Parul University
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1^{st} Year B.Tech Programme (All Branches)
Mathematics – 1(203191102)
Unit 5: Fourier Series

(Lecture Note)

Fourier Series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.

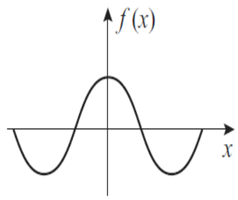
- Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series, which is in terms of sines and cosines.
- Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.
- Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives.
- Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.

Prerequisites:

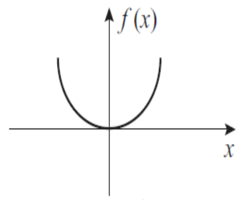
Even Function: A function $f(x)$ is said to be even, if $f(-x) = f(x)$, for all x.

- Even Function is always symmetric about $Y - axis$.

Eg: $cosx, xsinx,\ x^2$, etc



$f(x) = \cos x$

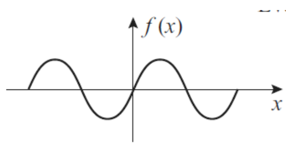


$f(x) = x^2$

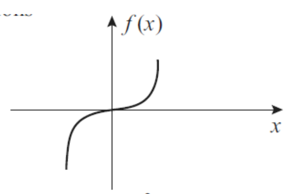
Odd Function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$, for all x.

- Odd function is always symmetric about $X - axis$.

Eg: x, x^3 , $x \cos x, \sin x$, etc



$f(x) = \sin x$



$f(x) = x^3$

Properties of Definite Integrals:

1. $\int_a^b f(x)dx = F(b) - F(a)$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$,if f(x) is even function
   $= 0$, if f(x) is odd function
4. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Basic Trigonometry

1. $sin\, n\pi = 0$
2. $cos\, n\pi = (-1)^n$
3. $cos\,(n+1)\,\pi = cos\,(n-1)\pi = (-1)^{n+1}$
4. $2sinA\, cosB = sin\,(A + B) + sin\,(A - B)$
5. $2sinA\, sinB = cos(A - B) - cos\,(A + B)$
6. $2cosA\, cosB = cos\,(A + B) + cos(A - B)$
7. $cos(^{n\pi}/_2) \neq 0$

Periodic Functions:

A function $f(x)$ is called periodic if it is defined for all real x and if there is some positive number p such that $f(x+p)=f(x)$ where p is known as **period** of $f(x)$. If a periodic function $f(x)$ has a smallest period p is called **fundamental period** of $f(x)$. eg $sinx, cosx$ has fundamental period of 2π. If l is a fixed number, then $sin(2\pi x/l)$ and $cos(2\pi x/l)$ have period l

Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c+2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, $i.e., f(x) = f(x+2l)$, where $2l$ is the period of the function $f(x)$.

(ii) $f(x)$ and its integrals are finite and single-valued.

(iii) $f(x)$ has a finite number of discontinuities, $i.e., f(x)$ is piecewise continuous in the interval $(c, c+2l)$.

(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as Dirichlet's conditions.

- **Fourier series:** Fourier series of periodic function $f(x)$ defined in interval $(c, c+2l)$ with fundamental period $2l$ **is given by**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)$$

$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx,\quad n=1,2,\ldots$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx\quad n=1,2,\ldots$$

Example:1 Find the Fourier Series of $f(x) = x$ in the interval $(0,2\pi)$

Solution: Here $c = 0\ and\ c + 2l = 2\pi$

$$\therefore l = \pi$$

The Fourier Series of given function is given by

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n\ \sin\frac{n\pi}{l}x\right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\,x + b_n\ \sin nx)$$

Now, find

$$\boldsymbol{a_0} = \frac{1}{\pi}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

Also,

$$\boldsymbol{a_n} = \frac{1}{\pi}\int_c^{c+2l} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx,$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos 2n\pi}{n^2} - 0 - \frac{1}{n^2}\ \right] = \frac{1}{\pi}(0) = 0$$

$$\boldsymbol{b_n} = \frac{1}{\pi}\int_c^{c+2l} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\right] = -\frac{2}{n}$$

Sub. The value of $a_0, a_n and\ b_n$ in $f(x)$

We get

$$f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Example:2 Find the Fourier Series of $f(x) = x^2$ in the interval $(0,2\pi)$.

