Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme (All Branches)

Mathematics – 1 (203191102)

Unit –1(B) Application of Integraion (Lecture Note)

Application of Integration

Overview: -

In this unit, we explore some applications of the definite integral by using it to compute length of plane curves, areas between curves, and volumes of solid by slicing, by rotation (revolution) about an axis and cylindrical shells.

Objective

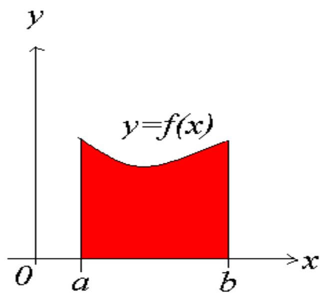
At the end of this unit you will be able to understand:

- Area bounded by curves in Cartesian and polar form
- Area of a region bounded by function
- Area of a region bounded by curves in parametric form
- Volume by slicing, volume of solid of revolution

1) Area bounded by curves in cartesian form:-

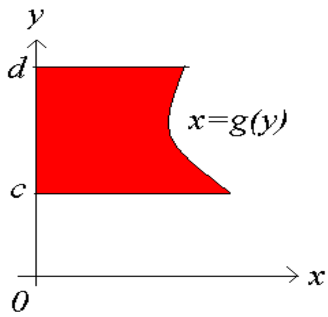
To compute the area between the x-axis and the graph $y = f(x)$ for $a \le x \le b$,

We, applied formula $A = \int_a^b f(x)dx$ -------------------------(1)



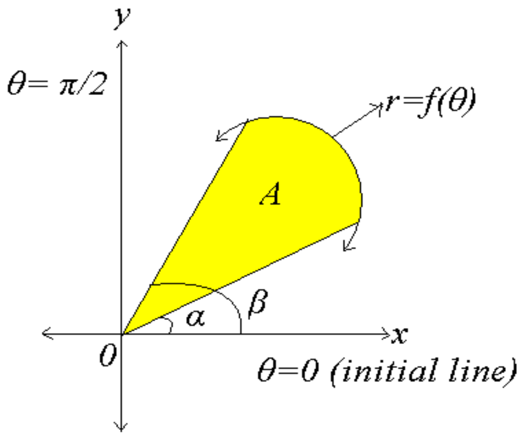
Similarly, to compute the area between the y-axis and the graph $x = g(y)$ for $c \le y \le d$,

We, applied formula $A = \int_c^d g(y)dy$ -------------------------(2)



2) Area bounded by a polar curve:-

The area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is given by



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \text{---------------------------(3)}$$

2) Area bounded by a curve expressed by parametric equations:-

If $x = f(t)$ and $y = g(t)$ are the parametric equations of a curve, then the area bounded by the curve, the x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = \int_a^b g(t)\frac{d}{dt}[f(t)]\,dt \text{-------------------------(4)}$$

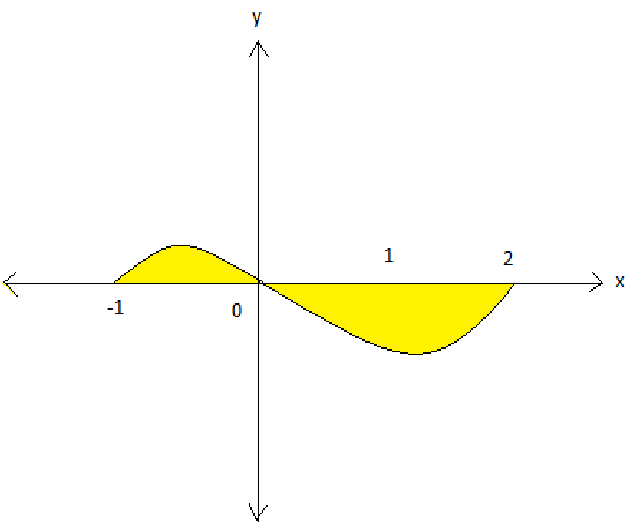
Similarly, the area bounded by the curve $x = f(t)$ and $y = g(t)$, the y-axis and the abscissas $y = c$ and $y = d$ is given by

$$A = \int_c^d f(t)\frac{d}{dt}[g(t)]\,dt \text{--------------------------(5)}$$

Ex:1 Find the area of the region between the x-axis the graph of

$$f(x) = x^3 - x^2 - 2x \ , -1 \le x \le 2$$

Sol. The graph of $f(x) = x^3 - x^2 - 2x$ intersects the x-axis where $f(x) = 0$



Therefore, $x^3 - x^2 - 2x \ = 0$

$\Rightarrow x(x^2 - x - 2) = 0$

$\Rightarrow x(x + 1)(x - 2) = 0$

$\Rightarrow x = 0, x = -1, x = 2$

If A is the required area, then

A=$|I_1|$+$|I_2|$

$$I_1 = \int_{-1}^{0}(x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$

$$= 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right]$$

$$= \frac{5}{12}$$

$$I_2 = \int_0^2(x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2$$

$$= \left[4 - \frac{8}{3} - 4\right] - 0$$

$= -\frac{8}{3}$

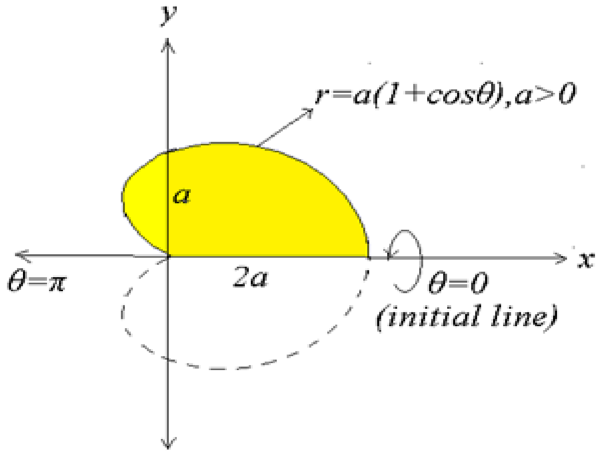
$A = \frac{5}{12} + \frac{8}{3}$

$= \frac{37}{12}$sq units

Ex: 2 Find the area enclosed by the cardioid $r = a(1 + cos\theta)$.

