Parul University
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1^{st} Year B.Tech Programme (All Branches)
Mathematics – 1 (203191102)
Unit – 3 MATRICES (Lecture Note)

<u>MATRIX</u>:

A Matrix is a rectangular array of numbers (or functions) enclosed in brackets. These number or functions are called entries or elements of the matrix.

For example:

$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 5 \end{bmatrix}$, $\begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$ are matrices.

<u>Trace of a matrix:</u>

If A is a square matrix, the trace of A, denoted by **tr(A)**, is defined to be the sum of entries on the main diagonal of A. The trace of A is undefined If A is not a square matrix.

For example:

$A = \begin{bmatrix} 4 & 5 \\ 10 & 6 \end{bmatrix}$**, tr(A)= 4+6=10**

<u>Symmetric matrix:-</u> For any **square** matrix A, if $A = A^T$ then it is known as symmetric matrix.

Example:- (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 5 \end{bmatrix}$ Here, we can see that $A = A^T$ so A is symmetric matrix.

<u>Skew-symmetric matrix:-</u> For any **square** matrix A, if $A = -A^T$ then it is known as Skew symmetric matrix.

Example:- (1) $A = \begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & -3 & -5 \\ 3 & 0 & 2 \\ 5 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix} = -A$

Here, we can see that $A = -A^T$ so A is skew-symmetric matrix.

<u>Singular and non-singular matrix:-</u>For any **square** matrix A, if $|A| \neq 0,$ then it is known as non-singular matrix and if $|A| = 0,$ then it is known as singular matrix.

Example:- (1) If $A = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \Rightarrow |A| = 8 - 8 = 0 \Rightarrow \text{Singular matrix.}$

(2) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0 \Rightarrow$ non-singular matrix

Orthogonal Matrix: The matrix is said to be an orthogonal matrix if the product of a matrix and its transpose gives an identity value.

Example: Determine if A is an orthogonal matrix.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

System of linear equation

Linear Equations: Any straight line in the xy-plane can be represented algebraically by equation of the form ax+by=c, where a & b are real numbers.

A **system of linear equation** is a collection of one or more linear equations involving the same variables.

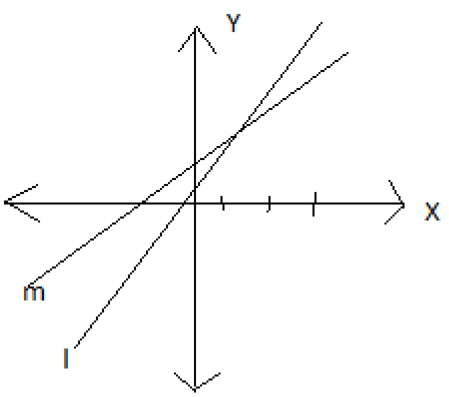
A linear system of m linear equations in n variables: An arbitrary system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ is a set of equations of the form

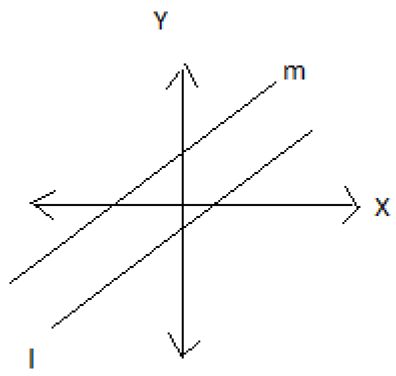
$$\boxed{\sum_{j=1}^{n} a_{ij}\, x_j = b_i \;\; (i = 1,2,3,\ldots,m,\; j = 1,2,3,\ldots.n)}$$

A system of linear equations has either

1. No solutions, or
2. Exactly one solutions, or
3. Infinitely many solutions

Geometrical representation:





$x - 2y = -1$ l

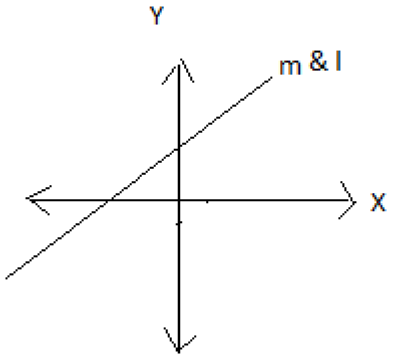
$-x + 3y = 3$ .... m

Exactly one solution

$x - 2y = -1$ l

$-x + 2y = 3$ .... m

No solution



$$x - 2y = -1 \quad .... \, l$$
$$-x + 2y = 1 \quad .... \, m$$

Infinitely many solutions

NOTE: (i) The system is said to be consistent if we get infinitely many solution or Unique solution.

(ii) The system is said to be inconsistent if we get No solution.

