Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B. Tech Programme (All Branches)

Mathematics– 1 (303191101)

Unit – 1 MATRICES (Lecture Note)

Matrix:

A Matrix is a rectangular array of numbers (or functions) enclosed in brackets. These number or functions are called entries or elements of the matrix.

For example:

$$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 5 \end{bmatrix}, \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$$

Trace of a matrix:

If A is a square matrix, the trace of A, denoted by $tr(A)$ and is defined to be the sum of entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

For example:

If $A = \begin{bmatrix} 4 & 5 \\ 10 & 6 \end{bmatrix}$, then $tr(A) = 4 + 6 = 10$.

Symmetric matrix: - For any **square** matrix A, if $A = A^T$,then it is known as symmetric matrix.

For example:

If $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 7 \\ 4 & 7 & 4 \end{bmatrix}$ $then\ A^T = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 7 \\ 4 & 7 & 4 \end{bmatrix}$

Here, we can see that so $A = A^T$; hence A is symmetric matrix.

Skew-symmetric matrix: - For any **square** matrix A, if $A = -A^T$then it is known as Skew symmetric matrix.

For example:

$$A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} = -A^T$$

Here, we can see that $A = -A^T$; so A is skew-symmetric matrix.

Singular and non-singular matrix: -

For any square matrix A, if $|A| \neq 0$, then it is known as non-singular matrix and if $|A| = 0$ then it is known as singular matrix.

Example 1: - If $A = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \Rightarrow |A| = 8 - 8 = 0 \Rightarrow \text{Singular Matrix}$

Example 2: - If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0 \Rightarrow \text{Non} - \text{Singular Matrix}$

Orthogonal Matrix: The matrix is said to be an orthogonal matrix if the product of a matrix and its transpose gives an identity value. i.e. $AA^T = I$

Example: Given A is an orthogonal matrix because

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ Then } A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

System of linear equation

Linear Equations: Any straight line in the xy-plane can be represented algebraically by equation of the form $ax + by = c$, where a & b are real numbers.

A **system of linear equation** is a collection of one or more linear equations involving the same variables.

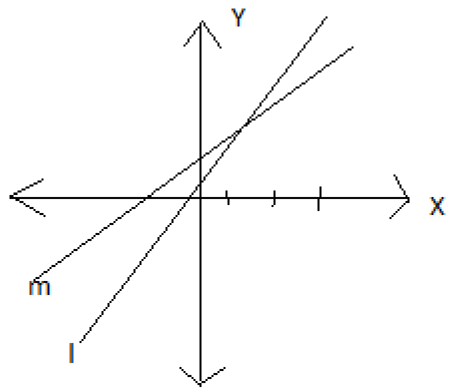
A linear system of m linear equations in n variables: An arbitrary system of m linear equations in n variables $x_1, x_2, x_3, \ldots\ldots x_n$ is a set of equations of the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \; (i = 1,2,3, \ldots, m, j = 1,2,3, \ldots, n)$$

A system of linear equations has either

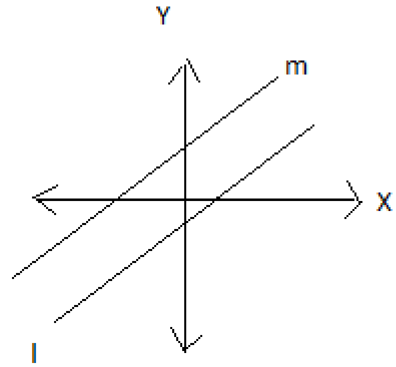
1. No solutions, or
2. Exactly one solution, or
3. Infinitely many solutions

Geometrical representation



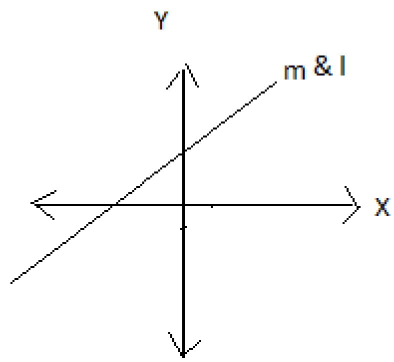
$$x - 2y = -1 \quad .... \quad l$$
$$-x + 3y = 3 \quad .... \quad m$$

Exactly one solution



$$x - 2y = -1 \quad .... \, l$$
$$-x + 2y = 3 \quad .... \, m$$

No solution



$$x - 2y = -1 \quad .... \, l$$
$$-x + 2y = 1 \quad .... \, m$$

Infinitely many solutions

Notes

(i) The system is said to be consistent if we get infinitely many solutions or unique solution.
(ii) The system is said to be inconsistent if we get No solution.

<u>Augmented matrix</u>

A system of m equations in n unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{bmatrix}$$

This is known as augmented matrix.

For example: Find the augmented matrix for each of the following system of linear equations:

$$\begin{aligned} 2x_1 + \quad +2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Then, augmented matrix is given by $\begin{bmatrix} 2 & 0 & 2 & | & 1 \\ 3 & -1 & 4 & | & 7 \\ 6 & 1 & -1 & | & 0 \end{bmatrix}$.

<u>Condition of Consistency for non-homogeneous system:</u>

(1) If there is a zero row to left of the augmentation bar but the last entry of this row is non-zero then the system has **no solution**.

For example: $\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 4 & | & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & | & \mathbf{4} \end{bmatrix}$

(2) If at least one of the columns on the left of the augmentation bar has zero element pivot entry, then the system has **infinitely many solutions**.

