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Optimal consumption and allocation in variable annuities with Guaranteed Minimum Death Benefits

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ABSTRACT

We determine the optimal allocation of funds between the fixed and variable subaccounts in a variable annuity with a GMDB (Guaranteed Minimum Death Benefit) clause featuring partial withdrawals by using a utility-based approach. The Merton method is applied by assuming that individuals allocate funds optimally in order to maximize the expected utility of lifetime consumption. It also reflects bequest motives by including the recipient's utility in terms of the policyholder's guaranteed death benefits. We derive the optimal transfer choice by the insured, and furthermore price the GMDB through maximizing the discounted expected utility of the policyholders and beneficiaries by investing dynamically in the fixed account and variable fund and withdrawing optimally.

1. Introduction and motivation

Variable annuities are insurance contracts which provide periodic payments to the policyholder, usually after a deferment period. In the United States, the funds within a variable annuity are held in subaccounts which are kept independent from the insurance company's other assets. Their benefits are based on the performance of the underlying bond or equity portfolio.

More than 1.2 trillion US dollars is invested in variable annuities as of early 2007 in the United States. Individuals buy variable annuities for many reasons. They are tax-deferred and protect the policyholder from outliving his assets during retirement. In addition, insurers offer various forms of option-like guarantees which provide protection against the downside risks of the subaccounts. A typical example of an option-like feature is the Guaranteed Minimum Death Benefit (GMDB), which can be viewed as a put option with a random exercise time at the moment of death. This rider helps protect the policy's beneficiary from negative market movements.

Many papers discuss GMDB riders. Milevsky and Posner (2001) apply risk-neutral option pricing methods to value GMDB riders embedded in annuity contracts. Milevsky and Salisbury (2001) notice that when the embedded options are out of the money, policyholders have a real option to lapse their policy and simultaneously repurchase the investment with higher death benefit. They assume that policyholders exercise this option optimally so that the lapse decision can be formulated as an optimal stopping problem. Based on this assumption, they calculate the surrender charge for the lapse to compensate the income loss of the insurance company. This surrender charge is derived by making the policyholder indifferent between keeping and lapsing the policy.

Another important option that is frequently available in these contracts is the option to transfer funds between a variable account and an attached fixed account that promises a guaranteed rate. Many insurance companies sell variable annuities with a fixed or guaranteed subaccount earning a fixed interest rate. The policyholders have two options for the allocation of their funds. One option is to leave the funds in the variable account, where the performance of the account will follow the market fluctuations. The other option would be to move the funds to the fixed account and forgo the market swings.

Ulm (2006) discusses the effect of the real option to transfer funds between fixed and variable accounts. He uses the no-arbitrage pricing methodology and gets the boundary between the

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area where all money is invested in the variable account and the area where all money is invested in the fixed account. He shows analytically that the option to transfer to the fixed fund has no value and will never be used unless the fixed growth rate is smaller than the risk free rate less any asset fees taken off the variable account. If the fixed growth rate is less than this, the value of the option can be calculated and the approximate location of the optimal exercise boundary can be determined.

Some GMDB contracts also contain a feature allowing the policyholder to withdraw from the invested capital at any time prior to the maturity of the contract. Bauer et al. (2008) suggest a general solution to the GMDB with optimal partial withdrawals at discrete time horizons in a Black–Scholes option pricing model. Belanger et al. (2009) develop a pricing model from the issuer's perspective based on partial differential equations to determine the no-arbitrage insurance charge for contracts with a GMDB clause featuring partial withdrawals. They demonstrate that higher fees are required for GMDB contracts with a partial withdrawal option.

In all of the prior work, the authors assume that the variable annuity markets are complete. However, life insurance and variable annuities are not really complete markets, as it is not possible to sell your annuity to a third party and there may be barriers to surrendering it. Prices cannot be determined from no-arbitrage considerations alone. Utility-based models are a theoretically defensible way of treating such markets (see Shreve (2003) p. 70). Leung and Sircar (2009) take this approach to employee stock options which have similar trading restrictions. Milevsky (2001) first applies a utility based model in annuity analysis to choose when and if to annuitize. Charupat and Milevsky (2002) derive the optimal utility maximizing asset allocation between fixed and variable subaccounts. They do not model the guarantee options in their research. In this paper, we show that the guarantee options really change the insured's allocation decision.

We determine the optimal allocation of funds between the fixed and variable subaccounts in a variable annuity using a utility-based approach. This paper differs from Ulm (2006) in several ways. First, we assume the insureds are risk averse, so partial transfers between variable and fixed accounts could be optimal. More precisely, we apply the Merton (1969) method in this paper by assuming that individuals allocate funds in order to maximize the expected utility of lifetime consumption. In this model, the insured gets utility from consumption and has bequest motives. We include the effect on asset allocation from both savings (accumulation) and dissavings (consumption). We also reflect bequest motives by including the utility of the recipient of the policyholders guaranteed death benefits. We derive the optimal transfer choice by the insured, and furthermore price the GMDB through maximizing the discounted expected utility of the policyholders and beneficiaries by investing dynamically in the fixed account and variable fund and withdrawing optimally.

In addition to GMDB riders, variable annuity contracts frequently contain guarantees on the living benefits as well. In particular, some contracts contain Guaranteed Minimum Withdrawal Benefits (GMWB) which are analyzed extensively in, for example, Bauer et al. (2008), Chen and Forsyth (2008), Chen et al. (2008), Dai et al. (2008), Belanger et al. (2009) and Steinorth and Mitchell (2011). For simplicity, we ignore the possibility of other benefits and instead focus on allocations in the presence of death benefits and bequest motives. An analysis of withdrawal behavior with GMWB riders can be found in Bauer and Moenig (2012).

The remainder of this paper is organized as follows. Section 2 introduces the model and assumptions we use in this paper. Section 3 discusses the numerical method used to solve the optimal allocation proportion in the model and presents some numerical examples. Section 4 prices the GMDB option, and Section 5 concludes the paper with some general remarks.

2. Models

In the model, we treat only return of premium and roll-up benefits. An individual purchases a variable annuity product and makes a lump sum deposit. We restrict ourselves to insurance contracts with GMDB options only. There are two subaccounts in the VA account. One subaccount is a fixed account, which provides a fixed interest return g_t, and the other subaccount is a variable account, which provides a return related to the stock market performance, with guaranteed minimum death benefit, i.e.

$$k_t = a_0 \prod_{i=1}^{t} \left\{ (1 + r_{p_i}) \frac{a_i - c_i}{a_i} \right\}, \tag{1}$$

$$b_t = \max(k_t, a_t - c_t) = \max\left(k_{t-1}(1 + r_{p_t}) \frac{a_t - c_t}{a_t}, a_t - c_t \right), \tag{2}$$

where a_t is the total account value at time t in both the fixed account (F) and the variable account (S), i.e. $a_t = F_t + S_t$; a_0 is the initial wealth; r_{p_t} is the guaranteed rate for GMDB at time t; k_t is the guaranteed payment in the GMDB; b_t is the investment amount with a minimum guaranteed; and $a_0 = b_0 = k_0$; c_t is the withdrawal at the beginning of time t, and the insured consumes c_t immediately. c_t is non-negative which means that deposits are not allowed in our model.

The money in the VA account is partitioned between these two sub-accounts. $dS_t = r_t S_t dt + \sigma_t S_t dB_t$, $dF_t = g_t F_t dt$, where g_t is the risk-free rate and the fixed account grows at a rate g_t; B_t is a standard Brownian motion. We denote by ω_t the percentage of wealth held in the variable subaccount and $1 - \omega_t$ the proportion of wealth allocated in the fixed rate subaccount. The amount of withdrawals is c_t, and it may vary with time:

$$da_t = a_t[\omega_t r_t + (1 - \omega_t) g_t]dt + \omega_t \sigma_t a_t dB_t - c_t. \tag{3}$$

We assume that the insureds have options to transfer money in between fixed and variable accounts. We also assume a constant r in the analysis that follows. Furthermore, to be more realistic, we assume $0 \le \omega_t \le 1$, which means that there are no short sales.

We assume the insured and the beneficiary are risk averse with the same utility function. We set the same risk aversion values for both the insured and the spouse for the following reasons. First, it is often the case that couples are similar people, so their risk aversion values will be similar. In addition, using different risk aversion values would increase the computational complexity of the problem and will not alter our conclusions significantly.

We apply a constant relative risk aversion (CRRA) utility which has the functional form

$$U(c) = \begin{cases} \dfrac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0,\ \gamma \neq 1, \\ ln(x), & \gamma = 1. \end{cases}$$

This utility has some special properties:

- it is a homogeneous function of degree $1 - \gamma$ for $\gamma \neq 1$;
- γ is the coefficient of relative risk aversion; $1/\gamma$ is the intertemporal substitution elasticity between consumption in any two periods, i.e., it measures the willingness to substitute consumption between different periods.

If there is no possibility of death and no partial withdrawals in the accumulation stage and in the absence of a GMDB, the individual maximizes the expected utility at retirement date T. According to Charupat and Milevsky (2002), the objective function is

$$\max_{\omega_t} E\left[\frac{1}{1-\gamma} a_T^{1-\gamma} \right]. \tag{4}$$

The solution to the objective function is equal to

$$\omega^* = \min\left[\frac{r-g}{\gamma\sigma^2}, 1 \right], \tag{5}$$

where r is the risky asset's expected rate of return; σ is the volatility of risky return; g is the risk free asset's rate of return; and γ is the coefficient of relative risk aversion.

During the term of the contract, there are several possible types of events: the insured can

- transfer the funds between these two subaccounts;
- perform a partial surrender;
- completely surrender the contract;
- or pass away.

We incorporate these events into our "without consumption" and "with consumption" models.