- **Condition of Consistency for non-homogeneous system:**

(1) If there is a zero row to left of the augmentation bar but the last entry of this row is non-zero then the system has **no solution**.

(2) If at least one of the columns on the left of the augmentation bar has zero element pivot entry, then the system has **infinitely many solutions**.

(3) Otherwise the system has **unique solution.**

Augmented matrix: A system of m equations in n unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & .... & a_{1n} & b_1 \\ a_{21} & a_{22} & .... & a_{2n} & b_2 \\ & .... & & & \\ a_{m1} & a_{m2} & .... & a_{mn} & b_1 \end{array}\right]$$ This is known as augmented matrix.

For example: Find the augmented matrix for each of the following system of linear equations:

$$\begin{aligned} 2x_1 + \quad + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$ then augmented matrix is given by $$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right].$$

ROW - ECHELON FORMS OF A MATRIX
(Gauss Elimination Method)

In the definitions that follow, a nonzero row or column in a matrix means a row or column that contains at least one nonzero entry, a leading entry of a row refers to the leftmost nonzero entry(in a nonzero row).

Definition:

A rectangular matrix is in row-echelon form (or echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros:
   If a matrix in echelon form satisfies the following additional conditions, then it is in reduced row- echelon form (or reduced echelon).
4. The leading entry in each nonzero row is 1.
5. Each column that contains a leading 1 has zeros everywhere else in that column.

OR

Properties of a Matrix in Row-Echelon Form

1) If there are any "all-0" rows, then they must be at the bottom of the matrix. Aside from these "all-0" rows,

2) Every row must have a "1" (called a "leading 1") as its leftmost non-0 entry.

3) The "leading 1"s must "flow down and to the right." More precisely: The "leading 1" of a row must be in a column to the right of the "leading 1"s of all higher rows.

REDUCED ROW-ECHELON (RRE) FORM FOR A MATRIX
(Gauss Jordan Elimination Methods)

A matrix is in **reduced row echelon form** (also called **row canonical form**) if it satisfies the following conditions:

- It is in row echelon form.
- Every leading coefficient is 1 and is the only nonzero entry in its column.

Example 1:

Solve the following system by gauss-Elimination method

$2x + 2y + 2z = 0,$

$-2x + 5y + 2z = 1,$

$8x + y + 4z = -1$

Solution:

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

R2 + (1)R1

R3 + (-4)R1

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$

R3 + (1)R2

Example 2 :

Solve the system of equation

$-2b + 3c = 1$

$3a + 6b - 3c = -2$

$6a + 6b + 3c = 5$ by Gauss elimination.

Solution:

Interchanging first and second equation we get

$3a + 6b - 3c = -2$ (1)

$0a - 2b + 3c = 1$ (2)

$6a + 6b + 3c = 5$ (3)

$$\begin{bmatrix} 3 & -6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$

Operate R3 – 2R1

$$\begin{bmatrix} 3 & -6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 3 & 9 \end{bmatrix}$$

Operate R3/3

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Back substitution: from the second we find y= 1/7 – 4/7z. from this and first equation we get x = -1/7 – 3/7z. since, z remain arbitrary, we have infinitely many solutions.

Example 3 :

Solve the following system by gauss elimination method.

$$\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9,$$

$$\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10,$$

$$\frac{-1}{x}+\frac{3}{y}+\frac{4}{z}=30,$$

Solution:

Let $\frac{1}{x}=u, \frac{1}{y}=v, \frac{1}{z}=w$

-u + 3v + 4w = 30 (1)

3u + 2v – 1w = 9 (2)

2u – 1v + 2w = 10 (3)

$$\begin{bmatrix} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{bmatrix}$$

R3 + 2R1

R2 + 3R1

$$\begin{bmatrix} -1 & 3 & 4 & 30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 79 \end{bmatrix}$$

R2 – R3

$$\begin{bmatrix} -1 & 3 & 4 & 30 \\ 0 & 6 & 1 & 29 \\ 0 & 5 & 10 & 70 \end{bmatrix}$$

6R3 – 5R2

$$\begin{bmatrix} -1 & 3 & 4 & 30 \\ 0 & 6 & 1 & 29 \\ 0 & 0 & 55 & 275 \end{bmatrix}$$

Back substitution gives

w=5, v=4, u=2

$\therefore\ x=\frac{1}{2},\ y=\frac{1}{4},\ z=\frac{1}{5}$ is required unique solution.

$$\begin{bmatrix} 3 & -6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -2 & 3 & 3 \end{bmatrix}$$

Operate R3 – R2

$$\begin{bmatrix} 3 & -6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Which is an echelon form. This shows that the system has no solution.