For example: $\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 4 & | & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & | & \mathbf{0} \end{bmatrix}$

(3) If all the rows having the leading entry 1 then the system has **unique solution.**

For example: $\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 4 & | & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & | & \mathbf{4} \end{bmatrix}$

<u>Row-Echelon (RE) form and Row-Reduced Echelon (RRE) form of a matrix</u>

Definition: A rectangular matrix is in row-echelon form (or echelon form) if it has the following three properties:

1. The **first** element in each row must be **non-zero** and equals to **1**, that is called **leading entry 1**.
2. All the **leading 1's** must be on the **right-hand** side of the matrix.
3. If any **zero row** is available, then it must be **below** to the **all-leading 1**.

If the matrix satisfies the **4th property** (i.e., In each column except leading 1 if all entries are zero) then **row-echelon form (RE form)** becomes **row-reduced echelon form (RRE form)**.

Example 1: Which of the following matrices are in row-echelon or echelon form?

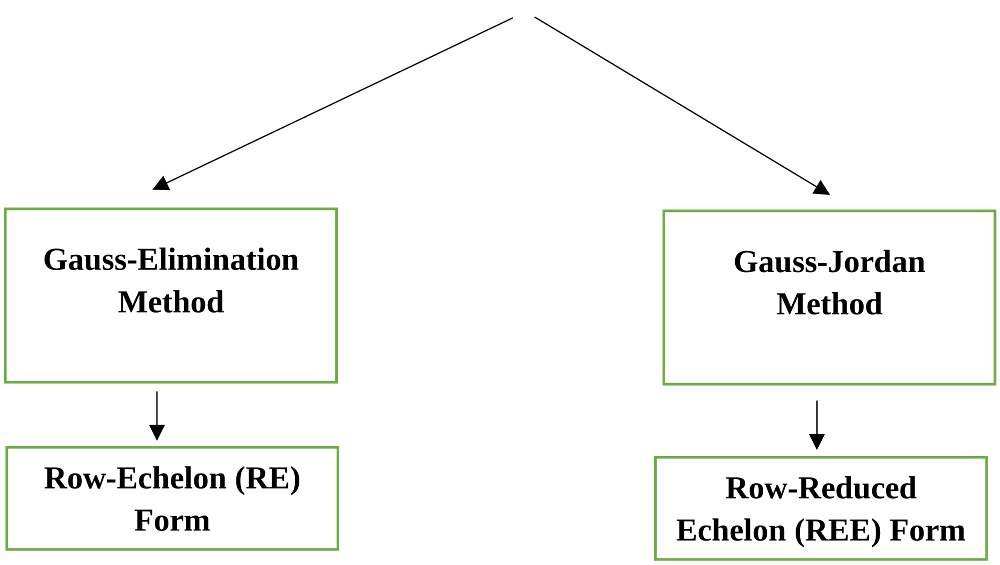
$$\text{(a)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (b)}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (c)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \text{ (d)}\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ (e)}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: Which of the following matrices are in reduced row-echelon or reduced echelon form?

$$\text{(a)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(b)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(c)}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(d)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(e)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{(f)}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (g)}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>Methods of solving system of linear equations</u>



Examples: Solve the following system using Gauss-Elimination Method

Case-1: Unique solution

Example 1: Solve the following system by gauss- Elimination method

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$2x + 4y + 7z = 30$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{array}\right]$$

Now, to convert the given augmented matrix in row-echelon form we apply elementary operations as following.

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equation is

$$x + y + z = 6$$
$$y + 2z = 8$$
$$z = 2$$

Case-2: No solution

Example 2: Solve the following system of equation by Gauss elimination.

$$-2b + 3c = 1$$
$$3a + 6b - 3c = -2$$
$$6a + 6b + 3c = 5$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 0 & -2 & 3 \\ 3 & 6 & -3 \\ 6 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_1 \to \left({}^{1}\!/_{3}\right) R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_3 \to R_3 - 6R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

By using back substitution of $z = 2$ in $y + 2z = 8$, we get $y = 4$ and $z = 2$ & $y = 4$ in $x + y + z = 6$ we get $x = 0$.

$x = 0, y = 4, z = 2$ is **unique solution** of given system.

Case-3: **Infinitely many solutions**

Example 3: Solve the following system by Gauss elimination method.

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 15R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$R_2 \to (-1/6)R_1, R_3 \to (-1/6)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

The system of linear equation is

$$a + 2b - c = -2/3$$

$$-2b + 3c = 1$$

$0a + 0b + 0c = 6$ is not possible.

This shows that the system has **no solution**.

Example 4:

Solve the following system by gauss elimination method.

$$\frac{-1}{x} + \frac{3}{y} + \frac{4}{z} = 30$$

$$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9$$

$$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10$$

Solution:

Let $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$

Then the system of equations

$$-u + 3v + 4w = 30$$

$$3u + 2v - w = 9$$

$$2u - v + 2w = 10$$

The matrix form of the system is

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & -3 & | -1 \\ 0 & 1 & -3 & | -2 \\ 0 & 0 & 0 & | \ 0 \end{bmatrix}$$

The corresponding system of equations is

$$x + y - 3z = -1$$

$$y - 3z = -2$$

Assigning the free variable z an arbitrary value t,

$y = 3t - 2,$

$x = -1 - 3t + 2 + 3t = 1$

Hence, $x = 1, y = 3t - 2, z = t$ is solution of the given system of equations.

Since t is arbitrary real number, The system has **infinitely many solutions.**

Example 5: Consider the following system

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

For what values of λ and μ the system has (i) infinitely many solutions (ii) unique solution and (iii) no solution.

