Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme (All Branches)

Mathematics – 1 (203191102)

Unit – 4 Sequence and Series (Lecture Note)

Sequence:

- Limit of a sequence
- Convergence & Divergence of a sequence
- Oscillatory sequence
- Sandwich/Squeezing theorem for sequences
- Convergence properties of sequence
- Monotonic sequence(Monotonic increasing & Monotonic decreasing)
- Alternating sequence
- Bounded & Unbounded sequence.

Series:

- Convergence, Divergence & Oscillatory series
- Some properties of infinite series
- Telescoping series
- Geometric series
- p-series, Integral test
- Comparison test
  (i)Direct
  (ii)Limit Comparison
- D'Alembert ratio test
- Cauchy's root test
- Alternating series
- Leibnitz test
- Absolute and conditionally convergent
- Power series
  - Interval of convergence
  - Radius of convergence

❖ Sequence:

A sequence is a function whose domain is the set of positive integers.

It is generally written as $a_1, a_2, a_3..........a_n,.........$

- If the number of terms in a sequence is infinite, it is called infinite sequence otherwise it is said to be finite sequence

$$e.g. 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad ; \quad 1, -1, 2, -2, \ldots$$

❖ Limit of a sequence:

Let $\{a_n\}$ be a sequence.

A real number l is said to be the limit of the sequence $\{a_n\}$; if for every $\varepsilon > 0$, there exist an integer N such that $n \geq N \Rightarrow |a_n - l| < \varepsilon$

If such a number exists then we write

$$\lim_{n\to\infty} a_n = l.$$

❖ Convergence, Divergence & oscillations of a sequence:

- A sequence $\{a_n\}$ issaid to be convergent if the sequence has finite limit.

$$i.e.\ if \quad \lim_{n\to\infty} a_n = finite.$$

- A sequence $\{a_n\}$ issaid to be divergent if the sequence has infinite limit.

$$i.e. \qquad if \ \lim_{n\to\infty} a_n = \pm\infty.$$

For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{n+1}{n} = 1$, $\lim_{n\to\infty} 2n = \infty$,

- A sequence $\{a_n\}$ issaid to be oscillatory if the sequence is neither convergent nor divergent. For example, let

$$\{u_n\} = \left\{(-1)^n + \frac{1}{2^{\frac{1}{n}}}\right\}$$

$$\overset{lim}{n \to \infty}\ u_n = 2\ if\ n\ is\ even$$
$$= 0\ if\ n\ is\ odd$$

Since the limit is not unique, the sequence is oscillatory.

❖ Convergence properties of sequences:

- Let $\{a_n\}$and $\{b_n\}$be two convergent sequences and k be any real number, then the following sequences will also converge.

1) $\{a_n + b_n\}$ With $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty}(b_n)$
2) $\{ka_n\}$ With $\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty}(a_n)$
3) $\{a_n b_n\}$ With $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty}(a_n)\right)\left(\lim_{n\to\infty}(b_n)\right)$

4) $\left\{\frac{a_n}{b_n}\right\}$ With $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$; $\left(if \lim_{n\to\infty}(b_n)\neq 0\right)$

❖ **Some Important Formula:**

$$\lim_{n\to\infty}\frac{ln(n)}{n}=0$$

$$\lim_{n\to\infty}\sqrt[n]{n}=1$$

$$\lim_{n\to\infty}x^{\frac{1}{n}}=1(x>0)$$

$$\lim_{n\to\infty}x^n=0(|x|<1)$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad (anyx)$$

$$\lim_{n\to\infty}\left(\frac{x^n}{n!}\right)=0 \quad (anyx)$$

Que. Applying the definition, show that $\left\{\frac{1}{n}\right\}$ converges 0 as $n\to\infty$.

To prove: Let $\epsilon>0$, we must show that there exists an integer N such that for all n,

$$n>N\Rightarrow\left|\frac{1}{n}-0\right|<\epsilon$$

Solution: Let $\epsilon>0$ be given.

Let N be an integer such that $N>\frac{1}{\epsilon}$.