Solution: Here $c = 0 \ and \ c + 2l = 2\pi$

$$\therefore l = \pi$$

The Fourier Series of given function is given by

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{l}x + b_n \ \sin\frac{n\pi}{l}x\right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\, x + b_n \ \sin nx)$$

Now, find

$$\boldsymbol{a_0} = \frac{1}{\pi}\int_0^{2\pi} x^2 \, dx$$

$$= \frac{8\pi^3}{3}$$

Also,

$$\boldsymbol{a_n} = \frac{1}{\pi}\int_c^{c+2l} f(x) \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{4}{n^2}$$

$$\boldsymbol{b_n} = \frac{1}{\pi}\int_c^{c+2l} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$= -\frac{4\pi}{n}$$

Hence,

$$f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos n\,x-\frac{\pi}{n}\sin nx\right)$$

Example:3 Obtain the Fourier series to represent f(x)= $e^{ax}(a\neq 0)$in the interval $0<x<2\pi$

Solution: Given function $f(x)$ may be expanded in Fourier series as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\ \ldots\ldots\ldots\ldots(i)$$

Where, $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$

$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$

$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$

Now, $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$

$=\frac{1}{\pi}\int_0^{2\pi}e^{ax}dx$

$=\frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$

$=\frac{1}{a\pi}[e^{2a\pi}-1]$

$\frac{a_0}{2}=\frac{e^{2a\pi}-1}{2a\pi}$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$=\frac{1}{\pi}\int_0^{2\pi}e^{ax}\cos nx\,dx$

Using $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\{a\cos bx+b\sin bx\}+c$

$$=\frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\{a\cos nx+n\sin nx\}\right]_0^{2\pi}$$

$$=\frac{1}{\pi(a^2+n^2)}[e^{ax}\{a\cos 2n\pi+n\sin 2n\pi\}-e^0\{a\cos 0+n\sin 0\}]$$

As $\cos 2\,n\pi=1$, $\sin 2n\pi=0$, $\cos 0=1$, $\sin 0=0$

$$\therefore\ a_n = \frac{a}{\pi(a^2+n^2)} e^{2a\pi} - 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx$$

Using $\int_0^{2\pi} e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\} + c$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\{a\sin nx - n\cos nx\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{2a\pi}\{a\sin 2n\pi - n\cos 2n\pi\} - \{a\sin 0 - n\cos 0\}\right]$$

But, $\cos 2\,n\pi = 1$, $\sin 2n\pi = 0$, $\cos 0 = 0$, $\sin 0 = 0$

$$= \frac{a}{\pi(a^2+n^2)}(-ne^{2a\pi} + n)$$

$$\therefore\ b_n = \frac{n}{\pi(a^2+n^2)}(1 - e^{2a\pi})$$

Substituting the values of $a_0 a_n$ and b_n in $(i), we\ have$

$$f(x) = \frac{e^{2a\pi} - 1}{\pi}\left[\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{\cos nx}{a^2+n^2} - \sum_{n=1}^{\infty}\frac{\text{n sin } nx}{a^2+n^2}\right]$$

Example:4 Find the Fourier series representation of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$.

Solution:

Given function $f(x)$ may be expanded in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x + |x| \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \, dx + \int_{-\pi}^{\pi} |x| \, dx \right]$$

$$= \frac{1}{\pi} \left[0 + 2 \int_{0}^{\pi} |x| \, dx \right]$$

$$= \frac{1}{\pi} [\pi^2] = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [x + |x|] \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx \, dx + \int_{-\pi}^{\pi} |x| \cos nx \, dx \right]$$

$$= \frac{2}{\pi} \int_{0}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} - (-1) \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} [x + |x|] \sin nx \, dx = -\frac{2}{n} (-1)^n$$

Substituting the values of a_0, a_n and b_n in $f(x)$, we have

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}[(-1)^n - 1]\ \cos nx - \frac{2}{n}(-1)^n \sin nx\right]$$

Example:5 Find a Fourier series for $\boldsymbol{f(x) = x + x^2}$, $\quad -\boldsymbol{\pi} < x < \pi$

Also deduce that

$$\frac{\mathbf{1}}{\mathbf{1^2}} + \frac{\mathbf{1}}{\mathbf{2^2}} + \frac{\mathbf{1}}{\mathbf{3^2}} + \cdots \ldots \ldots \ldots = \frac{\boldsymbol{\pi^2}}{\mathbf{6}}$$

Solution:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

be the required Fourier Series.