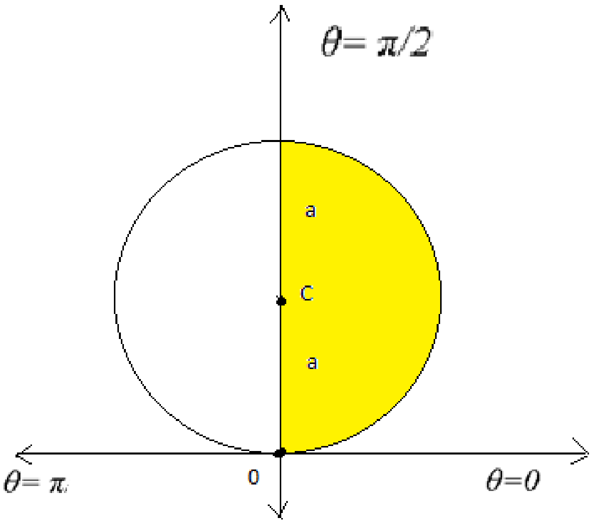
Sol. The given curve is symmetrical about the initial line.

$$A = 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{\pi} r^2 d\theta$$
$$= a^2 \int_0^{\pi} (1 + cos\theta)^2 d\theta$$
$$= a^2 \int_0^{\pi} (2cos^2\frac{\theta}{2})^2\, d\theta$$
$$= 4a^2 \int_0^{\pi} cos^4\frac{\theta}{2}\, d\theta$$
$$= 8a^2 \int_0^{\pi/2} cos^4 \varphi\, d\varphi \text{ (taking } \frac{\theta}{2} = \varphi \text{)}$$
$$= 8a^2 \frac{3.1}{4.2} . \frac{\pi}{2}$$
$$= \frac{3\pi a^2}{2}$$



Ex:3 Find the whole area of the circle $r = 2a\, sin\theta$.
Sol. When $\theta = 0 , r = 0$
and when $\theta = \frac{\pi}{2}, r = 2a$
Therefore,
Required area =2(Area of the shaded region)

$$= 2\int_0^{\pi/2} \frac{1}{2} r^2\, d\,\theta$$
$$= \int_0^{\pi/2} r^2\, d\theta$$
$$= \int_0^{\pi/2} 4a^2\, sin^2\theta\, d\theta$$
$$= 4a^2 \int_0^{\pi/2} sin^2\theta\, d\theta$$
$$= 4a^2 \int_0^{\pi/2} \left(\frac{1 - cos2\theta}{2}\right) d\theta$$
$$= 2a^2 \int_0^{\pi/2} (1 - cos2\theta)\, d\theta$$
$$= 2a^2 \left[\theta - \frac{sin2\theta}{2}\right]_0^{\frac{\pi}{2}}$$
$$= 2a^2 \left[\left(\frac{\pi}{2} - 0\right) - 0\right]$$
$$= \pi a^2$$



Exercise

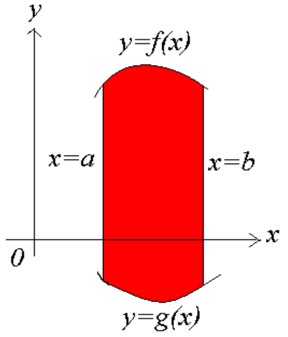
Ex:1 Find the whole area of the curve $r^2 = a^2 \cos 2\theta$.

Ex:2 Find the area of the loop of the curve $y^2 = x^2(1-x)$.
Ex:3 Find the area of one loop of the curve $y^2 = (x-a)(x-5a)^2$.
Ex: 4 Find the whole area of the curve $a^2x^2 = y^3(2a-y)$.
Ex:5 Find the area of a loop of the curve $r = a\sin 2\theta$.

3) Area between two curves:-

Consider the region that lies between two curves $= f(x)$, $y = g(x)$ and between the vertical lines $x = a$ and $x = b$ where, f and g are continuous functions and $f(x) \geq g(x)$ for all x in [a,b] We applied formula,

$A = \int_a^b [f(x) - g(x)]\, dx$--------------------------(6)



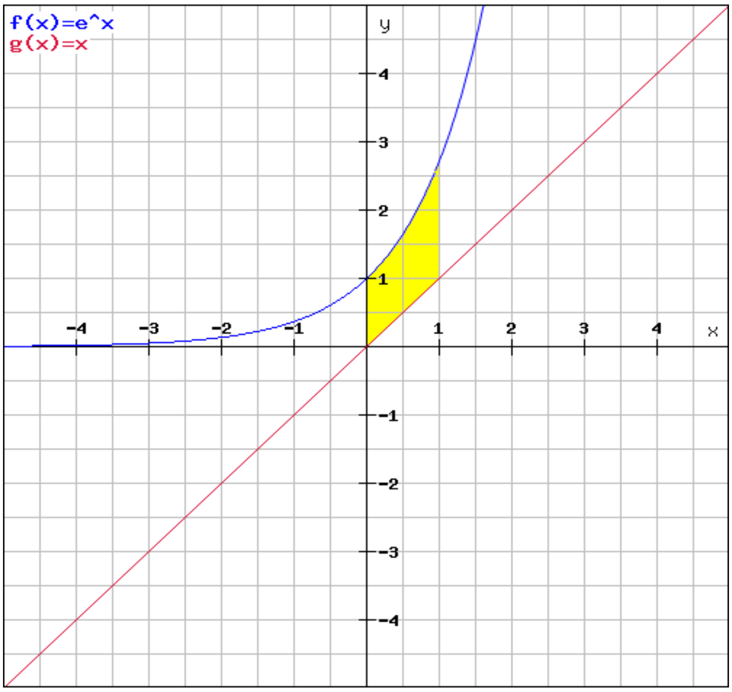
<u>**Ex:1**</u> Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$,and bounded on the sides by $x = 0$ and $x = 1$.

Sol:$A = \int_0^1 (e^x - x)\, dx$

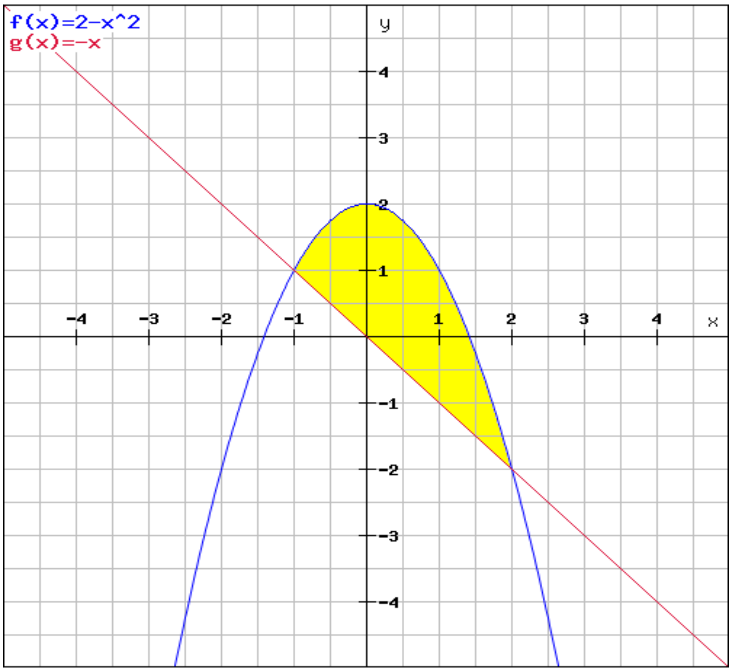
$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

$$= e - \frac{1}{2} - 1$$

$$= e - 1.5$$



Ex:2 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Sol: To find the points of intersection equating the two curves $f(x)$ and $g(x)$

$$2 - x^2 = -x$$
$$\Rightarrow x^2 - x - 2 = 0$$
$$\Rightarrow (x+1)(x-2) = 0$$
$$\Rightarrow x = -1\,, x = 2$$
$$x = -1 \Rightarrow y = 1$$
$$x = 2 \Rightarrow y = -2$$

$$A = \int_{-1}^{2} [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^{2} [2 - x^2 + x]dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2}.$$