$n\geq N \Rightarrow n\geq N>\frac{1}{\epsilon}$

$\Rightarrow n>\frac{1}{\epsilon}$

$\Rightarrow\left|\frac{1}{n}-0\right|=\frac{1}{n}<\epsilon$

$\Rightarrow\left|\frac{1}{n}-0\right|<\epsilon$

$$\therefore \lim_{n\to\infty}\frac{1}{n}=0$$

Que. Test the Convergence of the following sequences:

1) $\left\{ \dfrac{n^2+n}{2n^2-n} \right\}$

Solution :

$$Let\ n \xrightarrow{lin} \infty\ a_n = n \xrightarrow{\lim} \infty \frac{n^2+n}{2n^2-n}$$

$$= n \xrightarrow{\lim} \infty \frac{n^2(1+\frac{1}{n})}{n^2(2-\frac{1}{n})}$$

$$= \frac{1}{2}$$

As the value of limit is finite the sequence is convergent.

2) $\left\{ 2^n \right\}$

Solution :

$$Let\ n \xrightarrow{lin} \infty\ a_n = n \xrightarrow{\lim} \infty\ 2^n$$

$$= 2^{\infty}$$

As the value of the sequence is infinite the sequence is divergent.

3) $\left\{ 2-(-1)^n \right\}$

Solution :

$$Let\ a_n = 2-(-1)^n$$

$$n \xrightarrow{lin} \infty\, 2-(-1)^n$$

$$= 2+1 = 3 \quad if\ \ n\ is\ odd$$

$$0r = 2-1 = 1 \quad if\ \ n\ is\ even$$

As the value of limit is not unique the sequence is oscillating sequence.

4) $\left\{ \sqrt{n+1} - \sqrt{n} \right\}_{n=1}^{\infty}$

Solution :

$$n \xrightarrow{lin} \infty \sqrt{n+1} - \sqrt{n}$$

$$= n \xrightarrow{lin} \infty \sqrt{n+1} - \sqrt{n} \quad X\ \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= n \xrightarrow{lin} \infty \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= n \xrightarrow{lin} \infty \frac{1}{\infty} = 0$$

As the value of limit is finite the sequence is convergent.

- **Monotonic sequence**:
  - A sequence $\{a_n\}$ is said to be monotonically increasing if $a_n \leq a_{n+1}$ for each value of n.

$$a_n - a_{n+1} \leq 0$$

  - A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_n \geq a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be strictly increasing if $a_n < a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be strictly decreasing if $a_n > a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be monotonic if it is either increasing or decreasing.

- **Bounded & unbounded sequence:**
  - A sequence $\{a_n\}$ is said to be bounded above if there is a real number M such that $a_n \leq M, forall\ n \in \mathbb{N}$. M is said to be an upper bound of the sequence.
  - A sequence $\{a_n\}$ is said to be bounded below if there is a real number m such that
    $a_n \geq m, forall\ n \in \mathbb{N}.m$ is said to be a lower bound of the sequence.
  - A sequence $\{a_n\}$ is said to be bounded if it is both bounded above and bounded below.
  - A sequence $\{a_n\}$ is said to be unbounded if it is not bounded.

1) $a_n = n$

$$a_n = 1,2,3,4, \ldots \ldots$$

$$a_n \geq 1$$

$$a_n\ is\ bounded\ below.$$

2) $a_n = \frac{n}{n+1}$

$$= \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \ldots \ldots.$$

$$a_n \geq \frac{1}{2}, bounded\ below$$

$$a_n < 1, bounded\ above$$

$$\frac{1}{2} \leq a_n < 1$$

$$a_n \text{ is bounded.}$$

3) $a_n = \frac{1}{n}$

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots \dots$$

$$a_n \leq 1, bounede\ above$$

$$a_n > 0, bounded\ below$$

It is bounded.

4) $a_n = (-1)^n$

5) $a_n = (-1)^n . n$

unbounded.

❖ Note that

- If $\{a_n\}$ is bounded above and increasing then it is convergent.
- If $\{a_n\}$ is unbounded above and increasing then it is divergent to ∞.
- If $\{a_n\}$ is bounded below and decreasing then it is convergent.
- If $\{a_n\}$ is unbounded below and decreasing then it is divergent to $-\infty$.