2.1. "Without consumption" case

In the first step, let us assume there are no surrenders. If we only consider the insured and beneficiary utility without consumption, we can obtain the following objective function:

$$\max_{\omega_t} E\left[\sum_{t=1}^{T}\beta^{t-1}\left(\prod_{i=1}^{t-1}\phi_i\right)(1-\phi_t)\zeta v_t(b_t) + \beta^T\left(\prod_{i=1}^{T}\phi_i\right)V_{T+1}(a_{T+1})\right]. \tag{6}$$

The insured retires at the end of time T. β is the subjective discount factor. ϕ_t is the survival rate at time t. ζ denotes the strength of the bequest motive and ranges from 0 to 1. If $\zeta = 0$, the insured has no bequest motive and does not want to leave a bequest to his beneficiary; if $\zeta = 1$, the insured has the strongest bequest motive and will treat his beneficiary like himself. V is the policyholder's value function and v is the beneficiary's value function. If the insured dies before retirement, the beneficiary will get the larger of the account value or the guaranteed amount.

A number of papers have examined bequest motives empirically. A good overview is found in Arrondel et al. (1997) and Masson and Pestieau (1997). A quantitative estimate of this effect can be found in, for example, De Nardi (2004) or Ameriks et al. (2011).

Once the beneficiary receives the money, her objective function is

$$\max_{\omega_t^B} E\left[\beta^{T_B-t}\left(\prod_{i=t}^{T_B}\phi_i\right)v_{T_B+1}(a_{T_B+1})\right]. \tag{7}$$

When the insured purchases the VA product, the beneficiary has T_B years until retirement age. If the insured dies at time t, the beneficiary will receive the bequest and has $T_B - t$ years until retirement age. She will optimally allocate the amount between risky and risk-free investments, and periodically consume the amount after her retirement. However, the beneficiary's investment is not protected by the GMDB and we assume she has no bequest motive. If the insured survives until his retirement age, at the end of the policy period, he will get the entire account value without GMDB protection and annuitize it for his retirement life.

We get the Bellman equation for the insured:

$$V_t(a_t, b_t) = (1-\phi_t)\zeta v_B(b_t) + \max_{\omega} \beta\phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) \mid a_t], \tag{8}$$

subject to

$$V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}}\beta^{t-(T+1)}\left(\prod_{i=T+1}^{t-1}\phi_i\right)u(\bar{c}), \tag{9}$$

$$\bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}}\prod_{i=T+1}^{t-1}\phi_i(1+r_f)^{T+1-t}}, \tag{10}$$

$$a_{t+1} = a_t(\omega_t(1+r_{t+1}) + (1-\omega_t)(1+g_t)), \quad 0 \le \omega_t \le 1,$$

$$k_{t+1} = k_t(1+r_{p_t}),$$

$$b_{t+1} = \max\left(a_1\prod_{i=1}^{t}(1+r_{p_i}), a_{t+1}\right), = \max(k_{t+1}, a_{t+1}),$$

where r_t is the expected risky rate of return at time t, r_f is the risk-free rate of return, and $\bar{c}$ is annuity amount after retirement. In our model, the retired insured will receive a lifetime pay-out annuity,[2] and the monthly payout is $\bar{c}$.

Much of the economics literature has documented – and continues to puzzle over – the extremely low levels of voluntary annuitization exhibited among elderly retirees (Milevsky and Young (2002)). Some literature (Kotlikoff and Summers (1981), Bernheim (1991)) argues that individuals do not annuitize wealth because of their bequest motives. In our assumption, we assume there is no bequest motive after the individual retires. Therefore it is reasonable to assume that the insured converts his account value to a lifetime payout annuity.

2.2. "With consumption" case

For simplicity, we assume that the events (the consumption and the allocation) can only occur at discrete times. Therefore, the state variables only change at integer time points $t = 1, 2, \ldots, T$. The consumption amount, $c_t \in [0, a_t]$, is taken out of the two subaccounts in the same ratio as the existing account value and are consumed immediately. The objective function is

$$\max_{\omega_t, c_t} E\left[\sum_{t=1}^{T}\beta^{t-1}\left(\prod_{i=1}^{t-1}\phi_i\right)u(c_t) + \beta^T\left(\prod_{i=1}^{T}\phi_i\right)V_{T+1}(a_{T+1}) + \sum_{t=1}^{T}\zeta\beta^{t-1}\left(\prod_{i=1}^{t-1}\phi_i\right)(1-\phi_t)v_t(b_t)\right]. \tag{11}$$

Once the beneficiary receives the bequest b_t, which is protected by the GMDB, the objective function for the beneficiary is

$$\max_{\omega_t^B, c_t^B} E\left[\sum_{t_B=t}^{T_B}\beta^{t_B-t}\left(\prod_{i=t}^{t_B-1}\phi_i\right)u(c_t^B) + \beta^{T_B-t}\left(\prod_{i=t}^{T_B}\phi_i\right)v_{T_B+1}(a_{T_B+1})\right]. \tag{12}$$

The beneficiary will maximize her own utility by optimally choosing her own consumption c_t^B and investment allocation ω_t^B. As in the "without consumption" case, the beneficiary's investment is not protected by the GMDB and she has no bequest motive.

The derived Bellman equation for the insured is

$$V_t(a_t, b_t) = \max_{\omega_t, c_t}\{u_t(c_t) + \beta\phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) \mid a_t] + \zeta(1-\phi_t)v_t(b_t)\}, \tag{13}$$

subject to

$$V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}}\beta^{t-(T+1)}\left(\prod_{i=T+1}^{t-1}\phi_i\right)u(\bar{c}),$$

$$0 \le \omega_t \le 1,$$

[2] Lifetime payout annuity is an insurance product that converts an accumulated investment into income that the insurance company pays out over the life of the investor (Chen et al. (2006)). Many papers study lifetime payout annuities on pricing of these products, and how much and when to annuitize. The literature includes Yaari (1965); Richard (1975); Milevsky and Young (2002); Brown (2001); Poterba (1997); Brown et al. (1999); Brown and Poterba (2000); Brown and Warshawsky (2004); Kapur and Orszag (1999), and Blake et al. (2003).

$$0 < c_t \le a_t,$$
$$a_{t+1} = (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))(a_t - c_t),$$
$$\bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t}},$$
$$k_{t+1} = k_t(1 + r_{p_t})\frac{a_{t+1} - c_{t+1}}{a_{t+1}},$$
$$b_{t+1} = \max(k_{t+1}, a_{t+1}).$$

Following Hardy (2003), all state variables are denoted as $(\cdot)_{t^-}$, $(\cdot)_{t^+}$, i.e. the value immediately before and after the transactions at the discrete time t, respectively. Withdrawals and consumptions occur at t^-, then at t^+, which is still at time t but after withdrawal, the insured decides the amount to transfer between the fixed and the variable subaccounts. We also assume that the beneficiary gets the bequest immediately at t^+ just after the insured dies at t^+. Therefore, the Bellman equation will have 2 stages: at the 1st stage from t^- to t^+, the insured gets the utility from consumption of withdrawal:

$$V_{t^-}(a_{t^-}, b_{t^-}) = \max_{c_t}\{u(c_t)\} + \{V_{t^+}(a_{t^+}, b_{t^+})\}, \tag{14}$$

$$\Longrightarrow V_{t^-}(a_{t^-}, m_t) = \max_{c_t}\{u(c_t)\} + \{V_{t^+}(a_{t^+}, m_t)\}, \tag{15}$$

where

$$m_t = \frac{a_{t^-}}{b_{t^-}},$$

$$a_{t^+} = a_{t^-} - c_t = \left(1 - \frac{c_t}{a_{t^-}}\right)a_{t^-},$$

and

$$u(c_t) + V_{t^+}(a_{t^+}, m_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \left(1 - \frac{c_t}{a_{t^-}}\right)^{1-\gamma} V_{t^+}(a_{t^-}, m_t).$$

It is easy to see that m_t is the same at t^- and t^+. Because

$$b_{t^+} = \frac{a_{t^+}}{a_{t^-}} b_{t^-} = \left(1 - \frac{c_t}{a_{t^-}}\right) b_{t^-},$$

$$m_{t^+} = \frac{a_{t^+}}{b_{t^+}} = \frac{(1 - \frac{c_t}{a_{t^-}})a_{t^-}}{(1 - \frac{c_t}{a_{t^-}})b_{t^-}} = \frac{a_{t^-}}{b_{t^-}} = m_{t^-}.$$

At the second stage from t^+ to $(t + h)^-$, the insured chooses a proportion ω to invest in the variable account,

$$\begin{aligned} V_{t^+}(a_{t^+}, b_t) &= (1 - \phi_t)\zeta\, v(\max(a_{t+}, b_{t+})) \\ &\quad + \max_{\omega_t}\{\phi_t \beta^h E V_{t+h^-}(a_{t+h^-}, b_{t+h^-})\}, \end{aligned} \tag{16}$$

$$\begin{aligned} \Longrightarrow V_{t^+}(a_{t^+}, m_t) &= (1 - \phi_t)\zeta\, v(\max(a_{t+}, m_t)) \\ &\quad + \max_{\omega_t}\{\phi_t \beta^h E V_{t+h^-}(a_{t+h^-}, m_{t+h^-})\}, \end{aligned} \tag{17}$$

where

$$a_{t+h^-} = (\omega e^{rh} + (1 - \omega)e^{gh})a_{t^+},$$

then we can get

$$\begin{aligned} V_{t^+}(a_{t^+}, m_t) &= (1 - \phi_t)\zeta\, v(\max(a_{t+}, b_{t+})) \\ &\quad + \max_{\omega_t}\left\{\phi_t \beta^h E V_{t+h^-}\left(a_{t+}((\omega e^{rh} + (1 - \omega)e^{gh})),\right.\right. \\ &\quad \left.\left. \frac{a_{t+}(\omega e^{rh} + (1 - \omega)e^{gh})}{b_{t+}e^{r_p h}}\right)\right\}. \end{aligned} \tag{18}$$

3. Numerical methodology

We apply a trinomial lattice model in both "without consumption" and "with consumption" cases.

