PARUL UNIVERSITY
Faculty of Engineering & Technology
Department of Applied Sciences & Humanities
1st year B.Tech Programme (All branches)
Mathematics-II (Subject Code: 203191102)
UNIT-2 First Order Differential Equation

❖ **Differential Equation:**

<u>Ordinary differential equation</u>

An ordinary differential equation is an equation which contains derivatives of a dependent variable, $y(x)$, w.r.t. only one independent variable.

For example:

1. $\frac{dy}{dx} = \cos x$
2. $\frac{d^2y}{dx^2} + 4y = 0$
3. $x^2 \frac{d^3y}{dx^3} + 2e^x \frac{d^2y}{dx^2} = (x^2+2)y^2$

<u>Partial differential equation</u>

A partial deferential equation is an equation which contains partial derivatives of a dependent variables $f(x, y)$, w.r.t. two or more independent variables.

For example:

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
2. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

❖ **Order**

The order of a differential equation is the order of the highest derivative occurring in that equation.

❖ **Degree**

The degree of a differential equation is the highest index of the highest order derivative.

● **Examples:**

Sr. No	Differential equation	Order	Degree
1.	$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y=0$	**2**	**1**
2.	$3\left(\frac{d^3y}{dx^3}\right)^3+\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}=e^{-x}\sin x$	**3**	**3**
3.	$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=1$ **OR** $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3=\left(\frac{d^2y}{dx^2}\right)^2$	**2**	**2**

Initial Value Problem: A particular solution can be obtained from a general solution by an initial condition $y(x_0)=y_0$ which determines the value of the arbitrary constant c. An ordinary differential equation with initial condition is known as **Initial Value Problem.**

$$y'=f(x,y) \qquad y(x_0)=y_0$$

Exact Differential Equation: A first order differential equation of the form

$$M(x,y)dx+N(x,y)dy=0$$

is called an exact differential equation if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the solution of the exact differential equation is

$$\int_{y\,is\,const} M(x,y)dx+\int_{terms\,containing\,only\,y} N(x,y)dy=c$$

For example:

1. Solve $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$ Solution: $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$ $M=(x^3+3xy^2)$ and $N=(3x^2y+y^3)$ $\frac{\partial M}{\partial y}=6xy$ and $\frac{\partial N}{\partial x}=6xy$ Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is	2. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$ Solution: $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$ $(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0$ $M=(y\cos x+\sin y+y)$ and $N=(\sin x+x\cos y+x)$

$\int\limits_{y\,is\,const} M(x,y)dx + \int\limits_{terms\,containing\,only\,y} N(x,y)dy = c$ $i.e.\ \int (x^3+3xy^2)dx + \int y^3 dy = c$ $\therefore \frac{x^4}{4} + 3y^2\frac{x^2}{2} + \frac{y^4}{4} = c$	$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ and $\frac{\partial N}{\partial y} = \cos x + \cos y + 1$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is $\int\limits_{y\,is\,const} M(x,y)dx + \int\limits_{terms\,containing\,only\,y} N(x,y)dy = c$ $i.e.\ \int (y\cos x + \sin y + y)dx + \int 0dy = c$ $\therefore y\sin x + x\sin y + xy = c$
3.Solve $((x+1)e^x - e^y)dx - xe^y dy = 0, \quad y(1)=0$ Solution: $((x+1)e^x - e^y)dx - xe^y dy = 0$ $M = ((x+1)e^x - e^y)$ and $N = -xe^y$ $\frac{\partial M}{\partial y} = -e^y$ and $\frac{\partial N}{\partial y} = -e^y$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is $\int\limits_{y\,is\,const} M(x,y)dx + \int\limits_{terms\,containing\,only\,y} N(x,y)dy = c$ $i.e.\ \int ((x+1)e^x - e^y)dx + \int 0dy = c$ $\therefore (x+1)e^x - e^x - xe^y = c$ Now given that y(1)=0 Therefore substituting x=1, y=0 in the solution $e-1=c$ Therefore the solution is $\therefore (x+1)e^x - e^x - xe^y = e-1$	**Examples:** 1.Solve $(2xy+e^y)dx + (x^2+xe^y)dy = 0, \quad y(1)=1$ 2.Solve $ye^x dx + (2y+e^x)dy = 0, \quad y(0)=-1$ 3.Solve $\frac{dy}{dx} = \frac{y+1}{e^y(y+2)-x}$

Non exact Differential Equation OR Reducible to exact diff. Equation:

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the given equation is not exact.

Therefore, by multiplying the given equation with integrating factor reduces it to exact. There are four cases for finding the integrating factor.

