Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme (All Branches)

Mathematics – 1 (203191102)

Unit – 1 Improper Integral (Lecture Note)

OVERVIEW:

In most of applications of Engineering and Science there occurs special function, like gamma function, beta function etc, which are in the form of integrals which are of special types in which the **limits of integration are infinity** or **the integrand becomes unbounded within the limits**. Such types of integrals are known as **improper integrals**. Beta and gamma functions are very fundamental and hold great importance in various branches of Engineering and physics.

❖ **Course Outcome** :

- Identify various types of Improper Integration.
- Add together infinitely many numbers.
- Represent a differentiable function f(x) as an infinite sum of powers of x.
- Decide on convergence or divergence of a wide class of series.
- To answer at least about the convergence or divergence of integral when integral is not easily evaluated using techniques known.

❖ **Improper integrals:** The integral $\int_a^b f(x)dx$, is called improper integral if

i) one or both limits of integration are infinite. (Type-1)

ii) Function f(x) becomes infinite at a point within or at the end points of the interval of integration. (Type-2)

Examples:

1. $\int_1^\infty xe^{-x}dx$, is an improper integral due to infinite limit.

2. $\int_0^3 \frac{e^{-x}}{\sqrt{x}}dx$, is an improper integral as the integrand tends to ∞ as $x \to 0$.

3. $\int_{-1}^4 \frac{1}{x-1}dx$, is an improper integral as the integrand is unbounded as $x \to 1$.

4. $\int_0^5 xe^{3x}dx$, is a proper integral.

❖ **Improper integrals are classified into three kinds.**

- Type –I
- Type –II
- Type –III

❖ **Improper integrals of the first kind(Type-I):**

If in the definite integral $\int_a^b f(x)dx$,a or b or both a and b are infinite, then the integral is called improper integral of Type-I.

(1) If f(x) is continuous on [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$

(2) If f(x) is continuous on (−∞, b], then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

(3) If f(x) is continuous on (−∞, ∞), then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$,

Or

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to\infty} \int_0^b f(x)dx$$

Evaluate 1)

$$\boldsymbol{Solve}: \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$
$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$
$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx$$
$$= \lim_{a\to-\infty} [\tan^{-1} x]_a^0 + \lim_{b\to\infty} [\tan^{-1} x]_0^b$$
$$= \lim_{a\to-\infty} [\tan^{-1} 0 - \tan^{-1} a] + \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 0]$$
$$= 0 - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - 0$$
$$= \frac{\pi}{2} + \frac{\pi}{2}$$
$$= \pi$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x^2+1^2}dx = \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right) = \tan^{-1}x$$

$$\tan 0^0 = 0 \Leftrightarrow \tan^{-1}0 = 0$$

$$\tan\frac{\pi}{2} = \infty \Leftrightarrow \tan^{-1}\infty = \frac{\pi}{2}$$

$$\tan^{-1}(-\theta) = -\tan^{-1}\theta$$

Evaluate 2) $\int_0^{\infty} \frac{dx}{1+x^2}$

Sol. Here

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2} dx$$

$$= \lim_{b\to\infty} [tan^{-1} x]_0^b$$

$$= \lim_{b\to\infty} [tan^{-1} b - tan^{-1} 0]$$

$$= \frac{\pi}{2}$$

Evaluate 3) $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2} dx$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} - (-1)\right] \qquad Because\ \frac{1}{\infty} = 0$$

$$= 1$$

Evaluate 4) $\int_{-\infty}^{0} xsinxdx = \lim_{a\to-\infty} \int_a^0 xsinx\, dx$

Here u=x and v=sin x

$$Use\ \int_a^b uv\, dx = \left[u \int vdx\right]_a^b - \int_a^b \left(\frac{du}{dx} \int vdx\right) dx$$

$$= \lim_{a\to-\infty} \left[[x \int sinxdx]_a^0 - \int_a^0 \left(\frac{d}{dx} x \int sinxdx\right)\right] dx$$

$$= \lim_{a\to-\infty} \left[[-xcosx]_a^0 + \int_a^0 cosx\, dx\right]$$

$$= \lim_{a\to-\infty} [0 - (-acosa) + [sinx]_a^0]$$

$$= \lim_{a\to-\infty} [acosa + sino - sina]$$

$$= \lim_{a\to-\infty}[acosa - sina]$$

$$= -\infty$$

Evaluate 5) $\int_1^\infty \frac{1}{\sqrt{x}}dx$

$$= \lim_{a\to\infty}\int_1^a \frac{1}{\sqrt{x}}dx$$
$$= \lim_{a\to\infty}[2\sqrt{x}]_1^a$$
$$= \lim_{a\to\infty}[2\sqrt{a} - 2]$$
$$\to \infty$$

Note:

1) If value of the improper integral is finite then we can say that the integral is converges.