<u>Solution:</u> The Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & | \ 6 \\ 1 & 2 & 3 & | 10 \\ 1 & 2 & \lambda & | \ \mu \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

The augmented matrix is

$$\begin{bmatrix} -1 & 3 & 4 & | 30 \\ 3 & 2 & -1 & | \ 9 \\ 2 & -1 & 2 & | 10 \end{bmatrix}$$

$$R_1 \to (-1)R_1$$

$$\begin{bmatrix} 1 & -3 & -4 & | -30 \\ 3 & 2 & -1 & | \ 9 \\ 2 & -1 & 2 & | \ 10 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & -3 & -4 & | -30 \\ 0 & 11 & 11 & | \ 99 \\ 0 & 5 & 10 & | \ 70 \end{bmatrix}$$

$$R_2 \to \left(\frac{1}{11}\right) R_2, R_3 \to \left(\frac{1}{5}\right) R_3$$

$$\begin{bmatrix} 1 & -3 & -4 & | -30 \\ 0 & 1 & 1 & | \ 9 \\ 0 & 1 & 2 & | \ 14 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -3 & -4 & | -30 \\ 0 & 1 & 1 & | \ 9 \\ 0 & 0 & 1 & | \ 5 \end{bmatrix}$$

The corresponding system of equations is

$$u - 3v - 4w = -30$$

$$v + w = 9$$

$$w = 5$$

By doing back substitution we get

$$v + 5 = 9 \Rightarrow v = 4 \Rightarrow y = \frac{1}{4}$$

$$u - 12 - 20 = -30 \Rightarrow u = 2 \Rightarrow x = \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 1 & \lambda-1 & | & \mu-6 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & \lambda-3 & | & \mu-10 \end{bmatrix}$$

(i) If $\lambda - 3 = 0$ and $\mu - 10 = 0$, that is if $\lambda = 3$ and $\mu = 10$ then the system has infinitely many solutions.

(ii)If $\lambda - 3 = 0$ then the system has a unique solution. That is $\lambda \neq 3$ and μ can possess any real value.

(iii)If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, that is if $\lambda = 3$ and $\mu \neq 10$ then the system does not have any solution.

Hence, $x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{5}$ is required **unique solution** of the system.

Exercise: Solve the following system of equations by using Gauss elimination method.

(1) $x + y + 2z = 9$

$2x + 4y - 3z = 1$

$3x + 6y - 5z = 0$

Ans: $x = 1, y = 2, z = 3$

(2)$3x + y - 3z = 13$

$2x - 3y + 7z = 5$

$2x + 19y - 47z = 32$

Ans: No solution as the augmented matrix in row-echelon form is

$$\begin{bmatrix} 1 & 1/3 & -1 & | & 13/3 \\ 0 & 1 & -27/11 & | & 1 \\ 0 & 0 & 0 & | & 5 \end{bmatrix}$$

(3) $2x + 2y + 2z = 0$

$-2x + 5y + 2z = 1$

$8x + y + 4z = -1$

Ans: Infinitely many solutions. The solution set is $\{(\frac{-3k-1}{7}, \frac{1-4k}{7}, k)/$ k $\in$R}.

Examples: Solve the following system using Gauss-Jordan Method

Case-1: Unique Solution

(1) $x + y + 2z = 8$

$-x - 2y + 3z = 1$

$3x - 7y + 4z = 10$

<u>Solution:</u> The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix is

Case-2: Infinitely many Solutions

(2) $x + 2y - 3z = -2$

$3x - y - 2z = 1$

$2x + 3y - 5z = -3$

<u>Solution:</u> The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$R_2 \to R_2 + R_1, R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_2 \to (-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_3 \to R_3 + 10R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$R_3 \to (-1/52)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 \to R_2 + 5R_3, R_1 \to R_1 - 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equation is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \to (-1/7)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations gives

$$x - z = 0$$

$$y - z = -1$$

Assigning the free variable z an arbitrary value t,

$$z = t$$

$$x = z = t$$

$$y = z - 1 = t - 1$$

$x=3, y=1, z=2$ which is a unique solution of the given system of equations.

Case-3: No Solution

(3) $x+y+z=1$

$3x-y-z=4$

$x+5y+5z=-1$

Solution: The matrix form of the system is

$$\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & -1\\ 1 & 5 & 5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 4\\ -1\end{bmatrix}$$

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 3 & -1 & -1 & 4\\ 1 & 5 & 5 & -1\end{array}\right]$$

$$R_2 \to R_2-3R_1, R_3 \to R_3-R_1$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & -4 & -4 & 1\\ 0 & 4 & 4 & -2\end{array}\right]$$

$$R_3 \to R_3+R_2$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & -4 & -4 & 1\\ 0 & 0 & 0 & -1\end{array}\right]$$

Observe the 3rd row in last matrix gives

$0x+0y+0z=-1$ which is not possible.

This shows that the system has **no solution**.

Hence, $x=t, y=t-1, z=t$ is solution of the given system of equations and since t is arbitrary real number, The system has **infinitely many solutions.**

Exercise: Solve the following system of equations by using Gauss- Jordan method.

(1) $2y+3z=7$

$3x+6y-12z=-3$

$5x-2y+2z=-7$

Ans: Unique solution $x=-1, y=2, z=1$

(2) $-2y+3z=1$

$3x+6y-3z=-2$

$6x+6y+3z=5$

Ans: No Solution as the augmented matrix in row-echelon form is

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & -2/3\\ 0 & 1 & -3/2 & -1/2\\ 0 & 0 & 0 & 6\end{array}\right]$$

(3) $x-y+z=1$

$2x+y-z=2$

$5x-2y+2z=5$

Ans: Infinitely many solutions. The solution set is $\{(1,k,k)/\ k \in R\}$.

HOMOGENEOUS EQUATIONS

A system of linear equations in terms of $x_{1,}x_{2,}x_{3,} \ldots x_n$ having the matrix form AX=O, where A is $m \times n$ coefficient matrix, X is $n \times 1$ column matrix, O is a $m \times 1$ zero column matrix is called a system of homogeneous equations.