Case-:1

If the given differential equation is homogeneous with $Mx + Ny \neq 0$ then

I.F =

Case-2:

If the given differential equation is of the form $f(x,y)ydx + g(x,y)xdy = 0$ with $Mx - Ny \neq 0$, then

I.F =

Case-3:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say f(x), then

I.F =

Case-4:

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say g(y), then

I.F =

For examples:

1. Solve $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$

Solution: $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$

$M = (xy - 2y^2)$ and

$N = (x^2 - 3xy)$

$\frac{\partial M}{\partial y} = x - 4y$ and

$\frac{\partial N}{\partial x} = 2x - 3y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.
Therefore

$Mx + Ny = x^2y - 2xy^2 - x^2y + 3xy^2 = xy^2 \neq 0$

$\therefore I.F = \frac{1}{Mx + Ny} = \frac{1}{xy^2}$

Multiplying throughout by I.F, the equation becomes

2. Solve $\left(x^4 + y^4\right) dx - xy^3 \ dy = 0$

$M = x^4 + y^4$ $\qquad N = -xy^3$

$\frac{\partial M}{\partial y} = 4y^3$ $\qquad \frac{\partial N}{\partial x} = -y^3$

$$\therefore I.F. = \frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$$

$$\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}dy = 0$$

$$\int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx = c$$

$$\ln x - \frac{1}{4}\left(\frac{y}{x}\right)^4 = c$$

3. Solve $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$

Solution: $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$

$M = (x^2y^2 + 2)y$ and $N = (2 - x^2y^2)x$

$$\frac{(xy-2y^2)}{xy^2}dx-\frac{(x^2-3xy)}{xy^2}dy=0$$

$$\therefore\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}+\frac{3}{y}\right)dy=0$$

$$\therefore M'=\left(\frac{1}{y}-\frac{2}{x}\right) \qquad N'=\left(-\frac{x}{y^2}+\frac{3}{y}\right)$$

$$\therefore \frac{\partial M'}{\partial y}=-\frac{1}{y^2} \qquad \frac{\partial N'}{\partial x}=-\frac{1}{y^2}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\,is\,const} M'(x,y)dx+\int\limits_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\ \int\left(\frac{1}{y}-\frac{2}{x}\right)\ dx+\int\frac{3}{y}dy=c$$

$$\therefore \frac{x}{y}-2\log x+3\log y=c$$

$$\frac{\partial M}{\partial y}=3x^2y^2+2$$

and

$$\frac{\partial N}{\partial x}=2-3x^2y^2$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.
Therefore

$$Mx-Ny=x^3y^3+2xy-2xy+x^3y^3=2x^3y^3\neq 0$$

$$\therefore I.F=\frac{1}{Mx-Ny}=\frac{1}{2x^3y^3}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{(x^2y^2+2)y}{2x^3y^3}dx+\frac{(2-x^2y^2)x}{2x^3y^3}dy=0$$

$$\therefore \frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)dx+\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0$$

$$\therefore M'=\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right) \qquad N'=\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)$$

$$\therefore \frac{\partial M'}{\partial y}=-\frac{2}{x^3y^3} \qquad \frac{\partial N'}{\partial x}=-\frac{2}{x^3y^3}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\,is\,const} M'(x,y)dx+\int\limits_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\ \int\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)\ dx+\int-\frac{1}{y}dy=c$$

$$\therefore \frac{1}{2}\left(\log x-\frac{1}{x^2y^2}\right)-\frac{1}{2}\log y=c$$

$$\therefore \log x-\log y-\frac{1}{x^2y^2}=c$$

$$\therefore \log\left(\frac{x}{y}\right)-\frac{1}{x^2y^2}=c$$

3.Solve $(2x\log x-xy)dy+2ydx=0$

Solution: $2ydx+(2x\log x-xy)dy=0$

$M=2y$ and

$N=(2x\log x-xy)$

$$\frac{\partial M}{\partial y}=2$$

and

$$\frac{\partial N}{\partial x}=2\log x+2-y$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

4.Solve $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

Solution: $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

$M=\left(\frac{y}{x}\sec y-\tan y\right)$ and

$N=(\sec y\log x-x)$

Therefore

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2-2\log x-2+y}{2x\log x-xy}$$

$$=-\frac{1}{x}=f(x)$$

$$\therefore I.F=e^{\int -\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{1}{x}(2y)dx+\frac{1}{x}(2x\log x-xy)dy=0$$

$$\therefore \frac{2y}{x}dx+(2\log x-y)dy=0$$

$$\therefore M'=\frac{2y}{x} \qquad N'=2\log x-y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{2}{x} \qquad \frac{\partial N'}{\partial x}=\frac{2}{x}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.

Therefore the solution is

$$\int_{y\,is\,const} M'(x,y)dx+\int_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\ \int \frac{2y}{x}dx+\int -y dy=c$$

$$\therefore 2y\log x-\frac{y^2}{2}=c$$

$$\frac{\partial M}{\partial y}=\frac{1}{x}\sec y+\frac{y}{x}\sec y\tan y-\sec^2 y$$

and

$$\frac{\partial N}{\partial x}=\frac{\sec y}{x}-1$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{\frac{\sec y}{x}-1-\frac{\sec y}{x}-\frac{y}{x}\sec y\tan y+\sec^2}{\frac{\sec y}{x}-\tan y}$$

$$=-\tan y=f(y)$$

$$\therefore I.F=e^{\int -\tan y\,dy}=e^{-\log\sec x}=\sec^{-1}y=\cos y$$

Multiplying throughout by I.F, the equation becomes

$$\cos y\left(\frac{y}{x}\sec y-\tan y\right)dx+\cos y(\sec y\log x-x)$$

$$\therefore \left(\frac{y}{x}-\sin y\right)dx+(\log x-x\cos y)dy=0$$

$$\therefore M'=\left(\frac{y}{x}-\sin y\right) \qquad N'=(\log x-x\cos y)$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{1}{x}-\cos y \qquad \frac{\partial N'}{\partial x}=\frac{1}{x}-\cos y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.