2) If value of the improper integral is infinite then we can say that the integral is diverges.

Note: P integral $\int_1^\infty \frac{1}{x^p}dx$, converges when p >1 and diverges when p$\leq$ 1

Evaluate 6) Show that $\int_1^\infty \frac{logx}{x^2}$ Converges and obtain it value.

Sol. Let u=logx and v=$\frac{1}{x^2}$

$$Use \int_a^b uv\,dx = \left[u\int vdx\right]_a^b - \int_a^b \left(\frac{du}{dx}\int vdx\right)dx$$

$$= \lim_{a\to\infty}\left[\left[logx\int\frac{1}{x^2}dx\right]_1^a - \int_1^a\left(\frac{d}{dx}logx\int\frac{1}{x^2}dx\right)\right]dx$$

$$= \lim_{a\to\infty}\left[\left[logx\left(-\frac{1}{x}\right)\right]_1^a - \int_1^a\frac{1}{x}\left(-\frac{1}{x}\right)dx\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \int_1^a\frac{1}{x^2}\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \left[\left(-\frac{1}{x}\right)\right]_1^a\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \left[-\frac{1}{a} - \left(-\frac{1}{1}\right)\right]\right]$$

(**Use L'Hospitals rule: If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} = \frac{0}{0}$ then**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{a \to \infty} \left[\frac{\log a}{a}\right] = \lim_{a \to \infty} \frac{\frac{1}{a}}{1} = 0 \quad)$$

$=[-0 + 0 - 0 + 1]$

$= 1$

<u>EXERCISE:</u>

Evaluate:

1. $\int_2^\infty \frac{(x+3)}{(x-1)(x^2+1)} dx$
2. $\int_{-\infty}^\infty e^x dx$
3. $\int_0^\infty x^2 \, e^{-x} dx$
4. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

Sol-1 Use partial fraction

$$\therefore \frac{(x+3)}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\therefore \frac{(x+3)}{(x-1)(x^2+1)} = \frac{A(x^2+1) + (Bx+c)(x-1)}{(x-1)(x^2+1)}$$

$$\therefore x + 3 = A(x^2+1) + (Bx+C)(x-1)$$

Put x=1 in above equation, we get

$$\therefore 1 + 3 = A(1^2+1) + (B1+C)(1-1)$$

$$\therefore 4 = 2A$$

$$\therefore A = 2$$

$$\therefore x+3 = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$\therefore x+3 = (A+B)x^2 + (C-B)x + A - C$$

Comparing coefficients on both side

$0 = \text{A+B} \Leftrightarrow B = -A = -2$

$1 = \text{C - B} \Leftrightarrow 1 = C - (-2) \Leftrightarrow 1 = C + 2 \Leftrightarrow C = -1$

$$\therefore \frac{(x+3)}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{-2x-1}{x^2+1}$$

$$= \frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}$$

$$\int_2^\infty \frac{(x+3)}{(x-1)(x^2+1)} dx = \int_2^\infty \left(\frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \lim_{a\to\infty} \int_2^a \left(\frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \lim_{a\to\infty} \left[2\int_2^a \frac{1}{x-1} dx - \int_2^a \frac{2x}{x^2+1} - \int_2^a \frac{1}{x^2+1}\right]$$

$$= \lim_{a\to\infty} [(2ln|x-1|)_2^a - (ln(x^2+1))_2^a - (tan^{-1}x)_2^a]$$

$$= \lim_{a\to\infty} [2(ln|a-1| - ln|2-1|) - ln(a^2+1) + ln(2^2+1) - tan^{-1}a + tan^{-1}2]$$

$$= \infty$$

Sol-2 Here $\int_{-\infty}^{\infty} e^x dx = \int_{-\infty}^{0} e^x dx + \int_0^\infty e^x dx$

$$= \lim_{a\to-\infty} \int_a^0 e^x dx + \lim_{b\to\infty} \int_0^b e^x dx$$

$$= \lim_{a\to-\infty} [e^x]_a^0 + \lim_{b\to\infty} [e^x]_0^b$$

$$= \lim_{a\to-\infty} [e^0 - e^x] \quad + \lim_{b\to\infty} [e^b - e^0]$$

$$= [1-0] + [\infty - 1]$$

$$= \infty$$

Sol-3 *Here* $\int_0^\infty x^2 e^{-x} dx = \lim_{a\to\infty} \int_0^a x^2 e^{-x} dx$