For example: (i) $x + y + z = 0$

$$x + 2y - z = 0$$
$$x + 3y + 2z = 0$$

(ii) $x + y = 0$

$$x + 2y = 0$$

Homogeneous equations are never inconsistent. They always have the solution “all variables = 0”. The solution (0, 0, …, 0) is often called the **trivial solution**. Any other solution is called **nontrivial solution.**

Example-1: Solve the following system:

$$4x + 3y - z = 0$$

$$3x + 4y + z = 0$$

$$5x + y - 4z = 0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$[A|B]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] R_1 \to \frac{1}{4} R_1$$

=

Example-2: Solve the following system

$$\begin{aligned} -2x + 2y - 3z &= 0 \\ 2x + y - 6z &= 0 \\ -x - 2y + 2z &= 0 \\ 3x + y + 4z &= 0 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} -2 & 2 & -3 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] R_1 \to -\frac{1}{2} R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] R_2 \to \frac{1}{3} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 0 & 7/4 & 7/4 & 0 \\ 0 & -11/4 & -11/4 & 0 \end{array}\right] R_2 \to \frac{4}{7}R_2$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - z = 0,\ y + z = 0,\ 0 = 0.$

The last equation does not give any information about the equations**.**

Let

$z = k \Rightarrow y = -k \ and \ x = k$.

$\therefore the\ solution\ set\ is \{(k, -k, k) / k \in R\}$

Exercise: Solve the following system of equations.

(1) $x + y - z + w = 0$ $x - y + 2z - w = 0$ $3x + y \quad + w = 0$	Ans: Infinitely many solutions. The solution set is $\{(t/4, -7t/4, t) / \ t \in R\}$.
(2) $2x + y + 3z = 0$ $x + 2y \quad = 0$ $y + z = 0$	Ans: Trivial solution $x = 0, y = 0, z = 0$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2, \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -11/2 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] R_3 \to -\frac{2}{11}R_3$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 3/2R_3, \\ R_2 \to R_2 + 3R_3 \\ R_4 \to R_4 - 23/2R_3 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The required solution is $x = 0, y = 0, z = 0$ which is trivial solution.

Rank of a Matrix

The positive integer r is said to be a rank of a matrix A if it possesses the following properties:

(1) There is at least one minor of order r which is non-zero.

(2) Every minor of order greater than r is zero.

- **Notes:**

1. Rank of matrix A is denoted by $\rho(A)$
2. The rank of matrix remains unchanged by elementary transformation
3. $\rho(A^T) = \rho(A)$
4. The rank of the product of two matrices always less than or equal to the rank of either matrix (i.e., $\rho(AB) \leq \rho(A)$ or $\rho(AB) \leq \rho(B)$).

Methods for finding Rank of a Matrix

❖ **Method-1: Rank of a Matrix by Determinant Matrix**

Consider a square matrix A of order r.

- **Step-1:** Find the determinant of A. If $det(A) \neq 0$ then $\rho(A) = r$. Otherwise $\rho(A) < r$.
- **Step-2:** Find the all-possible minors of order $r-1$. If any one of them is non-zero then order is $r-1$, otherwise $\rho(A) < r-1$.
- **Step-3:** By continuing this process upto the non-zero determinant.

Example 1: Find the rank the following matrices by determinant method:

(1) $\boldsymbol{A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}}$

Solution: Given, $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ then $det(A) \neq 0$. Hence, the $\rho(A) = 3$

(2) $\boldsymbol{A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}}$

Solution: Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ then $det(A) = 0$. Hence, the rank of A is less than 3.

Now, minor of 1 $= \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1 \neq 0$. Hence, $\rho(A) = 2$.

(3) $\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{8} & \mathbf{4} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-\frac{3}{2}} \end{bmatrix}$

Solution: Given, $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$ then $det(A) = 0$. Hence $\rho(A) < 3$.

Consider all the minors of order 2, i.e.,

$$\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 0, \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ -2 & -\frac{3}{2} \end{vmatrix} = 0$$

Here, all the minors of order 2 are zero. There rank is less than 2. Hence, $\rho(A) = 1$.

(4) $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{5} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{6} & \mathbf{-7} \end{bmatrix}$

Solution: Here, the order of matrix A is 3×4. Hence the rank of A is maximum 3 as we can find the square matrix of order 3. Therefore, consider all the minors of order 3, i.e.,

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & -2 & 6 \end{vmatrix} = 0, \begin{vmatrix} 2 & -1 & -4 \\ 4 & 3 & 5 \\ -2 & 6 & -7 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & 5 \\ -1 & -2 & -7 \end{vmatrix} = 0, \begin{vmatrix} 1 & -1 & -4 \\ 2 & 3 & 5 \\ -1 & 6 & -7 \end{vmatrix} = -120$$

Here, one minor of rank 3 is not equal to zero. Hence, $\rho(A) = 3$.

❖ **<u>Method-2: Rank of a Matrix by Row Echelon Form</u>**

The Rank of a Matrix in Row Echelon Form is equal to the number of non-zero rows of the matrix.

For example: $A = \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{vmatrix}$; the matrix A is in Row Echelon form with two non-zero rows. Hence, rank of matrix A is 2.

Example 1: Find the rank the following matrices by reducing to Row Echelon Form:

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(1) $\mathbf{A}=\begin{bmatrix} \mathbf{5} & \mathbf{3} & \mathbf{14} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{0} \end{bmatrix}$

Solution: Given

$$A=\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

By applying row-operations

$$R_{13} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$$R_3-5R_1 \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$$

$$R_3-8R_2 \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

$$\left(-\frac{1}{12}\right)R_3 \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

The equivalent matrix is in Row-Echelon Form.