3.1. "Without consumption" case

Let us first assume that the insured buys the variable annuity with GMDB options in a lump sum at age 35. The insured can transfer money between fixed and variable subaccounts every month. If the insured dies during the month, his beneficiary will receive the bequest. At age 65, the insured retires and annuitizes the variable annuity to support his retirement life.

Based on Hull (1997), we use a trinomial lattice to do the calculation. Assume the ratio between the stock level at one lattice point and the next larger one is $u = e^{\sigma\sqrt{3\Delta t}}$; the reciprocal $d = 1/u$; the mean value in the variable account ($S = \omega a$); the mean of the continuous log-normal distribution $E[S] = \omega a e^{rh}$ (assume $h = \Delta t$), and the variance is $Var[S] = \omega^2 a^2 e^{2rh}[e^{\sigma^2 h} - 1]$; the mean value in the fixed account ($F = (1 - \omega)a$) is $E[F] = (1 - \omega)ae^{gh}$, and variance is $Var[F] = 0$; and the covariance $Cov[F, S] = 0$.

According to Boyle (1988), there are three conditions to apply to the trinomial lattice:

(1) The probabilities sum to one;
(2) The mean of the discrete distribution is equal to the mean of the continuous log-normal distribution;
(3) The variance of the discrete distribution is equal to the variance of the continuous distribution.

The above three conditions are,

$$p_u + p_m + p_d = 1, \tag{19}$$

$$p_u a u + p_m a + p_d a d = a[\omega e^{rh} + (1 - \omega)e^{gh}], \tag{20}$$

$$\begin{aligned} &p_u a^2 u^2 + p_m a^2 + p_d a^2 d^2 - (\omega e^{rh} + (1 - \omega)e^{gh})^2 \\ &\quad = \omega^2 a^2 e^{2rh}[e^{\sigma^2 h} - 1]. \end{aligned} \tag{21}$$

By some algebraic transformation, we can get

$$p_u = \frac{A\omega^2 + B\omega + C}{(u - 1)(u - d)}, \tag{22}$$

$$p_d = \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d - 1} - \frac{A\omega^2 + B\omega + C}{(d - 1)(u - d)}, \tag{23}$$

$$\begin{aligned} p_m &= 1 - \frac{A\omega^2 + B\omega + C}{(u - 1)(u - d)} - \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d - 1} \\ &\quad + \frac{A\omega^2 + B\omega + C}{(d - 1)(u - d)}, \end{aligned} \tag{24}$$

where

$$A = e^{(2r+\sigma^2)h} - 2e^{(r+g)h} + e^{2gh},$$
$$B = (e^{rh} - e^{gh})(2e^{gh} - d - 1),$$
$$C = (e^{gh} - 1)(e^{gh} - d).$$

Since

$$\begin{aligned} V_{t,j}(\omega) &= (1 - \phi_t)\zeta\, v(b_{t,j}) + \beta\phi_t(p_u V_{t+1,j+1} \\ &\quad + p_m V_{t+1,j} + p_d V_{t+1,j-1}), \end{aligned} \tag{25}$$

to maximize $V_{t,j}$, we take the first derivative on ω, and we get the optimal ω: See Eq. (26) in Box I.

By the no-short-selling restriction, we know that $\omega \in [0, 1]$, so we only need to check 3 possible values of ω : 0, $\omega^\star$, 1, and if $\omega^\star < 0$ or > 1, we only need to check 0 or 1.

We assume the insured can adjust his allocation at the beginning of each month, and starting wealth level at time 0 is 1. The algorithm to do the numerical values can be done as follows:

$$\omega^\star = -\frac{(d-1)BV_{t+1,j} - V_{t+1,j-1} + (u-1)(V_{t+1,j} - V_{t+1,j+1})[(u-d)(e^{rh} - e^{gh}) - B]}{2A[(d-1)(V_{t+1,j-1} - V_{t+1,j}) + (u-1)(V_{t+1,j} - V_{t+1,j+1})]}. \tag{26}$$

Box I.

Table 1
Common parameters in the base case.

Strength of Bequest Motive	ζ	0.5
Subjective Discount Rate	β	0.97
Risk Free Rate	r_f	3%
Coefficient of Relative Risk Aversion	γ	2
Growth Rate of Fixed Subaccount	r_g	4%
Expected Return of Risky Asset	r	7%
Volatility of Risky Return	σ	15%
GMDB roll-up rate	r_p	0
Annual Mortality Rate	q	1994 MGDB table

1. Initialize account value at time 1: $a_1 = 1$, and other parameter values;
2. Calculate the jump sizes $u = e^{\sigma\sqrt{3\Delta t}}$, $d = \frac{1}{u}$ and $m = 1$;
3. Build the tree for account value a by using jump sizes until age 65;
4. Set terminal value $V_{T+1}(a_{T+1})$ by using Eqs. (9) and (10);
5. For $t = T$ to 1, at each time period, use backward induction to maximize the insured's utility:
   5.1 Calculate the optimal allocation $\omega_t^\star$ by (26);
   5.2 Calculate the transition probabilities p_u, p_d and p_m by plugging $\omega_t^\star$ into (22)–(24);
   5.3 Derive V_t by (25) until $t = 1$.

Let us first assume the base case as follows (Tables 1 and 2).

We assume uniform distribution of deaths over the year, which is that any given month 1/12 of the people die. Correspondingly, we can get the survival rate $\phi_t = 1 - q_t$.

We will check the changes of allocation by giving some shocks: (1) $r = 0.055$; (2) $\sigma = 0.25$; (3) $\gamma = 2.5$; (4) $r_p = 0.03$; (5) $q =$ 150% of 1994 MGDB; and (6) $\zeta = 0.2$.

Fig. 1 (age 45 allocation under "without consumption" case) shows the amount of money allocated to the variable account at age 45 when the option is at-the-money. An "argument" between the beneficiary and the insured is a helpful way of looking at the results. The insured prefers the allocation determined by Merton (1969) at all times and benefit levels. At all stock-to-strike levels, the beneficiary prefers a more aggressive allocation than the insured, as he is protected against downside risk. This effect is most pronounced when the account is in-the-money, but expected to be at-the-money at the future moment the beneficiary expects to receive it. When the account is significantly out-of-the-money, the downside protection is not very valuable and the beneficiary prefers an allocation near the Merton level. Therefore, there is no argument. When the account is significantly in-the-money, the beneficiary does not have a strong preference as he receives the strike in (nearly) every case. Again, there is no argument as the beneficiary is (nearly) indifferent. It is only near the at-the-money level that the beneficiary has a preference that is both strong and aggressive.

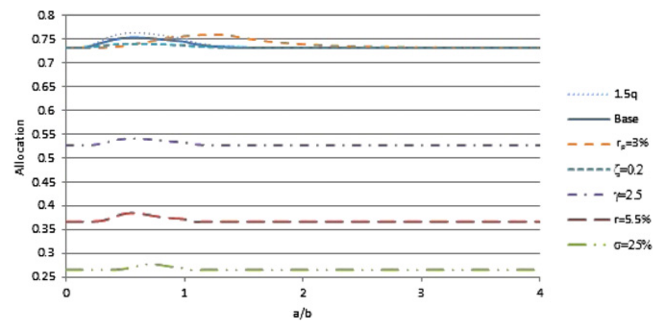

Fig. 1. Age 45 allocation under without consumption case.

The effect of the argument is clearly seen by the bumps in this figure for all parameter levels. Parameter changes primarily affect the level of the insured's preferred allocation rather than the size of the bump.

As the risky rate of return r decreases from 7% to 5.5%, the variable subaccount allocation reduces from 73.1% to around 37%. For $r = 7\%$, as the asset level goes up, the allocation increases from 73.1%, which is a Merton allocation, to 75.3% (goes up 2.2%) around the at-the-money area, and then goes down to Merton allocation again; for $r = 5.5\%$, the allocation increases from 36.7%, to 38.6% (goes up 1.9%), then goes down to 36.7%. As the risky rate of return decreases, the risky subaccount will lose some attraction to both insured and beneficiary.

Table 2
1994 MGDB table.

Age	Mortality rate	Age	Mortality rate	Age	Mortality rate	Age	Mortality rate
35	0.001013	55	0.005543	75	0.046121	95	0.285199
36	0.001037	56	0.006226	76	0.050813	96	0.305931
37	0.001082	57	0.007025	77	0.056327	97	0.325849
38	0.001146	58	0.007916	78	0.062629	98	0.344977
39	0.001225	59	0.008907	79	0.069595	99	0.363757
40	0.001317	60	0.010029	80	0.077114	100	0.382606
41	0.001424	61	0.011312	81	0.085075	101	0.401942
42	0.00154	62	0.012781	82	0.093273	102	0.422569
43	0.001662	63	0.014431	83	0.101578	103	0.445282
44	0.001796	64	0.016241	84	0.110252	104	0.469115
45	0.001952	65	0.018191	85	0.119764	105	0.491923
46	0.002141	66	0.020259	86	0.130583	106	0.51156
47	0.002366	67	0.022398	87	0.143012	107	0.526441
48	0.002618	68	0.024581	88	0.156969	108	0.536732
49	0.002900	69	0.026869	89	0.172199	109	0.543602
50	0.003223	70	0.029363	90	0.188517	110	0.547664
51	0.003598	71	0.032169	91	0.205742	111	0.54954
52	0.004019	72	0.035268	92	0.223978	112	0.55
53	0.004472	73	0.038558	93	0.243533	113	0.55
54	0.004969	74	0.042106	94	0.264171	114	0.55
						115	1

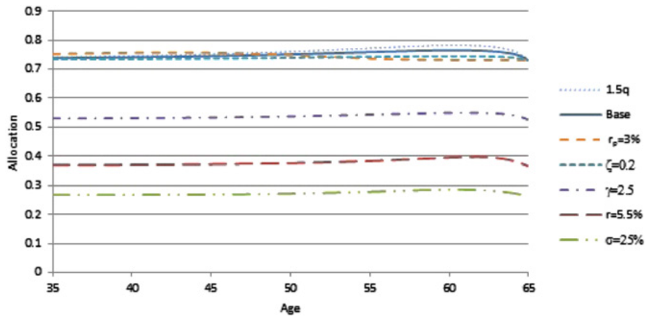

Fig. 2. At-the-money allocation under without consumption case.