Let u=x^2 $and\ v = e^{-x}$

$$\boldsymbol{Use\ \int_a^b uv\,dx = \left[u\int vdx\right]_a^b - \int_a^b \left(\frac{du}{dx}\int vdx\right)dx}$$

$$= \lim_{a\to\infty}\left[\left[logx\int\frac{1}{x^2}dx\right]_1^a - \int_1^a\left(\frac{d}{dx}logx\int\frac{1}{x^2}dx\right)\right]dx$$

$$= \lim_{a\to\infty}\left[\left[logx\left(-\frac{1}{x}\right)\right]_1^a - \int_1^a\frac{1}{x}\left(-\frac{1}{x}\right)dx\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \int_1^a\frac{1}{x^2}\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \left[\left(-\frac{1}{x}\right)\right]_1^a\right]$$

$$= \lim_{a\to\infty}\left[-\frac{loga}{a} - \left(-\frac{log1}{1}\right) + \left[-\frac{1}{a} - \left(-\frac{1}{1}\right)\right]\right]$$

❖ **<u>Improper Integrals of the second type (Type-II):</u>**

If in the definite integral $\int_a^b f(x)dx$, the integrand f(x) becomes infinite at x=a or x=b or at one or more points within the interval (a, b), then the integral is called improper integral of Type-II.

(1) If f(x) is unbounded at x=a then

$$\int_a^b f(x)dx = \lim_{c\to a}\int_c^b f(x)dx$$

(2) If f(x) is unbounded at x=b then

$$\int_a^{b} f(x)dx = \lim_{c\to b}\int_a^{c} f(x)dx$$

(3) If f(x) is unbounded at x=a and x=b then

$$\int_a^b f(x)dx = \lim_{c1\to a}\int_{c1}^{0} f(x)dx + \lim_{c2\to b}\int_0^{c2} f(x)dx$$

Note: Suppose our function becomes infinity at a point a<c<b

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b f(x)dx = \lim_{t\to c}\int_a^{t} f(x)dx + \lim_{t\to c}\int_t^{b} f(x)dx$$

The improper integral is said to converges (or exist) when limit in R.H. S. of 1), 2) ,3) exist or finite. Otherwise, it is said to diverge.

Evaluate 1)

$$\int_0^{3} \frac{1}{\sqrt{3-x}}\,dx$$

$$\textbf{\textit{Here}}\ f(x) = \frac{1}{\sqrt{3-x}}\ \textbf{\textit{at}}\ x = 3\ , f(x) = \infty$$

$$= \lim_{a\to 3}\int_0^{a} \frac{1}{\sqrt{3-x}}dx$$

$$= \lim_{a\to 3}\left[-2\sqrt{3-x}\right]_0^a$$

$$= \lim_{a\to 3}\left[-2\sqrt{3-a} - (-2\sqrt{3})\right]$$

$$= 2\sqrt{3}$$

The given improper integral is convergent.

Evaluate 2) $\int_0^{\frac{\pi}{2}} secxdx \ = \lim_{a\to\frac{\pi}{2}}\int_0^{a} secxdx$

$$= \lim_{a\to\frac{\pi}{2}}[log|secx + tanx|]_0^a$$

$$= \lim_{a\to\frac{\pi}{2}}[log|seca + tana| - log|sec0 + tan0|]$$

$$= \infty$$

The given improper integral is divergent.

Evaluate 3) $\int_0^5 \frac{1}{(x-2)^2} dx = \int_0^2 \frac{1}{(x-2)^2} dx + \int_2^5 \frac{1}{(x-2)^2} dx$

$$= \lim_{c \to 2} \int_0^c \frac{1}{(x-2)^2} dx + \lim_{c \to 2} \int_c^5 \frac{1}{(x-2)^2} dx$$

$$= \lim_{c \to 2} \left[-\frac{1}{x-2} \right]_0^c + \lim_{c \to 2} \left[-\frac{1}{x-2} \right]_c^5$$

$$= \lim_{c \to 2} \left[-\frac{1}{c-2} - \left(-\frac{1}{0-2} \right) \right] + \lim_{c \to 2} \left[-\frac{1}{5-2} - \left(-\frac{1}{c-2} \right) \right]$$