Ex:3_Find the area of the region bounded by the curve $y^2 = 4x$ and the line $y = 2x - 4$.

Sol: To find the points of intersection solving the two curves

$$y = 2\left(\frac{y^2}{4}\right) - 4$$

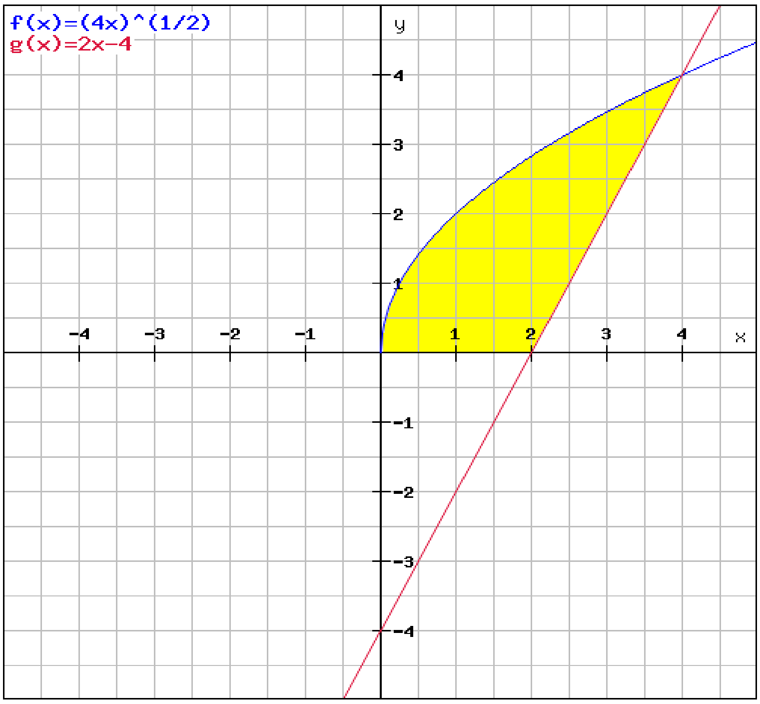
$$\Rightarrow y = \left(\frac{y^2}{2}\right) - 4$$

$$\Rightarrow 2y = y^2 - 8$$

$$\Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y+2)(y-4) = 0$$

$$\Rightarrow y = -2,\ y = 4$$



We have $f(y) = \frac{y+4}{2}$ and $g(y) = \frac{y^2}{4}$

$$A = \int_{-2}^{4} \left[\frac{y+4}{2} - \frac{y^2}{4}\right] dy$$

$$= \frac{1}{4}\int_{-2}^{4} [2y + 8 - y^2] dy$$

$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right]$$

$$= \frac{1}{4}\left[60 - \frac{72}{3}\right]$$

$$= \frac{1}{4}[60 - 24]$$

=9.

<u>Excercise</u>

<u>Ex:1</u> Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

<u>Ex:2</u> Find the area of the region enclosed by the curves $x = 1 - y^2$ and $x = y^2 - 1$.

<u>Ex:3</u> Find the area of the region enclosed by the curves $y = cosx$ and $y = 1 - \frac{2x}{\pi}$.

<u>Ex:4</u> Find the area of the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Ex:5 Find the area of the region enclosed by the parabola $y^2 - 4x = 4$ and the line $4x - y = 16$.

5)Volume by Slicing:-

This method is based on the formula

Volume=area x height

By this method, the volume of a solid of known cross-sectional area A(x) from $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\, dx \text{-------------------------(7)} \qquad \text{where, A(x) is integrable.}$$

Similarly, the volume of a solid of known cross-sectional area B(y) from $y = c$ to $y = d$ is

$$V = \int_c^d B(y)\, dy \text{--------------------------(8)} \qquad \text{where, B(y) is integrable.}$$

Ex:1 Find the volume of a cone with height 4 cm and radius of base 4 cm. Use the method of slicing.

Sol. Let us consider that the cone is centered at the origin with its axis as x-axis.

Given cone with height 4 cm and radius of base 4 cm.

The cross-sectional area at x is

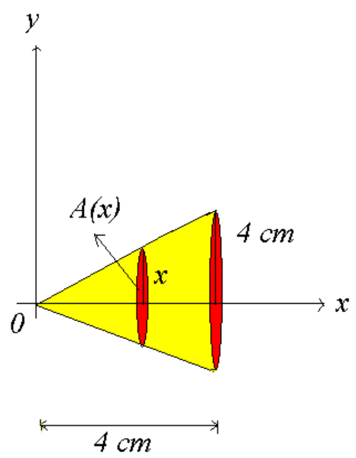
$$A(x) = \pi x^2$$

Therefore, the required volume is $V = \int_a^b A(x)\, dx$

$$= \int_0^4 \pi x^2\, dx$$

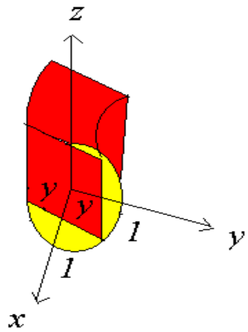
$$= \pi \left[\frac{x^3}{3}\right]_0^4$$

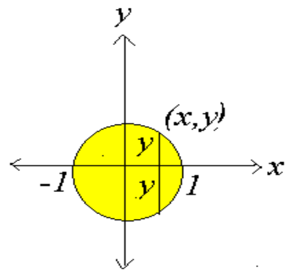
$$= \pi \left(\frac{64}{3}\right) cm^3 .$$



Ex:2 A solid has unit circle $x^2 + y^2 = 1$ as base. The cross-sections of the solid perpendicular to the base and to the x-axis are squares. Find the volume of the solid.

Sol: Figure shows the solid with base unit circle $x^2 + y^2 = 1$ and cross-section perpendicular to the base and to the x- axis as square.