1) The sequence n^2

1, 4, 9,16,……

Increasing sequence

2) $\frac{1}{2^n}$

Decreasing sequence

❖ Sandwich theorem:

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers such that

$$(i) c_n \leq a_n \leq b_n \ ; \ \forall\, n \geq n_0, for\ some\ n_0\ \ and$$

$$(ii) \lim_{n\to\infty} c_n = l = \lim_{n\to\infty} b_n$$

$$then\ \lim_{n\to\infty} a_n = l$$

Que. Show that the sequence $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Solution:

We know that $-1 \le sinn \le 1 \Rightarrow -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

Further, $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty}\frac{1}{n} = 0.$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty}\frac{\sin n}{n} = 0$

Example: Check the sequence

$\boldsymbol{a_n = \frac{n}{n^2+1}}$ ***is decreasing and bounded. Is cgt***?

Solution:

$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$a_n - a_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} > 0$$

$$a_n - a_{n+1} > 0$$

It is decreasing squence.

$$a_n = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \dots ..$$

$$a_n \le \frac{1}{2}, \quad a_n > 0$$

$0 < a_n \le \frac{1}{2}.$

It is bounded.

Every monotonically bounded sequence is cgt.

❖ <u>**Infinite Series:**</u>

The sum of an infinite sequence of numbers is called **infinite Series**

e.g. $a_1 + a_2 + a_3 + \dots\dots\dots\dots + a_n + \dots\dots = \sum_{n=1}^{\infty} a_n$

- $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is called n[th] partial sum of the series.
- The convergence of infinite series depends on the convergence of the corresponding infinite sequence of partial sums.
- The infinite series is

Convergent if $\lim_{n\to\infty} S_n = S\ (finite)$

Divergent	if	$\lim_{n\to\infty} S_n = \infty \ or \ -\infty$
Oscillatory	if	$\lim_{n\to\infty} S_n = niether finite nor \pm\infty$
Oscillating finitely	if	Value fluctuates within finite range
Oscillating infinitely	if	Value fluctuates within $\infty and -\infty$

- If a series $\sum_{n=1}^{\infty} a_n$ converges to S then we say that the sum of the series is S and we write $\sum_{n=1}^{\infty} a_n = S$

❖ Convergence properties of series:

Let $\sum a_n$ and $\sum b_n$ be two convergent series and k be any real number, then the following series will also converge.

1) $\sum(a_n \pm b_n)$ with $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$
2) $\sum ka_n$ With $\sum ka_n = k\sum a_n$

❖ Telescoping series:

A series is said to be telescoping if while writing the n^th partial sum all terms except first and last vanish.

Que Check the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

Solution: Here, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\frac{1}{1}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$
$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

It is cgt.

For example: $\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

Que. Find the Sum of the series $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \infty$

Solution:

$$S_n = \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \log\frac{n+1}{n}$$

$$= \log(2.\frac{3}{2}\frac{4}{3}\cdots\frac{n+1}{n})$$

$$S_n = \log(n+1)$$

$$n \xrightarrow{\lim} \infty S_n = n \xrightarrow{\lim} \infty \log(n+1)$$

$$= \log \infty$$

$$= \infty$$

As it is infinite therefore the series is divergent.

Que. Find the Sum of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \infty$

Solution: $a_n = \frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)$$

$$+ \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$

$$= 1 - \frac{1}{(n+1)!}$$

$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1$$

$$\therefore \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$$

❖ **<u>Geometric Series:</u>**

An infinite series in the form $a + ar + ar^2 + \ldots\ldots\ldots\ldots + ar^{n-1} + \ldots\ldots$ is said to be a geometric series.

It converges to $\frac{a}{1-r}$ if $|r| < 1$ i.e. $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $|r| < 1$.

If $|r| \geq 1$ then the series diverges.

If $r = -1$ then series is oscillatory.

Que. Discuss the convergence of $\sum_{n=0}^{\infty} 2^n$

Solution:

Given series, $\sum_{n=0}^{\infty} 2^n = 2^0 + 2^1 + 2^2 + \cdots$ is a geometric series with $a = 1$ and $r = 2$

$$r = \frac{2}{1} = 2, \;\; r = \frac{4}{2} = 2$$

Since $r = 2 > 1$, the series is divergent.

Que. Check the convergence of a series $\frac{1}{3^0} - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \cdots$ Also find sum

Solution:

$$S_n = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

$$r = \frac{a_2}{a_1} = -\frac{\frac{1}{3}}{1} = -\frac{1}{3}$$

$$r = \frac{a_3}{a_2} = \frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3}$$

$$r = -\frac{1}{3}$$

Here the series is geometric series with $a = 1$ and and $|r| = \frac{1}{3}$

Since $|r| = \frac{1}{3} < 1$, the series is convergent.