Example 4: Consider the following system

$x+y+z=6$

$x+2y+3z=10$.

$x+2y+\lambda z=\mu$

For what values of λ and μ do the system has (i) unique solution (ii) no solution (iii) infinitely many solutions.

Solution: The Augmented matrix is

$$[A \mid B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] R_3 \to R_3 - R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) If λ -3=0 and μ -10 $\neq$ 0, that is if λ =3 and $\mu \neq 10$ then the system does not have any solution.

(ii)If λ -3$\neq$0 then the system has a unique solution. That is $\lambda \neq 3$ and μ can possess any real value.

(iii)If λ -3=0 and μ -10 =0, that is if λ =3 and μ =10 then the system has infinite solutions.

❖ GAUSS-JORDAN METHOD (Reduced Row-Echelon Form):

Example: Solve the following system using Gauss-Jordan Method:

(1)
$$\begin{aligned} x+y+z&=3\\ x+2y-z&=4\\ x+3y+2z&=4 \end{aligned}$$

Solution: The augmented matrix is

$$[A \mid B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & -1 & 4\\ 1 & 3 & 2 & 4 \end{array}\right] \begin{array}{l} R_2 \to R_2-R_1,\\ R_3 \to R_3-R_1 \end{array}$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & -2 & 1\\ 0 & 2 & 5 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1-R_2,\\ R_3 \to R_3-2R_2 \end{array}$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2\\ 0 & 1 & -2 & 1\\ 0 & 0 & 5 & -1 \end{array}\right] R_3 \to \frac{1}{5}R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2\\ 0 & 1 & -2 & 1\\ 0 & 0 & 1 & -1/5 \end{array}\right] \begin{array}{l} R_1 \to R_1-3R_3,\\ R_2 \to R_2+2R_3 \end{array}$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 13/5\\ 0 & 1 & 0 & 3/5\\ 0 & 0 & 1 & -1/5 \end{array}\right]$$

$\therefore$ **The solution is unique because** $x=\frac{13}{5}, y=\frac{3}{5}, z=\frac{-1}{5}$.

(2)
$$\begin{aligned} 2x+y+3z\quad &=16\\ 3x+2y\quad +w&=16\\ 2x\quad +2z-5w&=5 \end{aligned}$$

Solution: The augmented matrix is

$$[A \mid B]=\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16\\ 3 & 2 & 0 & 1 & 16\\ 2 & 0 & 12 & -5 & 5 \end{array}\right] R_1 \to \frac{1}{2}R_1$$

$$=\left[\begin{array}{cccc|c} 1 & 1/2 & 3/2 & 0 & 8\\ 3 & 2 & 0 & 1 & 16\\ 2 & 0 & 12 & -5 & 5 \end{array}\right] \begin{array}{l} R_2 \to R_2-3R_1,\\ R_3 \to R_3-2R_1 \end{array}$$

$$=\left[\begin{array}{cccc|c} 1 & 1/2 & 3/2 & 0 & 8\\ 0 & 1/2 & -9/2 & 1 & -8\\ 0 & -1 & 9 & -5 & -11 \end{array}\right] R_2 \to 2R_2$$

$$=\left[\begin{array}{cccc|c} 1 & 1/2 & 3/2 & 0 & 8\\ 0 & 1 & -9 & 2 & -16\\ 0 & -1 & 9 & -5 & -11 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1-1/2R_2,\\ R_3 \to R_3+R_2 \end{array}$$

$$=\left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16\\ 0 & 1 & -9 & 2 & -16\\ 0 & 0 & 0 & -3 & -27 \end{array}\right] R_3 \to -1/3R_3$$

$$=\left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16\\ 0 & 1 & -9 & 2 & -16\\ 0 & 0 & 0 & 1 & 9 \end{array}\right] \begin{array}{l} R_1 \to R_1+R_3,\\ R_2 \to R_2-2R_3 \end{array}$$

$$=\left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & 25\\ 0 & 1 & -9 & 0 & -34\\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

Back Substitution: w=9, y-9z=-34, x+6z=25.
z is treated as independent variable , therefore we suppose z=k, then w=9, y=-34+9k & x=25-6k.

$\therefore$ **The solution set is {(25-6k, -34+9k, k, 9)/ k $\in$ R}.**

❖ HOMOGENEOUS EQUATIONS:

A system of linear equations in terms of $x_1, x_2, \ldots, x_n$ having the matrix form AX=O, where A is m x n coefficient matrix, X is n x 1 column matrix, O is a m x 1 zero column matrix is called a system of homogeneous equations.