$$= -\infty$$

Evaluate 4) $\int_{-1}^1 \frac{1}{x^{\frac{2}{3}}} dx = \int_{-1}^0 \frac{1}{x^{\frac{2}{3}}} dx + \int_0^1 \frac{1}{x^{\frac{2}{3}}} dx$

$$= \lim_{a \to 0} \int_{-1}^{a} \frac{1}{x^{\frac{2}{3}}} dx + \lim_{b \to 0} \int_{b}^{1} \frac{1}{x^{\frac{2}{3}}} dx$$

$$= \lim_{a \to 0} \left[3x^{\frac{1}{3}} \right]_{-1}^{a} + \lim_{b \to 0} \left[3x^{\frac{1}{3}} \right]_{b}^{1}$$

$$= \lim_{a \to 0} \left[3a^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} \right] + \lim_{b \to 0} \left[3(1)^{\frac{1}{3}} - 3b^{\frac{1}{3}} \right]$$

$$= -3(-1)^{\frac{1}{3}} + 3$$

$$= -3(-1) + 3$$

$$= 3 + 3$$

$$= 6$$

<u>**Improper Integral of third kind (Type-III):**</u> It is a definite integral in which one or both limits of integration are infinite , and the integrand become infinite at one or more points within or at the end points of the interval of integration. Thus it is a combination of the first kind and the second kind.

For example : $\int_0^\infty \frac{1}{x^2} dx$ is an improper integral of the third kind as the upper limit of integration is infinite and integrand $\frac{1}{x^2}$ is infinite at x = 0.

Evaluate : $\int_0^\infty \frac{1}{x^2} dx = \int_0^1 \frac{1}{x^2} dx + \int_1^\infty \frac{1}{x^2} dx$

$= \textbf{(Type-2)+(Type-1)}$

$$= \lim_{a\to 0}\int_a^1 \frac{1}{x^2}dx + \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$$

$$= \lim_{a\to 0}\left[-\frac{1}{x}\right]_a^1 + \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{a\to 0}\left[-1+\frac{1}{a}\right] + \lim_{b\to\infty}\left[-\frac{1}{b}+1\right]$$

$$= \infty + 1$$

$$= \infty$$

Direct Comparison Test:

1) If f(x) and g(x) are two continuous functions on [a,∞) and $0 \le f(x) \le g(x)$ for all $x \ge a$, then

$$if \int_a^\infty g(x)dx converges\ then \int_a^\infty f(x)dx\ converges,\ .$$

2) If f(x) and g(x) are two continuous functions on [a,∞) and $f(x) \ge g(x)$, then

$$\int_a^\infty f(x)dx\ diverges, if \int_a^\infty g(x)dx\ diverges\ .$$

Limit Comparison Test:

If f(x) and g(x) are positive and continuous on [a,∞) and if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = l, 0 < l < \infty$

Then $\int_a^\infty f(x)dx and \int_a^\infty g(x)dx$ *both converges or diverges together.*

Example-1: Test the convergence of improper integral $\int_1^\infty \frac{cosx}{x^2}dx$.

Solution: We know that

$$\cos x \le 1 \Leftrightarrow \frac{\cos x}{x^2} \le \frac{1}{x^2} \quad \text{for } x \ge 1$$

f(x) = $\frac{\cos x}{x^2}$ and let g(x) = $\frac{1}{x^2}$

$\therefore 0 \le f(x) \le g(x)$

P integral $\int_1^{\infty} \frac{1}{x^p} dx$, converges when p >1 and diverges when p$\le$ 1

$$\int_1^{\infty} g(x)dx = \int_1^{\infty} \frac{1}{x^2} dx$$

Thus, $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.

By comparison test, $\int_1^{\infty} \frac{cosx}{x^2} dx$ is convergent.

Example 2: Check the convergence of $\int_4^{\infty} \frac{3x+5}{x^4+7} dx$

f(x) = $\frac{3x+5}{x^4+7}$ and let g(x) = $\frac{1}{x^3}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{3x+5}{x^4+7}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3x^4+5x^3}{x^4+7} = \lim_{x\to\infty} \frac{x^4\left(3+\frac{5}{x}\right)}{x^4\left(1+\frac{7}{x^4}\right)} = \lim_{x\to\infty} \frac{\left(3+\frac{5}{x}\right)}{\left(1+\frac{7}{x^4}\right)}$ = 3>0

$\int_4^{\infty} g(x)dx = \int_4^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \int_4^{b} \frac{1}{x^3} dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_4^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} - \left(-\frac{1}{2(4)^2}\right)\right]$

=1/32

Thus, $\int_4^{\infty} \frac{1}{x^3} dx$ is convergent.