As the stock market volatility increases, i.e. σ increases from 15% to 25%, the variable subaccount allocation reduces from 73.1% to around 26.6%. For $\sigma = 25\%$, as the asset level goes up, the allocation increases from 26.6%, which is Merton allocation, to 27.8% (goes up 1.2%) around at-the-money area, and then goes down to Merton allocation again. As the volatility increases, the risky subaccount will not be as attractive to either the insured or the beneficiary. Consistent with Merton (1969) and Charupat and Milevsky (2002), identical Sharpe Ratios produce identical results.

As the insured's risk aversion level increases, i.e. the coefficient of relative risk aversion γ increases from 1.8 to 2.5, the allocation at all asset levels decreases about 27%, because the more risk averse the policyholder is, the more conservative allocation decision they would make. At $\gamma = 2.5$, the allocation increases from 52.7% to 54.1% (goes up 1.4%) around the at-the-money area.

When the roll-up rate r_p increases from 0% to 3%, the bump will move to a higher asset level. At age 45, the highest "argument" point moves from $a = 0.5916$ to 1.25, which is a significant increase. $a = 1.25$ is the new strike level produced by a 3% roll-up compounded over 10 years.

When the mortality rate is increased by 50% at all ages, the variable subaccount allocation will stay the same at almost all asset levels (73.1%), but the maximum allocation increases to 77.3%, which is higher than the base case. With a higher mortality rate, it is more likely that the beneficiary, rather than the policyholder, will consume the assets. Therefore, the "argument" moves in favor of the beneficiary, and the bequest motive makes the insured allocate more aggressively around the at-the-money area.

Decreasing the bequest motive from $\zeta = 0.5$ to 0.2 will keep the allocation level the same at almost all asset levels, but reducing the bequest motive makes the insured care less about beneficiary. The bump level around the at-the-money area is reduced and increases from 73.1% to 74% (i.e. goes up 0.9%).

Fig. 2 plots at-the-money allocation at all ages. It shows that at any parameter level, as the insured gets older, he will first care more about his beneficiary and then, as he continues to age, he will care more about himself. As the policyholder ages, he is more likely to die immediately, but less and less likely to die before retirement age and it is more likely that he, rather than his beneficiary, will consume the assets. As a result, the "argument" first moves in favor of the beneficiary, increasing the amount in the risky asset, and then in favor of the insured, decreasing the amount in the risky asset.

As the risky rate of return r decreases from 7% to 5.5%, the risky account will be less attractive, and as a result the insured puts less money into it. As the stock market volatility increases from 15% to 25%, a risk averse insured will transfer money from risky subaccount to the risk-free account. As the insured's risk aversion level increases from 1.8 to 2.5, the insured is more concerned about the safety of the investment and will allocate less into risky subaccount. The effect of changes in the roll-up rate r_p is somewhat more complicated: Increasing the roll-up rate r_p increases the at-the-money allocation when the policyholder is at a younger age; as the insured ages, the roll-up rate becomes less and less useful to protect the beneficiary, and as a result, the risky asset allocation drops more quickly than the base case, and finally converges to the base case allocation. With the increase in mortality rate q_t, the beneficiary is more likely to inherit the assets, and therefore the at-the-money allocation is more aggressive than the base case. The bequest motive ζ also has an effect, as the bequest motive decreases, the risky account allocation will slightly decrease, which also means the "argument" between the policyholder and the beneficiary decreases at all ages. From the figure, we can observe that changes in roll-up rate, mortality rate and bequest motive will only twist the allocation slightly and the allocation strategies converge as the insured ages; while changes in the risky rate of return, coefficient of relative risk aversion or volatility of stock market cause nearly parallel shifts.

3.2. "With consumption" case

We now include the effect of optimal consumption in our model. The insured buys the variable annuity with GMDB options in a lump sum at age 35. He can transfer and withdraw money every month. The withdrawals from the account take place between t^- to t^+. Taking the first order condition on this first stage, and we obtain:

$$\frac{\partial V_{t^-}(a_{t^-}, m_t)}{\partial c_t} = c_t^{-\gamma} - \frac{(1-\gamma)(1-\frac{c_t}{a_{t^-}})^{-\gamma} V_{t^+}(a_{t^-}, m_t)}{a_{t^-}}. \tag{27}$$

Let $D = \left(\frac{1}{(1-\gamma)V_{t^+}(1, m_t)}\right)^{\frac{1}{\gamma}}$, we can get

$$\frac{c_t}{a_{t^-}} = \frac{D}{1+D}.$$

To match the second stage, we need to modify the expression of D:

$$\begin{aligned} D &= \left(\frac{1}{(1-\gamma)V_{t^+}(1, m_t)}\right)^{\frac{1}{\gamma}} \\ &= \left(\frac{(m_t e^{r_p t})^{1-\gamma}}{(1-\gamma)V_{t^+}(m_t e^{r_p t}, m_t)}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

At the second stage, immediately after the consumption, the insured may be dead and the insurer will pay the GMDB amount to the beneficiary at t^+. If the insured still survives, he will choose allocation to the separate subaccounts. The numerical procedure for this stage is the same as the "without consumption" case. With the partial withdrawal option, one will see the behavior of the insureds change from the previous case. To do the sensitivity tests, we assume the same values of base parameters in "without consumption" case.

3.2.1. Risk aversion sensitivity

The purpose of this sensitivity test is to discover the optimal choices for the policyholder if all but one coefficient, relative risk aversion (γ), were kept constant. Fig. 3 shows that as the risk aversion level increases, the policyholder will invest less money in the variable subaccount. For all $\gamma > 1$, the proportion in the variable subaccount will increase at lower age and then decrease as the insureds age. As γ increases, the proportion in the variable subaccount will decrease. All of these are consistent with the "without consumption" case. When γ is small, the insured will put all the money into his variable subaccount.

Fig. 4 shows the pre-retirement consumption ratio changes for different levels of risk aversion. As the risk aversion level increases, the consumption ratio in each period will increase, because people

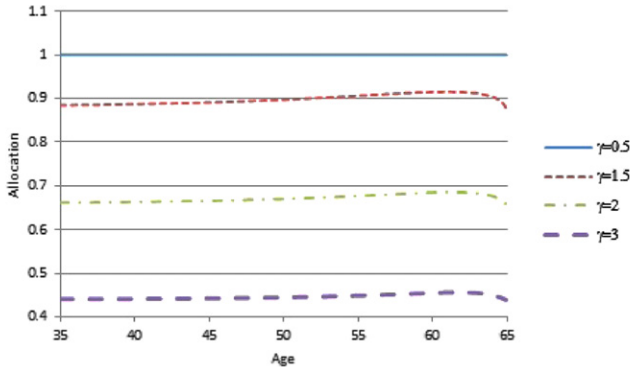

Fig. 3. At-the-money allocation with different γ under consumption case.

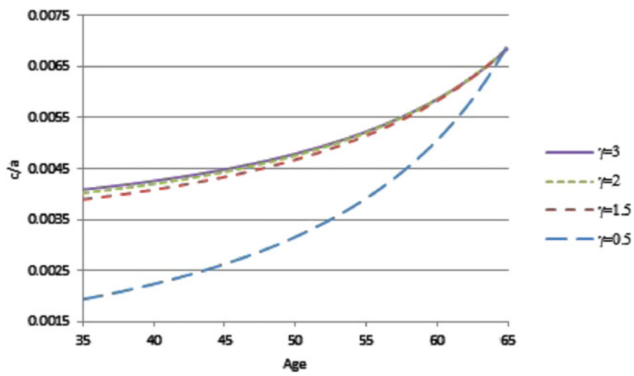

Fig. 4. At-the-money withdrawal ratio with different γ under consumption case.

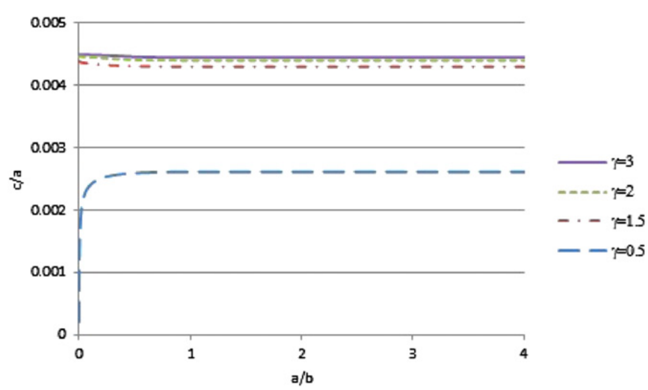

Fig. 5. Age 45 proportion of consumption with different γ under consumption case.

will be increasingly concerned about the investment risk in the future. A bird in the hand is worth two in the bush, so they will prefer consuming now rather than investing for the future. Also as the insured ages, it becomes less and less likely that the beneficiary will inherit the money. As in the "without consumption" case, the "argument" moves in favor of the insured. Therefore, the consumption ratios increase with time and eventually converge at the Merton level.

