Parul University
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1^{st} Year B.Tech Programme (All Branches)
Mathematics – 1 (203191102)
Unit – 6 Multivariable Calculus(Lecture Note)

MULTIVARIABLE CALCULUS

Overview:

In this unit, we will study about Limit and Continuity, partial differentiation of first order and higher order, chain rule, implicit function, jacobian . We will study further about Application for Partial Derivative includes tangent plane and normal line, extreme values of a function, use of Lagrange's Multiplier and Taylor's and Maclaurin'sseries for two variables.

Objective:

At the end of this unit, you will be able to understand

- Functions of Several Variables
- Limit and Continuity
- Use of Partial Differentiation
- Application of Chain Rule
- Properties of Jacobian
- How to find out Tangent Plane and Normal Line
- Method to find extreme values of a function
- Method of Lagrange's multipliers
- Expansion using Taylor Series

Functions of Two Variables

Until now, we have only considered functions of a single variable (y) = $f(x)$.
For example, the area function of a square is its side square which include only one variable.
However, many real-world functions consist of two (or more) variables.

Example:

- The area function of a rectangular shape depends on both its width and its height.
- The pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume.

- Volume of cylinder is $v = \pi r^2 h$, where r and h are two independent variables and v is dependent variable.

  Therefore, we require partial derivative for the function, which depends more than one independent variable.

Domain:

It is the set of inputs and it is denoted by D.

Range:

The range is the set of all possible output values, it is denoted by R.

Example:

Find the domain and range of the following function.

1) $f(x,y) = \ln(xy - 2)$

   $f(x,y)$ is defined iff $xy - 2 > 0$

$$xy > 2$$

Domain of function=$\{(x,y) \in R^2 / \, xy > 2\}$

Range of function= R

2) $f(x,y) = \sqrt{9 - x^2 - y^2}$

For domain of the function $f(x,y)$ is, $9 - x^2 - y^2 \geq 0$

$$9 \geq x^2 + y^2$$

Domain of function=$\{(x,y) \in R^2 / \, 9 \geq x^2 + y^2\}$

Range of function= [0,3]

Exercise:

3) $f(x,y) = \ln(y - x^2)$
4) $f(x,y) = x^2 - y$

Limit of a Function of Two Variables

If for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \text{ wherever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

then, the function $f(x, y)$ is said to approach the **limit** L as (x, y) approaches (x_0, y_0) , and we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

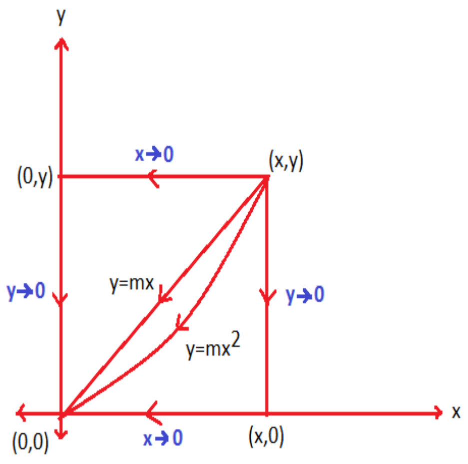
To evaluate limit following methods are applicable: -

(i)By direct substitution: - If example is regarding $(x, y) \to (x_0, y_0)$ and on direct substitution you obtain finite value.

(ii) By definition of limit: - If in example it is mentioned.

(iii)By path if example is regarding $(x, y) \to (0,0)$ and on direct substitution you obtain an indeterminate form (mostly $\left(\frac{0}{0}\right)$)

Explanation:



Path 1: Evaluate $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x, y)\right\}$

Path 2: Evaluate $\lim_{y\to 0}\left\{\lim_{x\to 0} f(x, y)\right\}$

If limit along path 1 and path 2 are same then

proceed further otherwise limit doesn’t exist.

Path 3: Evaluate the limit along $y = mx$as $x \to 0$.

If limit along path 1, path 2 and path 3 are same then proceed further otherwise limit doesn't exist.

Path 4: Evaluate the limit along $y = mx^2$ as $x \to 0$.

If limit along all the paths are same then limit exist.

Solved Examples:

1. Evaluate $\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$ **------------> Simple example by direct substitution**

Sol: $\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$

$$= \frac{3(1)^2(2)}{(1)^2 + 2^2 + 5}$$

$$= \frac{6}{10}$$

$$= \frac{3}{5}$$

$$\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2 - y^2 - 5}$$

2. Applying the definition of limit, show that $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$.

Sol.

Let $\varepsilon > 0$ be given.

We want to find a $\delta > 0$ such that

$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

Since, $y^2 \le x^2 + y^2$

$$=> \frac{y^2}{x^2+y^2} \leq 1$$

$$=> \frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}$$

Now, $\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \frac{4|x|y^2}{x^2+y^2} \leq 4\sqrt{x^2+y^2} < 4\delta$

On taking, $4\delta = \varepsilon$

$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$whenever$0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$ by definition.

3. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ --------------> By Path

Sol:

Path 1:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right) = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1$$

Path 2:

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right) = \lim_{y\to 0}(-\frac{y}{y}) = \lim_{y\to 0}(-1) = -1$$

Since both the limits are different, the limit does not exist.

4. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ ------------> By Path

Sol: Path 1:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^3+y^3}{x^2+y^2}\right) = \lim_{x\to 0} x = 0$$

Path 2:

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3+y^3}{x^2+y^2}\right) = \lim_{y\to 0} y = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x^3 + (mx)^3}}{\boldsymbol{x^2 + (mx)^2}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x^3 + m^3x^3}}{\boldsymbol{x^2 + m^2x^2}}\right) = \lim_{x\to 0} x\left(\frac{\boldsymbol{1 + m^3}}{\boldsymbol{1 + m^2}}\right) = 0$$

Path 4:

Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x^3 + (mx^2)^3}}{\boldsymbol{x^2 + (mx^2)^2}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x^3 + m^3x^6}}{\boldsymbol{x^2 + m^2x^4}}\right) = \lim_{x\to 0} x\left(\frac{\boldsymbol{1 + m^3x^3}}{\boldsymbol{1 + m^2x^2}}\right) = 0$$

Since, limit along all the paths are same.

Hence, the limit exists and its value is 0.

5. Evaluate $\lim\limits_{(x,y)\to(\mathbf{0},\mathbf{0})} \frac{2xy}{x^2+y^2}$ **------------> By Path**

Sol:

Path 1:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{\boldsymbol{2xy}}{\boldsymbol{x^2 + y^2}}\right) = 0$$

Path 2:

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{\boldsymbol{2xy}}{\boldsymbol{x^2 + y^2}}\right) = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{2x(mx)}}{\boldsymbol{x^2 + (mx)^2}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{2mx^2}}{\boldsymbol{x^2 + m^2x^2}}\right) = \frac{\boldsymbol{2m}}{\boldsymbol{1 + m^2}}$$

As the limit depends on m and m is not fixed, the limit doesn't exist.

Exercise:

1. Find $\lim_{(x,y)\to(1,2)} \frac{x^2+y}{3x+y^2}$.

2. Applying the definition of limit, show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

3. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

4. Find $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

Continuity of function of two Variables

A function $f(x,y)$ is **continuous at the point**$(x_{0,}y_0)$ if it satisfies following properties,

1. f is defined at (x_0, y_0) ,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

Solved Examples:

1. Discuss the continuity of $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Sol: i) Here,f is defined at $(0,0)$.

f(0,0)=0

ii)

Path 1:

$$\lim_{x\to 0}\left(\lim_{y\to o} \frac{x^2y}{x^3+y^3}\right) = 0$$

Path 2:

$$\lim_{y\to 0}\left(\lim_{x\to o} \frac{x^2y}{x^3+y^3}\right) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$\lim_{x\to 0}\left(\frac{\boldsymbol{x^2(mx)}}{\boldsymbol{x^3+(mx)^3}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{mx^3}}{\boldsymbol{x^3+m^3x^3}}\right) = \frac{\boldsymbol{m}}{\boldsymbol{1+m^3}}.$

As the limit depends on m and m is not fixed, the limit doesn't exist. Therefore, limit doesn't exist. Hence, $f(x,y)$ is discontinuous at (0,0).

2. Discuss the continuity of $\boldsymbol{f(x,y)} = \begin{cases} \frac{\boldsymbol{x^2-y^2}}{\sqrt{\boldsymbol{x^2+y^2}}}, & \boldsymbol{(x,y) \neq (0,0)} \\ \boldsymbol{0}, & \boldsymbol{(x,y) = (0,0)} \end{cases}$

Sol: Here,f is defined at (0,0).

F(0,0)=0

<u>Path 1</u>:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{\boldsymbol{x^2-y^2}}{\sqrt{\boldsymbol{x^2+y^2}}}\right] = \lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0$$

<u>Path 2</u>:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{\boldsymbol{x^2-y^2}}{\sqrt{\boldsymbol{x^2+y^2}}}\right] = \lim_{y\to 0}\left(-\frac{y^2}{y}\right) = \lim_{y\to 0}(-y) = 0$$

<u>Path 3</u>: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x^2-(mx)^2}}{\sqrt{\boldsymbol{x^2+(mx)^2}}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x^2(1-m^2)}}{\boldsymbol{x\sqrt{1+m^2}}}\right) = 0$$

<u>Path 4</u>: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x^2-(mx^2)^2}}{\sqrt{\boldsymbol{x^2+(mx^2)^2}}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x^2(1-m^2x^2)}}{\boldsymbol{x\sqrt{1+m^2x^2}}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exist.

Also, $\lim_{(x,y)\to(0,0)}\frac{\boldsymbol{x^2-y^2}}{\sqrt{\boldsymbol{x^2+y^2}}} = 0 = f(0,0)$

Hence, $f(x,y)$ is continuous at (0,0).

3. Check whether the given function is continuous at origin or not, if yes then find point of continuity.

$$\boldsymbol{f(x,y)} = \begin{cases} \dfrac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}} & (\boldsymbol{x},\boldsymbol{y}) \neq (\boldsymbol{0},\boldsymbol{0}) \\ \boldsymbol{-1} & (\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{0},\boldsymbol{0}) \end{cases}$$

Sol: Here, f is defined at $(0,0)$.