By Limit comparison test, $\int_4^{\infty} \frac{3x+5}{x^4+7} dx$ is convergent.

<u>Try this:</u> : Check the convergence of $\int_0^1 \frac{1-e^{-x}}{x^3} dx$

We know that

$$\therefore \mathbf{1 - e^{-x} \leq 1 \Leftrightarrow \frac{1 - e^{-x}}{x^3} \leq \frac{1}{x^3}}$$

Let f(x)= $\mathbf{\frac{1-e^{-x}}{x^3}}$ **and g(x)=** $\mathbf{\frac{1}{x^3}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} (\mathbf{1 - e^{-x}}) = 1 - 0 = 1>0 \qquad (Because\ e^{-\infty} = 0)$$

$$\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^3} dx = \lim_{b\to 0} \int_b^1 \frac{1}{x^3} dx$$

$$= \lim_{b\to 0} \left[-\frac{1}{2x^2}\right]_b^1$$

$$= \lim_{b\to 0} \left[-\frac{1}{2(1)^2} - \left(-\frac{1}{2b^2}\right)\right]$$

$$= \lim_{b\to 0} \left(-\frac{1}{2} + \frac{1}{2b^2}\right)$$

$$= \infty$$

Thus, $\int_0^1 \frac{1}{x^3} dx$ is diverges.

By Limit comparison test, $\mathbf{\int_0^1 \frac{1-e^{-x}}{x^3} dx}$ is diverges.

<u>Gamma function</u>

The function of n (n>0) defined by the integral $\int_0^\infty e^{-x}\, x^{n-1} dx$ is called gamma function and is denoted by Γn i.e $\Gamma n = \int_0^\infty e^{-x}\, x^{n-1} dx.$

❖ **<u>Properties of gamma function:</u>**

1) $\Gamma n+1 = n\Gamma n$
2) $\Gamma 1 = 1$
3) $\Gamma n+1 = n!$ n is a positive integer.
4) $\Gamma n = 2\int_0^\infty e^{-x^2}\, x^{2n-1} dx.$
5) $\Gamma_n = (n-1)\Gamma_{n-1}$
6) $\Gamma_n \Gamma_{(1-n)} = \frac{\pi}{\sin(n\pi)}$, Let n=$\frac{1}{2}$ $\therefore \Gamma_{\frac{1}{2}} \Gamma_{(1-\frac{1}{2})} = \frac{\pi}{\sin(\pi\frac{1}{2})}$

$$\therefore \Gamma_{\frac{1}{2}} \Gamma_{\frac{1}{2}} = \pi$$

$$\therefore {\Gamma_{\frac{1}{2}}}^2 = \pi$$

$$\therefore \Gamma_{\frac{1}{2}} = \sqrt{\pi}$$

2) Take $n = \frac{3}{2}$, $\Gamma_n = (n-1)\Gamma_{n-1}$

$$\Gamma_{\frac{3}{2}} = (\frac{3}{2} - 1)\Gamma_{\frac{3}{2}-1}$$

$$\Gamma_{\frac{3}{2}} = \frac{1}{2}\Gamma_{\frac{1}{2}}$$

$$\Gamma_{\frac{3}{2}} = \frac{1}{2}\sqrt{\pi}$$

1) Use $\Gamma n+1 = n!$

$$\Gamma_2 = \Gamma_{1+1} = 1! = 1$$

$$\Gamma_3 = \Gamma_{2+1} = 2! = 2$$

❖ **Examples**

Evaluate 1) $\int_0^\infty e^{-\sqrt{x}}\sqrt[4]{x}dx = \int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}dx$