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2 dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} xdx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$

As x is an odd function, $\int_{-\pi}^{\pi} xdx = 0$ and

$x^2$is an even function, therefore $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = 2\ \int_0^{\pi} x^2 dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$

$$a_0 = \frac{2\pi^2}{3}$$

$$\frac{a_0}{2} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

As $x\cos nx$ is an odd function $\int_{-\pi}^{\pi} x\cos nx\,dx = 0$ and $x^2\cos nx$ is an even function, therefore

$$\int_{-\pi}^{\pi} x^2\cos nx\,dx = 2\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2\sin nx}{n}\right) + \left(\frac{2x\cos nx}{n^2}\right) - \left(\frac{2\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3} - 0\right]_0^{\pi}$$

But $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$$\therefore\ a_n = \frac{4\,(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

As $x^2\sin nx$ $is\ an\ odd\ function$, $\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$ and $x\sin nx$ is an even function , therefore

$$\int_{-\pi}^{\pi} x\sin nx\,dx = 2\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{x\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - 0\right]$$

But $\cos n\pi = (-1)^n \ and \ \sin n\pi = 0$

$$\therefore \ b_n = -\frac{2\,(-1)^n}{n}$$

Substituting above value in $f(x)$, we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

As a required Fourier Series,

At, $x = \pi$,

The sum of the series is

$$f(\pi) = \frac{f(\pi+0) + f\,(\pi-0)}{2}$$

$$= \frac{\pi^2 + \pi + \pi^2 - \pi}{2}$$

$$f(\pi) = \pi^2$$

Putting $x = \pi$ in series (ii), we have

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\pi}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi}{n^2}$$

But $\cos n\pi = (-1)^n \, and \sin n\pi = 0$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\ldots\ldots\ldots\ldots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\ldots\ldots\ldots = \frac{\pi^2}{6} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example:6 Expand $f(x) = x\cos x, 0 < x < 2\pi$ as Fourier series

Solution: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

Where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$\frac{1}{\pi}\int_0^{2\pi} x\cos x\, dx$

Integrating by parts

$= \frac{1}{\pi}[(x)(sinx) - (1)(-\cos x)]_0^{2\pi}$

$= \frac{1}{\pi}[x\sin x + \cos x]_0^{2\pi}$-4

$= \frac{1}{\pi}[2\pi\sin 2\pi + \cos 2\pi - son\ 0 - \cos 0]$

$\cos 2\pi = 1, \sin 2\pi = 0, \cos 0 = 1, \sin 0 = 0$

$= \frac{1}{\pi}[1-1]$

$a_0 = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} xcos\ x\ \cos nx\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2cosnx\ \cos x)\, dx$

But 2CC= C+C

$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n+1)x + \cos(n-1)x\}dx$

$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos(n+1)x\, dx + \int_0^{2\pi} x\{\cos(n-1)x\, dx\}\right]$

Integrating both terms by parts,

$$= \frac{1}{2\pi}\left[(x)\left(\frac{\sin(n+1)x}{n+1}\right) - (1)\left(-\frac{\cos(n+1)x}{(n+1)^2}\right) + (x)\left(\frac{\sin(n-1)x}{n-1}\right) - (1)\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

Where $n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{\text{x}\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2} + \frac{\text{x}\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

As $\sin(n \pm 1)\pi = 0$, $\cos(n \pm 1)\pi = 1$

$$= \frac{1}{2\pi}\left[\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right]$$

$$\therefore\ a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xcos\ x\ \sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2sinnx\ \cos x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\sin(n+1)x + \sin(n-1)x\}dx$$

But , 2SC=S+S,

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x\, dx\ + \int_0^{2\pi} x\sin(n-1)x\, dx\ \right]$$

Integrating both terms by parts,

$$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2}\right) + (x)\left(-\frac{\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

Where $n \neq 1$

$$= \frac{1}{2\pi}\left[-\frac{\text{x}\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} - \frac{\text{x}\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

As $\cos 2(n \pm 1)\pi = 1$, $\sin 2(n \pm 1)\pi = 0$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1} - \frac{2\pi}{n-1}\right]$$

$$\therefore\ b_n = -\frac{2n}{n^2-1}\ where\ (n \neq 1)$$

Here $a_n\ and\ b_n$ cannot be calculated for $n = 1$.