F(0,0) = -1

Path 1:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}}\right] = \lim_{x\to 0}\frac{x}{\sqrt{\boldsymbol{x}}} = \lim_{x\to 0}\sqrt{\boldsymbol{x}} = 0$$

Path 2:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}}\right] = \lim_{y\to 0}\left(\frac{y}{-\sqrt{\boldsymbol{y}}}\right) = \lim_{y\to 0}(-\sqrt{\boldsymbol{y}}) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x+mx}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{mx}}}\right) = \lim_{x\to 0}\sqrt{\boldsymbol{x}}\left(\frac{\boldsymbol{1+m}}{\boldsymbol{1}-\sqrt{\boldsymbol{m}}}\right) = 0$$

Path 4: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x+mx^2}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{mx^2}}}\right) = \lim_{x\to 0}\left(\frac{\boldsymbol{x+mx^2}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{m}}\boldsymbol{x}}\right) = \lim_{x\to 0}\frac{x}{\sqrt{\boldsymbol{x}}}\left(\frac{\boldsymbol{1+mx}}{\boldsymbol{1}-\sqrt{\boldsymbol{mx}}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exist.

Also, $\lim_{(x,y)\to(0,0)}\frac{\boldsymbol{x+y}}{\sqrt{\boldsymbol{x}}-\sqrt{\boldsymbol{y}}} = 0 \neq -1 = f(0,0)$

Hence, $f(x,y)$ is discontinuous at (0,0).

<u>Exercise:</u>

1.Discuss the continuity of $f(x,y) = \begin{cases} \frac{x}{3x+5y}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$

2.Show that $f(x,y) = \begin{cases} \frac{x^2y}{y^2-x^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is continuous at origin.

Partial Derivatives:

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

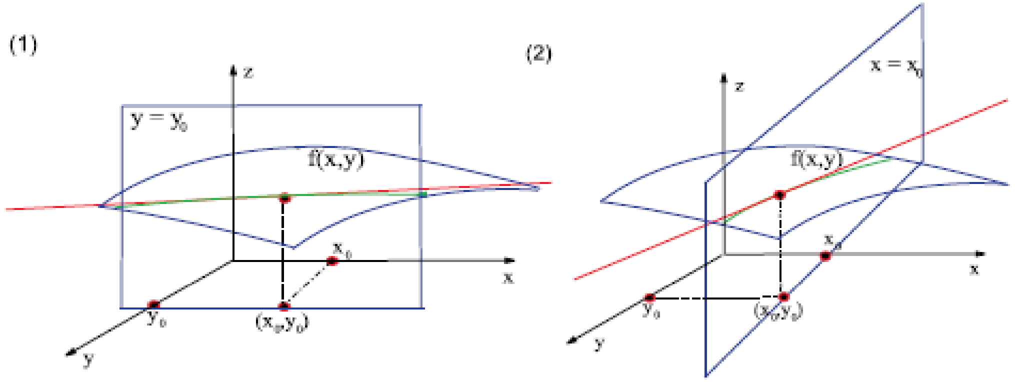
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h, y_0)-f(x_0,y_0)}{h}, \quad \text{provided the limit exists.}$$

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0, y_0+h)-f(x_0,y_0)}{h}, \quad \text{provided the limit exists.}$$

Geometric Interpretation of Partial Derivative:

Geometrical definition of f_x and f_y: The partial derivative $\partial f/\partial x$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the vertical plane $y = y_0$ at $x = x_0$. Likewise, the partial derivative $\partial f/\partial y$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the horizontal plane $x = x_0$ at $y = y_0$. Graphically:



Second order partial derivative:

Two successive partial differentiations of *f(x, y)* with respect to *x* (holding *y* constant) is denoted by $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}(x, y)$. That is, we define

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

Similarly, two successive partial differentiations of *f(x, y)* with respect to *y* (holding *x* constant) is denoted by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yy}(x, y)$. That is, we define

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We use $\frac{\partial^2 f}{\partial x \partial y}$ to mean differentiate first with respect to y then with respect to x, and we use $\frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}\,\partial \mathrm{x}}$ to mean differentiate first with respect to x then with respect to y.

Note:

$f_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x}$ and $f_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y}$ are known as mixed order partial derivatives.

The Mixed Derivative /Clairaut's Theorem

If $f(x, y)$ and its partial derivative f_x, f_y, f_{xy}, f_{yx} are

(i) defined throughout an open region containing a point (a, b) and

(ii) they are all continuous at (a, b), then

$f_{xy}(a, b) = f_{yx}(a, b)$.

THEOREM: If a function *f (x,y)* is differentiable at (x_0, y_0) then *f* is continuous at (x_0, y_0).

Solved Examples:

1. **Find $\frac{\partial f}{\partial z}$ at (1,2,3) for f(x,y,z)=x^2yz^2 using the definition.**

Sol: Here, $\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h}$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,2,3)} = \lim_{h\to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{2(9+6h+h^2) - 18}{h}$$

$$= \lim_{h\to 0} (12 + 2h)$$

$$= 12$$

2. **If $f(x, y) = x^3 + y^3 - 2xy^2$. Find all second order partial derivatives of f at (1,-1)**

Sol: Here, $f_x(x,y) = 3x^2 - 2y^2 f_y(x,y) = 3y^2 - 4xy$

$$f_{xx}(x,y) = 6x f_{yy}(x,y) = 6y - 4x$$

$$f_{xy}(x,y) = -4y, f_{yx}(x,y) = -4y$$

Then, $f_{xx}(1,-1) = 6 f_{yy}(1,-1) = -10$

$$f_{xy}(1,-1) = 4, f_{yx}(1,-1) = 4$$

3. **If $u = log(tanx + tany + tanz)$, then show that**

$$\mathbf{\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2}$$

Sol: u = log (tan x + tan y + tan z)