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$$

$assume\ \sqrt{x} = t$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$dx = 2t\,dt$$

$$\int_0^\infty e^{-t}\left(t^2\right)^{1/4}dt = \int_0^\infty e^{-t}\left(t\right)^{1/2}dt$$

$$= \int_0^\infty e^{-t}\left(t\right)^{3/2-1}dt$$

$$= \Gamma\frac{3}{2} = \frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty e^{-\sqrt{x}}\sqrt[4]{x}dx = \int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}dx$$

$$= \int_0^\infty e^{-t}(t^2)^{\frac{1}{4}}2tdt$$

$$= 2\int_0^\infty e^{-t}(t)^{\frac{1}{2}}tdt$$

$$= 2\int_0^\infty e^{-t}(t)^{\frac{3}{2}}dt$$

$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$, n-1=$\frac{3}{2} \Longrightarrow n = \frac{5}{2}$

$$= 2\int_0^\infty e^{-t}\,(t)^{\frac{5}{2}-1}dt$$

$$= 2\Gamma_{\frac{5}{2}}$$

$$= 2\frac{3}{4}\sqrt{\pi}$$

$$= \frac{3}{2}\sqrt{\pi}$$

$$\therefore \Gamma_n = (n-1)\Gamma_{n-1}$$

$$\therefore \Gamma_{\frac{5}{2}} = (\frac{5}{2}-1)\Gamma_{\frac{5}{2}-1}$$

$$= \frac{3}{2}\Gamma_{\frac{3}{2}}$$

$$= \frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$= \frac{3}{4}\sqrt{\pi}$$

$$\therefore \Gamma_{\frac{3}{2}} = (\frac{3}{2}-1)\Gamma_{\frac{3}{2}-1}$$

$$= \frac{1}{2}\Gamma_{\frac{1}{2}}$$

$$= \frac{1}{2}\sqrt{\pi}$$

Evaluate 2) $\int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}\,dx$

$$\Gamma n = \int_0^\infty e^{-x}\,x^{n-1}dx$$

Sol. Let $\sqrt[3]{x} = t \quad \Longrightarrow x = t^3$

$$\therefore dx = 3t^2 dt$$

Also, if x=0 then t=0

x=∞ then t=∞

Now

$$\therefore \int_0^\infty \sqrt{x} e^{-\sqrt[3]{x}} \, dx = \int_0^\infty x^{\frac{1}{2}} e^{-\sqrt[3]{x}} \, dx$$

$$= \int_0^\infty (t^3)^{\frac{1}{2}} e^{-t} \, 3t^2 dt$$

$$= 3 \int_0^\infty t^{\frac{7}{2}} e^{-t} \; dt$$

$$= 3 \int_0^\infty t^{\frac{9}{2}-1} e^{-t} \; dt$$

$$= 3\Gamma_{\frac{9}{2}}$$

$$= 3 \frac{105}{16} \sqrt{\pi}$$

$$= \frac{315}{16} \sqrt{\pi}$$

$\therefore \Gamma_n = (n-1)\Gamma_{n-1}$

$\therefore \Gamma_{\frac{7}{2}} = \left(\frac{7}{2} - 1\right) \Gamma_{\frac{7}{2}-1}$

$= \left(\frac{5}{2}\right) \Gamma_{\frac{5}{2}}$

$= \frac{5}{2} \left(\frac{5}{2} - 1\right) \Gamma_{\frac{5}{2}-1}$

$= \frac{5}{2} \frac{3}{2} \Gamma_{\frac{3}{2}}$

$= \frac{5}{2} \frac{3}{2} \left(\frac{3}{2} - 1\right) \Gamma_{\frac{3}{2}-1}$

$= \frac{5}{2} \frac{3}{2} \frac{1}{2} \Gamma_{\frac{1}{2}}$

$= \frac{15}{8}\sqrt{\pi}$

Evaluate 3) $\int_0^\infty x^4\, e^{-x^2} dx$

Sol. Let $x^2 = t$ ⟹2xdx=dt

⟹dx=$\frac{1}{2}t^{-\frac{1}{2}}dt$

If x=0 then t=0

If x=∞ then t=∞

$$\therefore \int_0^\infty x^4\, e^{-x^2} dx = \int_0^\infty t^2 e^{-t} \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{3}{2}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{5}{2}-1} dt$$

$$= \frac{1}{2}\Gamma_{\frac{5}{2}}$$

$$= \frac{1}{2}\frac{3}{4}\sqrt{\pi}$$

$$= \frac{3}{8}\sqrt{\pi}$$

<u>EXERCISE:</u>

1) $\int_0^\infty e^{-x^4}\, dx$

$Let\ \ u = x^4 \Leftrightarrow x = u^{\frac{1}{4}} \Leftrightarrow dx = \frac{1}{4}u^{-\frac{3}{4}}du$

$$\therefore \int_0^\infty e^{-x^4}\, dx = \int_0^\infty e^{-u} \frac{1}{4} u^{-\frac{3}{4}} du$$