Therefore the solution is

$$\int_{y\,is\,const} M'(x,y)dx+\int_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\ \int\left(\frac{y}{x}-\sin y\right)dx+\int 0dy=c$$

$$\therefore y\log x-x\sin y=c$$

Example:

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$
2. $x^2ydx-(x^3+y^3)dy=0$
3. $(x^2+y^2+1)dx-2xydy=0$
4. $xe^x(dx-dy)+e^xdx+ye^ydy=0$

❖ Linear Differential Equation

A first order differential Equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ or $\frac{dx}{dy}+P(y)x=Q(y)$

is called Linear Differential Equation.

Differential equation	Integrating factor	General solution
$\frac{dy}{dx}+P(x)y=Q(x)$	$I.F=e^{\int P(x)dx}$	$y(I.F)=\int Q(x)(I.F)dx+c$
$\frac{dx}{dy}+P(y)x=Q(y)$	$I.F=e^{\int P(y)dy}$	$x(I.F)=\int Q(y)(I.F)dy+c$

For Example:

1.Solve $\frac{dy}{dx}+2y\tan x=\sin x$ Solution: $\frac{dy}{dx}+2y\tan x=\sin x$ The equation is linear equation Therefore, comparing with $\frac{dy}{dx}+P(x)y=Q(x)$ $P(x)=2\tan x, \qquad Q(x)=\sin x$ $IF=e^{\int P(x)dx}=e^{\int 2\tan x dx}=e^{\log\sec^2 x}=\sec^2 x$ Therefore the general solution is $y(I.F)=\int Q(x)(I.F)dx+c$ $\therefore y(\sec^2 x)=\int \sin x\sec^2 xdx+c$ $\therefore y(\sec^2 x)=\int \tan x\sec xdx+c$ $\therefore y(\sec^2 x)=\sec x+c$	2.Solve $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$ Solution: $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$ $\frac{dy}{dx}-\frac{y}{(x+1)}=e^{3x}(x+1)$ The equation is linear equation Therefore, comparing with $\frac{dy}{dx}+P(x)y=Q(x)$ $P(x)=-\frac{1}{x+1}, \qquad Q(x)=e^{3x}(x+1)$ $IF=e^{\int P(x)dx}=e^{\int -\frac{1}{x+1}dx}=e^{\log(x+1)^{-1}}=\frac{1}{x+1}$ Therefore the general solution is $y(I.F)=\int Q(x)(I.F)dx+c$ $\therefore y\frac{1}{x+1}=\int e^{3x}(x+1)\frac{1}{x+1}dx+c$ $\therefore y\frac{1}{x+1}=\int e^{3x}dx+c$ $\therefore y\frac{1}{x+1}=\frac{e^{3x}}{3}+c$
3.Solve $\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(x^2+1)^3}$ Solution: $\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(x^2+1)^3}$ The equation is linear equation	4.Solve $y'+y\tan x=\sin 2x, \qquad y(0)=1$ Solution: $y'+y\tan x=\sin 2x$ The equation is linear equation Therefore comparing with $\frac{dy}{dx}+P(x)y=Q(x)$ $P(x)=\tan x, \qquad Q(x)=\sin 2x$

Therefore comparing with $\frac{dy}{dx}+P(x)y=Q(x)$ $$P(x)=\frac{4x}{1+x^2}, \qquad Q(x)=\frac{1}{(1+x^2)^3}$$ $$IF=e^{\int P(x)dx}=e^{\int\frac{4x}{1+x^2}dx}=e^{\log(1+x^2)^2}=(1+x^2)^2$$ Therefore the general solution is $$y(I.F)=\int Q(x)(I.F)dx+c$$ $$\therefore y(1+x^2)^2=\int\frac{1}{(1+x^2)^3}(1+x^2)^2dx+c$$ $$\therefore y(1+x^2)^2=\int\frac{1}{(1+x^2)}dx+c$$ $$\therefore y(1+x^2)^2=\tan^{-1}x+c$$	$$IF=e^{\int P(x)dx}=e^{\int\tan x dx}=e^{\log\sec x}=\sec x$$ Therefore the general solution is $$y(I.F)=\int Q(x)(I.F)dx+c$$ $$\therefore y(\sec x)=\int\sin 2x\sec x dx+c$$ $$\therefore y(\sec x)=\int\sin x dx+c$$ $$\therefore y(\sec x)=-\cos x+c$$ Given $y(0)=1$ $$\therefore 1=-1+c$$ $$\therefore c=2$$ $$\therefore y(\sec x)=-\cos x+2$$

For example:

1. Solve $\frac{dy}{dx}+\frac{1}{x^2}y=6e^{\frac{1}{x}}$
2. Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$
3. Solve $\frac{dy}{dx}+\frac{3y}{x}=\frac{\sin x}{x^3}$

❖ Non Linear Differential Equation or Bernoulli Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad (1)$$

where P and Q are functions of x or constants is known as Bernoulli's Equation.