Sum$= \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$.

Que. Discuss the convergence of $\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}}$

Solution: Since,

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}} = \sum_{n=1}^{\infty} \frac{(3^2)^n}{(4^2)^n} = \sum_{n=1}^{\infty} \frac{(9)^n}{(16)^n} = \sum_{n=1}^{\infty} \left(\frac{9}{16}\right)^n$$

is a geometric series with $a = \frac{9}{16}$ and $r = \frac{9}{16}$.

Since $r = \frac{9}{16} < 1$, it is convergent. Further it converges to $\frac{a}{1-r} = \frac{\left(\frac{9}{16}\right)}{\left(1-\left(\frac{9}{16}\right)\right)} = \frac{9}{7}$

Que. Check the convergence of $\sum_{n=1}^{\infty}\frac{4^n+5^n}{6^n}$

Solution:

$$\sum C_n = \sum_{n=1}^{\infty}\left[\left(\frac{4}{6}\right)^n + \left(\frac{5}{6}\right)^n\right] = \sum_{n=1}^{\infty}\left(\frac{4}{6}\right)^n + \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n = \sum a_n + \sum b_n$$

where $a_n = \left(\frac{4}{6}\right)^n$ *and* $b_n = \left(\frac{5}{6}\right)^n$

For $\sum a_n$, $r = \left(\frac{4}{6}\right) < 1$, hence $\sum a_n$ is convergent. And $\sum a_n = \frac{\left(\frac{4}{6}\right)}{\left(1-\frac{4}{6}\right)} = \frac{4}{2} = 2.$

Similarly, for $\sum b_n$, $r = \left(\frac{5}{6}\right) < 1$ so $\sum b_n$ is also convergent.

And $\sum b_n = \frac{\left(\frac{5}{6}\right)}{\left(1-\frac{5}{6}\right)} = 5$

Thus, the sum of $\sum a_n + \sum b_n$ is also convergent. $i.e.\sum c_n$ is convergent.
Further, $\sum c_n = \sum a_n + \sum b_n = 2 + 5 = 7$

Exercise:

1) Find the sum of $\sum_{n\to 1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}}$
2) Find the sum of $\sum_{n\to 1}^{\infty}\frac{4^n+1}{6^n}$
3) $prove\ that\ 1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\cdots..converges\ and\ find\ its\ sum.$
4) $prove\ that\ 5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots..converges\ and\ find\ its\ sum.$

❖ <u>**P-Series Test**</u>

The Series $\sum_{n=0}^{\infty}\frac{1}{n^P}$ converges if $P>1$, and diverges if $P\le 1$

1. $\sum\frac{1}{x^3}$ *is cgt or dgt?*
2. $\sum\frac{1}{x^{-3}}$ *is cgt or dgt?*
3. $\sum\frac{1}{x}$ *is cgt or dgt?*
4. $\sum\frac{1}{x^{\frac{3}{4}}}$ *is cgt or dgt?*

❖ Zero test of Divergence (Divergence test):

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ must be divergent

Note: If $\lim_{n\to\infty} a_n = 0$ then nothing can be said about convergence of the series $\sum_{n=1}^{\infty} a_n$. We have to apply another test for convergence

Que. Test the convergence of following series

1) $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

Hence, by zero test, the series is divergent.

2) $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \cdots\infty$

Solution:

Here, $a_n = \sqrt{\frac{n}{n+1}}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \lim_{n\to\infty}\sqrt{\frac{n}{n\left(1+\frac{1}{n}\right)}} = \lim_{n\to\infty}\sqrt{\frac{1}{\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1}{(1+0)}}$$

$$= 1 \neq 0$$

Hence, by zero test, the series is divergent.

Que. Prove that $\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$ *is divergent.*

Solution:

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2-1}{n^2+1} = \lim_{n\to\infty}\frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)} = \frac{(1-0)}{(1+0)} = 1$$

$$\neq 0$$

Hence, by zero test, the series is divergent.

❖ The Integral Test:

Let $\{a_n\}$ be a sequence of positive terms.