Here, for each x, $-1 < x < 1$, the base has width 2y.

Therefore, the area of the square cross-section at x is

$$A(x) = 4y^2 = 4(1 - x^2)$$

Therefore, the required volume is

$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} 4(1 - x^2)\,dx$$

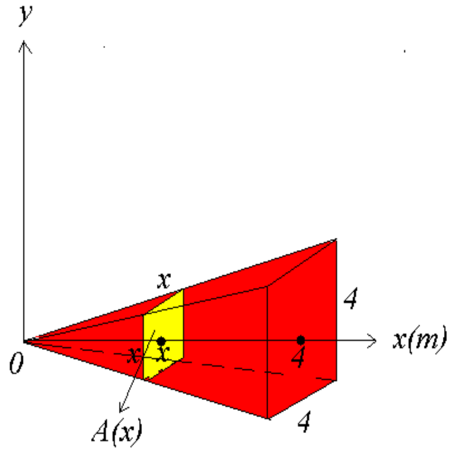
$$= 4\left[\, x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 4\left[\, 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right]$$

$$= 4\left[\, 2 - \frac{2}{3}\right]$$

$$= \tfrac{16}{3}\ .$$

<u>Ex:3</u> A pyramid having 4 m height has a square base with sides of length 4 m. The pyramid having a cross-section at a distance x m down from the vertex and perpendicular to the altitude is a square with sides of length x m. Find the volume of the pyramid.



Sol. The figure shows the pyramid with its altitude along the x-axis and its vertex at the origin.

Given that the cross-section of the pyramid at x m from vertex is a square with length x m.

So, its area is

$$A(x) = x^2$$

Therefore, the required volume is

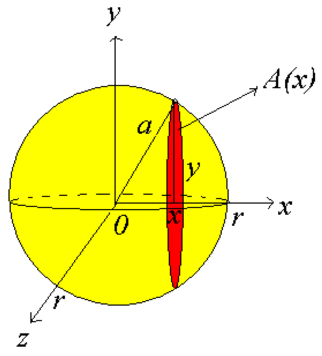
$$V = \int_a^b A(x)\,dx = \int_0^4 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^4$$

$$= \tfrac{64}{3} m^3\ .$$

<u>Ex:4</u> Using slicing method find the volume of a solid ball of radius a.

Sol.The figure shows the solid ball with radius a and a typical cross-section which is a circle having radius y.



The cross-sectional area at x is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Therefore, the required volume is

$$V = \int_{-a}^{a} A(x)\, dx$$
$$= \int_{-a}^{a} \pi(a^2 - x^2)\, dx$$
$$= 2\pi \int_{0}^{a} (a^2 - x^2)\, dx$$
$$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$$
$$= 2\pi \left[a^3 - \frac{a^3}{3}\right]$$
$$= \frac{4}{3}\pi a^3.$$

<u>Ex:5</u> Using slicing method,find the volumne of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{2x}$ from 0 to 1.

Sol. For $y = \sqrt{2x}$

x	0	1
y	0	$\sqrt{2}$

The region under the curve $y = \sqrt{2x}$ from 0 to 1 is as shown in figure 1.

The generated volume is as shown in figure 2.

The cross-sectional area of the generated volume is a circle with radius $\sqrt{2x}$, that is

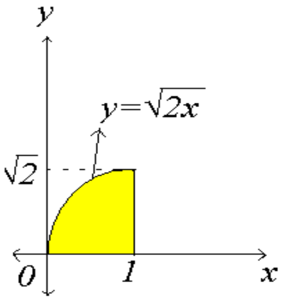


figure 1

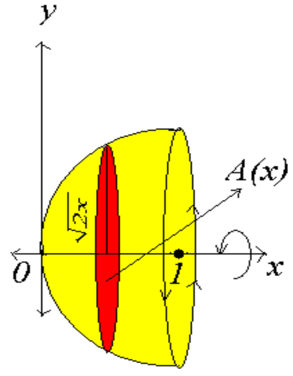


figure 2

$$A(x) = \pi(\sqrt{2x})^2 = 2\pi x$$

Therefore, the required volume is

$$V = \int_{0}^{1} A(x)\, dx$$
$$= \int_{0}^{1} 2\pi x\, dx$$
$$= 2\pi \left[\frac{x^2}{2}\right]_0^1$$
$$= \pi.$$

<u>Exercise:</u>

Ex:1Find the volume of a right pyramid whose base is a square with sides of length l and whose altitude is h.

Ex:2 Find the volume of a right tetrahedron whose altitude is h and whose base is a right triangle with sides of length a and b at right angle.

Ex:3 Prove that the volume of a right circular cylinder of base radius a and height h is $\pi a^2 h$.

Ex:4 Find the volume of the solid generated by rotating the plane region bounded by $y = \frac{1}{x}$, $x = 1$ and $x = 3$ about the x-axis.

Ex:5 Find the volume of the solid whose base is a triangle with vertices (0,0),(2,0) and (0,2) and whose cross-sections perpendicular to the base and parallel to the y-axis are semicircles.

6) <u>Volume by rotation and cylindrical shells:</u>

In this section, we will study the application of integration to find the volume of solid of revolution. A solid of revolution is generated when we revolve a plane region R about a line L. The line L is called the **<u>axis of revolution</u>**.

For example, as shown in the figure if we rotate, a plane region R bounded by $y = f(x)$, x-axis, $x = a$ and $x = b$,about x-axis then we get a solid.

Such a solid is called **<u>solid of revolution</u>**.