Differentiating u partially w.r.t. x, y and z,

$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x$

$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 y$

$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 z$

Hence,

$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z\ 2x \frac{\partial u}{\partial z} = \frac{2sinxcosxsec^2x + 2sinycosysec^2y + 2sinzcoszsec^2z}{\tan x + \tan y + \tan z}$

$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$

= 2

4. **If $u = e^{3xyz}$ show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$**

Sol: u $= e^{3xyz}$

Differentiating u w.r.t z,

$$\frac{\partial u}{\partial z} = 3xye^{3xyz}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t y,

$$\frac{\partial^2 u}{\partial y \partial z} = 3x \frac{\partial}{\partial y}(ye^{3xyz})$$

$$= 3x(e^{3xyz}.1 + ye^{3xyz}.3xz)$$

$$= e^{3xyz}(3x + 9x^2yz)$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ w.r.t x,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial \mathrm{x}}[[\, e^{3xyz}(3x + 9x^2yz)]$$

$$= \mathrm{e^{3xyz}(3 + 18xyz)} + (3x + 9x^2yz).\mathrm{e^{3xyz}.3yz}$$

$$= \mathrm{e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)}$$

$$= \mathrm{e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)}$$

Exercise:

1. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point (4,-5) if $f(x, y) = x^2 + 3xy + y - 1$.
2. If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$
3. Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Homogeneous function:

A function $u = f(x, y)$ is said to be homogeneous function of degree 'n' in x and y if degree of each term of $u = f(x, y)$ is n.

Thus, for a homogeneous function f of degree n; $\mathbf{f(tx, ty) = t^n f(x, y)}$

Note:

Also, if the function$u = f(x, y)$ is a homogeneous function of degree 'n' in x and y

then it can be written as $u = x^n \emptyset\left(\frac{y}{x}\right) or u = y^n \emptyset\left(\frac{x}{y}\right)$

Euler's theorem for the function of two independent variables:

If u is a homogeneous function of degree n in x and y, then

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Proof: Let $u=f(x,y)$ be a homogeneous function of degree 'n' in x and y, then it can be written as $u=x^n\emptyset\left(\frac{y}{x}\right)$ ____________(1)

Differentiate (1) partially w.r.t x, we get

$$\frac{\partial u}{\partial x}=x^n\frac{\partial}{\partial x}\left[\emptyset\left(\frac{y}{x}\right)\right]+\emptyset\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)$$

$$=x^n\emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right)+\emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$=x^n\emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+\emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$=-yx^{n-2}\emptyset'\left(\frac{y}{x}\right)+nx^{n-1}\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}=-yx^{n-1}\emptyset'\left(\frac{y}{x}\right)+nx^n\emptyset\left(\frac{y}{x}\right) \text{ ________________(2)}$$

Differentiate (1) partially w.r.t y, we get

$$\frac{\partial u}{\partial y}=x^n\frac{\partial}{\partial y}\left[\emptyset\left(\frac{y}{x}\right)\right]$$

$$=x^n\emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$=x^n\emptyset'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$=x^{n-1}\emptyset'\left(\frac{y}{x}\right)$$

$$\Rightarrow y\frac{\partial u}{\partial y}=yx^n\emptyset'\left(\frac{y}{x}\right) \text{ ____________________________(3)}$$

Adding (2) and (3) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-yx^{n-1}\emptyset'\left(\frac{y}{x}\right)+nx^{n}\emptyset\left(\frac{y}{x}\right)+yx^{n}\emptyset'\left(\frac{y}{x}\right)=nx^{n}\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Euler's theorem for the function of three independent variables:

If u is a homogeneous function of degree n in x, y and z, then

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$$

[Similar to **Euler's theorem for the function of two independent variables]**

Cor.1 If u is a homogeneous function of degree n in x and y, then

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.}$$

Proof: Let $u=f(x,y)$ be a homogeneous function of degree 'n' in x and y, then by Euler's theorem for homogeneous functions

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(1)$$

Differentiate (1) partially w.r.t x, we get

$$\Rightarrow x\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]+\frac{\partial u}{\partial x}\left[\frac{\partial}{\partial x}(x)\right]+y\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right]=n\frac{\partial}{\partial x}(u)$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\ +y\left[\frac{\partial^2 u}{\partial x\partial y}\right]=n\frac{\partial u}{\partial x}$$

On multiplying both sides by x, we get

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}\ +xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]=nx\frac{\partial u}{\partial x}\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)$$

Differentiate (1) partially *w.r.t.* y, we get

$$\Rightarrow x\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]+\frac{\partial u}{\partial y}\left[\frac{\partial}{\partial y}(y)\right]+y\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right]=n\frac{\partial}{\partial y}(u)$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}\left[since,\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

On multiplying both sides by y, we get

$$\Rightarrow xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=ny\frac{\partial u}{\partial y}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(3)$$

Adding (2) and (3) we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu=n^2u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n^2u-nu$$

$$\Rightarrow\boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.}$$

<u>Cor.2</u> If $\emptyset(u)=f(x,y)$ is a homogeneous function of degree *n* in *x* and *y*, then

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}}\text{[Without proof]}$$

<u>Cor.3</u> If $\emptyset(u)=f(x,y)$ is a homogeneous function of degree *n* in *x* and *y*, then

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]}$$

where, $\boldsymbol{g(u)=n\frac{f(u)}{f'(u)}}$**.** [Without proof]

<u>Note</u>: Corollary 2 and 3 are also known as Modified Euler's theorem of first and second order respectively.