Suppose that $a_n = f(n)$,where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N is a positive integer).

Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

If $\int_1^{\infty} f(x)\, dx = \textit{finite than it is convergent}.$ and if it is infinite than it is divergent.

Que. Test the convergence of $\sum \frac{1}{n \log n}\,; n \geq 2$

Solution:

Let $f(x) = \frac{1}{x \log x}\ ; x \geq 2$. Then $f(x)$ is a continuous, positive, decreasing function of x for all $x \geq 2$.

Also, $a_n = \frac{1}{n \log n} = f(n)\, ; n \geq 2$.

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{x \log x} dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \log x} dx = \lim_{b \to \infty} \int_2^{b} \frac{\left(\frac{1}{x}\right)}{\log x} dx$$

$$= \lim_{b \to \infty} \log\,(log x)$$

$$= \lim_{b \to \infty} [\log(\log b) - \log(\log 2)]$$

$$= \infty - 2 = \infty$$

$i.e. \int_2^{\infty} f(x)\, dx$ is divergent.

Hence, by integral test, the given series also diverges.

Que.. Test the convergence of

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Solution:

Let $f(x) = xe^{-x^2}\ ; x \geq 1$. Then $f(x)$ is a continuous, positive, decreasing function of x for all $x \geq 1$.

Also, $a_n = \ ne^{-n^2} = f(n)\, ; n \geq 1$.

$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} xe^{-x^2}\ dx = \lim_{b \to \infty} \int_1^{b} xe^{-x^2}\ dx = \lim_{b \to \infty} \int_1^{b^2} e^{-t} \left(\frac{dt}{2}\right)$$

$\left[\text{taking } x^2 = t \text{ we get } xdx = \dfrac{dt}{2} \text{ and } x = 1 \Rightarrow t = 1 \text{ and } x = b \Rightarrow t = b^2\right]$

$$= \frac{1}{2} \lim_{b\to\infty} \left[\frac{e^{-t}}{(-1)}\right]_1^{b^2}$$

$$= \frac{-1}{2} \lim_{b\to\infty} \left[e^{-b^2} - e^{-1}\right]$$

$= \frac{-1}{2}\left[0 - \frac{1}{e}\right] = \frac{1}{2e} =$ finite

$i.e. \int_2^\infty f(x)\,dx$ is convergent.

Hence, by integral test, the given series also converges.

Que. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n[1+\log^2 n]}$$

Solution:

Let $f(x) = \frac{1}{x[1+\log^2 x]}$; $x \geq 1$.Then $f(x)$ is a continuous, positive, decreasing function of x for all $x \geq 1$.

Also, $a_n = \frac{1}{n[1+\log^2 n]} = f(n)\,; n \geq 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x[1+\log^2 x]}\;dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x[1+\log^2 x]}\;dx$$

Taking $\log x = t$ then $\frac{1}{x}dx = dt$ and $x = 1 \Rightarrow t = \log 1 = 0$ and $x = b \Rightarrow t = \log b$

$$\therefore \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x[1+\log^2 x]}\;dx = \lim_{b\to\infty} \int_0^{\log b} \frac{1}{1+t^2}\;dt$$

$$= \lim_{b\to\infty} [\tan^{-1} t]_0^{\log b} = \lim_{b\to\infty} [\tan^{-1}(\log b) - \tan^{-1} 0]$$

$$= [\tan^{-1}\infty - \tan^{-1} 0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$i.e. \int_2^\infty f(x)\,dx$ is convergent.

Hence, by integral test, the given series also converges.

Que. Test the convergence of the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Sol: $\int_1^\infty x^2 e^{-x^3} dx = \lim_{b\to\infty} \int_1^b x^2 e^{-x^3} dx = \lim_{b\to\infty} -\frac{1}{3}\int_1^b -3x^2 e^{-x^3} dx =$

$= \lim_{b\to\infty} -\frac{1}{3}\,[e^{-x^3}]_1^b$

$=-\frac{1}{3}[e^{-\infty}-e^{-1}]$

$=-\frac{1}{3}\left[0-\frac{1}{e}\right]$

$=\frac{1}{3e}$= finite

By Integral test , it is convergent.