Number of non-zero rows $=3$ .Hence, $\rho(A)=3$

</td>
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(2) $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$

Solution: Given

$$A=\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

By applying row-operations

$$R_2+2R_1, R_3-R_1 \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_{24} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$$

$$R_3+2R_2, R_4-3R_2 \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equivalent matrix is in Row-Echelon Form.

Number of non-zero rows $=2$. Hence, $\rho(A)=2$

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Exercise: (1) Find the ranks of A, B, AB and verify $\rho(AB) \leq \rho(A)$ or $\rho(AB) \leq \rho(B)$) where $A = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{vmatrix}$, $B = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$

(2) Find the rank the following matrices by reducing to Row Echelon Form:

(I) $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$, (II) $A = \begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

Important Results

(1) If $\rho(A) \neq \rho(A|B)$ then the system is inconsistent.
(2) If $\rho(A) = \rho(A|B)$ then the system is consistent.
(3) If $\rho(A) < n$ then there are infinitely many solutions (n is the number of unknowns)
(4) If $\rho(A) = n$ then there is a unique solution.

Example: Find the number of parameters in the general solution of $AX = O$ if A is a 5×7 matrix of rank 3.

Solution: Here, $\rho(A) = 3$ and $n = 7$. Hence, number of parameters = $n - \rho(A) = 7 - 3 = 4$.

Eigen values and Eigen vectors

Let A be $n \times n$ matrix, then there exists a real number λ and a nonzero vector X such that

$$AX = \lambda X$$

then, λ is called as the eigen value or characteristic value or proper roots of the matrix A, and X is called as eigen vector or characteristic vector or real vector corresponding to eigen value λ of the matrix A.

Notes

1. An eigen vector is never the zero vector.
2. The matrix [A – λI_n] is known as the **characteristic matrix** of A.
3. The determinant of (A – λI_n) after expansion gives the polynomial in λ, it is known as the **characteristic polynomial** of the matrix A of order $n \times n$ and is of degree n.
4. $|A - \lambda I_n|$=0 is called the **characteristic equation** of matrix A.
5. The root of the characteristic equation is known as **characteristic value** or **eigenvalue** of the matrix.
6. The set of all characteristic roots (eigen values) of the matrix A is called the **spectrum of A.**
7. Let A be $n \times n$ matrix and λ be an eigen value for A. Then the set $E_\lambda = \{X/AX = \lambda X\}$ is called the **eigen space of λ.**

Results

1. The eigen values of a diagonal matrix are its diagonal elements.
2. The sum of eigen values of an $n \times n$ matrix is its trace and their product is $|A|$.
3. For the upper triangular (lower triangular) $n \times n$ matrix A, the eigen values are its diagonal elements.

Example 1: If $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$, find the eigen values for the given matrices:

(i) A, (ii) A^T, (iii) A^{-1}, (iv) $4A^{-1}$, (v) A^2, (vi) $A^2 - 2A + I$,

(vii) $A^3 + 2I$

Solution: Given, $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$

The characteristic equation of matrix A is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0$$

$$\therefore \lambda = 5 \text{ or } \lambda = -2$$

Eigenvalues of $A = \lambda$	$5, -2$
Eigenvalues of $A^T = \lambda^T$	$5, -2$
Eigenvalues of $A^{-1} = \lambda^{-1}$	$\frac{1}{5}, -\frac{1}{2}$
Eigenvalues of $4A^{-1} = 4\lambda^{-1}$	$\frac{4}{5}, -2$
Eigenvalues of $A^2 = \lambda^2$	$25, 4$
Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$	$16, 9$
Eigenvalues of $A^3 + 2I = \lambda^3 + 2$	$127, -6$

Example 2: Find the eigen values of $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$

Solution: Given $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$, then the characteristic equation of matrix A is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(8-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 11\lambda + 18 = 0 \Rightarrow (\lambda - 9)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 9 \text{ or } \lambda_2 = 2$$

<u>Types of Eigen Values</u>

Type-1	Type-2	Type-3
Eigenvalues are non-repeated, whether matrix is symmetric or non-symmetric	Eigenvalues are repeated and the matrix is non-symmetric	Eigenvalues are repeated and the matrix is symmetric

Example 3: Find the eigen values and eigen vector of the matrix A= $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Solution:</u>

The characteristic equation is $|A-\lambda I_n|=0$

$$\begin{vmatrix} -2-\lambda & -8 & -13 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$S_1 = tr(A) = -2+4+1 = 3$

$S_2 =$ Sum of minors of diagonal entries

$$= \begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -12 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 4-2+0 = 2$$

$$|A| = -2(4) + 8(1) - 12(0) = -8+8 = 0$$

$\therefore$ characteristic equation is

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$
$$\Rightarrow \lambda\left(\lambda^2 - 3\lambda + 2\right) = 0$$
$$\Rightarrow \lambda = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0$$
$$\Rightarrow \lambda = 0 \text{ or} (\lambda-2)(\lambda-1) = 0$$
$$\Rightarrow \lambda = 0 \text{ or } \lambda = 1 \text{or } \lambda = 2$$

Here, one can observe that all eigenvalues are non-repeated and matrix is non-symmetric.