Solved Examples:

1. **If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2}$, prove that**

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u \quad (ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$$

Sol: $u = f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}$

Replacing x by tx and y by ty,

$$f(tx, ty) = \frac{1}{t^2x^2} + \frac{1}{txty} + \frac{log\frac{tx}{ty}}{t^2x^2+t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}\right]$$

$$= t^{-2}f(x, y)$$

Hence, u is a homogeneous function of degree -2

By Euler's Theorem,

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$$

$$(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2(-2-1)u = 6u$$

2. **If $u = tan^{-1}(\frac{x^2+y^2}{x+y})$, prove that**

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin\,u\,cos\,u$$

$$(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2sin^3ucosu.$$

Sol: $u = tan^{-1}(\frac{x^2+y^2}{x+y})$

Replacing x by tx and y by ty,

$u = tan^{-1}[t(\frac{x^2+y^2}{x+y})]$

u is a non-homogeneous function. But $tan\,u = (\frac{x^2+y^2}{x+y})$ is a homogeneous function of degree 1.

By Modified Euler's Theorem,

Let $f(u) = tanu$

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.\frac{tanu}{sec^2u} = \sin u \cos u$$

$$(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)} = 1.\frac{tanu}{sec^2u} = \sin u \cos u$

$$\Rightarrow g'(u) = cos^2u - sin^2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin u \cos u\,(cos^2u - sin^2u - 1)$$

$$= sinucosu(cos2u - 1)$$

$$= sinucosu(-2sin^2u)$$

$$= -2sin^3ucosu$$

Exercise:

1. If $u = y^2 e^{\frac{y}{x}} + x^2 \tan^{-1}\left(\frac{x}{y}\right)$,show that

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

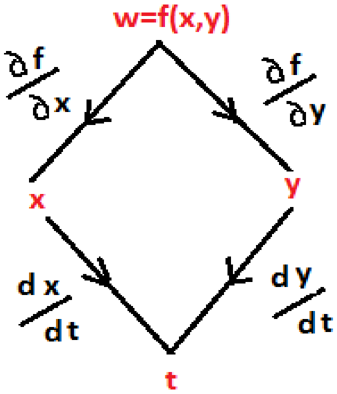
2. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$$

$$(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}(tan^3u - tanu)$$

Chain Rule for Function of Two Independent Variable

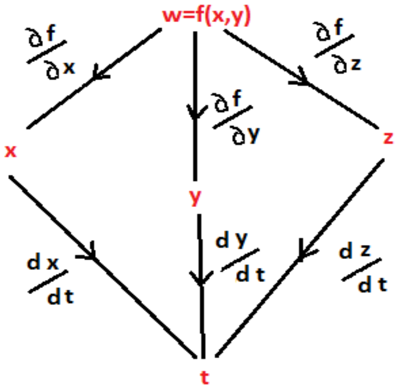
If $w = f(x, y)$ has continuous partial derivative f_x, f_y and if $x = x(t),\ \ y = y(t)$ are differentiable function of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t then,



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule for Function of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and $x, y\ and\ z$ are differentiable function of t then w is a differentiable function of t then,



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

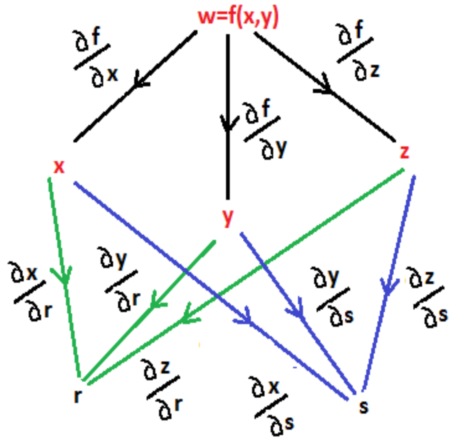
Chain Rule for Function of Two Independent Variable and Three Intermediate Variable

Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s)\ \ and$

$z = k(r, s)$.If all four functions are differentiable then w has partial derivative with respect to $rands$ givenby the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$



Solved Examples:

1.Let $z = x^2y^3$, where $x = t^2$and $y = t$, then verify chain rule by expressing z in terms of t.

Sol: Here, the chain rule is $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= (2xy^3)(2t) + (3x^2y^2)(1)$$

$$= (2t^2t^3)(2t) + (3t^4t^2)$$

$$= 4t^6 + 3t^6$$

$= 7t^6 - - - - - - - -(1)$

$$Also, z = x^2y^3$$
$$z = (t^4)(t^3) = t^7$$

$\frac{dz}{dt} = 7t^6 - - - - - - - -(2)$

Hence, from (1) and (2), the chain rule is verified.

2**. If $\boldsymbol{u = xy^2 + yz^3, x = logt, y = e^t, z = t^2}$find $\boldsymbol{\frac{du}{dt}}$ at $\boldsymbol{t = 1}$.**

Sol:$u = xy^2 + yz^3, x = logt, y = e^t, z = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y^2)\frac{1}{t} + (2xy + z^3)e^t + (3yz^2)2t$$

Substituting x ,y and z,

$$\frac{du}{dt} = 2(e^{2t})\frac{1}{t} + \left(2(logt) \quad e^t + t^6\right)e^t + 3e^t t^4 .2t$$

Substituting $t = 1$,

$$\frac{du}{dt} = 2e^2 + e^1 + 6e^1$$

$$= 2e^2 + 7e^1$$

3.If $\boldsymbol{u = f(x - y, y - z, z - x)}$, show that $\boldsymbol{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$