Fig. 5 shows, at age 45, the proportion of consumption to account value corresponding to different levels of risk aversion. The proportion of funds consumed is roughly constant when the GMDB is in the far out-of-the-money area, and the value of the consumption proportion is consistent with the value derived in the Merton model. If γ is less than 1, the consumption ratio decreases as the GMDB becomes more deeply in the money; while if γ is greater than 1, the consumption level increases as the asset level decreases. The first result seems intuitive, as an in-the-money withdrawal costs the beneficiary far more than the insured and

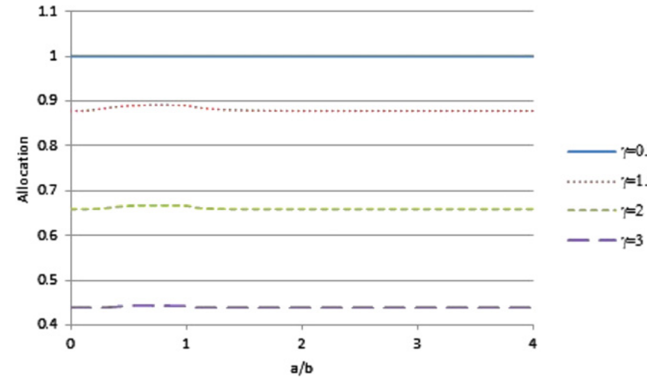

Fig. 6. Age 45 allocation with different γ under consumption case.

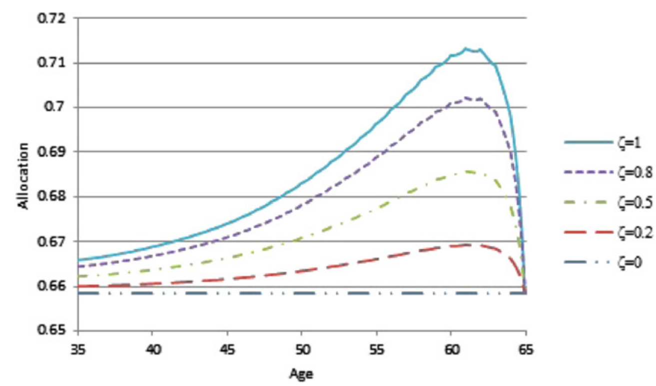

Fig. 7. At-the-money allocation with different ζ under consumption case.

even a mild bequest motive will cause the insured to maintain the account.

The counterintuitive result for $\gamma > 1$ in Fig. 5 comes from the fact that withdrawals from the variable annuity are the only source of consumption. For most currently available variable annuities, the GMDB strike is reduced proportionally with the reduction of the VA account value. When $\gamma = 1$, utility is logarithmic. A proportional reduction in the higher strike value reduces the beneficiary's utility by exactly the same amount as a reduction of the lower account value harms the insured's utility. When $\gamma < 1$, the beneficiary's utility is reduced by more than the insured's utility. When $\gamma > 1$, the beneficiary's utility is actually reduced less than the insured's utility is. In the most extreme case, the insured's utility can approach $-\infty$. The insured sells his beneficiary up the river for a loaf of bread. A more realistic model would include an outside consumption source in addition to the partial withdrawals from the variable annuity account which would reduce the effective CRRA parameter.

Fig. 6 shows the optimal allocation choice at age 45 at different risk aversion levels. It shows the same general pattern as in Fig. 1.

3.2.2. *Bequest motive sensitivity*

There are two competing effects. The first effect (increased probability of survival to 65 with age) makes the beneficiary argue less and the allocation go down with age. The second effect (increasing mortality with age) causes the beneficiary to be more likely to receive money immediately as the policyholder ages. This causes him to argue more and the allocation to go up with age. The first effect wins at low ages, and the second effect wins at ages near 65.

Fig. 7 shows the at-the-money asset allocation for different bequest motives with $\gamma = 2$. When the bequest motive $\zeta = 0$, the asset allocation level agrees with the Merton result at all ages. As

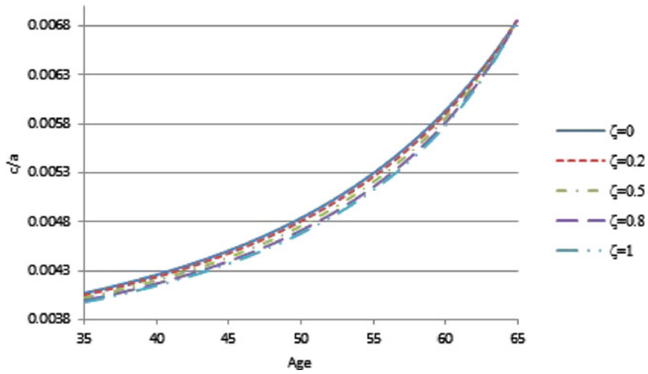

Fig. 8. At-the-money withdrawal ratio with different ζ under consumption case.

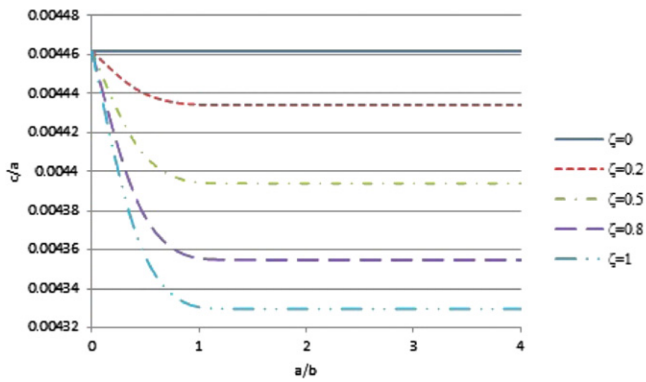

Fig. 9. Age 45 proportion of consumption with different ζ under consumption case.

the bequest motive increases, the proportion in the risky account will increase. Consistent with the previous tests, the allocation to the risky account goes up with age first and then decreases at ages near 65 to the Merton allocation at age 65 for any non-zero levels of the policyholder's bequest motive.

As the bequest motive level ζ increases, the pre-retirement consumption ratio in each period will decrease as shown in Fig. 8. With a larger bequest motive, the insured prefers to reduce his consumption to leave more money to his beneficiary. The consumption will increase as the insured ages at all levels of bequest motive. One can see that the higher the bequest motive, the lower the consumption ratio. The insured cares more for himself as he grows older. One can also observe that the consumption levels converge as the insured grows older and consumption ratios become the same at retirement age because after the insured retires, the wealth in VA account will become a lifetime payout annuity, and there is no longer any bequest motive.

Fig. 9 shows the consumption ratio vs. account value at age 45 with different levels of bequest motive. As the bequest motive increases, the proportion of funds consumed will decrease. As the GMDB goes deeper in-the-money, the consumption ratios at all non-zero bequest motives converge to the zero bequest level. In addition, consumption ratios stay constant as the GMDB goes out-of-the-money. This is also counter-intuitive, and the reason is the same as in the risk aversion sensitivity test: we assume that withdrawals from the variable annuity are the only source of the policyholder's consumption. Since $\gamma = 2$ in this test, the beneficiary's utility is actually reduced less than the insured's utility for a given withdrawal level, so the insured becomes selfish. The problem should disappear if an outside income is included in addition to the partial withdrawals from the variable annuity account.

Fig. 10 shows the proportion allocated to the risky asset vs. account value at age 45 with different levels of bequest motive.

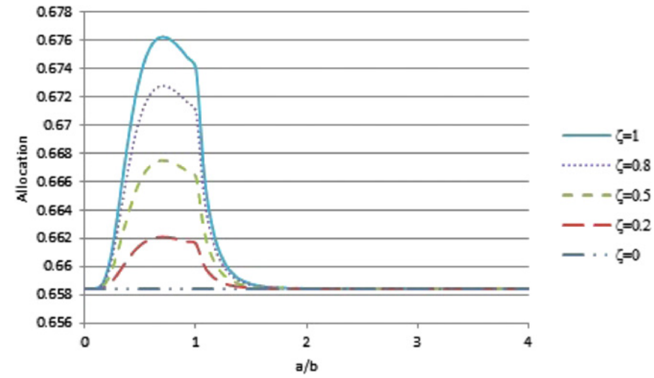

Fig. 10. Age 45 allocation with different ζ under consumption case.

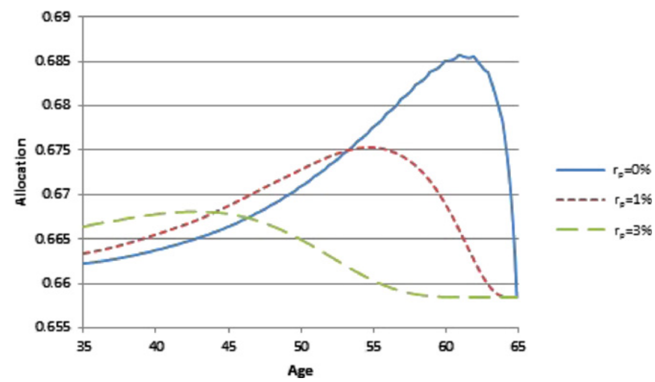

Fig. 11. At-the-money allocation with different r_p under consumption case when $\gamma = 2$.

When $\zeta = 0$, the insured's decision follows the Merton rule. When the insured has a bequest motive, he begins to make choices to maximize the joint utility. Therefore, around the at-the-money area, the insured breaks Merton's rule and takes more risk to help the beneficiary. One can observe the spikes around at-the-money area corresponding to different levels of bequest motive: the stronger the bequest motive, the larger the spike in risky allocation.

3.2.3. Roll-up rate sensitivity

Fig. 11 shows at-the-money asset allocations under different roll-up rates. This figure is somewhat counter-intuitive: At lower ages, larger roll-up rate provides more downside protection and makes the insured allocate more into a risky subaccount. However, as the insured ages, the risky allocation is largest for $r_p = 0\%$, even though this case provides the least downside protection. This occurs because for $r_p = 1\%$ and $r_p = 3\%$, the level of return of premium and roll-up benefits is increasing period by period. Therefore $a = 1$ is not the real "at-the-money" area for $r_p > 0$, and the argument area for $r_p > 0$ will go higher as the insured ages.

