, r]$ a n-vector
$\Omega \equiv [\sigma_{ij}]$, the $n \times n$ variance-covariance matrix of the risky assets
Ω is symmetric and positive definite.

Then, the general form of (35) for m-assets is, in matrix notation,

$$0 = \max_{\{C,w\}} \left[U(C) - \rho J(W) + J'(W)\,\{[w'(\alpha - \hat{r}) + r]\, W - C\} + \frac{1}{2} J''(W)\, w'\,\Omega\, w W^2\right] \tag{58}$$

and instead of two, there will be m first-order conditions corresponding to a maximization of (35) with respect to $w_1, \ldots, w_n$ and C. The optimal decision rules corresponding to (42) and (43) in the two-asset case, are

$$C_\infty^*(t) = \left\{ \frac{\rho}{1-\gamma} - \gamma \left[\frac{(\alpha - \hat{r})'\Omega^{-1}(\alpha - \hat{r})}{2\,(1-\gamma)^2} + \frac{r}{1-\gamma} \right] \right\} W(t) \tag{59}$$

[14] Clearly, if there were more than one certain asset, the one with the highest rate of return would dominate the others.

and

$$w_\infty^*(t) = \frac{1}{(1-\gamma)}\Omega^{-1}(\alpha - \hat{r}) \tag{60}$$

where $w_\infty^{*\prime}(t) = [w_1^*(t), \ldots, w_n^*(t)]$.

IX Constant Absolute Risk Aversion

System (*) of section III, can be solved explicitly for a second special class of utility functions of the form yielding constant absolute risk-aversion. Let $U(C) = -e^{-\eta C}/\eta$, $\eta > 0$, where $-U''(C)/U'(C) = \eta$ is Pratt's [17] measure of absolute risk-aversion. For convenience, I return to the two-asset case and infinite-time horizon form of system (*) which can be written in this case as,

$$(*'')\quad \begin{cases} 0 = \dfrac{-J'(W)}{\eta} - \rho J(W) + J'(W)rW \\ \quad + \dfrac{J'(W)}{\eta}\log[J'(W)] \\ \quad - \dfrac{(\alpha-r)^2}{2\sigma^2}\dfrac{[J'(W)]^2}{J''(W)} & (17'') \\ C^*(t) = -\dfrac{1}{\eta}\log\ [J'(W)] & (18) \\ w^*(t) = -J'(W)(\alpha-r)/\sigma^2 W J''(W) \\ \text{subject to } \lim\limits_{t\to\infty} E[e^{-\rho t}J(W(t))] = 0 & (19) \end{cases}$$

where $J(W) \equiv e^{\rho t} I[W(t),t]$ as defined in section VI.

To solve (17″) of (*″), take as a trial solution,

$$\bar{J}(W) = \frac{-p}{q}e^{-qW}. \tag{61}$$

By substitution of the trial solution into (17″), a necessary condition that $\bar{J}(W)$ be a solution to (17″) is found to be that p and q must satisfy the following two algebraic equations:

$$q = \eta r \tag{62}$$

and

$$p = e^{\left(\frac{r - \rho - (\alpha-r)^2/2\sigma^2}{r}\right)} \tag{63}$$

The resulting optimal decision rules for portfolio selection and consumption are,

$$C^*(t) = rW(t) + \left[\frac{\rho - r + (\alpha-r)^2/2\sigma^2}{\eta r}\right] \tag{64}$$

and

$$w^*(t) = \frac{(\alpha-r)}{\eta r\sigma^2 W(t)}. \tag{65}$$

Comparing equations (64) and (65) with their counterparts for the constant relative risk-aversion case, (42) and (43), one finds that consumption is no longer a constant proportion of wealth (i.e., marginal propensity to consume does not equal the average propensity) although it is still linear in wealth. Instead of the proportion of wealth invested in the risky asset being constant (i.e., $w^*(t)$ a constant), the total dollar value of wealth invested in the risky asset is kept constant (i.e., $w^*(t)W(t)$ a constant). As one becomes wealthier, the proportion of his wealth invested in the risky asset falls, and asymptotically, as $W \to \infty$, one invests all his wealth in the certain asset and consumes all his (certain) income. Although one can do the same type of comparative statics for this utility function as was done in section VII for the case of constant relative risk-aversion, it will not be done in this paper for the sake of brevity and because I find this special form of the utility function behaviorially less plausible than constant relative risk aversion. It is interesting to note that the substitution effect in this case, $\left[\dfrac{\partial C}{\partial \theta}\right]_{\bar{I}_0}$; is zero except when $r = \theta$.

X Other Extensions of the Model

The requirements for the general class of probability distributions which could be acceptable in this model are,

(1) the stochastic process must be Markovian.

(2) the first two moments of the distribution must be $O(\triangle t)$ and the higher-order moments $o(\triangle t)$ where $o(\cdot)$ is the order symbol meaning "smaller order than."

So, for example, the simple Wiener process postulated in this model could be generalized to include $\alpha_i = \alpha_i(X_1, \ldots, X_m, W, t)$ and $\sigma_i = \sigma_i(X_1, \ldots, X_m, W, t)$, where X_i is the price of the i^{th} asset. In this case, there will be $(m+1)$ state variables and (17′) will be generated from the general Taylor series expansion of $I[X_2], \ldots, X_m, W, t]$ for many variables. A particular example would be if the i^{th} asset is a bond which fluctuates in price for $t < t_i$, but will be called at a fixed price at time $t = t_i$. Then $\alpha_i = \alpha_i(X_i,t)$ and $\sigma_i = \sigma_i(X_i,t) > 0$ when $t < t_i$ and $\sigma_i = 0$ for $t > t_i$.

A more general production function of a neoclassical type could be introduced to replace the simple linear one of this model. Mirrlees [5] has examined this case in the context of a growth model with Harrod-neutral technical progress a random variable. His equations (19) and (20) correspond to my equations (35) and (37) with the obvious proper substitutions for variables.

Thus, the technique employed for this model can be extended to a wide class of economic models. However, because the optimality equations involve a partial differential equation, computational solution of even a slightly generalized model may be quite difficult.

REFERENCES

[1] Arrow, K. J., "Aspects of the Theory of Risk-Bearing," Helsinki, Finland, Yrjö Jahnssonin Säätio, 1965.

[2] Dreyfus, S. E., *Dynamic Programming and the Calculus of Variations* (New York: Academic Press, 1965).

[3] Graaff, J. De V., "Mr. Harrod on Hump Saving," *Economica* (Feb. 1950), 81–90.

[4] Itô, K., "On Stochastic Differential Equations," *Memoirs, American Mathematical Society*, No. 4 (1965), 1–51.

[5] Mirrlees, J. A., "Optimum Accumulation Under Uncertainty," Dec. 1965, unpublished

[6] Phelps, E. S., "The Accumulation of Risky Capital: A Sequential Utility Analysis," *Econometrica*, 30 (1962), 729–743.

[7] Pratt, J., "Risk Aversion in the Small and in the Large," *Econometrica*, 32 (Jan. 1964), 122–136.

[8] Samuelson, P. A., "Lifetime Portfolio Selection by Dynamic Stochastic Programming," this REVIEW, L (Aug. 1969).

[9] Tobin, J., "The Theory of Portfolio Selection," *The Theory of Interest Rates*, F. H. Hahn and F. P. R. Brechling, (ed.) (London: MacMillan Co., 1965).