For example:(i) $\begin{matrix} x+y+z=0 \\ x+2y-z=0 \\ x+3y+2z=0 \end{matrix}$ (ii) $\begin{matrix} x+y=0 \\ x+2y=0 \end{matrix}$

Homogeneous equations are never inconsistent. They always have the solution "all variables = 0". The solution (0, 0,…., 0) is often called the **trivial solution**. Any other solution is called **nontrivial solution.**

Example1: Consider the following system:

$$4x+3y-z=0$$
$$3x+4y+z=0$$
$$5x+y-4z=0$$

Solution:

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] R_1 \to \frac{1}{4}R_1$$

$$=\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$=\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 0 & 7/4 & 7/4 & 0 \\ 0 & -11/4 & -11/4 & 0 \end{array}\right] R_2 \to \frac{4}{7}R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x-z=0,\ y+z=0,\ 0=0.$

The last equation does not give any information about the equations. let $z=k \Rightarrow y=-k \ and \ x=k$.

$\therefore the\ solution\ set\ is \{(k,-k,k)/k \in R\}$

Example2: Consider the following system

$$-2x+2y-3z=0$$
$$2x+y-6z \quad =0$$
$$-x-2y+2z \quad =0$$
$$3x+y+4z \quad =0$$

Solution:

$$\left[\begin{array}{ccc|c} -2 & 2 & -3 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] R_1 \to -\frac{1}{2}R_1$$

$$=\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$=\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] R_2 \to \frac{1}{3}R_2$$

$$=\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] \begin{matrix} R_1 \to R_1 + R_2, \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{matrix}$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -11/2 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] R_3 \to -\frac{2}{11}R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] \begin{matrix} R_1 \to R_1 + 3/2R_3, \\ R_2 \to R_2 + 3R_3 \\ R_4 \to R_4 - 23/2R_3 \end{matrix}$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The required solution is x=0, y= 0, z=0.

Rank of the matrix: Let A be an m x n matrix, the **rank** of A is number of nonzero rows in reduced-row-echelon form of A and is denoted by rank (A) or $\rho(A)$.

Example: Determine the rank of the matrix A, if $A=\begin{bmatrix}1&1&1&1&1\\4&5&6&7&8\\5&6&7&8&9\\10&11&12&13&14\\15&16&17&18&19\end{bmatrix}\begin{matrix}R_2\to R_2-4R_1\\R_3\to R_3-5R_1\\R_4\to R_4-10R_1\\R_5\to R_5-5R_1\end{matrix}$ $=\begin{bmatrix}1&1&1&1&1\\0&1&2&3&4\\0&1&2&3&4\\0&1&2&3&4\\0&1&2&3&4\end{bmatrix}\begin{matrix}R_3\to R_3-R_2\\R_4\to R_4-R_2\\R_5\to R_5-R_2\end{matrix}$	$=\begin{bmatrix}1&1&1&1&1\\0&1&2&3&4\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$ $\therefore rank(A)=2$

RESULT:

(1) If rank (A) $\neq$ rank (A, B) then the system is inconsistent.

(2) If rank (A) = rank (A, B) = ρ then the system is consistent.

(a) if ρ < n then there are infinitely many solutions. (n is the no of unknowns).

(b) if ρ =n then there is a unique solution.

Example: Find the number of parameters in the general solution of AX = O if A is a 5 x 7 matrix of rank 3.

Solution: Here rank(A) = ρ (A)= 3 and n= 7 then number of parameters = n - ρ (A) = 7-3 = 4.

Eigen values and Eigen vectors

Let A be n x n matrix, then there exists a real number λ and a nonzero n-vector X such that AX = λ X then, λ is called as the eigen value or characteristic value or proper roots of the matrix A, and X is called as eigen vector or characteristic vector or real vector corresponding to eigen value λ of the matrix A.

Note: An eigen vector is never the zero vector.

1.The matrix [A - λ I_n] is known as the **characteristic matrix** of A.

2.The determinant of (A - λ I_n) after expansion gives the polynomial in λ, it is known as the **characteristic polynomial** of the matrix A of order n x n and is of degree n.

3. $|A - \lambda\ In|$=0 is called the **characteristic equation** of matrix A.

4. The root of the characteristic equation is known as **characteristic value** or **eigenvalue** of the matrix.

5. The set of all characteristic roots (eigen values) of the matrix A is called the **spectrum of A.**

6. Let A be n x n matrix and λ be an eigen value for A. Then the set $E_\lambda = \{X / AX = \lambda X\}$ is called the **eigen space of** λ **.**

Result: 1. The eigen values of a diagonal matrix are its diagonal elements.

2. The sum of eigen values of an n x n matrix is its trace and their product is $|A|$.

3. For the upper triangular (lower triangular) n x n matrix A, the eigen values are its diagonal elements.

Example 1: If $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ then find the eigen values of A^T, A^9, 5A & A^{-1}.