$$= \frac{1}{4}\int_0^\infty e^{-u}\, u^{-\frac{3}{4}} du$$

$$= \frac{1}{4}\int_0^\infty e^{-u}\, u^{\frac{1}{4}-1} du$$

$$= \frac{1}{4}\Gamma_{\frac{1}{4}}$$

$$= \Gamma_{\frac{5}{4}}$$

Because $\Gamma(n+1) = n\Gamma_n$

If we put n=$\frac{1}{4}$ then

$$\therefore \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\,\Gamma_{\frac{1}{4}}$$

$$\therefore \Gamma_{\frac{5}{4}} = \frac{1}{4}\Gamma_{\frac{1}{4}}$$

2) $\int_0^\infty e^{-x^2}\sqrt{x^3}\,dx$

$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1} dx$

Sol. $Let \quad x^2 = u$ ⟹2xdx=du

⟹dx=$\frac{1}{2}u^{-\frac{1}{2}}du$

when x=0 , t=0

when x=∞, t=∞

We get,

$$\therefore \int_0^\infty e^{-x^2}\sqrt{x^3}dx = \int_0^\infty e^{-x^2}\, x^{\frac{3}{2}}dx$$

$$= \int_0^\infty e^{-u}\left(\sqrt{u}\right)^{\frac{3}{2}}\frac{1}{2}u^{-\frac{1}{2}}du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}\left(u^{\frac{1}{2}}\right)^{\frac{3}{2}}u^{-\frac{1}{2}}du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}\,(u)^{\frac{3}{4}-\frac{1}{2}}du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}\,u^{\frac{1}{4}}du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}\,u^{\frac{5}{4}-1}du$$

We know that $\Gamma n = \int_0^\infty e^{-x}\, x^{n-1} dx$

$$= \frac{1}{2}\Gamma_{\frac{5}{4}}$$

$$=\frac{1}{8}\Gamma_{\frac{1}{4}}$$

$Because\ \Gamma(x+1)=x\Gamma_x$

If we put x=$\frac{1}{4}$ then

$$\therefore \Gamma\left(\frac{1}{4}+1\right)=\frac{1}{4}\ \Gamma_{\frac{1}{4}}$$

$$\therefore \Gamma_{\frac{5}{4}}=\frac{1}{4}\Gamma_{\frac{1}{4}}$$

3) $\int_0^\infty 5^{-4x^2}\,dx$ t=5^{-4x^2} $\Longrightarrow lnt=-4x^2ln5$

$$\Longrightarrow t=e^{-4x^2ln5}$$

$$\int_0^\infty 5^{-4x^2}\,dx=\int_0^\infty e^{-4x^2ln5}\,dx$$

Sol. $Let\ \ x^2=u$ ⟹2xdx=du

⟹dx=$\frac{1}{2}u^{-\frac{1}{2}}du$

when x=0 , u=0

when x=∞, u=∞

We get,

$$\int_0^\infty e^{-4x^2ln5}\,dx=\int_0^\infty e^{-4uln5}\frac{1}{2}u^{-\frac{1}{2}}du$$

$$=\frac{1}{2}\int_0^\infty e^{-4uln5}\,u^{-\frac{1}{2}}du$$

Let $4uln5=v \Longrightarrow u=\frac{v}{4ln5} \Longrightarrow du=\frac{dv}{4ln5}$

$$=\frac{1}{2}\int_0^\infty e^{-v}\left(\frac{v}{4ln5}\right)^{-\frac{1}{2}}\frac{dv}{4ln5}$$

$$=\frac{1}{2\sqrt{4ln5}}\int_0^\infty e^{-v}\,(v)^{-\frac{1}{2}}dv$$

$$=\frac{1}{2\sqrt{4ln5}}\int_0^\infty e^{-v}\,(v)^{\frac{1}{2}-1}dv$$

$$= \frac{1}{2\sqrt{4ln5}} \Gamma_{\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{4ln5}}$$

Beta function

The function of m and n defined by the integral (m, n>0) $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ is called the Beta function and is denoted by $\beta(m, n)$.

i.e $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Properties of Beta Function:

1) $\boldsymbol{\beta(m, n) = \beta(n, m)}$
2) $\boldsymbol{\beta(m, n) = 2 \int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta}$