Again, $a_n = \frac{1}{\pi}\int_0^{2\pi} xcos\, x\ \cos nx\, dx$

Putting $n = 1$

$a_1 = \frac{1}{\pi}\int_0^{2\pi} xcos^2x\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x(1 + cos2x)\, dx$

$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\, dx + \int_0^{2\pi} xcos2x)\right]$

$= \frac{1}{2\pi}\left[\frac{x^2}{2} + (x)\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-cos2x}{4}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{cos2x}{4}\right]_0^{2\pi}$

As $\sin 4\pi = 0$, $\cos 4\pi = 1$

$= \frac{1}{2\pi}\left[2\pi^2 + \frac{1}{4} - \frac{1}{4}\right]$

$a_1 = \pi$

And, $b_n = \frac{1}{\pi}\int_0^{2\pi} xcos\, x\ \sin nx\, dx$

Putting $n = 1$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos x)\, dx$

$\frac{1}{2\pi}\int_0^{2\pi} xsin2x\, dx$

Integrating by Parts,

$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{sin2x}{4}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$

As $\sin 4\pi = 0$, $\cos 4\pi = 1$

$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]$

$b_1 = -\frac{1}{2}$

Now, $f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$

Substituting the above values, we get

$$f(x) = \pi \cos x - \frac{1}{2}\sin x - 2\ \sum_{n=2}^{\infty} \frac{n}{n^2-1} \sin nx$$

$$\therefore\ x\cos x = \pi\cos x - \frac{\sin x}{2} - 2\left[\frac{2}{3}\sin 2x + \frac{3}{8}\sin 3x + \frac{4}{15}\sin 4x + \cdots.\right]$$

Which is desired Fourier series of $f(x) = x\cos x$

Example:7 Find the Fourier Series of the function

$$\boldsymbol{f(x)} = \begin{cases} x & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$$

Solution: Here $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\pi x + b_n \sin n\pi x]$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)dx$$

$$= \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} x\,dx + \int_{0}^{1} 2\,dx$$

$$= -\frac{1}{2} + 2 = 3/2$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1} 2\cos n\pi x\,dx$$

$$= \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} + \left(\frac{\sin n\pi x}{n\pi}\right)_{0}^{1}$$

$$= \frac{1}{n^2\pi^2}[1 - (1)^n]$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\frac{\sin n\pi x}{n\pi}dx$$

$$= \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} + \left(\frac{\cos n\pi x}{n\pi}\right)_{0}^{1}$$

$$= \frac{1}{n\pi}[2 - 3(-1)^n]$$

Hence,

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}[1-(1)^n]\cos n\pi x + \frac{1}{n\pi}[2-3(-1)^n]\sin n\pi x\right]$$

Exercise:

1. **Find the Fourier series of the periodic functions $f(x)$ with period 2π defined as follows: f(x) =**$\begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Answer: $f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 1]\frac{\cos nx}{n^2} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$

2. **Find the Fourier Series of the function $f(x) = 4 - x^2$ in the interval $(0, 2)$**

Answer: $a_0 = \frac{4}{3}, a_n = -\frac{4}{n^2\pi^2}, b_n = \frac{4}{n\pi}$

3. **Find the Fourier Series of the function $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$. Hence deduced that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$**

   Answer: $a_0 = a_n = 0, b_n = \frac{1}{n}$

4. Find the Fourier series of periodic functions with period 2, which are given below:

$$f(x) = \begin{cases} \pi; \ for 0 \le x \le 1 \\ \pi(2-x); \ for 1 \le x \le 2 \end{cases}$$

ANS: $\left[a_0 = \frac{3\pi}{4} \ , \ a_n = \frac{1}{\pi n^2}((-1)^n - 1) \ , \ b_n = \frac{1-2(-1)^n}{n}\right]$

Fourier Series for even and odd Function.

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if $f(x)$ is even function

$= 0$, if $f(x)$ is odd function

Example:7 Find the Fourier Series of the function $f(x) = |x|$ in the interval $(-\pi, \pi)$.