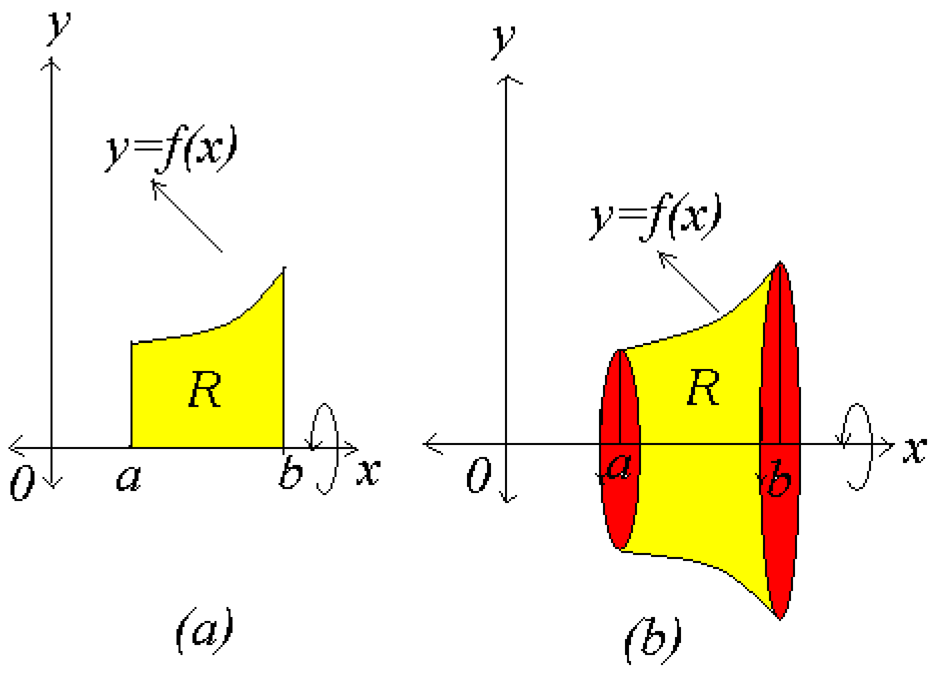


Figure (a) Plane region R

Figure (b) Solid of revolution by plane region R about x-axis

<u>7) Volume of Solid of revolution in Cartesian form</u>

(1) Cylindrical Disc (C.D.) method

(a)Revolution about x-axis:-

When the area bounded by the curve $y = f(x)$, the ordinates $x = a$,$x = b$and the x-axis is revolved about x-axis then the volume of the solid generated is given by

$$V = \pi \int_a^b y^2\, dx \text{-----------------------------}(9)$$

(b)Revolution about y-axis:-

Similarly, when the area bounded by the curve $x = g(y)$,the abscissas $y = c$, $y = d$ and the y-axis is revolved about y-axis then the volume of the solid generated is given by

$$V = \pi \int_c^d x^2\, dy \text{----------------------------}(10)$$

(c)Volume of a solid generated by revolution of the area about any axis:-

When some area is revolved about any axis then the volume of the solid generated is given by

$$V = \pi \int_a^b r^2\, dh \text{----------------------------}(11)$$

where r is the perpendicular distance from the curve to the axis of revolution PQ.

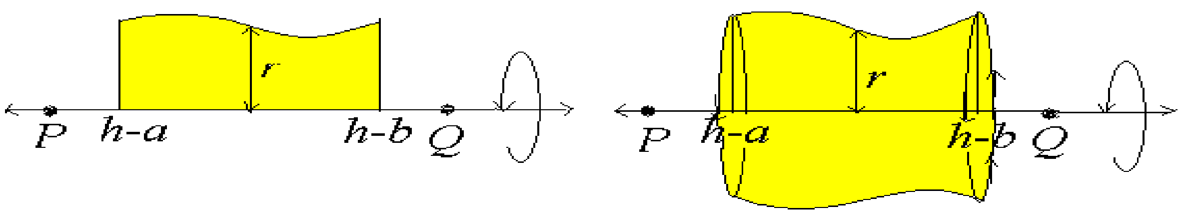


Fig.Volume generated by revoluting area about any axis

(2)**<u>(The Washer method) Volume of a solid generated by two intersecting curves</u>**:-

Suppose A be the area bounded by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ as shown in figure . Let $x = a$ and $x = b$ be the x-coordinates of their point of intersection. Then the volume of the solid generated by the revolution of area A about the x-axis between the ordinates $x = a$ and $x = b$ is given by

$$V = \pi \int_a^b (y_2)^2\, dx - \pi \int_a^b (y_1)^2\, dx = \pi \int_a^b ((y_2)^2 - (y_1)^2)\, dx \text{-------------------------}(12),$$

provided $f_2(x) \geq f_1(x)$.

Therefore, the revolution of the area bounded between two circles with outer radius R and inner radius r, that is

$$V = \pi \int_a^b (R^2 - r^2)\, dx \text{----------------------------}(13)$$

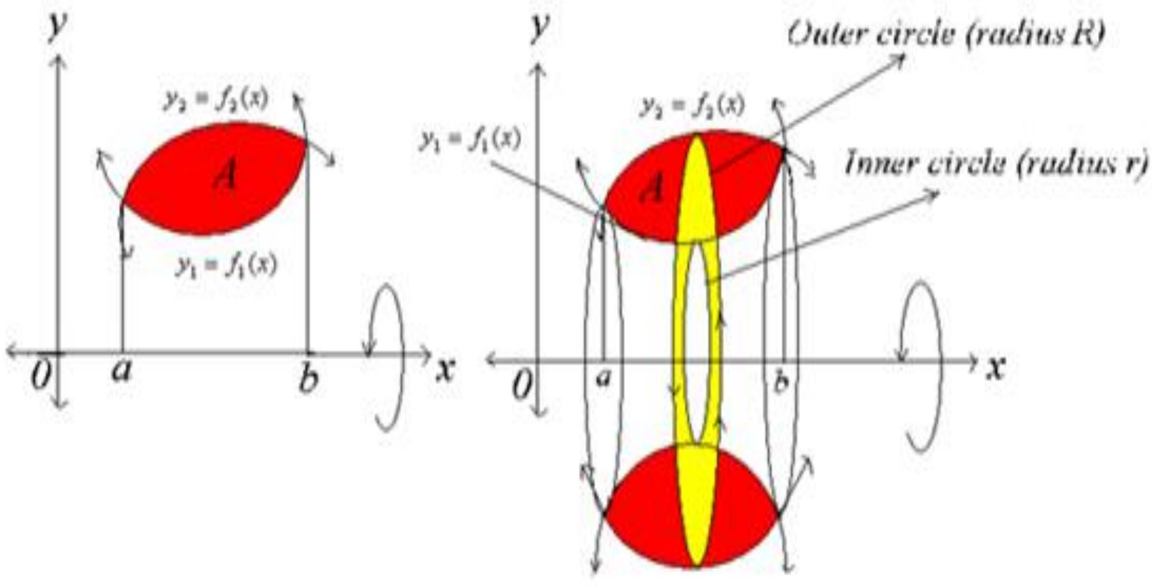


Fig. Volume generated by revolving area A about x-axis

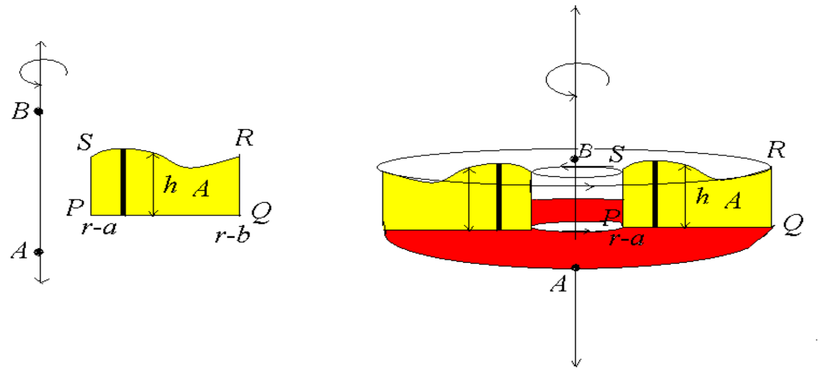
8) Cylindrical Shell (C.S) method:-

This method is used when the axis of rotation is not a part of the boundary of the plane area A.