Examples

Evaluate 1) $\int_0^3 (3-x)^{\left(\frac{1}{2}\right)} x^{\frac{5}{2}} dx$

$assume\ x = 3t$

$dx = 3dt$

$x = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow t = 1$

$$\int_0^1 (3-3t)^{1/2} (3t)^{5/2}\, 3\, dt$$

$$= 3^4 \int_0^1 (1-t)^{1/2} (t)^{5/2}\ dt$$

$$= 3^4 \int_0^1 (1-t)^{3/2-1} (t)^{7/2-1}\ dt$$

$$= 3^4 \beta\left(3/2, 7/2\right)$$

Evaluate 2) $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \int_0^1 x(1-x^5)^{-\frac{1}{2}}dx$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$x^5 = t \Longrightarrow x = t^{\frac{1}{5}}$$

$$5x^4dx = dt \Longrightarrow dx = \frac{1}{5}\left(t^{\frac{1}{5}}\right)^{-4} dt$$

$$\Longrightarrow dx = \frac{1}{5}t^{-\frac{4}{5}}dt$$

If x=0 then t=0

If x=1 then t=1

$$\int_0^1 x(1-x^5)^{-\frac{1}{2}}dx = \int_0^1 t^{\frac{1}{5}}(1-t)^{-\frac{1}{2}}\frac{1}{5}t^{-\frac{4}{5}}dt$$

$$= \frac{1}{5}\int_0^1 t^{-\frac{3}{5}}(1-t)^{-\frac{1}{2}}dt$$

$$= \frac{1}{5}\int_0^1 t^{\frac{2}{5}-1}(1-t)^{\frac{1}{2}-1}dt$$

e$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$$

Evaluate 3) $\int_0^{2a} x^2\sqrt{2ax-x^2}dx$

Let $x = at \Longrightarrow dx = adt$

$$If\ x = 0\ then\ t = 0, \quad If\ x = 2a\ then\ t = 2$$

$$\int_0^{2a} x^2\sqrt{2ax-x^2}dx = \int_0^2 (at)^2\sqrt{2a(at)-(at)^2}\,adt$$

$$= a^4\int_0^2 (t)^2\sqrt{2t-t^2}\,dt$$

$$= \boldsymbol{a^4 \int_0^2 (t)^2 \sqrt{1-(1-t)^2}\, dt}$$

$$\boldsymbol{1 - t = sin\theta \Rightarrow t = 1 - sin}$$

$$\boldsymbol{-dt = cos\theta d\theta \Rightarrow dt = -cos\theta d\theta}$$

$$\boldsymbol{If\ \ t = 0\ then\ \theta = \frac{\pi}{2}, \qquad if\ t = 2\ then\ \theta = -\frac{\pi}{2}}$$

$$= \boldsymbol{a^4 \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (1 - sin\theta)^2 \sqrt{1-(sin^2\theta)}\, (-cos\theta) d\theta}$$

$$= \boldsymbol{a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - sin\theta)^2\, cos^2\theta d\theta}$$

$$= \boldsymbol{2a^4 \int_0^{\frac{\pi}{2}} (1 - sin\theta)^2\, cos^2\theta d\theta}$$

$$= \boldsymbol{2a^4 \int_0^{\frac{\pi}{2}} (1 - sin\theta)^2\, cos^2\theta d\theta}$$

$$= \boldsymbol{2a^4 \int_0^{\frac{\pi}{2}} (1 - 2sin\theta + sin^2\theta)\, cos^2\theta d\theta}$$

$$= \boldsymbol{2a^4 \left[\int_0^{\frac{\pi}{2}} sin^0\theta\, cos^2\theta d\theta - 2\int_0^{\frac{\pi}{2}} sin\theta\, cos^2\theta d\theta + \int_0^{\frac{\pi}{2}} sin^2\theta\, cos^2\theta d\theta \right]}$$

$$\boldsymbol{\int_0^{\pi/2} sin^p\theta cos^q \quad \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

$$= 2a^4 \left[\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) + \right]$$

❖ **EXERCISE:**

1) $\int_0^{\frac{\pi}{4}} \cos^3 2x \sin^4 4x \, dx$

2) $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

3) Prove that $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}} dx$

Properties of Beta and Gamma Function:

1) $\boldsymbol{\beta(m,n)} = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

2) $\frac{1}{2}\boldsymbol{\beta}\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \boldsymbol{sin^p\theta cos^q \quad \theta d\theta}$

3) $\int_0^{\pi/2} \boldsymbol{sin^p\theta cos^q\theta d\theta} = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \quad \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\Pi}$

Examples

Evaluate 1) $\int_3^7 \sqrt[4]{(x-3)(7-x)}$ dx

Evaluate 2) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{sin\theta}\, d\theta$

Evaluate 3) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$