<u>Solution:</u>The eigen values for A^T are 1, 2, 2. The eigen values for A^9 are 1, (2^9), (2^9).

The eigen values for 5A are 5, 10, 10. The eigen values for A^{-1} are 1, ½, ½.

Type 1: When the eigen values are nonrepeated, whether the matrix is symmetric or non-symmetric.

Example: Find the eigen values and eigen vector of the matrix A= $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Solution:</u>The characteristic equation is $|A - \lambda In| = 0$

$$\begin{vmatrix} -2-\lambda & -8 & -13 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1= trace(A)= -2+4+1 = 3

S_2= sum of minors of diagonal entries= 4-2+0 = 2.

$|A|$ = 1(-8+ 8) = 0

Characteristic equation is $\therefore \lambda^3 - 3\lambda^2 + 2\lambda = 0$

$$\therefore \lambda(\lambda^2 - 3\lambda + 2) = 0$$

$$\therefore \lambda(\lambda - 1)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$$

When $\lambda_1 = 0$

$$[A - \lambda I \mid O] = \left[\begin{array}{ccc|c} -2 & -8 & -12 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to -1/2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 8/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We suppose z=k, y=0, x+4z=0

Z= k , y = 0, x = -4k

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Therefore eigen vector space for $\lambda_2 = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda_3 = 2$

$$[A - \lambda I \mid O] = \left[\begin{array}{ccc|c} -2-2 & -8 & -12 & 0 \\ 1 & 4-2 & 4 & 0 \\ 0 & 0 & 1-2 & 0 \end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1 & 4 & 6 & 0\\0 & 0 & -2 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

Therefore we suppose x+4y+6z = 0, -2z = 0, y = k.
z = 0, y = k, x = -4k.
Therefore eigen vector space is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4k\\k\\0\end{bmatrix}=k\begin{bmatrix}-4\\1\\0\end{bmatrix}$$

Therefore eigen vector space for $\lambda_1=0$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4\\1\\0\end{bmatrix}$$

When $\lambda_2=1$

$$[A-\lambda I\mid O]=\left[\begin{array}{ccc|c}-2-1 & -8 & -12 & 0\\1 & 4-1 & 4 & 0\\0 & 0 & 1-1 & 0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}-3 & -8 & -12 & 0\\1 & 3 & 4 & 0\\0 & 0 & 0 & 0\end{array}\right] R_1\to -1/3R_1$$

$$=\left[\begin{array}{ccc|c}1 & 8/3 & 4 & 0\\1 & 3 & 4 & 0\\0 & 0 & 0 & 0\end{array}\right] R_2\to R_2-R_1$$

$$=\left[\begin{array}{ccc|c}1 & 8/3 & 4 & 0\\0 & 1/3 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right] R_2\to 3R_2$$

$$=\left[\begin{array}{ccc|c}-4 & -8 & -12 & 0\\1 & 2 & 4 & 0\\0 & 0 & -1 & 0\end{array}\right] R_1\to -1/4R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\1 & 2 & 4 & 0\\0 & 0 & -1 & 0\end{array}\right] R_2\to R_2-R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & 0 & 1 & 0\\0 & 0 & -1 & 0\end{array}\right] R_3\to -R_3$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & 0 & 1 & 0\\0 & 0 & 1 & 0\end{array}\right] \begin{array}{l}R_1\to R_1-3R_3\\R_2\to R_2-R_3\end{array}$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

We suppose z=0, y=k, x+2y=0
Z= 0 , y = k, x = -2k

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2k\\k\\0\end{bmatrix}=k\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Therefore eigen vector space for $\lambda_3=2$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

- **Algebraic multiplicity:** Let A be n x n matrix, let λ be an eigen value for A. If λ occurs times (k ≥ 1) then k is called the **algebraic multiplicity** of λ, and the number of basis vectors is called **Geometric multiplicity**.

Type 2: When the roots are repeated and the matrix is non-symmetric.

Example: Find eigen values and eigen vectors of the matrix. A= $\begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$$

$= (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)]-2[2(-\lambda)-(-1)(-6)]-3[2(-2)-(-1)(1-\lambda)]$

$= (-2-\lambda)[-\lambda+\lambda^2-2]-2[-2\lambda-6]-3[-4+1-\lambda]$

$= -\lambda^3-\lambda^2+21\lambda+45$

$= -(\lambda^3+\lambda^2-21\lambda-45)$

$\therefore -(\lambda^3+\lambda^2-21\lambda-45)=0$

$\therefore \lambda^3+\lambda^2-21\lambda-45=0$

$\therefore (\lambda+3)(\lambda^2-2\lambda-15)=0$

$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$

$\Rightarrow \lambda_1=5, \lambda_2=-3, \lambda_3=-3$

Algebraic Multiplicity of $\lambda=-3$ is 2 and of $\lambda=5$ is 1.