Example: Test the convergence of $\sum_{n\to1}^{\infty}\frac{1}{\sqrt{n}}$

Sol: $\int_1^{\infty} f(x)\,dx = \int_1^{\infty}\frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}\int_1^{b}\frac{1}{\sqrt{x}}dx = [2\sqrt{x}]_1^{\infty} = \infty - 2 = \infty$

By Integral test , it is divergent.

❖ **Direct Comparison Test**

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$,for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \ge d_n$ for all $n > N$, for some integer N.

❖ **Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $forall\ n \ge N$ (N an integer).

(a) If $\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0,$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) If $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(c) If $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Note: $b_n = \frac{Highest\ power\ term\ in\ numerator}{Highest\ power\ term\ in\ denomerator}$

Que.for what value of p does the series $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots$ is convergent?

Solution:

Here, $a_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{\left(1+\frac{1}{n}\right)}{n^{(p-1)}} = \frac{1}{n^{(p-1)}}\left(1+\frac{1}{n}\right).$

Let $b_n = \frac{1}{n^{(p-1)}}$. Then $\frac{a_n}{b_n} = \left(1+\frac{1}{n}\right)$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 + 0 = 1 \ \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$\sum b_n = \sum \frac{1}{n^{(p-1)}}$ converges for $p - 1 > 1$, $i.e.$ for $p > 2$ and it diverge otherwise.

$\therefore \sum a_n = \sum \frac{n+1}{n^p}$ converges for $p \geq 2$ and it diverge otherwise.

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{2n^2 + 2n}{5 + n^5}$$

Solution:

Here, $a_n = \frac{2n^2+2n}{5+n^5} = \frac{n^2\left(2+\frac{2}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$.

Let $b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)} = \frac{(2+0)}{(0+1)} = 2 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{2n^2+2n}{5+n^5}$ converges. [by comparison test]

Que: Test the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Sol: $a_n = \frac{\sqrt{n}}{n^2+1}$

$$b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} n^{\frac{3}{2}} \frac{\sqrt{n}}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = 1 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$$b_n = \frac{1}{n^{\frac{3}{2}}}, By\ p - series, p = \frac{3}{2} > 1, it\ is\ convergent.$$

By Limit comparision Test, $\sum a_n$ *is convergent*.

Que. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1} + \sqrt{n}}$$

Solution:

Here, $a_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}} = \frac{n^p}{n^{\frac{1}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{\frac{1}{2}-p}} \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$.

Let $b_n = \frac{n^p}{n^{\frac{1}{2}}} = \frac{1}{n^{\frac{1}{2}-p}}$. Then $\frac{a_n}{b_n} = \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)} = \frac{1}{\left(1+\sqrt{1+0}\right)} = \frac{1}{2} \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^{\frac{1}{2}-p}}$ is a $p-series$ which converges for $\frac{1}{2}-p > 1$, $i.e.$ for $p < -\frac{1}{2}$ and diverges otherwise.

$\therefore \sum a_n = \sum \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$ also converges for $p < -\frac{1}{2}$ and diverges otherwise.[by comparison test]

Que. Test the convergence of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

Solution:

Here, $a_n = \frac{1}{n.(n+1)} = \frac{1}{n^2} \frac{1}{\left(1+\frac{1}{n}\right)}$.

Let $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = \frac{1}{\left(1+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{(1+0)} = 1 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^2}$ is a $p-series$ with $p = 2 > 1.$ Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{n(n+1)}$ converges. [by comparison test]

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\cdots+n^2}$$

Solution:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{1}{n^3} \frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$= \frac{6}{n.n.n\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$(n^2+n)(2n+1) = (2n^3+n^2+2n^2+n) = 2n^3+3n^2+n$$

$$= n^3(2+\frac{3}{n}+\frac{1}{n^2})$$

Let $b_n = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{6}{(1+0)(2+0)} = 3 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p-series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

❖ <u>Ratio Test(D' Alembert Ratio Test)</u>

Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

Then (a) the series converges if $L < 1$

(b)the series diverges if $L > 1$,

(c) the test is fail if $L = 1$

Que. Test the convergence of a series $\sum \frac{1}{n!}$

Solution:

Here $a_n = \frac{1}{n!} \Longrightarrow a_{n+1} = \frac{1}{(n+1)!}$ and

$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots$

Solution:

Here $a_n = \frac{n}{(n+1)!}$

$$\Longrightarrow a_{n+1} = \frac{n+1}{(n+2)!} and \ \frac{a_{n+1}}{a_n} = \frac{n+1}{(n+2)!}\frac{(n+1)!}{n} = \frac{(n+1)!}{(n+2)(n+1)!}\frac{n+1}{n}$$

$$= \frac{1}{n+2}\left(1+\frac{1}{n}\right)$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{n+2}\left(1+\frac{1}{n}\right) = 0(1+0) = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=0}^{\infty}\frac{4^n-1}{3^n}$

Solution:

Here $a_n = \frac{4^n-1}{3^n}$

$$\Longrightarrow a_{n+1} = \frac{4^{n+1}-1}{3^{n+1}} and$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}-1}{3^{n+1}}\frac{3^n}{4^n-1} = \frac{3^n}{3^{n+1}}\frac{4^n\left(4-\frac{1}{4^n}\right)}{4^n\left(1-\frac{1}{4^n}\right)} = \frac{1}{3}\left(\frac{4-\frac{1}{4^n}}{1-\frac{1}{4^n}}\right)$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{4-\frac{1}{4^n}}{1-\frac{1}{4^n}}\right) = \frac{1}{3}\left(\frac{4-0}{1-0}\right) = \frac{4}{3} > 1$$

Hence, by ratio test, given series is divergent.

Que. Example: Test the convergence of the series $\sum_{n=0}^{\infty}\frac{n3^n(n+1)!}{2^n n!}$

Solution:

$$a_n = \frac{n3^n(n+1)!}{2^n n!}$$

$$= n(n+1)\left(\frac{3}{2}\right)^n$$

$$\Longrightarrow a_{n+1} = (n+1)(n+2)\left(\frac{3}{2}\right)^{n+1} \ and$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+2)\left(\frac{3}{2}\right)^{n+1}}{n(n+1)\left(\frac{3}{2}\right)^n} = \frac{(n+2)}{n}\left(\frac{3}{2}\right)$$

$$= \left(1+\frac{2}{n}\right)\left(\frac{3}{2}\right)$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\left(\frac{3}{2}\right) = (1+0)\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by ratio test, given series is divergent.

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\cdots+n^2}$$

Solution:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{6}{n(n+1)(2n+1)}$$

$$\Longrightarrow a_{n+1} = \frac{6}{(n+1)(n+2)(2(n+1)+1)} = \frac{6}{(n+1)(n+2)(2n+3)} \, and$$

$$\frac{a_{n+1}}{a_n} = \frac{6}{(n+1)(n+2)(2n+3)} \frac{n(n+1)(2n+1)}{6} = \frac{n(2n+1)}{(n+2)(2n+3)}$$

$$= \frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$= \frac{(2+0)}{(1+0)(2+0)} = 1$$

Hence, by ratio test fails.

We need to use some other test to check the convergence of the series.

Using comparison test as follows:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)} = \frac{1}{n^3} \frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let $b_n = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{6}{(1+0)(2+0)} = 3 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

Que. Test the convergence of the series $2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$

Solution: Here $a_n = \frac{n+1}{n} x^{n-1}$

$$\Rightarrow a_{n+1} = \frac{n+2}{n+1}x^n \text{ and } \frac{a_{n+1}}{a_n} = \frac{(n+2)x^n}{n+1}\frac{n}{(n+1)x^{n-1}} = \frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}x$$

$$= \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}x$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}x$$

$$= \frac{(1+0)}{(1+0)^2}x = x$$

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x > 1$

For $x = 1$.

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = (1+0)$$

$$= 1 \neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x \geq 1$

❖ <u>Root Test (Cauchy Root Test)</u>

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ for some N and suppose that

$$\lim_{n\to\infty}|a_n|^{1/n} = L$$

Then (a) the series converges if $L < 1$

(b) the series diverges if $L > 1$,

(c) the test fails if $L = 1$

Que. Test the convergence of series $\sum_{n=1}^{\infty}\frac{3^n}{2^{n+3}}$

Solution:

$$a_n = \frac{3^n}{2^{n+3}} = \frac{1}{8}\left(\frac{3}{2}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\frac{1}{8}\left(\frac{3}{2}\right)^n\right|^{\frac{1}{n}} = \frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{8^0}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by root test, given series is divergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+5}\right)^n$

Solution:

$$a_n = \left(\frac{n}{2n+5}\right)^n = \left(\frac{1}{2+\frac{5}{n}}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{1}{2+\frac{5}{n}}\right)^n\right|^{\frac{1}{n}} = \left(\frac{1}{2+\frac{5}{n}}\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{2+\frac{5}{n}}\right) = \left(\frac{1}{2+0}\right)$$

$$= \frac{1}{2} < 1$$

Hence, by root test, given series is convergent.