When $\lambda_1 = 0$

$$[A-\lambda I \mid O] = \left[\begin{array}{ccc|c} -2 & -8 & -12 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to -1/2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 8/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We suppose $z = k, y = 0, x + 4z = 0$

$$\therefore z = k, y = 0, x = -4z = -4k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda_3 = 2$

$$[A-\lambda I \mid O] = \left[\begin{array}{ccc|c} -2-2 & -8 & -12 & 0 \\ 1 & 4-2 & 4 & 0 \\ 0 & 0 & 1-2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -4 & -8 & -12 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] R_1 \to -1/4R_1$$

$$=\left[\begin{array}{ccc|c}1 & 4 & 6 & 0\\ 1 & 4 & 4 & 0\\ 0 & 0 & 1 & 0\end{array}\right] R_2 \to R_2 - R_1$$

$$=\left[\begin{array}{ccc|c}1 & 4 & 6 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

Therefore, we suppose

$x+4y+6z=0, -2z=0, y=k$

$\therefore z=0, y=k, x=-4k$

Therefore, eigen vector space is

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-4k\\ k\\ 0\end{bmatrix}=k\begin{bmatrix}-4\\ 1\\ 0\end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1=0$ is

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-4\\ 1\\ 0\end{bmatrix}$$

When $\lambda_2=1$

$$[A-\lambda I \mid O]=\left[\begin{array}{ccc|c}-2-1 & -8 & -12 & 0\\ 1 & 4-1 & 4 & 0\\ 0 & 0 & 1-1 & 0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}-3 & -8 & -12 & 0\\ 1 & 3 & 4 & 0\\ 0 & 0 & 0 & 0\end{array}\right] R_1 \to -1/3R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 1 & 2 & 4 & 0\\ 0 & 0 & -1 & 0\end{array}\right] R_2 \to R_2 - R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\end{array}\right] R_3 \to -R_3$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \begin{array}{l}R_1 \to R_1 - 3R_3\\ R_2 \to R_2 - R_3\end{array}$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

We suppose $z=0, y=k, x+2y=0$

$\therefore z=0, y=k, x=-2z=-2k$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-2k\\ k\\ 0\end{bmatrix}=k\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$$

Therefore, eigen vector space for $\lambda_3=2$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$$

$=\left[\begin{array}{ccc\|c} 1 & 8/3 & 4 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to R_2 - R_1$ $=\left[\begin{array}{ccc\|c} 1 & 8/3 & 4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to 3R_2$	

Algebraic multiplicity and Geometric multiplicity

Let A be $n \times n$ matrix and λ be an eigen value for A. If λ occurs $(k \geq 1)$ times then k is called the **Algebraic multiplicity** of λ, and the number of basis vectors is called **Geometric multiplicity**.

Example: Find eigen values and eigen vectors of the matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$$

$$= (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)] - 2[2(-\lambda)-(-1)(-6)] - 3[2(-2)-(-1)(1-\lambda)]$$

$$= (-2-\lambda)[-\lambda+\lambda^2-2] - 2[-2\lambda-6] - 3[-4+1-\lambda]$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45)$$

$$\therefore -(\lambda^3 + \lambda^2 - 21\lambda - 45) = 0$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$$

Algebraic Multiplicity of $\lambda = -3$ is 2 and of $\lambda = 5$ is 1. We solve the following homogeneous system: $\therefore [A-\lambda I]X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	
Case I: When $\lambda_1 = 5$ $\therefore [A-\lambda I \mid O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ = $\left[\begin{array}{ccc\|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] R_1 \leftrightarrow R_3$ = $\left[\begin{array}{ccc\|c} -1 & -2 & -5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] R_1 \to -R_1$ = $\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$ = $\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right] R_2 \to -1/8R_2$ =	Case II : When $\lambda_2 = -3, \lambda_3 = -3$ $\therefore [A-\lambda I \mid O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ = $\left[\begin{array}{ccc\|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$ = $\left[\begin{array}{ccc\|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is in Row-Echelon from. We suppose $x_2 = k_1, x_3 = k_2, x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2k_1 + 3k_2$ Therefore, eigen space is for $\lambda_2 = -3, \lambda_3 = -3$ is $\left\{k_1(-2,1,0) + k_2(3,0,1) / k_1, k_2 \in R\right\}$ Hence, Geometric multiplicity of $\lambda_2 = -3$ is 2 and of $\lambda = 5$ is 1.

$\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right]$ $\begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 16R_2 \end{array}$ = $\left[\begin{array}{ccc\|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is in Row-Echelon from. We suppose $\begin{array}{l} x_3 = k, x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k, \\ x_1 + x_3 = 0 \Rightarrow x_1 = -k \end{array}$ Therefore, eigen space is for $\lambda_1 = 5$ is $\{k(-1,-2,1) / k \in R\}$	

Example: Find eigen values and eigen vectors of the matrix. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.
Solution: The characteristic equation is $\|A - \lambda I_n\| = 0$. $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$ $= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$ $= -\lambda^3 + 3\lambda + 2 = -(\lambda^3 - 3\lambda - 2)$ $\therefore -(\lambda^3 - 3\lambda - 2) = 0$ $\therefore \lambda^3 - 3\lambda - 2 = 0$ $\Rightarrow (\lambda - 2)(\lambda + 1)^2 = 0$ $\Rightarrow \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$