Consider the figure , where the axis of rotation AB is not a part of the plane area PQRS. Then the volume generated by revolving strip about AB is given by

$dV = (mean\ circumference) * (height) * (thickness) = (2\pi r)(h)(dr)$. So,

$V = \int_{r=a}^{b} 2\pi rh\, dr$---------------------------(14)



Volume generated when axis of rotation is not a part of the region

If the area PQRS is rotated about y-axis (AB) is

$$V = \int_{x=a}^{b} (2\pi x)(y)\, dx \text{ -----------------------(15)}$$

And if the area PQRS is rotated about x-axis,then

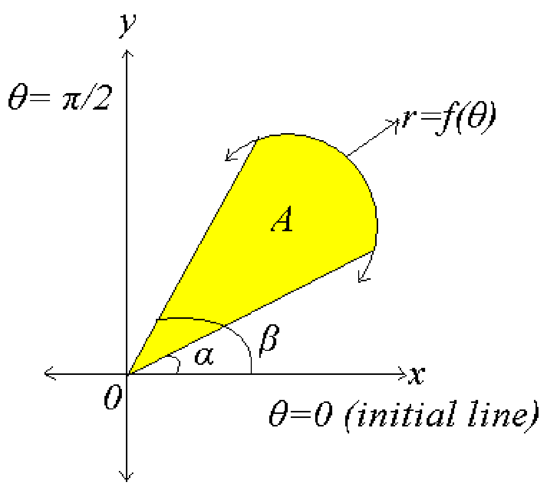
$$V = \int_{y=c}^{d} (2\pi y)(x)\, dy \text{ -----------------------(16)}$$

When the area bounded by the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$; provided $f_2(x) \geq f_1(x)$,the x-axis and the ordinates $x = a$ and $x = b$ is revolved about y-axis then the volume of the solid generated is given by

$$V = 2\pi \int_a^b (y_2 - y_1)(x)\, dx \text{-------------------(17)}$$

9)<u>Volume of Solid of revolution in Polar form</u>:-

The volume of solid of revolution generated by revolving the plane area A,as shown in the figure bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$



(i) about the initial line(positive x-axis or $\theta = 0$) is given by

$$V = \frac{2\pi}{3} \int_{\theta=\alpha}^{\beta} r^3 \sin\theta\, d\theta \text{ ---------------------(18)}$$

(ii)about the line $\theta = \frac{\pi}{2}$ is given by

$$V = \frac{2\pi}{3} \int_{\theta=\alpha}^{\beta} r^3 \cos\theta\, d\theta \text{ ---------------------(19)}$$

10)<u>Volume of Solid of revolution in Parametric form</u>:-

If the equation of the curve by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, then the volume of the solid of revolution about the x -axis can be found out by substituting x and y in the formula (9) , by $f(t)$ and $g(t)$ respectively, we get

$$V = \pi \int_a^b [g(t)]^2 \frac{dx}{dt}\, dt \qquad \text{or} \qquad V = \pi \int_a^b [g(t)]^2 f'(t)\, dt \text{ ----------------------(20)}$$

Similarly, the volume of the solid of revolution about the y –axis can be found out by substituting x and y in the formula (10) , by $f(t)$ and $g(t)$ respectively, we get

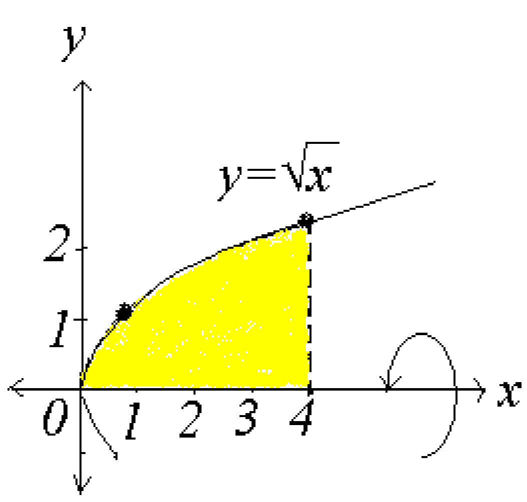
$$V = \pi \int_a^b [f(t)]^2 \frac{dy}{dt}\, dt \qquad \text{or} \qquad V = \pi \int_a^b [f(t)]^2 g'(t)\, dt \text{ ----------------------(21)}$$

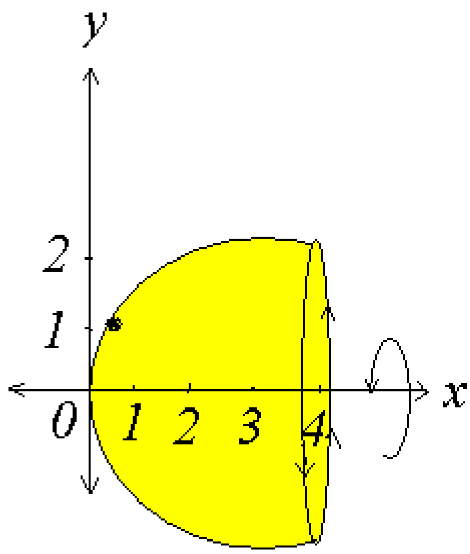
<u>Ex:1</u>The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$ and the x -axis is revolved about the x –axis to generate a solid.Find its volume.