Dividing (1) by y^n

$$y^{-n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \qquad (2)$$

Put $\frac{1}{y^{n-1}}=v$

$$\frac{1-n}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dv}{dx}$$

Substituting in (2)

$$\frac{dv}{dx} + (1-n)Pv = Q$$

which is linear form.

For Example

1.Solve $x\frac{dy}{dx} + y = x^3y^6$

Solution: $x\frac{dy}{dx} + y = x^3y^6$

Dividing both the sides by y^6

$$y^{-6}x\frac{dy}{dx} + y^{-5} = x^3 \qquad (1)$$

Taking $y^{-5} = v$

$$\therefore -5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx} - \frac{5}{x}v = -5x^2$$

$$\therefore IF = e^{\int -\frac{5}{x}dx} = x^{-5}$$

Therefore the general solution is

$$v(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore vx^{-5} = \int -5x^2x^{-5}dx + c$$

$$\therefore vx^{-5} = \int -5x^{-3}dx + c$$

$$\therefore vx^{-5} = \frac{-5x^{-2}}{-2} + c$$

$$\therefore y^{-5}x^{-5} = \frac{5x^{-2}}{2} + c$$

2.Solve $x\frac{dy}{dx} + y = y^2\log x$

Solution:

$$x\frac{dy}{dx} + y = y^2\log x$$

Dividing both the sides by xy^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x} \qquad (1)$$

Taking $y^{-1} = v$

$$\therefore -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx} - \frac{1}{x}v = -\frac{\log x}{x}$$

$$\therefore IF = e^{\int -\frac{1}{x}dx} = x^{-1} = \frac{1}{x}$$

Therefore the general solution is

$$v(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore v\frac{1}{x} = \int\left(-\frac{\log x}{x}\right)\frac{1}{x}dx + c$$

$$\therefore v\frac{1}{x} = -\int \log x\frac{1}{x^2}dx + c$$

$$\therefore y^{-1}x^{-1} = \frac{1}{x}\log x + \frac{1}{x} + c$$

2. Solve

$$x\frac{dy}{dx} = 4x^3y^2 + y \qquad y(0) = 2$$

Solution: $x\frac{dy}{dx} = 4x^3y^2 + y$

$$\therefore \frac{dy}{dx} - \frac{1}{x}y = 4x^2y^2$$

Dividing both the sides by y^2

$$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = 4x^2 \qquad (1)$$

Taking $y^{-1} = v$

$$\therefore y^{-1}\frac{dy}{dx} = -\frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx} + \frac{v}{x} = -4x^2$$

$$\therefore IF = e^{\int \frac{1}{x}dx} = x$$

Therefore the general solution is

$$v(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore vx = \int -4x^2 x dx + c$$

$$\therefore vx^{-5} = \int -4x^3 dx + c$$

$$\therefore vx = \frac{-4x^4}{4} + c$$

$$\therefore vx = -x^4 + c$$

$$\therefore \frac{x}{y} = -x^4 + c$$

$$\because \ y(0) = 2$$

$$\therefore 0 = 0 + c \Rightarrow c = 0$$

$\therefore$ General Solution is $\frac{x}{y} = -4x^2$

3. Solve $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Solution: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing both the sides by e^y

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad (1)$$

Taking $e^{-y} = v$

$$\therefore e^{-y}\frac{dy}{dx} = -\frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$$

$$\therefore IF = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

Therefore the general solution is

$$v(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore v\frac{1}{x} = \int -\frac{1}{x^2}\frac{1}{x}dx + c$$

$$\therefore v\frac{1}{x} = \int -x^{-3}dx + c$$

$$\therefore v\frac{1}{x} = \frac{x^{-2}}{2} + c$$

$$\therefore v\frac{1}{x} = x^{-2} + c$$

$$\therefore \frac{e^{-y}}{x} = x^{-2} + c$$

Examples:

1. Solve $\frac{dy}{dx} + y\tan x = y^3 \sec x$
2. Solve $y^4 dx = \left(x^{-\frac{3}{4}} - y^3 x\right)dy$

<u>Differential equations of first order but not of first degree:</u>

We shall study first order differential equation of higher degree. We shall denote the derivative $\frac{dy}{dx} = p$. For a given differential equation of first order but of higher degree, three cases may arises.

Case1- First-Order Equations of Higher Degree Solvable for p

Let F(x, y, p) = 0 can be solved for p and can be written as

$(p-q_1(x,y))\ (p-q_2(x,y))\ \ldots\ldots\ldots\ (p-q_n(x,y)) = 0$

Equating each factor to zero we get equations of the first order and first degree.

One can find solutions of these equations by the methods discussed in the previous chapter. Let their solution be given as:

$\square_i(x,y,c_i)=0,\ i=1,2,3\ \ldots\ldots\ldots n$ (1)

Therefore the general solution can be expressed in the form

$\square_1(x,y,c_1)\ \square_2(x,y,c_2)\ldots\ldots\ldots\square_n(x,y,c) = 0$ (2)

where c in any arbitrary constant.