Fig. 12 (at-the-money consumption ratios under different roll-up rates with $\gamma = 2$) and 13 (at-the-money consumption ratios under different roll-up rates with $\gamma = 0.5$) show the consumption ratios under different roll-up rates. As $\gamma > 1$ (Fig. 12), the consumption ratio with the lower roll-up rate is larger than that with the higher roll-up rate. A higher roll-up rate gives the insured more incentive to keep the money for the beneficiary, because GMDB strike price goes up, which makes the withdrawal hurt the beneficiary much more than the policyholder himself. Since the policyholder has bequest motives, he will reduce his consumption for the sake of his beneficiary. When the roll-up rate is large enough (e.g. $r_p = 3\%$), the consumption ratio decreases for the whole pre-retirement period, while as $\gamma < 1$ (Fig. 13) the consumption ratio

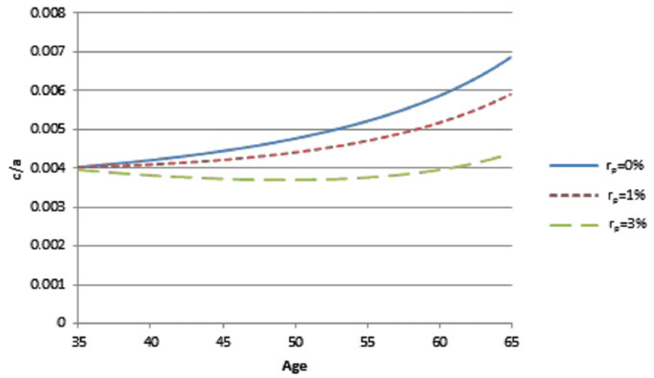

Fig. 12. At-the-money proportion of consumption with different r_p under consumption case when $\gamma = 2$.

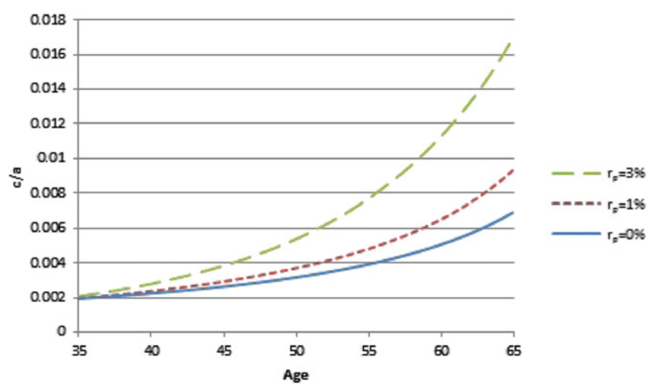

Fig. 13. At-the-money proportion of consumption with different r_p under consumption case when $\gamma = 0.5$.

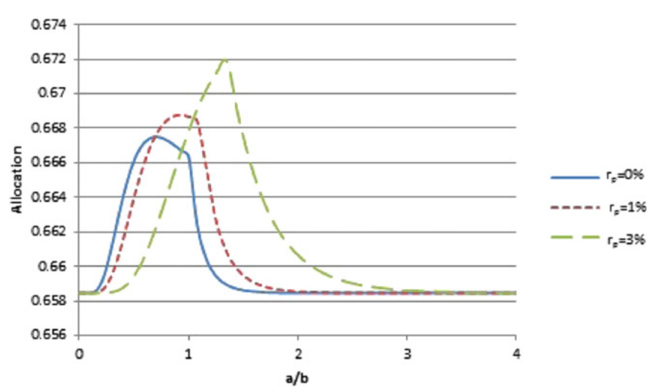

Fig. 14. Age 45 allocation with different r_p under consumption case.

with the lower roll-up rate is lower than that with the higher roll-up rate.

Fig. 14 shows the asset allocation at age 45 for different roll-up rates. A higher roll-up rate moves the at-the-money area to a higher asset level. Correspondingly, the "argument" between the insured and the beneficiary occurs at a higher asset level corresponding to the new at-the-money level. One can also observe that the size of the GMDB effect on the allocation increases roughly proportionately to the increase in roll-up rate.

Fig. 15 shows the proportion of funds consumed vs. account value at age 45 for different roll-up rates. The insured consumes less when the roll-up rate is higher, because the consumption will hurt the beneficiary much more.

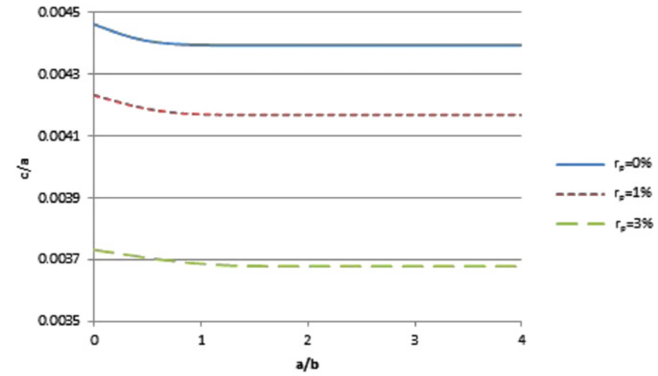

Fig. 15. Age 45 proportion of consumption with different r_p under consumption case.

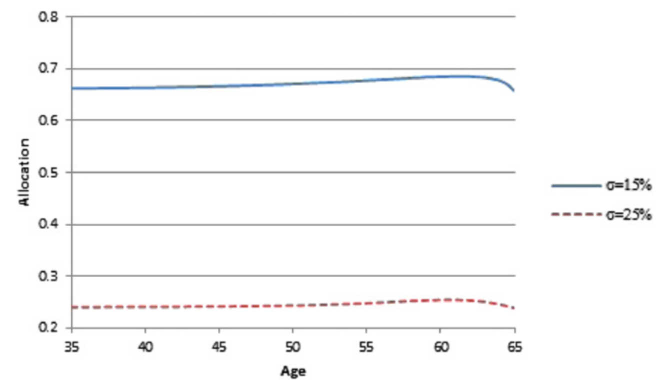

Fig. 16. At-the-money allocation with different σ under consumption case.

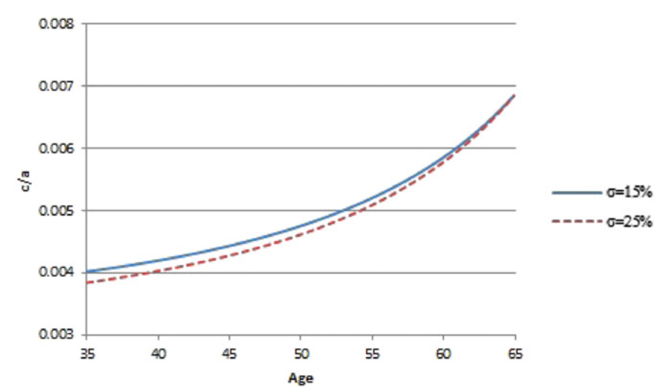

Fig. 17. At-the-money proportion of consumption with different σ under consumption case.

3.2.4. Volatility sensitivity

Fig. 16 shows that, as in the "without consumption" case, higher stock market volatility will make the insured more conservative and invest less money into the variable subaccount during the whole test period. Greater volatility also reduces the policyholder's consumption ratio as seen in Figs. 17 and 18.

3.2.5. Risky rate of return sensitivity

Fig. 19 shows at-the-money asset allocation with different risky rates of returns. As expected, the percentage of wealth held in the variable subaccount first increases, then decreases as the insured ages.

With a higher expected risky rate of return, the insured will get more returns from the variable subaccount, and will have more money to consume. Therefore the at-the-money consumption ratio

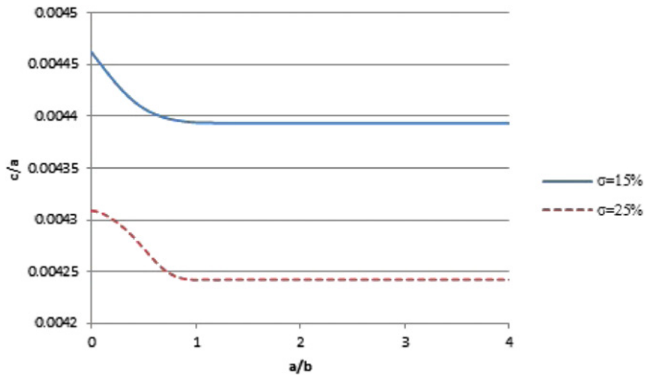

Fig. 18. Age 45 proportion of consumption with different σ under consumption case.

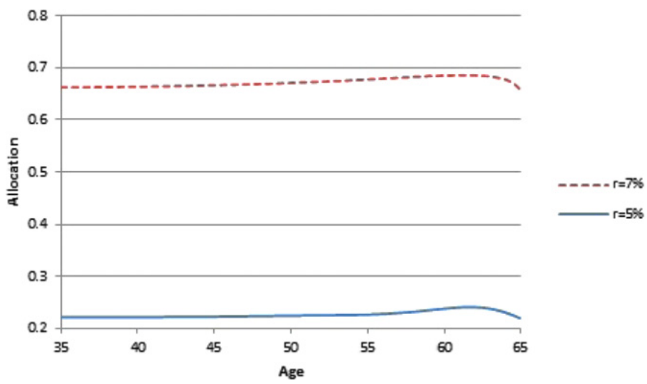

Fig. 19. At-the-money allocation with different r under consumption case.

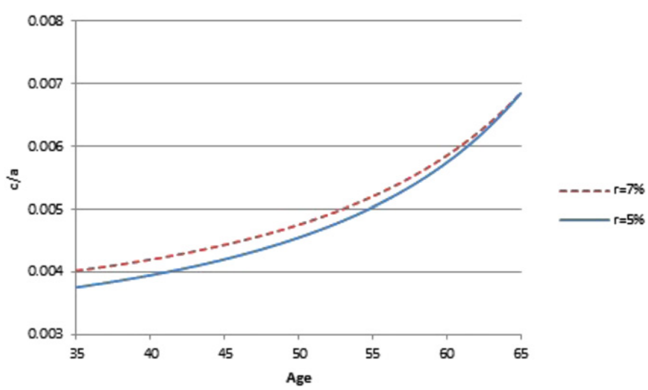

Fig. 20. At-the-money proportion of consumption with different r under consumption case.