Sol.For $y = \sqrt{x}$,

x	0	1	4
y	0	1	2

Therefore, the region bounded by $y = \sqrt{x}$, $0 \leq x \leq 4$ and the x -axis with its revolution about the x –axis is shown in figure



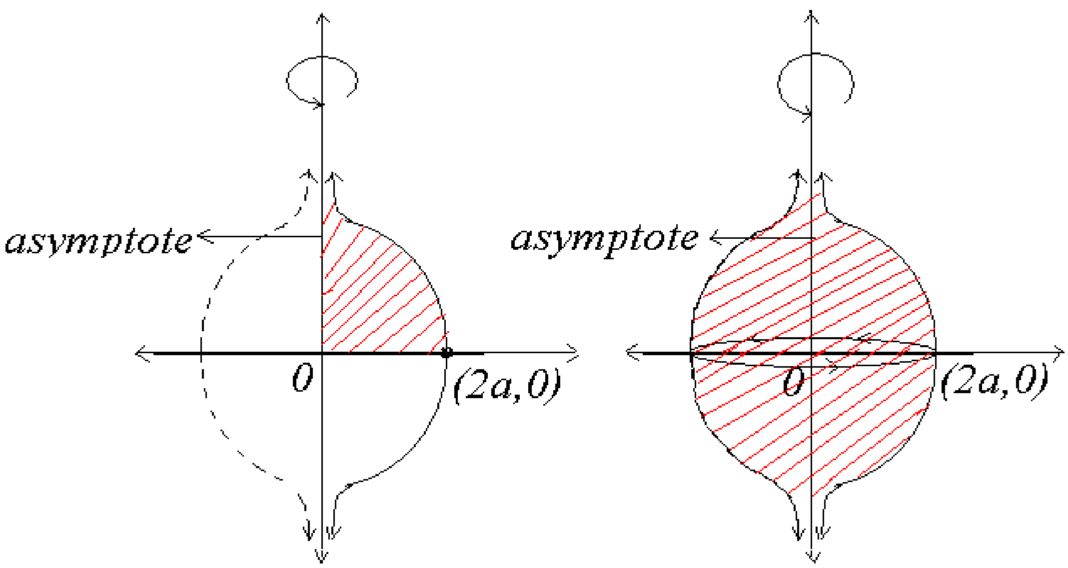


The required volume is given by

$$V = \pi \int_a^b y^2 \, dx = \pi \int_0^4 y^2 \, dx$$
$$= \pi \int_0^4 x \, dx$$
$$= \pi \left[\frac{x^2}{2}\right]_0^4$$
$$= 8\pi$$

<u>Ex:2</u>Find the volume of the solid generated by revolving the area between the curve $xy^2 = 4a^2(2a - x)$, a>0 and its asymptote about its asymptote.

Sol. The volume V generated by revolving the area bounded between the $xy^2 = 4a^2(2a - x)$ and



its asymptote about its asymptote is as shown in figure.

The required volume is given by

$$V = 2\pi \int_0^{\infty} x^2\, dy = 2\pi \int_0^{\infty} \left(\frac{8a^3}{y^2+4a^2}\right)^2 dy$$

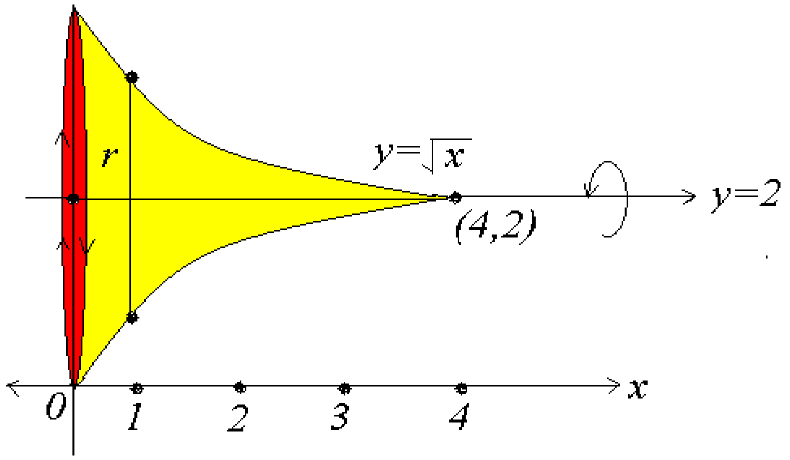
$$= 2\pi \int_0^{\pi/2} \frac{64\, a^6}{(4a^2 sec^2\theta)^2}\, 2a\, sec^2\theta\, d\theta \qquad \text{(Putting } y = 2a\, tan\theta\text{)}$$

$$= 16\pi a^3 \int_0^{\pi/2} cos^2\theta\, d\theta$$

$$= 4\pi^2 a^3$$

Ex:3 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2,\ x = 0$ about the line $y = 2$.

Sol.The region bounded by$y = \sqrt{x}$ and the lines $y = 2,\ x = 0$ is as shown in figure .The generated volume is as shown in figure



The required volume V using the formula (8)

$$V = \pi \int_a^b r^2\, dh$$

$$= \pi \int_0^4 (2-y)^2\, dx$$

$$= \pi \int_0^4 (2-\sqrt{x})^2\, dx$$

$$= \pi \int_0^4 (4 - 4\sqrt{x} + x)\, dx$$

$$= \pi \left[4x - \frac{8}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_0^4$$

$$= \pi \left[16 - \frac{64}{3} + \frac{16}{2}\right]$$

$$= \frac{8}{3}\pi$$

<u>Ex:4</u> The region enclosed by the parabola $y = x^2 + 1$ and the straight line $y = 2x + 1$ is revolved about x –axis. Find the volume of the solid of revolution.

Sol. Solving $y = x^2 + 1$ and $y = 2x + 1$, we get

$x^2 + 1 = 2x + 1$

$\Rightarrow x(x - 2) = 0$

$\Rightarrow x = 0$ or $x = 2$.

when $x = 0$, $y = 1$ and $x = 2$, $y = 5$.

Therefore, the points of intersection are (0,1) , (2,5).