Solution:

Here $f(x) = |x|$ is even function , therefore for the given interval $(-\pi, \pi)$, $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^L f(x)dx$$

$$= \frac{2}{\pi}\int_0^\pi |x|\, dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi = \pi$$

$$a_n = \frac{2}{l}\int_0^L f(x)\cos\frac{n\pi x}{l}\, dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi |x|\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{\cos nx}{n^2} - \frac{\cos 0}{n^2}\right)$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

Hence

$$f(x) = |x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx$$

Example:8 Expand $f(x) = |\sin x|, -\pi \le x \le \pi$ as Fourier series.

Solution:

The function $f(x) = |\sin x|$is an even function

$\therefore\ b_n = 0$

For even function the required Fourier Series is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots\dots\dots\dots\dots\dots..(i)$

Where,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$f(x) = |\sin x| = \sin x\, , 0 < x < \pi$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

As $f(x)$ is even function

$a_0 = \frac{2}{\pi}\int_{0}^{\pi} |\sin x| dx$

$= \frac{2}{\pi}\int_{0}^{\pi} \sin x\, dx$

$= \frac{2}{\pi}[-\cos x]_0^{\pi}$

$a_0 = \frac{2}{\pi}(2)$

$\therefore \frac{a_0}{2} = \frac{2}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

Since $f(x)\cos nx$ is even function

$a_n = \frac{2}{\pi}\int_{0}^{\pi} \cos nx \sin x\, dx$

But 2CS= S-S

$= \frac{1}{\pi}\int_{0}^{\pi} [\sin(n+1)x = \sin(n-1)x]dx$

$= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]_0^{\pi}$, (Where, $n \neq 1$)

Since $\cos(n \pm 1)\pi = (-1)^{n+1}$

$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right] \dots\dots\dots\dots\dots\dots..(1)$

Where n is even, $(-1)^{n+1} = -1$

$\therefore\ a_n = \frac{2}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$

$a_n = -\frac{4}{\pi(n^2-1)}$

Where n is odd, $(-1)^{n+1} = 1$, from (1)

$= \frac{1}{\pi}\left[\frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right]$

$\therefore\ a_n = 0$

Thus ,$a_n = 0$,if n is odd

$= -\frac{4}{\pi(n^2-1)}$, if $n$is even

When $n = 1$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\, dx$

$= \frac{2}{\pi}\int_0^{\pi} |\sin x| cosx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin 2x\, dx$

$= \frac{2}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$

$a_1 = 0$

$\therefore\ series\ (i) becomes,$

$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\substack{n=2 \\ n-even}}^{\infty} \frac{\cos nx}{n^2-1}$

Half Range Cosine – Sine Series

The Fourier series of an even function of period of 2*l* is a "Fourier cosine series" $(-\boldsymbol{l}, \boldsymbol{l})$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l} x$$

$$a_0 = \frac{2}{l}\int_0^L f(x)dx \quad , \quad a_n = \frac{2}{l}\int_0^L f(x)\cos\frac{n\pi x}{l}\, dx$$

The Fourier series of an odd function of period of 2L is a "Fourier sine series" $(-\boldsymbol{l}, \boldsymbol{l})$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l} x$$

$$b_n = \frac{2}{l}\int_0^L f(x)\sin\frac{n\pi x}{l}\, dx$$

The length of the interval (a, b) is $l = b - a$.

Example:9 Find the Fourier sine series of $f(x) = \pi - x, for\ 0 < x < 3$

Solution: let the required half range sine series be,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$

Integrating by parts

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left(-\frac{\pi}{n}\right)$$

$$b_n = \frac{2}{n}$$

Substituting above values in (i), we get

$$f(x) = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$f(x) = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \ldots\ldots..\right]$$

Example:10. Find the half range of cosine series for $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$

Solution: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Now, $a_0 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)dx\right]$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\frac{\pi}{2}} + \left(\pi x - \frac{x^2}{2}\right)_{\frac{x^2}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(x^2)_0^{\frac{\pi}{2}} + (2\pi x - x^2)_{\frac{x^2}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{4} + \pi^2 - \frac{3\pi^2}{4}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{2}$$

$$\therefore \frac{a_0}{2} = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x cos\ nx\ \ dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left((x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\left((\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{x\ \sin nx}{n} + \frac{\cos nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\left((\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$\therefore a_n = \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

Where n is odd

$$\therefore a_n = \frac{2}{n^2\pi}\cos\frac{n\pi}{2} = 0\left(\cos\frac{n\pi}{2} = 0, where\ n\ is\ odd\right)$$