We solve the following homogeneous system:

$$\therefore [A-\lambda I]X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case I: When $\lambda_1=5$

$$\therefore [A-\lambda I|O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} -1 & -2 & -5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] R_1 \to -R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 7R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right]$$

$R_2 \to -1/8R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 16R_2 \end{array}$$

Case II :When $\lambda_2=-3, \lambda_3=-3$

$$\therefore [A-\lambda I|O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let

$x_3 = k_1, x_2 = k_2, x_1 + 2x_2 - 3x_3 = 0,$.

$x_1 = -2k_1 + 3k_2$

Therefore eigen space is for $\lambda_2=-3, \lambda_3=-3$ is {k1 (-2, 1, 0) + k2 (3, 0, 1) / k $\in$R}

Geometric multiplicity of $\lambda=-3$ is 2 and of $\lambda=5$ is 1.

$=\left[\begin{array}{ccc\|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Let $x_3 = k, x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k, x_1 + x_3 = 0$ $\Rightarrow x_1 = -k$ Therefore eigen space is for $\lambda_1 = 5$ is {k (-1, -2, 1)/ k $\in$R}	

Type 3: When eigen values are repeated and matrix is symmetric.

Example: Find eigen values and eigen vectors of the matrix. A=$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A-\lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1+\lambda)$

$= -\lambda^3 + 3\lambda + 2 = -(\lambda^3 - 3\lambda - 2)$

$\therefore -(\lambda^3 - 3\lambda - 2) = 0$

$\therefore \lambda^3 - 3\lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1)^2 = 0$

$\Rightarrow \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$

Algebraic Multiplicity of $\lambda = -1$ is 2 and of $\lambda = 2$ is 1.

CaseI: $\lambda_1 = 2$

$\therefore [A - \lambda I | O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$

$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad R_2 \to -1/3R_2$

CaseI: $\lambda_2 = -1, \lambda_3 = -1$

$\therefore [A - \lambda I | O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_3 = k_1, x_2 = k_2, x_1 + x_2 + x_3 = 0,\ x_1 = -k_1 - k_2$. Therefore eigen space is for $\lambda_2 = -1, \lambda_3 = -1$ is {k_1 (-1, 0, 1) + k_2 (-1, 1, 0) / k $\in$ R}

$=\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$ $\begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$ $=\left[\begin{array}{ccc\|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Let $x_3 = k, x_2 - x_3 = 0, \Rightarrow x_2 = k, x_1 - x_3 = 0, x_1 = k$. Therefore eigen space is for $\lambda_1 = 2$ is {k (1,1,1) / k $\in$ R}	Geometric Multiplicity of $\lambda = -1$ is 2 and of $\lambda = 2$ is 1.

Example: Determine algebraic and geometric multiplicity of $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.
Answer: $\lambda = 1, 2, 2$ therefore algebraic multiplicity of $\lambda = 2$ is 2 and geometric multiplicity is 1. For $\lambda = 1$ A.M. is 1 and G.M. is 1.

- Every square matrix can be decomposed as a sum of symmetric and skew-symmetric matrices.
  Proof: Let A be m x n matrix.
  Let $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$ be two matrices.
  Obviously A= B+C.
  Now, $B = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left[(A + A^T)\right]^T = \frac{1}{2}\left[(A)^T + (A^T)^T\right] = \frac{1}{2}\left[(A^T + A)\right] = B$
  Therefore, $B = B^T$, B is symmetric.

$$C = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}\left[(A - A^T)\right]^T = \frac{1}{2}\left[(A)^T - (A^T)^T\right] = \frac{1}{2}\left[(A^T - A)\right] = -C$$

Therefore, $-C = C^T$, C is skew- symmetric.

Therefore A is a sum of symmetric and skew-symmetric matrices.

<u>**Caley –Hamilton Theorem:**</u> Every square matrix satisfies its characteristic equation i.e. The theorem states that, for a square matrix A of order n, if $|A - \lambda I| = 0$.

Example (i): Using Caley-Hamilton theorem find inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
<u>Solution</u>: The characteristics equation is $\|A - \lambda I\| = 0$ $\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$ $\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$ $\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

By caley-Hamilton theorem

$A^2 - 4A - 5I = 0$

$\Rightarrow A^2A^{-1} - 4AA^{-1} - 5IA^{-1} = 0$

$\Rightarrow A - 4I - 5A^{-1} = 0$

$\Rightarrow 5A^{-1} = 4I - A$

$\Rightarrow A^{-1} = \frac{1}{5}(4I - A) = \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4/5 & 0 \\ 0 & 4/5 \end{bmatrix} - \begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ -2/5 & 1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

Example (ii): Find the characteristics equation of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence prove that

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$.