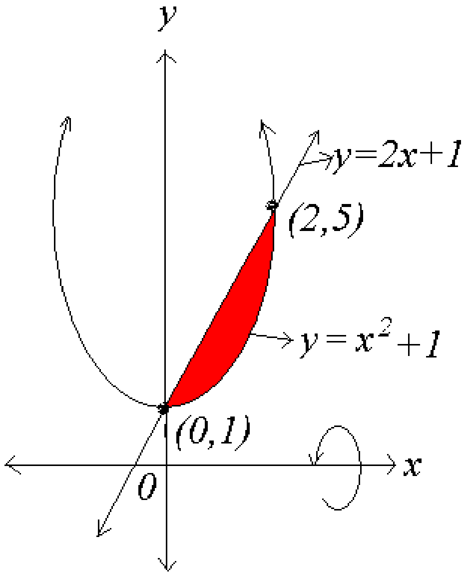
For $y = x^2 + 1$,

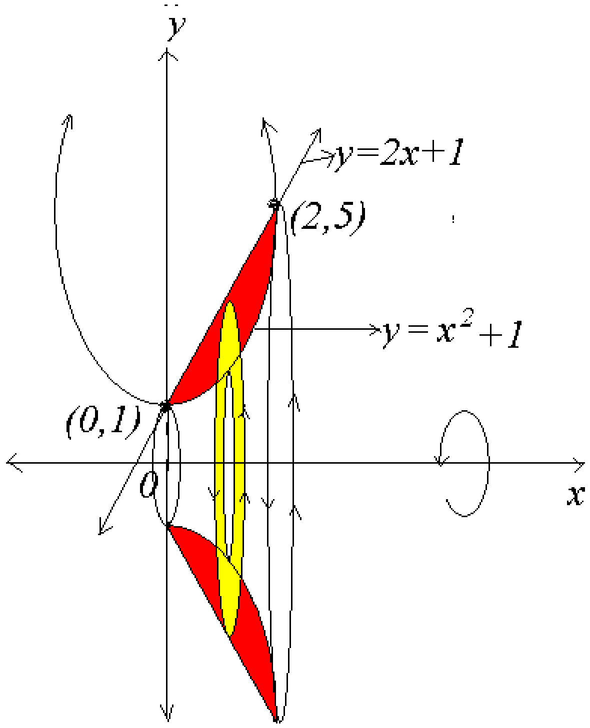
x	0	1	2
y	1	2	5

and for $y = 2x + 1$,

x	0	2
y	1	5

The volume V generated by revolving the area between $y = x^2 + 1$ and $y = 2x + 1$ about x – axis is as shown in the figure





The required volume is given by

$$V = \pi \int_a^b ({y_2}^2 - {y_1}^2)\ dx$$

$$= \pi \int_0^2 ({y_2}^2 - {y_1}^2)\ dx$$

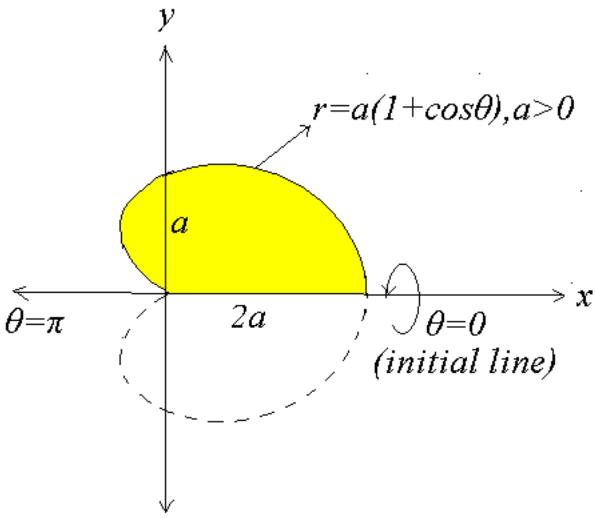
$$= \pi \int_0^2 [(2x + 1)^2 - (x^2 + 1)^2]\ dx$$

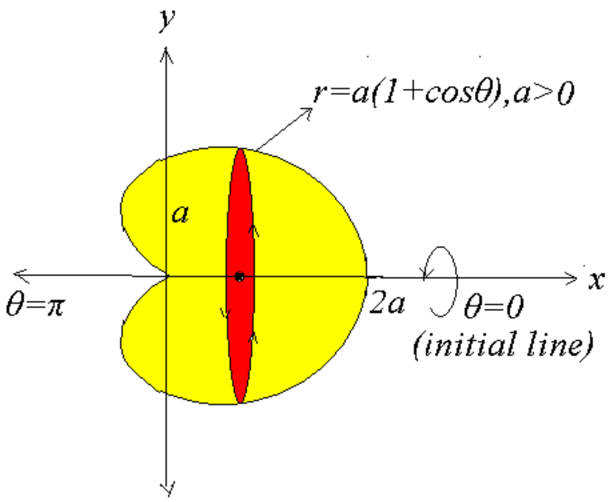
$$= \pi \int_0^2 [(2x + 1)^2 - (x^4 + 2x^2 + 1)]\ dx$$

$$= \pi\left[\frac{(2x+1)^3}{6} - \frac{x^5}{5} - \frac{2x^3}{3} - x\right]_0^2$$
$$= \frac{71\pi}{10}$$

Ex:5 Find the volume of the solid generated by revolving the cardiod $r = a(1 + cos\theta), a > 0$ about the initial line.

Sol.The volume V generated by revolving the cardiod $r = a(1 + cos\theta), a > 0$ about the initial line is as shown in the figure





The required volume is given by

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta\, d\theta$$
$$= \frac{2\pi}{3}\int_{0}^{\pi} r^3 \sin\theta\, d\theta$$
$$= \frac{2\pi}{3}\int_{0}^{\pi} a^3(1 + cos\theta)^3 \sin\theta\, d\theta$$

Let $1 + cos\theta = t$

$-sin\theta d\theta = dt$

Also, when $\theta = 0$, $t = 2$

and when $\theta = \pi$, $t = 0$.

Thus, $$V = \frac{2\pi a^3}{3}\int_0^2 t^3\, dt$$
$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2$$
$$= \frac{8\pi a^3}{3}.$$

Ex:6 Find the volume of the solid of revolution generated by revolving the curve$x = 2t + 3$,

$y = 4t^2 - 9$ about the x-axis for t = $-3/2$ to t = $3/2$.

Sol. $x = 2t + 3$

$$\frac{dx}{dt} = 2$$

The volume of solid is generated by revolving the curve about the x-axis. For the required region, t varies from $-3/2$ to $3/2$.

$$V = \int_{-3/2}^{3/2} \pi\, y^2 \frac{dx}{dt}\, dt$$

$$= \pi \int_{-3/2}^{3/2} (4t^2 - 9)^2 (2) dt$$

$$= 4\pi \int_0^{3/2} (16t^4 - 72t^2 + 81) dt$$

$$= 4\pi [16\frac{t^5}{5} - 72\frac{t^3}{3} + 81t]_0^{3/2}$$

$$=1296\,\pi$$

<u>Ex.7</u> The region bounded by the curve $y = \sqrt{x}$, the x-axis , and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid using the method of cylindrical shells.

Sol. Circumference=$2\pi x$

Let the volume of the solid be

$$V = \int_a^b (circumference)(height) dx$$

$$= \int_0^4 (2\pi\, x)\left(\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx$$

$$= 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^4$$

$$= \frac{128\pi}{5}$$

<u>Exercise</u>

Ex:1 Using the method of cylinderical shells,find the volume of the solid generated by rotating about the line $y = 3$.The region bounded by the curves $y = \sqrt{x-1}$,$y = 0$ and $x = 5$.
Ex:2 Using the method of cylinderical shells,find the volume of the solid generated by revolving about y-axis , the region R enclosed by $y = x^3, x = 1, y = 0$.
Ex:3 Find the volume of the solid generated by revolving the curve $r = a + bcos\theta$,(a>b) about the initial line.
Ex:4 If the ellipse $x = acos\theta$, $y = bsin\theta$ is revolved about the line $x = 2a$, show that the volume of the solid generated is $4\pi^2 a^2 b$
Ex:5 Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.