Algebraic Multiplicity of $\lambda=-1$ is 2 and of $\lambda=2$ is 1.	
Case-1: $\lambda_1=2$ $\therefore [A-\lambda I \mid O]=\begin{bmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}$ $=\left[\begin{array}{ccc\|c}-2 & 1 & 1 & 0\\ 1 & -2 & 1 & 0\\ 1 & 1 & -2 & 0\end{array}\right]$ $R_2 \leftrightarrow R_1$ $=\left[\begin{array}{ccc\|c}1 & -2 & 1 & 0\\ -2 & 1 & 1 & 0\\ 1 & 1 & -2 & 0\end{array}\right]$ $R_2 \to R_2+2R_1$, $R_3 \to R_3-R_1$ $=\left[\begin{array}{ccc\|c}1 & -2 & 1 & 0\\ 0 & -3 & 3 & 0\\ 0 & 3 & -3 & 0\end{array}\right]$ $R_2 \to -1/3R_2$ $=\left[\begin{array}{ccc\|c}1 & -2 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 3 & -3 & 0\end{array}\right]$ $R_1 \to R_1+2R_2$, $R_3 \to R_3-3R_2$ $=\left[\begin{array}{ccc\|c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$ Let $x_3=k, x_2-x_3=0, \Rightarrow x_2=k, x_1-x_3=0, x_1=k$. Therefore, eigen space is for $\lambda_1=2$ is $\left\{k\left(1,1,1\right)/\, k\in R\right\}$	Case-2: $\lambda_2=-1, \lambda_3=-1$ $\therefore [A-\lambda I \mid O]=\begin{bmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}$ $=\left[\begin{array}{ccc\|c}1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\end{array}\right]$ $R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$ $=\left[\begin{array}{ccc\|c}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$ Let $x_3=k_{1,} x_2=k_{2,}$ $\Rightarrow x_1+x_2+x_3=0 \Rightarrow x_1=-k_1-k_{2,}.$ Therefore, eigen space is for $\lambda_2=-1, \lambda_3=-1$ is $\left\{k_1\left(-1,0,1\right)+k_2\left(-1,1,0\right)/\, k_1,k_2\in R\right\}$ Hence, Geometric Multiplicity of $\lambda_2=-1$ is 2 and $\lambda_1=2$ of is 1.

Example: Determine algebraic and geometric multiplicity of matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

Answer: $\lambda = 1,2,2$ therefore algebraic multiplicity of $\lambda = 2$ is 2 and geometric multiplicity is 1. For $\lambda = 1$ A.M. is 1 and G.M. is 1.

Note

Theorem: Every square matrix can be decomposed as a sum of symmetric and skew-symmetric matrices.

Proof: Let A be $m \times n$ matrix.

Let $B = \frac{1}{2}(A + A^T)$ and $C = = \frac{1}{2}(A - A^T)$ be two matrices.

Obviously, $A = B + C$

Now, $B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}[(A + A^T)]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T + A) = B$

As $B^T = B$, B is symmetric.

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[(A - A^T)]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -C$$

Therefore, $C^T = -C$, C is skew- symmetric.

Therefore, A is a sum of symmetric and skew-symmetric matrices.

Caley –Hamilton Theorem

Every square matrix satisfies its own characteristic equation i.e. The theorem states that, for a square matrix A of order n, if $|A - \lambda I_n| = 0$.

Example (i): Verify Caley-Hamilton theorem and hence find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and A^4.
Solution: The characteristic equation for given matrix is $|A - \lambda I_2| = 0.$ $\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$ $\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$ $\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

Now, by putting $\lambda = A$, we have

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, Cayley-Hamilton theorem verified.

Now, by using Cayley-Hamilton theorem, we have

$A^2 - 4A - 5I = 0$, by applying A^{-1} on both the sides

$$A^{-1}\left(A^2 - 4A - 5I\right) = A^{-1}(0)$$
$$\Rightarrow A - 4I - 5A^{-1} = 0$$
$$\Rightarrow 5A^{-1} = A - 4I$$
$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

And for A^4, applying A^2 both the sides

$$A^2\left(A^2 - 4A - 5I\right) = A^2(0)$$
$$\Rightarrow A^4 - 4A^3 - 5A^2 = 0$$
$$\Rightarrow A^4 = 4A^3 + 5A^2$$
$$\Rightarrow A^4 = 4\begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix} + 5\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} 209 & 416 \\ 208 & 417 \end{bmatrix}$$

Example (ii): Find the characteristics equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence prove that

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

Solution: The characteristics equation is

$$|A-\lambda I|=\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$$

$$\therefore \lambda^3-5\lambda^2+7\lambda-3=0$$

By Caley-Hamilton Theorem

$$\therefore A^3-5A^2+7A-3I=0 \qquad \text{..............(1)}$$

Now,

$$A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$$

$$=A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A-3I)+A^2+A+I$$

$$=A^2+A+I \qquad using\ (1)$$

$$\therefore A^2+A+I=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$$

Exercise: (1) Verify Caley-Hamilton theorem and hence find the inverse of $A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$ and A^4.

(2) Compute $A^9-6A^8+10A^7-3A^6+A+I$, where $A=\begin{bmatrix}1&2&3\\-1&3&1\\1&0&2\end{bmatrix}$ (Answer: $\begin{bmatrix}2&2&3\\-1&4&1\\1&0&3\end{bmatrix}$)

Diagonalization of a matrix:

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

OR

If $n \times n$ matrix A has a basis of eigenvectors, then $D = P^{-1}AP$ is diagonal, with the eigenvalues of A as the entries on the main diagonal. Here, P is the matrix with these eigenvectors as column vectors.

Also, $D^n = P^{-1}A^nP$ and $A^n = PD^nP^{-1}$

Example (i): Find a matrix P that diagonalizes matrix A and determine $P^{-1}AP$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Solution (i): $$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I_n| = 0$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1,2,3$$

For $\lambda = 1$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=

$$= \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 4/3R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore z = k, y - z = 0 \;\&\; x - 2/3z = 0$$

$$\Rightarrow z = k, y = k, x = 2/3k$$

$$\therefore (x,y,z) = k(\frac{2}{3},1,1); k \in R$$

$$\therefore (x,y,z) = 3k(2,3,3); k \in R \quad (\square\ 3k = k')$$

$$E_1 = \{k'(2,3,3) / k' \in R\}$$

For $\lambda = 3$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] R_1 \to -1/2R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] R_2 \to -1/3R_2$$

=

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 5R_2 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - z = 0 \;\&\; x - z = 0$

$\Rightarrow z = k, y = k, x = k$

$\therefore (x, y, z) = k(1,1,1); k \in R$

$E_1 = \{k(1,1,1) / k \in R\}$

For $\lambda = 2$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=

=

$$\left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] R_1 \to -1/4R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] R_2 \to -1/2R_2$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - 3/4z = 0 \;\&\; x - 1/4z = 0$