Example 1 **Solve** $xy\left(\frac{dy}{dx}\right)^2 + (x^2+y^2)\frac{dy}{dx} + xy = 0$ **(1)**

Solution: This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

Equation (1) can be written as

$xy\ p^2+(x^2+y^2)\ p+xy=0$

$xyp^2 + x^2p + y^2p + xy = 0$

$xp(yp + x) + y(yp + x) = 0$

$(xp + y)(yp + x) = 0$

This implies that

$xp+y=0,\ yp+x=0$

$p = \frac{-y}{x}$ $\qquad$ $p = \frac{-x}{y}$

$\frac{dy}{dx} = \frac{-y}{x}$ $\qquad$ $\frac{dy}{dx} = \frac{-x}{y}$

By solving equations, we get

$xy=c_1$ and

$x^2+y^2=c_2$ respectively

[$x\frac{dy}{dx} + y = 0$ or $\frac{dy}{dx} + \frac{1}{x}y = 0,$ Integrating factor

$I(x) = e^{\int \frac{1}{x}dx} = e^{\log x}.$ This gives

y.x = +o.x dx $+c_1$ or $xy=c_1$]

[$y\frac{dy}{dx} + x = 0,$ or $ydy + xdx = 0$

By integration we get $\frac{1}{2}y^2 + \frac{1}{2}x^2 = c$

or $x^2+y^2 = c_2$, $c_2 > 0$, $-\sqrt{c2} \le x \le \sqrt{c2}$]

The general solution can be written in the form

$(x^2+y^2-c_2)(xy-c_1)=0$

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

Example 2: Solve $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}x + y = 0$

Example 3: Solve $\left(\frac{dy}{dx}\right)^2 - 5y + 6 = 0$

Example 4: Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Case 2: Differential equations Solvable for y

Let the differential equation given by (1) be solvable for y. Then y can be expressed as a function x and p, that is,

$$y = f(x, p)$$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx}$$

is a first order differential equation of first degree in x and p.

Let solution be expressed in the form

$$\varphi(x, p, c) = 0$$

The solution of equation is obtained by eliminating p from equation. If elimination of p is not possible then together may be considered parametric equations of the solutions with p as a parameter.

Example 1: Solve y^2-1-p^2=o

Solution: It is clear that the equation is solvable for y, that is

$$y = \sqrt{1+p^2}$$

By differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1+p^2}}.2p\frac{dp}{dx}$$

or $$p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or $$p\left[1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right] = 0$$

gives p=o or $1-\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

By solving p=0 we get

y=1

By $1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

we get a separable equation in variables p and x.

$$\frac{dp}{dx}=\sqrt{1+p^2}$$

By solving this we get

p=sinh (x+c)

By eliminating p we obtain

y=cos h (x+c) is a general solution.

Solution y=1 of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in solution.

Example 2: $y = 2px + p^4x^4$

Example 3: $y = x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}$

Example 4: $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = e^y$

<u>Case 3: Differential Equations Solvable for x</u>

Let equation F(x, y, p) = 0 be solvable for x, that is x=f(y,p) ..

Then as argued in the previous section for y we get a function ¬ such that

¬(y, p, c) = 0

By eliminating p from F(x, y, p) = 0 we get a general solution.

<u>Example 1:</u>

Solve $x\left(\frac{dy}{dx}\right)^3-12\frac{dy}{dx}-8=0$

Solution: Let $p=\frac{dy}{dx},\ \ then$

$xp^3-12p-8=0$

It is solvable for x, that is,

$$x = \frac{12p+8}{p^3} = \frac{12}{p^2} + \frac{8}{p^3} \qquad \ldots \qquad (1)$$

Differentiating (3.18) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$$

$$or \frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$$

$$or \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$or \quad y = +\frac{24}{p} + \frac{12}{p^2} + c \qquad (2)$$

(1) and (2) constitute parametric equations of solution of the given differential equation.

Example 2: $y = 2px + y^2 p^3$

Example 3: $y^2 \log y = xyp + p^2$

<u>Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation</u>.

Let Equation F(x, y, p) = 0 be of the first degree in x and y, then

$y = x\prod_1(p) + \prod_2 (p) \qquad \ldots \qquad (1)$

Equation (1) is known as Lagrange's equation.

If $\prod_1(p) = p$ then the equation

$y = xp + \prod_2 (p) \qquad .. \qquad (2)$

is known as Clairaut's equation

By differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \varphi_1(p) + x\varphi_1'(p)\frac{dp}{dx} + \varphi_2'(p)\frac{dp}{dx}$$

or $$p - \varphi_1(p) = (x\varphi_1'(p) + \varphi_2'(p))\frac{dp}{dx} \qquad \ldots \qquad (3)$$

From (3) we get

$$(x+\varphi_2'(p))\frac{dp}{dx}=0$$ for $\prod_1(p)=p$

This gives

$$\frac{dp}{dx}=0$$ or $x+\prod_2'(p)=0$

$$\frac{dp}{dx}=0$$ gives p = c and

by putting this value in (2) we get

$y=cx+\prod_2(c)$

This is a general solution of Clairaut's equation.