Sol: Let $x - y = l, y - z = m, z - x = n$

$$\frac{\partial l}{\partial x} = 1, \qquad \frac{\partial m}{\partial x} = 0, \qquad \frac{\partial n}{\partial x} = -1,$$

$$\frac{\partial l}{\partial y} = -1, \qquad \frac{\partial m}{\partial y} = 1, \qquad \frac{\partial n}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0, \qquad \frac{\partial m}{\partial z} = -1, \qquad \frac{\partial n}{\partial z} = 1$$

$$u = f(x - y, y - z, z - x) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}.1 + \frac{\partial u}{\partial m}.0 + \frac{\partial u}{\partial n}.-1$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \quad --------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}.-1 + \frac{\partial u}{\partial m}.1 + \frac{\partial u}{\partial n}.0$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} --------(2)$$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{z}} = \frac{\partial \mathrm{u}}{\partial \mathrm{l}}\frac{\partial \mathrm{l}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u}}{\partial \mathrm{m}}\frac{\partial \mathrm{m}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u}}{\partial \mathrm{n}}\frac{\partial \mathrm{n}}{\partial \mathrm{z}}$$

$$= \frac{\partial u}{\partial l}.0 + \frac{\partial u}{\partial m}.-1 + \frac{\partial u}{\partial n}.1$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} --------(3)$$

Adding Eq.(1),(2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Exercise:

1. If $u = f(x^2 + 2yz, y^2 + 2xz)$, then find the value of

$(y^2 - xz)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$.

2. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

3. If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

Implicit Differentiation:

Suppose that $f(x,y)$ is differentiable and that the equation $f(x,y)=0$ defines y as a

Differentiable function of x. Then at any point where $f_y \neq 0$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Solved Examples:

1. **If $ysinx = xcosy$, find $\frac{dy}{dx}$.**

Sol: Let $f(x,y) = \ ysinx - xcosy$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= \ -\frac{ycosx - cosy}{sinx + xsiny}$$

$$= \frac{cosy - ycosx}{sinx + xsiny}$$

2. **If $(cosx)^y = \ (siny)^x$, find $\frac{dy}{dx}$.**

Sol: $(cosx)^y = \ (siny)^x$

Taking log on both the sides,

$$ylogcosx = xlogsiny$$

Let $f(x,y) = ylogcosx - xlogsiny$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{cosx}(-sinx) - logsiny}{logcosx - \frac{x}{siny}(cosy)}$$

$$= \frac{ytanx + logsiny}{logcosx - xcoty}$$

Exercise:

1. If $x^3 + y^3 + 3xy = 1$ then, find $\frac{dy}{dx}$.

2. If $x^y + y^x = c\ then,$ find $\frac{dy}{dx}$.

Jacobian:

If u and v are continuous and differentiable functions of two independent variables x and y, then the Jacobian of u,v with respect to x,y and is denoted by

J=$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly, if u,v and w are continuous and differentiable functions of three independent variables x,yand z, then the Jacobian of u,v,w with respect to x,y,z and is denoted by

J =$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of Jacobian:

1. If u and v are functions of x and y, then $\boldsymbol{J.J' = 1}$, where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

2. If u,v are functions of r,s and r,s are functions of x,y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}$

Solved Examples:

1. Find the Jacobian$\frac{\partial(u,v)}{\partial(x,y)}$ for $\boldsymbol{u = x^2 - y^2, v = 2xy}$.

Sol:$u = x^2 - y^2,\ v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2 + y^2)$$

2. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x - y, v = x + y$. Also, verify that $J.J' = 1$.

Sol: $u = x - y,\ v = x + y$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -1, \quad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2$$

Now, $x = \frac{u+v}{2}, y = \frac{v-u}{2}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \qquad \frac{\partial y}{\partial u} = -\frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}, \qquad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}$$

$$JJ' = 1.$$

Exercise:

1. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = xsiny, v = ysinx$.

2. If $u = 2xy$, $v = x^2 - y^2$ and $x = rcos\theta$, $y = rsin\theta$ then, evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$.

APPLICATIONS OF PARTIAL DERIVATIVES

Tangent Plane and Normal Line

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of the differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. It is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$. It is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}$$

Solved Examples:

1. Find the equation of the tangent plane and normal line to the surface $z + 8 = xe^y cosz$ at the point (8,0,0).

Sol: Let $f(x, y, z) = xe^y cosz - z - 8$

$f_x(x, y, z) = e^y cosz$, $f_x(8,0,0) = 1$

$f_y(x, y, z) = xe^y cosz$, $f_y(8,0,0) = 8$

$f_z(x, y, z) = sinz - 1$, $f_z(8,0,0) = -1$

Hence, the equation of the tangent plane at (8,0,0) is

$$1(x - 8) + 8(y - 0) - 1(z - 0) = 0$$

$$x - 8 + 8y - z = 0$$

$$x + 8y - z - 8 = 0$$

The equation of normal line is $\frac{x-8}{1} = \frac{y-0}{8} = \frac{z-0}{-1}$

2. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ atthe point (1,1,1).

Sol: Here, $f(x, y, z) = x^2 + y^2 + z^2 - 3$

$f_x(x, y, z) = 2x$, $f_x(1,1,1) = 2$

$f_y(x, y, z) = 2y$, $f_y(1,1,1) = 2$

$f_z(x, y, z) = 2z$, $f_z(1,1,1) = 2$

Hence, the equation of the tangent plane at (1,1,1) is

$$(x-1)2 + (y-1)2 + (z-1)2 = 0$$

$$x + y + z = 3$$

The equation of normal line is $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$.