When n is even from (ii)

$$a_n = \frac{2}{n^2\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

For $n = 2$, $a_2 = \frac{1}{\pi}[-2] = -\frac{2}{\pi}$

For $n = 4$, $a_4 = \frac{1}{4\pi}[0] = 0$

For $n = 6$, $a_6 = \frac{1}{9\pi}[-2] = -\frac{2}{9\pi}$

Substituting above values in (i), we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots \ldots \ldots \ldots .\right]$$

Example:11. If $f(x) = \begin{cases} mx & , 0 \le x \le \frac{\pi}{2} \\ m(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$, then show that,

$$f(x) = \frac{4m}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots \ldots \ldots\right\}$$

Solution:

Here we have to find half range sine series in $(0, \pi)$

Let the required half range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ldots \ldots \ldots \ldots \ldots \ldots \ldots (i)$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} mx \sin nx \, dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} m(\pi - x) \sin nx \, dx$$

Integrating by parts

$$= \frac{2m}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2m}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{2m}{\pi}\left[x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}} - \frac{2m}{\pi}\left[(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{2m}{\pi}\left[\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] + \frac{2m}{\pi}\left[\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{4m}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

Substituting above value in (i), it takes the form

$$f(x) = \frac{4m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx$$

$$f(x) = \frac{4m}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots \ldots \ldots\right\}$$

Example:12. Obtain half range sine series to represent $f(x) = lx - x^2$ in the range $(0, l)$

Solution: Let $f(x) = \sum_{n=1}^{\infty} f(x) \sin\frac{n\pi x}{l}$

Where, $b_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l} \, dx$

$$= \frac{2}{l}\int_0^{l} lx - x^2 \sin\frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4l^2}{n^3\pi^3}[\cos n\pi - 1]$$

$$= \frac{-4l^2}{n^3\pi^3}[(-1)^n - 1]$$

$$= 0 \; if \; n \; is \; even$$

$$= \frac{8l^2}{n^3\pi^3} \; if \; n \; is \; odd$$

Hence, half range sine series for $f(x)$ is

$$f(x) = \frac{8l^2}{n^3\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots \ldots..\right]$$

Example:13 Find Fourier series of $f(x) = x^2, 0 < x < c$

Solution:

Let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Comparing $0 < x < c \; with \; 0 < x < l, we \; get \; l = c$

Now, $a_0 = \frac{2}{c}\int_0^c f(x)dx$

$$= \frac{2}{c}\int_0^c x^2 dx$$

$$= \frac{2}{c}\left[\frac{x^3}{3}\right]_0^c$$

$$= \frac{2}{c}\left[\frac{c^3}{3}\right]$$

$$a_0 = \frac{2c^2}{3}$$

$$\frac{a_0}{2} = \frac{c^2}{3}$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{2}{c}\int_0^c x^2\cos\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{2}{c}\left[(x^2)\left(\frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right) - (2x)\left(-\frac{\cos\left(\frac{n\pi x}{c}\right)}{\frac{n^2\pi^2}{c^2}}\right) + (2)\left(-\frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n^3\pi^3}{c^3}}\right)\right]_0^c$$

$$= \frac{2}{c}\left[\frac{2c^2}{n^2\pi^2}x\cos\left(\frac{n\pi x}{c}\right)\right]_0^c$$

$$= \frac{4c}{n^2\pi^2}\left[x\cos\left(\frac{n\pi x}{c}\right)\right]_0^c$$

$$= \frac{4c}{n^2\pi^2}[c\cos n\pi]$$

$$\therefore\ a_n = \frac{4c^2(-1)^n}{n^2\pi^2}$$

Substituting above values in (i), we get

$$f(x) = \frac{c^2}{3} + \frac{4c^2}{n^2\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{\cos\left(\frac{n\pi x}{c}\right)}{n^2}$$

- **EXERCISE:**

1. Find a Fourier sine series for $f(x) = k$ in $0 < x < \pi$.
   [ANS: $f(x) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{sinnx}{n}$]

2. Find half range cosine series for

$$f(x) = kx \qquad , 0 \le x \le \frac{l}{2}$$

$$= k\,(l - x) \quad , \frac{l}{2} \le x \le l$$

[ANS: $f(x) = \frac{\pi}{2}\sin x - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{2n\sin 2nx}{\pi(4n^2-1)^2}$]