Solution: The characteristics equation is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Caley-Hamilton Theorem

$\therefore A^3 - 5A^2 + 7A - 3I = 0$ ....(1)

Now,

$A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^2 + A + I$ using (1)

$$\therefore A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Diagonalization of a matrix:

An n x n matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

OR

If n x n matrix A has a basis of eigenvectors, then $D = P^{-1}AP$ is diagonal, with the eigenvalues of A as the entries on the main diagonal. Here P is the matrix with these eigenvectors as column vectors.

Also, $D^n = P^{-1}A^nP$ and $A^n = PD^nP^{-1}$

Example (i): Find a matrix P that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Solution (i): $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\therefore \lambda = 1,2,3$

For $\lambda = 1$

$$\therefore [A - \lambda I | O] = \left[\begin{array}{ccc|c} -1-\lambda & 4 & -2 & 0 \\ -3 & 4-\lambda & 0 & 0 \\ -3 & 1 & 3-\lambda & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \quad R_1 \to -1/2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \quad R_2 \to -1/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 5R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 4/3R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - z = 0 \;\&\; x - 2/3z = 0$

$\Rightarrow z = k, y = k, x = 2/3k$

$\therefore (x, y, z) = k(\frac{2}{3},1,1); k \in R$

$\therefore (x, y, z) = 3k(2,3,3); k \in R \quad (\because 3k = k')$

$E_1 = \{k'(2,3,3) / k' \in R\}$

For $\lambda = 3$

$$\therefore [A - \lambda I | O] = \left[\begin{array}{ccc|c} -1-\lambda & 4 & -2 & 0 \\ -3 & 4-\lambda & 0 & 0 \\ -3 & 1 & 3-\lambda & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \quad R_1 \to -1/4R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \quad R_2 \to -1/2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$$=\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

$\therefore z=k, y-z=0 \,\&\, x-z=0$

$\Rightarrow z=k, y=k, x=k$

$\therefore (x,y,z)=k(1,1,1); k\in R$

$E_1=\{k(1,1,1)/k\in R\}$

For $\lambda=2$

$$\therefore [A-\lambda I|O]=\begin{bmatrix}-1-\lambda&4&-2\\-3&4-\lambda&0\\-3&1&3-\lambda\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$=\left[\begin{array}{ccc|c}-3&4&-2&0\\-3&2&0&0\\-3&1&1&0\end{array}\right] R_1\to -1/3R_1$$

$$=\left[\begin{array}{ccc|c}1&-4/3&2/3&0\\-3&2&0&0\\-3&1&1&0\end{array}\right] \begin{array}{l}R_2\to R_2+3R_1\\R_3\to R_3+3R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1&-4/3&2/3&0\\0&-2&2&0\\0&-3&3&0\end{array}\right] R_2\to -1/2R_2$$

$$=\left[\begin{array}{ccc|c}1&0&-1/4&0\\0&1&-3/4&0\\0&0&0&0\end{array}\right]$$

$\therefore z=k, y-3/4z=0 \,\&\, x-1/4z=0$

$\Rightarrow z=k, y=3/4k, x=1/4k$

$\therefore (x,y,z)=k(\frac{1}{4},\frac{3}{4},1); k\in R$

$\therefore (x,y,z)=4k(1,3,4); k\in R \quad (\because 4k=k')$

$E_1=\{k'(1,3,4)/k'\in R\}$

$$\therefore P=\begin{bmatrix}1&2&1\\1&3&3\\1&3&4\end{bmatrix}$$

$$\therefore P^{-1}AP=D=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$$

Example (ii): Find a matrix P that diagonalizes A and determine P^{-1} AP

$A=\begin{bmatrix}1&0\\6&-1\end{bmatrix}$. Also find A^{10} and find eigenvalues of A^2.

<u>Solution (ii):</u> $A=\begin{bmatrix}1&0\\6&-1\end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda&0\\6&-1-\lambda\end{vmatrix}=0$$