Exercise:

1. Find the equations of tangent plane and normal line to the surface $z = x^2 + 3y^2 - 4$ at $(1,1,0)$.

2. Find the equations of tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at $(2,1,-3)$.

Local Maximum and Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b).Then

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivative are continuous throughout

a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then, for

$$r = \frac{\partial^2 f}{\partial x^2}, \ s = \frac{\partial^2 f}{\partial x \partial y}, \ t = \frac{\partial^2 f}{\partial y^2}$$

(i) f has a local maximum at (a, b) if $r < 0 \ and \ rt - s^2 > 0$ at (a, b)

(ii) f has a local minimum at (a, b) if $r > 0 \ and \ rt - s^2 > 0$ at (a, b)

(iii) f has a saddle point at (a, b) if $rt - s^2 < 0$ at (a, b)

(iv) The test has no conclusion at $(a, b) if \ rt - s^2 = 0$ at (a, b).

Solved Examples:

1.Find the points on the surface $z^2 = x^2 + y^2$ that are closed to P(1,1,0).

Sol: Let A(x,y,z) be any point on the surface, then by distance formula, the distance d

betweenA and P is given by $d = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$

$$d^2 = (x-1)^2 + (y-1)^2 + z^2$$

$$= 2x^2 + 2y^2 - 2x - 2y + 2 = f \ (say)$$

$f_x(x, y) = 4x - 2, \qquad f_y(x, y) = 4y - 2$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$4x - 2 = 0, \quad 4y - 2 = 0$$

$$x = \frac{1}{2}, \quad y = \frac{1}{2}$$

So, $(\frac{1}{2}, \frac{1}{2})$ is a stationary point.

$$r = \frac{\partial^2 f}{\partial x^2} = 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 4$$

$$rt - s^2 = (4)(4) - 0 = 16 > 0$$

Hence, function is minimum at $(\frac{1}{2}, \frac{1}{2})$

Minimum value of $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

So, $z = \pm\frac{1}{2}$.

2. Find the extreme value of $\boldsymbol{x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x}$

Sol: Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$and $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 30x + 72 = 0and6xy - 30y = 0$

$x^2 + y^2 - 10x + 24 = 0$and$6y(x - 5) = 0$

$x^2 + y^2 - 10x + 24 = 0$and$y = 0, x = 5$

When $y = 0$, we have $x^2 - 10x + 24 = 0$

$$x = 4,6$$

and when $x = 5$, we have $25 + y^2 - 50 + 24 = 0$

$$y = \pm 1$$

Therefore, the stationary points are $(4,0), (6,0), (5,1), (5,-1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

$(\boldsymbol{x}, \boldsymbol{y})$	$\boldsymbol{r}$	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{rt - s^2}$	**Conclusion**	$\boldsymbol{f(x,y)}$
$(4,0)$	-6<0	0	-6	36>0	Maximum	112
$(6,0)$	6>0	0	6	36>0	Minimum	108
$(5,1)$	0	6	0	-36<0	Saddle Point	--
$(5,-1)$	0	-6	0	-36<0	Saddle Point	--

Exercise:

1. Discuss the maxima and minima of the function$(x, y) = x^2 + y^2 + 6x + 12$.

2. Find the extreme values of the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

The Method of Lagrange Multipliers:

The method of Lagrange multipliers allows us to maximize or minimize function with a constraint.

Given a function $f(x, y, z)$ subject to the constraint $\emptyset(x, y, z) = 0$__________(1), we must solve by following

Steps:

1.Construct an equation $f(x, y, z) + \lambda\emptyset(x, y, z) = 0$____________(2)

where, λ is a variable called Lagrange multiplier.

2. Differentiate Eq. (2) partially w.r.t x, y, z to obtain

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 0\text{\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 0 \_\_\_\_\_\_\_\_\_\_\_\_ (4)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z} = 0 \_\_\_\_\_\_\_\_\_\_\_\_ (5)$$

3.Solve and eliminate λ from Eqs. $(1), (3), (4), (5)$ to obtain the stationary points (x, y, z).

4. Substitute the stationary points (x, y, z) into f to see where, f attains its maximum and minimum values.

Note: For function of two independent variables, the condition is similar, but without thevariable z.

Solved Examples:

1.Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Sol: Let $f(x, y, z) = xy$

and $\emptyset(x, y, z) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$__________(1)

Let the equation be $xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) = 0$__________(2)

Differentiating Eq.(2) partially w.r.t x,

$$y + \frac{\lambda x}{4} = 0 \_\_\_\_\_\_\_\_\_\_ (3)$$

Differentiating Eq.(2) partially w.r.t y,

$$x + \lambda y = 0 \_\_\_\_\_\_\_\_\_\_ (4)$$

From Eqs.(3),(4),

$$\frac{-4y}{x} = \frac{-x}{y}$$

$$\Rightarrow 4y^2 = x^2$$

Substituting x^2 in Eq.(1),

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \pm 1$$

$$\Rightarrow x = \pm 2$$

Therefore, the function $f(x, y) = xy$ takes extreme values on the ellipse at four points $(2,1), (-2,1), (-2,-1), (2,-1)$

The maximum value is $xy = 2$ and minimum value is$xy = -2$.