$\Rightarrow z = k, y = 3/4k, x = 1/4k$

$\therefore (x, y, z) = k(\frac{1}{4}, \frac{3}{4}, 1); k \in R$

$\therefore (x, y, z) = 4k(1,3,4); k \in R \quad (\because 4k = k^{'})$

$E_1 = \{k^{'}(1,3,4) / k' \in R\}$

$\left[\begin{array}{ccc\|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] R_1 \to -1/3R_1$ = $\left[\begin{array}{ccc\|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$ = $\left[\begin{array}{ccc\|c} 1 & -4/3 & 2/3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] R_2 \to -1/2R_2$	$\therefore P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ $\therefore P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Example (ii): Find a matrix P that diagonalizes A and determine $P^{-1}AP$ where $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ Also find A^{10} and find eigenvalues of A^2.	
<u>Solution (ii):</u> $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ The characteristic equation is $\|A - \lambda I_n\| = 0$ $\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$ $\therefore (1-\lambda)(-1-\lambda) - 0 = 0$ $\therefore \lambda = 1, -1$	For $\lambda = -1$ $\therefore [A - \lambda I \| O] = \left[\begin{array}{cc\|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$ = $\left[\begin{array}{cc\|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right]$ $y = k, 6x = 0 \Rightarrow x = 0$ $\therefore (x, y) = \{k(0,1)/k \in R\}$

For $\lambda = 1$ $\therefore [A - \lambda I \mid O] = \left[\begin{array}{cc\|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$ $=$ $\left[\begin{array}{cc\|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right]$ Suppose $x = k, 6x - 2y = 0$ $x = k, y = 3k$ $\therefore (x, y) = \{k(1,3)/k \in R\}$	Now, $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$ $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = PDP^{-1}$ $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Eigenvalues of A^2 are: $1^2 = 1$ and $(-1)^2 = 1$.

Quadratic Forms

A homogeneous polynomial of second degree in real variables $x_1, x_2, x_3, \dots, x_n$ is called Quadratic form.

For example:

(i) $ax^2 + 2hxy + by^2$ is a quadratic form in the variables x and y

(ii) $2x_1x_2 + 2x_2x_3 + 2x_3x_1 + {x_3}^2$ is a quadratic form in the variables x_1, x_2, x_3.

A quadratic on R^n is a function Q define on R^n whose value at a vector x in R^n can be computed in n variables $x_1, x_2, x_3, \dots, x_n$ by an expression of the form.

$$\boxed{Q(x) = x^T Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ x_i\ \ x_j}$$

Here, A is known as the coefficient matrix. Where A is an $n \times n$ symmetric matrix and is called matrix of the quadratic form.

Matrix Representation of Quadratic Forms

A quadratic form can be represented as a matrix product.

For example:

(I)

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(II)

$$2x_1x_2 + 2x_2x_3 + 2x_3x_1 + x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example:

(i) Find a real symmetric matrix C of the quadratic form

$$Q = x_1{}^2 + 3x_2{}^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3.$$

Solution: The coefficient matrix of Q is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

So, C = symmetric matrix =

$$\left[\frac{1}{2}(A + A^T)\right] = \frac{1}{2}1\left\{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

(ii) Express the following quadratic forms in matrix notation

$$Q = x^2 - 4xy + y^2$$

Solution:

$$x^2 - 4xy + y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

-

Transformation (Reduction) of Quadratic form to canonical form OR Diagonalizing Quadratic Forms:

Procedure to Reduce Quadratic form to canonical form:

1. Identity the real symmetric matrix associated with the quadratic form.
2. Determine the eigenvalues of A.

3. The required canonical form is given by

$$Q(x) = y^T D y = \lambda_1 {y_1}^2 + \lambda_2 {y_2}^2 + \lambda_3 {y_3}^2 + \ldots\ldots + \lambda_n {y_n}^2 \quad \ldots\ldots\ldots\ldots\ (1)$$

where $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are the eigenvalues of A and $D = P^T A P$. The matrix P is said to orthogonally diagonalize the quadratic form.
and equation (1) is known as canonical form.

4. Form modal matrix P (where $x = Py$) containing the n eigenvectors of A as n column vectors.

Example: Reduce the quadratic form into canonical form

$$Q = 3x^2 + 3z^2 + 4xy + 8xz + 8yz$$

<u>Solution:</u>

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$$

Eigenvalues for A are $3, -\frac{4}{3}, -1$.

The canonical form of the given quadratic form is

$$y^T B y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3{y_1}^2 - 4/3{y_2}^2 - {y_3}^2$$

Nature of quadratic form *Q*

a. Positive definite if Q(x) > 0 for all x $\neq$ 0,
b. Negative definite if Q(x) < 0 for all x $\neq$ 0,
c. Indefinite if Q(x) assumes both positive and negative values.
d. Positive semidefinite if Q(x) $\geq$ 0 for all x.
e. Negative semidefinite if Q(x) $\leq$ 0 for all x.

OR

a. Positive definite if and only if the eigenvalues of A are positive,
b. negative definite if and only if the eigenvalues of A are positive,
c. Indefinite if and only if A has both positive and negative eigenvalues.
d. Positive semi-definite if and only if A has only non-negative eigenvalues.
e. Indefinite if and only if A has only non-positive eigenvalues.

Example: Describe the nature of quadratic forms.

1. $Q = 3x_1^{\ 2} + x_2^{\ 2} + x_3^2 + 4x_1x_2 + 4x_2x_3$
2. $Q = 2x_1x_2 + 2x_2x_3 + 2x_2x_1$

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