$\therefore (1-\lambda)(-1-\lambda)-0=0$

$\therefore \lambda=1,-1$

For $\lambda=1$

$$\therefore [A-\lambda I|O]=\left[\begin{array}{cc|c}1-\lambda&0&0\\6&-1-\lambda&0\end{array}\right]$$

$$=\left[\begin{array}{cc|c}0&0&0\\6&-2&0\end{array}\right]$$

x=k, 6x-2y=0

x=k, y=3k

(x, y) = { k (1, 3) / k$\in$R }

For $\lambda=-1$

$$\therefore [A-\lambda I|O]=\left[\begin{array}{cc|c}1-\lambda&0&0\\6&-1-\lambda&0\end{array}\right]$$

$$=\left[\begin{array}{cc|c}2&0&0\\6&0&0\end{array}\right]$$

y=k, 6x =0 = 2x

x=0, y=k

(x, y) = { k (0, 1) / k$\in$R }

$$P=\begin{bmatrix}1&0\\3&1\end{bmatrix}\Rightarrow P^{-1}=\begin{bmatrix}1&0\\-3&1\end{bmatrix}$$

$$P^{-1}AP=\begin{bmatrix}1&0\\0&-1\end{bmatrix}=D$$

	$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = PDP^{-1}$ $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Eigenvalues of A^2 are : $1^2=1$ and $(-1)^2=1$.

Quadratic Forms:

A homogeneous polynomial of second degree in real variables $x_1, x_2, x_3, \ldots\ldots, x_n$ is called Quadratic form.

For example:

(i) $ax^2 + 2hxy + by^2$ is a quadratic form in the variables x and y

(ii) $2x_1x_2 + 2x_2x_3 + 2x_3x_1 + x_3^2$ is a quadratic form in the variables x_1, x_2, x_3.

A quadratic on R^n is a function Q define on R^n whose value at a vector x in R^n can be computed in n variables $x_1, x_2, x_3, \ldots\ldots, x_n$ by an expression of the form.

$$\boxed{Q(x) = x^T Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ x_i\ \ x_j}$$

Here A is known as the coefficient matrix. Where A is n x n symmetric matrix and is called matrix of the quadratic form.

Matrix Representation of Quadratic Forms:

A quadratic form can be represented as a matrix product.

For example:

(i) $ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $2x_1x_2 + 2x_2x_3 + 2x_3x_1 + x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Example:

(i) Find a real symmetric matrix C of the quadratic form $Q = x_1^{\ 2} + 3x_2^{\ 2} + 2x_3^2 + 2x_1x_2 + 6x_2x_3$.

Solution: The coefficient matrix of Q is $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

So, C = symmetric matrix = $\left[\frac{1}{2}(A+A^T)\right] = \frac{1}{2}1\left\{\begin{bmatrix}1 & 2 & 0\\0 & 3 & 6\\0 & 0 & 2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\2 & 3 & 0\\0 & 6 & 2\end{bmatrix}\right\} = \begin{bmatrix}1 & 1 & 0\\1 & 3 & 3\\0 & 3 & 2\end{bmatrix}$

(ii) Express the following quadratic forms in matrix notation $Q = x^2 - 4xy + y^2$.

<u>Solution:</u> $x^2 - 4xy + y^2 = [x \quad y]\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Transformation (Reduction) of Quadratic form to canonical form OR Diagonalizing Quadratic Forms:

Procedure to Reduce Quadratic form to canonical form:

1. Identity the real symmetric matrix associated with the quadratic form.
2. Determine the eigenvalues of A.
3. The required canonical form is given by
   $Q(x) = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + ... + \lambda_n y_n^2$ ....(1)
   where $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are the eigenvalues of A and $D = P^T AP$. The matrix P is said to orthogonally diagonalize the quadratic form.
   And equation (1) is known as canonical form.
4. Form modal matrix P (where x = Py) containing the n eigenvectors of A as n column vectors.

Example: Reduce the quadratic form $Q = 3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form .

<u>Solution:</u> $A = \begin{bmatrix}3 & 2 & 4\\2 & 0 & 4\\4 & 4 & 3\end{bmatrix}$

Eigenvalues for A is 3, -4/3, -1.

The canonical form of the given quadratic form is

$$y^T By = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}3 & 0 & 0\\0 & -4/3 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = 3y_1^2 - 4/3y_2^2 - y_3^2$$

Nature of quadratic form Q :

a. Positive definite if Q(x) > 0 for all x $\neq$ 0,
b. Negative definite if Q(x) < 0 for all x $\neq$ 0,
c. Indefinite if Q(x) assumes both positive and negative values.
d. Positive semidefinite if Q(x) $\geq$ 0 for all x.
e. Negative semidefinite if Q(x) $\leq$ 0 for all x.

OR

a. Positive definite if and only if the eigenvalues of A are positive,
b. negative definite if and only if the eigenvalues of A are positive,
c. Indefinite if and only if A has both positive and negative eigenvalues.
d. Positive semi-definite if and only if A has only non-negative eigenvalues.
e. Indefinite if and only if A has only non-positive eigenvalues.

Example: Describe the nature of quadratic forms.

1. $Q = 3x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$
2. $Q = 2x_1x_2 + 2x_2x_3 + 2x_2x_1$