The elimination of p between

$x+\prod_2'(p) = 0$ and (2) gives a singular solution.

If $\prod_1(p) \neq p$ for any p, then we observe from (3) that

$$\frac{dp}{dx}\neq 0$$ everywhere. Division by

$$[p-\varphi_1(p)]\frac{dp}{dx}$$ in (3) gives

$$\frac{dx}{dp}-\frac{\varphi_1'}{p-\varphi_1(p)}x=\frac{\varphi_2'(p)}{p-\varphi_1(p)}$$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (2) will form a parametric representation of the general solution of (2)

<u>Example 1</u>: Solve $\left(\frac{dy}{dx}-1\right)\left(y-x\frac{dy}{dx}\right)=\frac{dy}{dx}$

Solution: Let $p=\frac{dy}{dx}$ then,

(p-1) (y-xp) =p

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Thus either $\frac{dp}{dx} = 0$ or

$$x - \frac{1}{(p-1)^2} = 0$$

$\frac{dp}{dx} = 0$ gives p=c

Putting p=c in the equation we get

$$y = cx + \frac{c}{c-1}$$

(y-cx)(c-1)=c

which is the required solution.

Exercises

Solve the following differential equations

1. $$\left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx}e^{2x}$$

2. $y(y-2)p^2 - (y-2x+xy)p+x=0$

3. $$-\left(\frac{dy}{dx}\right)^2 + 4y - x^2 = 0$$

4. $$\left(\frac{dy}{dx} + y + x\right)\left(x\frac{dy}{dx} + y + x\right)\left(\frac{dy}{dx} + 2x\right) = 0$$

5. $$y + x\frac{dy}{dx} - x^4\left(\frac{dy}{dx}\right)^2 = 0$$

6. $$\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right)=h^2\frac{dy}{dx}$$

7. $$y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}=x$$

8. $$x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}+ax=0$$

9. $$\left(\frac{dy}{dx}\right)^2=y-x$$

10. $$xy\left(y-x\frac{dy}{dx}\right)=x+y\frac{dy}{dx}$$

MATHEMATICAL MODELING

POPULATION GROWTH

One of the simplest models of population growth is based on the observation that when populations (people, plants, bacteria, and fruit flies, for example) are not constrained by environmental limitations, they tend to grow at a rate that is proportional to the size of the population—the larger the population, the more rapidly it grows.

To translate this principle into a mathematical model, suppose that $y = y(t)$ denotes the population at time t. At each point in time, the rate of increase of the population with respect to time is dy/dt, so the assumption that the rate of growth is proportional to the population is described by the differential equation

$$\frac{dy}{dt} = ky \tag{1}$$

where k is a positive constant of proportionality that can usually be determined experimentally. Thus, if the population is known at some point in time, say $y = y_0$ at time $t = 0$, then a general formula for the population $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

PHARMACOLOGY

When a drug (say, penicillin or aspirin) is administered to an individual, it enters the bloodstream and then is absorbed by the body over time. Medical research has shown that the amount of a drug that is present in the bloodstream tends to decrease at a rate that is proportional to the amount of the drug present—the more of the drug that is present in the bloodstream, the more rapidly it is absorbed by the body.

To translate this principle into a mathematical model, suppose that $y = y(t)$ is the amount of the drug present in the bloodstream at time t. At each point in time, the rate of change in y with respect to t is dy/dt, so the assumption that the rate of decrease is proportional to the amount y in the bloodstream translates into the differential equation

$$\frac{dy}{dt} = -ky \tag{2}$$

where k is a positive constant of proportionality that depends on the drug and can be determined experimentally. The negative sign is required because y decreases with time. Thus, if the initial dosage of the drug is known, say $y = y_0$ at time $t = 0$, then a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0$$

EXAMPLE 5 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 16.7)?

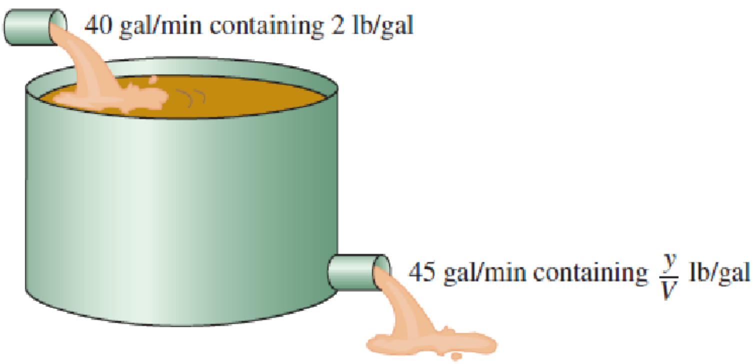


FIGURE 16.7 The storage tank in Example 5 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (9)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min and } v = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. && \text{Eq. (10)} \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

in pounds per minute.