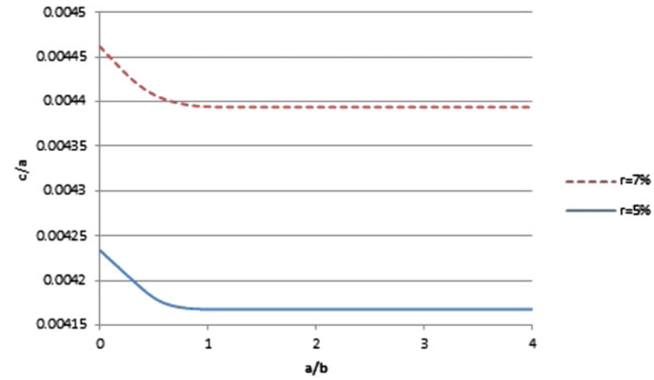

Fig. 21. Age 45 proportion of consumption with different r under consumption case.

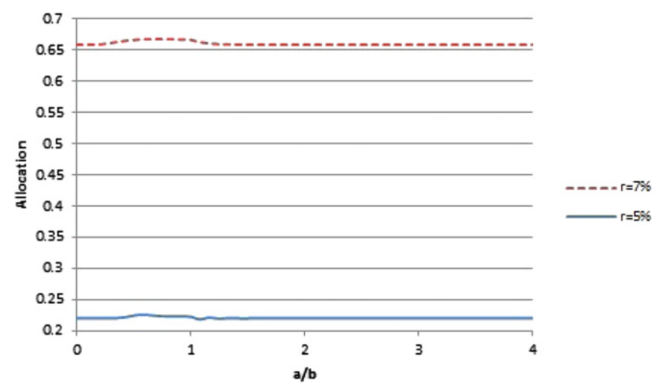

Fig. 22. Age 45 allocation with different r under consumption case.

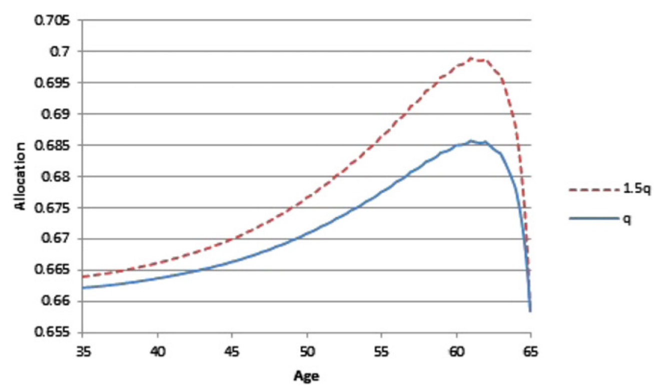

Fig. 23. At-the-money allocation with different q under consumption case.

is higher when the risky asset return is higher as shown in Fig. 20. Fig. 21 shows that at age 45, the consumption ratio is higher at all asset levels when the risky rate of return is higher.

At any given age, for example at age 45, the allocation choice will obey the Merton rule except around the at-the-money area (see Fig. 22), which is consistent with the previous discussion.

3.2.6. Mortality sensitivity

The purpose of this sensitivity test is to discover the optimal choices for the policyholder if all but one coefficient, mortality rate (q), were kept constant. In the test, one can observe the comparison between two scenarios (the normal mortality case: 1994 MGDB mortality and the high mortality case: 150% of 1994 MGDB mortality). Fig. 23 shows at-the-money allocation with different mortality rates. If mortality rates are increased 50% uniformly, the beneficiary is more likely to inherit the assets, therefore the "argument" moves in favor of the beneficiary and the allocation in the risky asset increases. As the insured ages, it becomes more likely the beneficiary will inherit the money quickly, but less and less likely that the beneficiary will inherit the money at all. Therefore the risky asset allocation in the high mortality case converges to the base case and the risky asset allocation goes back to the Merton level at the retirement age in both cases.

Fig. 24 shows at-the-money proportion of consumption with the two different mortality scenarios. One can observe that if the insured expects he is not healthy and will live shorter than a normal person, he will consume more in the current period; while a low mortality rate gives the insured more incentive to keep the money for the future.

Fig. 25 shows the proportion of funds consumed vs. account value at age 45 for the two different mortality scenarios. The

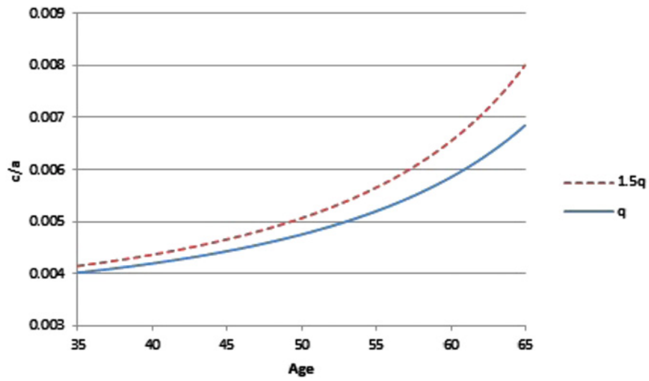

Fig. 24. At-the-money proportion of consumption with different q under consumption case.

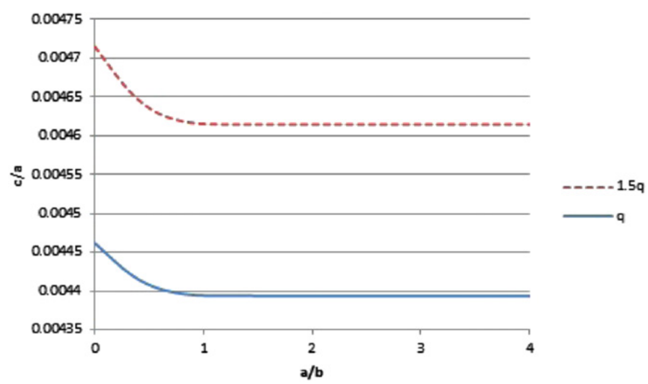

Fig. 25. Age 45 proportion of consumption with different q under consumption case.

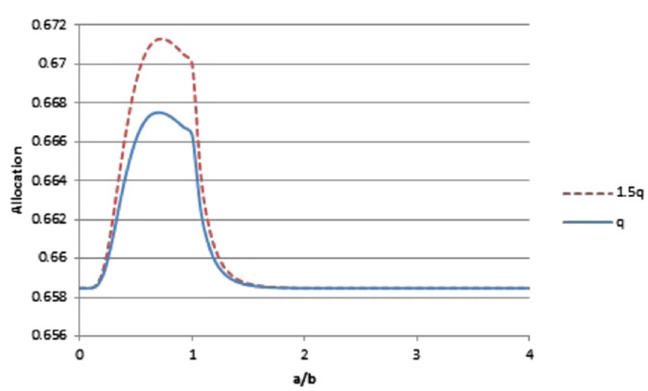

Fig. 26. Age 45 allocation with different q under consumption case.

insured will consume more at any asset levels if he knows he is not healthy.

Fig. 26 shows the asset allocation at age 45 for the two different mortality scenarios. Consistent with previous analyses, as the mortality rate goes up 50%, the variable subaccount allocation will be the same as the base case (65.85%) at almost all asset levels, but around the at-the-money area, the allocation increases to 67.13%, which is higher than the base case (66.75%). With a higher mortality rate, it is more likely that the beneficiary, rather than himself, will consume the assets. Therefore, the "argument" moves in favor of the beneficiary, and the bequest motive makes the insured allocate more aggressively around the at-the-money area.

4. GMDB pricing and delta ratio

We now turn our attention to the effect of the policyholder behavior derived in the previous sections on the pricing and hedging of the guarantees. A number of papers have examined pricing and hedging of options embedded in variable annuity contracts. Examples include Coleman et al. (2006), Bauer et al. (2008), Chen and Forsyth (2008), Chen et al. (2008), Dai et al. (2008), Belanger et al. (2009), Kling et al. (2010) and Steinorth and Mitchell (2011). The following analysis involves two steps. First, policyholder actions are determined by maximizing utility under the realistic measure, as in the previous sections. Next, the policyholder actions obtained in the first step are used to price the option after replacing the realistic probabilities with risk-neutral ones. Finally, the lattice is perturbed in order to find hedge ratios through numerical differentiation.

4.1. Fees and expenses of GMDB contracts

Like other investments, fees and expenses are incurred in variable annuities:

1. Investment Management Fee is a payment to the management company for the services and investment portfolio recommendations. These fees will vary depending on the various subaccount options within the annuity.
2. Administrative charges cover the paperwork, record keeping, and periodic reports to the annuity policyholders.
3. Surrender charges are charged if withdrawals occur before a specified period of time. It is usually a percentage of the account value and declines over time. Most variable annuities permit partial withdrawals each year without a surrender charge.
4. Mortality and Expense Charges are also called "M & E" fee. They are used to pay: (1) the mortality risk related with the guaranteed death benefit; (2) a guarantee that annual expenses will not exceed a certain percentage of assets; and (3) an allowance for profit.

All these fees and expenses except surrender charges, reduce the rate of return on the variable account and are implicitly included in our model.

4.2. GMDB pricing and delta ratio in the "with consumption" case

In the "with consumption" case, the optimal allocation ω and optimal withdrawal amount c were derived. By assuming the insured and beneficiary have equivalent CRRA parameters, and they both make (from their perspective) optimal choices, the insurer can use the strategy of applying the insured's optimal allocation ω and lapse c into a risk neutral model to price the GMDB options and implement a delta hedging strategy. Therefore the trinomial lattice equation is modified as follows,

$$V_{t,j} = q_t h \max(k_{t,j} - a_{t,j}) + e^{-r_f h}\left(1 - q_t h - \frac{c_{t,j}}{a_{t,j}} h\right) \times (p_u V_{t+1,j+1} + p_m V_{t+1,j} + p_d V_{t+1,j-1}). \tag{28}$$

Here the value function represents dollar amount paid out rather than utility. Consumptions and allocations are not "optimized" but given from risk averse analysis in the "with consumption" case, and probabilities p_u, p_m and p_d are derived in (22)–(24) respectively.