2. **Find the maximum value of $x^2y^3z^4$, subject to the condition $x + y + z = 5$.**

Sol: Let $f(x, y, z) = x^2y^3z^4$

and $\emptyset(x, y, z) = x + y + z - 5 = 0$___________(1)

Let the equation be $x^2y^3z^4 + \lambda(x + y + z - 5) = 0$___________(2)

Differentiating Eq.(2) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0$$

$\lambda = -2xy^3z^4$___________(3)

Differentiating Eq.(2) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0$$

$\lambda = -3x^2y^2z^4$___________(4)

Differentiating Eq.(3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0$$

$\lambda = -4x^2y^3z^3$___________(5)

From Eqs.(3), (4),(5),

$$-2xy^3z^4 = -3x^2y^2z^4 = -4x^2y^3z^3$$

$$\Rightarrow 2yz = 3xz = 4xy$$

$\Rightarrow y = \frac{3}{2}x$ and $z = 2x$

Substituting y and z in Eq.(1),

$$x + \frac{3}{2}x + 2x = 5$$

$$\Rightarrow 9x = 10$$

$$\Rightarrow x = \frac{10}{9}$$

$$y = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

$$z = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(2^{10})(5^9)}{3^{15}}$.

<u>Exercise:</u>

1. Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$on the

circle $x^2 + y^2 = 1$ using the method of Lagrange's multipliers.

2. A soldier placed at a point (3,4) wants to shoot the fighter plane of an enemy which is flying along the curve $y = x^2 + 4$ when it is nearest to him. Find such distance.

Taylor's Formula for f(x,y) at the point (a,b)

Suppose $f(x, y)$ and its partial derivative are continuous throughout an open rectangular region R centered at a point (a, b).Then, throughout R ,

$$f(x,y) = f(a,b) + xf_x(a,b) + yf_y(a,b) + \frac{1}{2!}\left(x^2 f_{xx}(a,b) + 2xyf_{xy}(a,b) + y^2 f_{yy}(a,b)\right)$$

$$+\frac{1}{3!}\left(x^3 f_{xxx}(a,b) + 3x^2 y f_{xxy}(a,b) + 3xy^2 f_{xyy}(a,b) + y^3 f_{yyy}(a,b)\right) + \cdots$$

Taylor's Formula for $f(x, y)$ at origin (Also known as Maclaurin's series)

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$

$$+\frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right) + \cdots$$

Note: For quadratic expansion, find the taylor's series upto second degree terms and for cubic expansion , find the taylor's series upto third degree terms.

Solved Examples:

1. Expand $x^2 y + 3y - 2$ in powers of $(x - 1)$and $(y + 2)$ upto second degree terms.

Sol: Let $f(x, y) = x^2 y + 3y - 2$

By Taylor's Expansion,

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Here, $a = 1, b = -2$

$f(x,y) = x^2y + 3y - 2,$ $\quad f(1,-2) = (1)^2(-2) + 3(-2) - 2 = -10$

$f_x(x,y) = 2xy,$ $\quad f_x(1,-2) = 2(1)(-2) = -4$

$f_y(x,y) = x^2 + 3,$ $\quad f_y(1,-2) = (1)^2 + 3 = 4$

$f_{xx}(x,y) = 2y,$ $\quad f_{xx}(1,-2) = 2(-2) = -4$

$f_{xy}(x,y) = 2x,$ $\quad f_{xy}(1,-2) = 2(1) = 2$

$f_{yy}(x,y) = 0,$ $\quad f_{yy}(1,-2) = 0$

Substituting these values in Taylor's Expansion,

$$f(x,y) = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \ldots$$

$$x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + \cdots$$

2. Expand $e^x\log(1+y)$<u>in powers of x and y</u> upto third degree. -------------><u>which means Maclaurin's series.</u>

Sol: Let $f(x,y) = e^x\log(1+y)$

By Maclaurin's series,

$$f(x,y) = f(0,0) + \left[xf_x(0,0) + yf_y(0,0)\right] + \frac{1}{2!}\left[(x)^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + (y)^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[(x)^3 f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0)3xy^2 f_{xyy}(0,0) + (y)^3 f_{yyy}(0,0)\right] + \ldots$$

$f(x,y) = e^x\log(1+y),$ $\quad f(0,0) = 0$

$f_x(x,y) = e^x\log(1+y),$ $\quad f_x(0,0) = 0$

$f_y(x,y) = \frac{e^x}{1+y},$ $\quad f_y(0,0) = 1$

$f_{xx}(x,y) = e^x\log(1+y),$ $\quad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \frac{e^x}{1+y},$ $\quad f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -\frac{e^x}{(1+y)^2},$ $\quad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x\log(1+y),$ $\quad f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \frac{e^x}{1+y},$ $\quad f_{xxy}(0,0) = 1$

$$f_{xyy}(x,y) = -\frac{e^x}{(1+y)^2}, \qquad f_{xxy}(0,0) = -1$$

$$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3}, \qquad f_{yyy}(0,0) = 2$$

Substituting these values in Taylor's series,

$$f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(-1)]$$
$$+\frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)] + \cdots$$

$$e^x \log(1+y) = y + \frac{1}{2!}(2xy - y^2) + \frac{1}{3!}(3x^2y - 3xy^2 + 2y^3) + \cdots$$
$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \cdots$$

Exercise:

1. Expand $x^2 + xy + y^2$in powers of $(x-1)$and $(y-2)$ upto second- degree terms.

2. Expand $e^x cosy$ in powers of x and y upto third degree.