To solve this differential equation, we first write it in standard form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t}\, y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) &= e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}\,dt} \\ &= e^{-9\ln(2000-5t)} \qquad 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$(2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}\, y\right) = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9}\frac{dy}{dt} + 45(2000 - 5t)^{-10}\, y = 80(2000 - 5t)^{-9}$$

$$\frac{d}{dt}\left[(2000 - 5t)^{-9} y\right] = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9} y = \int 80(2000 - 5t)^{-9}\, dt$$

$$(2000 - 5t)^{-9} y = 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C.$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^{9}.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$100 = 2(2000 - 0) + C(2000 - 0)^{9}$$

$$C = -\frac{3900}{(2000)^{9}}.$$

The particular solution of the initial value problem is

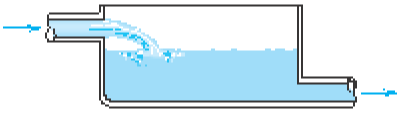
$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$

Mixing problem

Example: The tank in the figure contains 200 litres of water in which 40 lb of salt are dissolved. Five litres of brine, each containing 2lb of dissolved salt, run into the tank per minute, and the mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt $y(t)$ in the tank at any time t.



Tank

Solution: The time rate of change $y' = \frac{dy}{dt}$ of $y(t)$ equals the inflow of salt minus the outflow.

The inflow is 10lb/min. $y(t)$ is the total amount of salt in the tank. The tank always contains 200 litres because 5 litres flow in and 5 litres flow out per minute.

Thus 1 litre contains $\frac{y(t)}{200}$ lb of salt. Hence the 5 out flowing litres contains $\frac{5y(t)}{200} = \frac{y(t)}{40} = 0.025y(t)$ lb of salt. This is the outflow. The time rate of change y' is the balance:

$$y' = \text{salt inflow rate - salt outflow rate}$$

Since the inflow of salt is 10lb/min and the outflow is $0.025y(t)\ lb/\min$, this equation becomes

$$y' = 10 - 0.025y$$

By separation of variables, the equation will become

$$y' = 10 - 0.025y = -0.025(y - 400)$$

$$\therefore \frac{dy}{y-400} - 0.025dt$$

$$\ln(y - 400) = 0.025t + a$$

$$y - 400 = ce^{-0.025t}$$

Initially $y(0) = 40$

$$40 - 400 = c$$
$$c = -360$$

$$y - 400 = -360e^{-0.025t}$$

Newton's Law of Cooling:

According to this law, the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surroundings and heat of the body itself.

If θ_0 is the temperature of the surroundings and θthat of the body at any time t, then

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where k is a constant .

Example: Suppose that you turn off the heat in your home at night 2 hours before you go to bed; call this time t=0. If the temperature T at t=0 is $66^\circ F$ and at the time you go to bed (t=2) has dropped to $63^\circ F$, what temperature can you expect in the morning, say, 8 hours later (t=10)? Of course, this process of cooling off will depend on the outside temperature T_A , which we assume to be constant at $32^\circ F$.

Solution: The equation $\frac{dT}{dt} = k(T - T_A) = k(T - 32)$ is the equation of Newton's Law of cooling.

$$\frac{dT}{(T-32)} = kdt$$
$$\therefore \ln|T - 32| = kt + c$$
$$\therefore T(t) = 32 + ce^{kt}$$

Given initial condition T(0)=66

$$\therefore T(0) = 32 + c = 66$$
$$\therefore c = 34$$

Also T(2)=63

$$\therefore T(2) = 32 + 34e^{2k} = 63$$
$$\therefore e^{2k} = \frac{63-32}{34} = 0.9117$$
$$\therefore k = -0.0461$$

Now after 10 hours of shutdown t=10, the temperature will be

$$\therefore T(10) = 32 + 34e^{-0.0461(10)} = 53.4^0 F$$

Self study

Modelling with Exponential Functions

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modelled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

REMARK: Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of $1,000,000 will increase more in one year than an investment of $1000.

EXPONENTIAL GROWTH MODEL: A population that experiences exponential growth increases according to the model

$n(t) = n_0e^{rt}$where t is time,

$n(t)$ is the population size at time t, n_0 is the initial size of the population, and r is the relative rate of growth (expressed as a proportion of the population).

Example: Exponential growth or decay: The differential equation of exponential growth or decay is governed by

$$y' = ky$$

Solution:

$y' = ky$ is in variable separable form

$$\therefore \frac{1}{y}\frac{dy}{dx} = k$$

$$\therefore \frac{1}{y}dy = kdx$$

Integrating both the sides

$$\therefore \int \frac{1}{y}dy = k\int dx + \log(c)$$

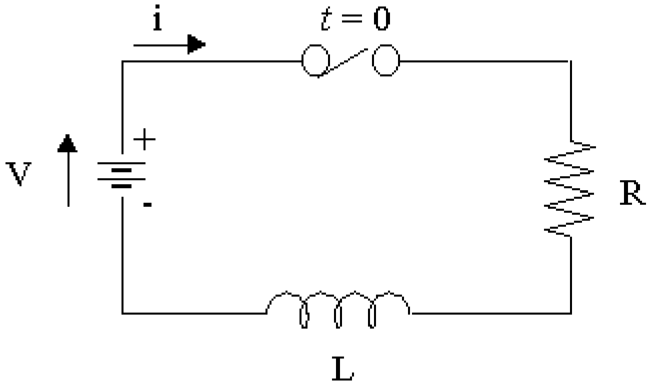
$$\therefore \log(y) = kx + \log(c)$$

$$\therefore y = ce^{kx}$$

Electric Circuit

1. **RL series circuit :**

Consider a circuit containing resistance R and inductance L in series with a voltage source E.