We present several findings in pricing the GMDB from Tables 3 through 7. First, as the risk aversion level γ decreases, the insured will increase the allocation of money in the variable subaccount, and the GMDB will become more valuable in protecting downside risk. The delta ratio experiences a corresponding increase. Second,

Table 3
GMDB price and delta ratio δ with different risk aversion level γ under $\sigma = 15\%$.

Delta ratio	GMDB price	Risk aversion	Bequest
−0.0012725	0.0001116	2	0
−0.0012798	0.0001127	2	0.2
−0.0012906	0.0001143	2	0.5
−0.0013012	0.0001159	2	0.8
−0.0013083	0.0001170	2	1
−0.0014637	0.0001548	1.8	0
−0.0014732	0.0001566	1.8	0.2
−0.0014871	0.0001591	1.8	0.5
−0.0015012	0.0001617	1.8	0.8
−0.0015105	0.0001634	1.8	1
−0.0018481	0.0002689	1.5	0
−0.0018629	0.0002726	1.5	0.2
−0.0018847	0.0002781	1.5	0.5
−0.0019071	0.0002835	1.5	0.8
−0.0019218	0.0002872	1.5	1
−0.0021865	0.0003981	1.2	0
−0.0021938	0.0003999	1.2	0.2
−0.0022045	0.0004025	1.2	0.5
−0.0022150	0.0004051	1.2	0.8
−0.0022218	0.0004068	1.2	1
−0.0025002	0.0004731	0.5	0
−0.0025191	0.0004778	0.5	0.2
−0.0025467	0.0004847	0.5	0.5
−0.0025735	0.0004913	0.5	0.8
−0.0025909	0.0004957	0.5	1

Other parameter values: annual risky rate of return $r = 7\%$, annual volatility $\sigma = 15\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual roll-up rate $r_p = 0\%$ and annual mortality rate $q = 1994$ MGDB.

Table 4
GMDB price and delta ratio δ with different risk aversion level γ under $\sigma = 25\%$.

Delta ratio	GMDB price	Risk aversion	Bequest
−0.0007544	0.0000293	2	0
−0.0007582	0.0000296	2	0.2
−0.0007638	0.0000300	2	0.5
−0.0007695	0.0000304	2	0.8
−0.0007733	0.0000306	2	1
−0.0008968	0.0000427	1.8	0
−0.0009021	0.0000432	1.8	0.2
−0.0009101	0.0000438	1.8	0.5
−0.0009180	0.0000445	1.8	0.8
−0.0009232	0.0000449	1.8	1
−0.0012052	0.0000811	1.5	0
−0.0012157	0.0000823	1.5	0.2
−0.0012314	0.0000840	1.5	0.5
−0.0012475	0.0000858	1.5	0.8
−0.0012581	0.0000870	1.5	1
−0.0017144	0.0001758	1.2	0
−0.0017349	0.0001793	1.2	0.2
−0.0017664	0.0001847	1.2	0.5
−0.0017981	0.0001902	1.2	0.8
−0.0018200	0.0001939	1.2	1
−0.0052387	0.0026180	0.5	0
−0.0053825	0.0027303	0.5	0.2
−0.0055990	0.0029017	0.5	0.5
−0.0058026	0.0030558	0.5	0.8
−0.0059226	0.0031448	0.5	1

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual roll-up rate $r_p = 0\%$ and annual mortality rate $q = 1994$ MGDB.

the Sharpe Ratio matters in deciding the GMDB prices and delta ratios: identical Sharpe Ratios produce identical results. When the expected risky rate of return r and other parameters are fixed, the GMDB price and delta ratio go down as the stock market volatility σ increases. One would normally expect that a riskier asset would produce a higher GMDB price, but instead the value decreases because a higher equity market volatility will lower the amount invested in the variable subaccount, and a less risky investment will need less GMDB protection. Equivalently, given fixed values of equity market volatility and other parameters, the GMDB price

Table 5
GMDB price and delta ratio δ with different risk aversion level γ under $\sigma = 35\%$.

Delta ratio	GMDB price	Risk aversion	Bequest
−0.0004556	0.0000097	2	0
−0.0004578	0.0000098	2	0.2
−0.0004608	0.0000099	2	0.5
−0.0004638	0.0000100	2	0.8
−0.0004657	0.0000101	2	1
−0.0005527	0.0000144	1.8	0
−0.0005552	0.0000145	1.8	0.2
−0.0005588	0.0000147	1.8	0.5
−0.0005624	0.0000149	1.8	0.8
−0.0005648	0.0000150	1.8	1
−0.0007673	0.0000285	1.5	0
−0.0007720	0.0000288	1.5	0.2
−0.0007790	0.0000293	1.5	0.5
−0.0007861	0.0000297	1.5	0.8
−0.0007908	0.0000301	1.5	1
−0.0011257	0.0000648	1.2	0
−0.0011374	0.0000659	1.2	0.2
−0.0011557	0.0000676	1.2	0.5
−0.0011747	0.0000694	1.2	0.8
−0.0011876	0.0000707	1.2	1
−0.0043466	0.0013031	0.5	0
−0.0044840	0.0013699	0.5	0.2
−0.0046957	0.0014756	0.5	0.5
−0.0049175	0.0015879	0.5	0.8
−0.0050701	0.0016666	0.5	1

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual roll-up rate $r_p = 0\%$ and annual mortality rate $q = 1994$ MGDB.

Table 6
GMDB price and delta ratio δ with different roll-up rates r_p and different volatilities σ.

Delta ratio	GMDB price	Bequest ζ	Roll-up rate (%)	Volatility σ (%)
−0.0012725	0.0001116	0	0	15
−0.0012798	0.0001127	0.2	0	15
−0.0012906	0.0001143	0.5	0	15
−0.0013012	0.0001159	0.8	0	15
−0.0013083	0.0001170	1	0	15
−0.0019216	0.0002079	0	1	15
−0.0019361	0.0002106	0.2	1	15
−0.0019580	0.0002147	0.5	1	15
−0.0019800	0.0002188	0.8	1	15
−0.0019946	0.0002216	1	1	15
−0.0062687	0.0011402	0	3	15
−0.0063201	0.0011584	0.2	3	15
−0.0063945	0.0011857	0.5	3	15
−0.0064678	0.0012127	0.8	3	15
−0.0065154	0.0012307	1	3	15
−0.0007544	0.0000293	0	0	25
−0.0007582	0.0000296	0.2	0	25
−0.0007638	0.0000300	0.5	0	25
−0.0007695	0.0000304	0.8	0	25
−0.0007733	0.0000306	1	0	25
−0.0012437	0.0000631	0	1	25
−0.0012537	0.0000639	0.2	1	25
−0.0012687	0.0000651	0.5	1	25
−0.0012840	0.0000664	0.8	1	25
−0.0012944	0.0000673	1	1	25
−0.0066604	0.0007234	0	3	25
−0.0067590	0.0007450	0.2	3	25
−0.0069065	0.0007773	0.5	3	25
−0.0070497	0.0008097	0.8	3	25
−0.0071460	0.0008313	1	3	25

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, risk aversion level $\gamma = 2$ and annual mortality rate $q = 1994$ MGDB.

and delta ratio go down as the expected risky rate of return r goes down. The roll-up rate r_p is a major factor in the GMDB price and the delta ratio. A higher roll-up rate means the beneficiary can keep more money in bad market states, and it encourages the insured to take a more aggressive allocation when the GMDB

Table 7
GMDB price and delta ratio δ with different mortality rates: 1994 MGDB vs. 150% of 1994 MGDB.

Delta ratio	GMDB price	Mortality rate	Bequest ζ
−0.0012725	0.0001116	Real	0
−0.0012798	0.0001127	Real	0.2
−0.0012906	0.0001143	Real	0.5
−0.0013012	0.0001159	Real	0.8
−0.0013083	0.0001170	Real	1
−0.0018840	0.0001643	1.5x	0
−0.0018993	0.0001666	1.5x	0.2
−0.0019221	0.0001700	1.5x	0.5
−0.0019444	0.0001734	1.5x	0.8
−0.0019593	0.0001757	1.5x	1

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual roll-up rate $r_p = 0\%$, annual volatility $\sigma = 15\%$ and risk aversion level $\gamma = 2$.

is at-the-money. Therefore the GMDB price and delta ratio will go up as we increase the roll-up rate. The bequest motive is also very important in determining the GMDB price. A higher bequest motive increases the allocation to the variable subaccount and discourages withdrawals from the VA account. As a result, a higher bequest motive makes the GMDB more valuable. Last but not least, one cannot disregard the mortality effects in the GMDB price and the delta ratio. Higher mortality rates mean that the beneficiary will have more chance to inherit the assets from the insured. Therefore, the insured may take a more risky allocation especially when the GMDB is at-the-money and the insurer should charge more for the GMDB.

5. Conclusions

In this paper, we examine the allocation and withdrawal choices of CRRA individuals who own Variable Annuity contracts with Guaranteed Minimum Death Benefits and attempt to maximize their utility taking into account bequest motives. We find that individuals follow the Merton allocation and consumption rules when the GMDB is very far out-of-the-money, but find noticeable deviations in allocation when the GMDB is near-the-money and noticeable deviations in consumption when the GMDB is in-the-money. These deviations can be intuitively explained as an ''argument'' between the insured, who prefers the Merton allocation and consumption rules, and the beneficiary who prefers a more aggressive allocation and stands to lose more when in-the-money withdrawals are made. These deviations produce noticeable effects on the price and hedging parameters of the options and should be taken into account by issuers of these riders.

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