RL circuit diagram

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and L=E

i.e., $Ri + L\frac{di}{dt}$=E or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ .... (1)

This is Leibnitz's linear equation.

I.F.=$e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

therefore, its solution is

$$i(I.F.) = \int \frac{E}{L}(I.F.)dt + c$$

Or $ie^{\frac{Rt}{L}} = \int \frac{E}{L}E^{\frac{Rt}{L}}dt + c = \frac{E}{L}\frac{L}{R}e^{\frac{Rt}{L}} + c$ whence $i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$ .... (2)

If initially there is no current in the circuit, i.e. i=0, when t=0, we have $c = -\frac{E}{R}$

Thus (2) becomes $i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$ which shows that i increases with t and attains the maximum value E/R.

Example: The current *i(t)* flowing in an R-L circuit is governed by the equation $\left[L\frac{di}{dt} + Ri = E_0 \sin \omega t\right]$ **where R is the constant resistance, L is the constant inductance and** $E_0 \sin \omega t$ **is the voltage at time *t*,** E_0 **and** ω **being constants. Find the current at any time assuming that initially it is zero.**

Solution: The given equation can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin \omega t$

$$\therefore I.F = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

Therefore general solution is

$$i(t)e^{\frac{Rt}{L}} = \int \frac{E_0}{L}\sin \omega t e^{\frac{Rt}{L}}dt + c$$

$$\therefore i(t) = \frac{E_0}{R^2 + w^2L^2}\left[R\sin \omega t - \omega L\cos \omega t\right] + ce^{-\frac{Rt}{L}}$$

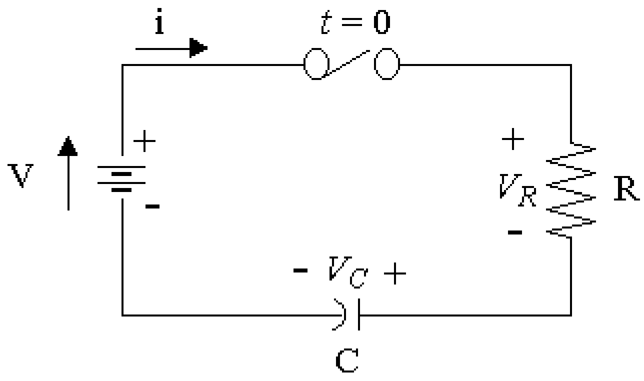
Given the initial condition is $i(0)=0$, which gives

$$c = \frac{\omega E_0 L}{R^2 + w^2 L^2}$$

Therefore the current i at any time t is given by

$$\therefore i(t) = \frac{E_0}{R^2 + w^2 L^2}\left[R\sin\omega t - \omega L\cos\omega t + \omega L e^{-\frac{Rt}{L}} \right]$$

2. RC series circuit :



In an RC circuit, the **capacitor** stores energy between a pair of plates. When voltage is applied to the capacitor, the charge builds up in the capacitor and the current drops off to zero.

Case: 1 Constant Voltage

The voltage across the resistor and capacitor are as follows

$V_R = Ri$ and $V_C = \frac{1}{C}\int idt$

Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int idt = V$$

One way to solve this equation is to turn it into a **differential equation**, by differentiating throughout with respect to t:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}e^{-\frac{t}{RC}}$$

Case:2 Variable Voltage

We need to solve variable voltage cases in q, rather than in i, since we have an integral to deal with if we use i.

We have $i = \frac{dq}{dt}$ and $q = \int idt$

So the equation $Ri + \frac{1}{C}\int \quad idt = V$ becomes

$$R\frac{dq}{dt} + \frac{q}{C} = V$$

Example: The current *i(t)* flowing in an R-C circuit is governed by the equation $\left[R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}\right]$ **where R is the constant resistance, C is the capacitance and E(t) is the periodic electromotive force at time *t*. Find the current at any time assuming that E is constant and** $E(t) = E_0 \sin \omega t$.

Solution: $\left[R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}\right]$

$\left[\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dE}{dt}\right]$ which is linear equation.

$$\therefore I.F = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

Therefore general solution is

$$I(t)e^{\frac{t}{RC}} = \int \frac{1}{R}e^{\frac{t}{RC}}\frac{dE}{dt}dt + c$$

Given E=0, then $\frac{dE}{dt} = 0$, which gives

$$I(t) = ce^{-\frac{t}{RC}}$$

Given $E(t) = E_0 \sin \omega t$, then $\frac{dE}{dt} = \omega E_0 \cos \omega t$, which gives

$$\therefore I(t) = ce^{-\frac{t}{RC}} + \frac{\omega E_0 C}{1+(wRC)^2}\left[\cos \omega t + \omega RC \sin \omega t\right]$$