Que: $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots ..$

Sol: $\boldsymbol{a_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}}$

$$L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n}\right) - \frac{n+1}{n}\right]^{-1}$$

$$= \left[\left(\frac{1+\frac{1}{n}}{1}\right)^n\left(\frac{1+\frac{1}{n}}{1}\right) - \frac{1+\frac{1}{n}}{1}\right]^{-1}$$

$$= [e.1-1]^{-1}$$

$= \frac{1}{e-1}$<1

Hence, by root test, given series is convergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Solution:

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}\right|^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\left(n^{\frac{3}{2}}\right)(n^{-1})} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1}$$

$$(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x} \quad (any x))$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1} = (e^{1})^{-1} = \frac{1}{e} < 1$$

Hence, by root test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+2}{n+3}\right)^{n} x^{n}$

Solution:

$$a_n = \left(\frac{n+2}{n+3}\right)^{n} x^{n}$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{n+2}{n+3}\right)^{n} x^{n}\right|^{\frac{1}{n}} = \left(\frac{n+2}{n+3}\right)x$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)x = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$= \left(\frac{1+0}{1+0}\right)x = x$$

Hence, by root test, given series is (i) convergent if $x < 1$

(ii) divergent if $x > 1$.

For $x = 1$.

$$a_n = \left(\frac{n+2}{n+3}\right)^{n} = \left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^{n} = \frac{\left(1+\frac{2}{n}\right)^{n}}{\left(1+\frac{3}{n}\right)^{n}}$$

$$\Rightarrow \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^{n}}{\left(1+\frac{3}{n}\right)^{n}} = \frac{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}}{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{n}} = \frac{(e)^{2}}{(e)^{3}} = \frac{1}{e}$$

$$\neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by root test, given series is (i) convergent if $x < 1$

(ii) divergent if $x \geq 1$.

Alternative series

A series in which the terms are alternatively positive and negative is called an alternating Series. $e.g.\ 1 - 4 + 9 - 16 + \cdots$

❖ **Leibnitz Test**

The infinite Series $a_1 - a_2 + a_3 - \dots$ in which the terms are alternatively positive and negative is convergent if (i) $a_n \geq a_{n+1}$ i.e. series is decreasing (ii) $\lim_{n\to\infty} a_n = 0$

Note: If $\lim_{n\to\infty} a_n \neq 0$ **then** $\sum_{n=1}^{\infty} a_n$ **is oscillatory.**

The series is Alternative

if $|a_n|$ is convergent & a_n is convergent then series is **absolutely convergent**

if $|a_n|$ is divergent but a_n is convergent then series is **Conditionally convergent**

Que. Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}}{n}$ $\quad u_{n+1} = \frac{(-1)^{n+2}}{n+1}$

$|u_n| = \frac{1}{n}$ $\quad |u_{n+1}| = \frac{1}{n+1}$

1)

$$|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$$
$$= \frac{n+1-n}{n(n+1)}$$
$$= \frac{1}{n(n+1)} \succ 0$$

$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$

Thus each term is less than its preceding term.

Now

2)

$$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{1}{n} = 0$$

Thus by Leibnitz's test the alternating series is convergent.

Que. Test the convergence of the series $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}(n+1)}{n} \quad u_{n+1} = \frac{(-1)^{n+2}(n+2)}{n+1}$

$$|u_n| = \frac{(n+1)}{n} \quad |u_{n+1}| = \frac{n+2}{n+1}$$

1)

$$|u_n| - |u_{n+1}| = \frac{n+1}{n} - \frac{n+2}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \succ 0$$

$$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Thus each term is less than its preceding term.

Now

2)

$$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{n+1}{n}$$

$$= \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n}$$

$$= 1 \neq 0$$

Thus by Leibnitz's test the alternating series is oscillating.

Que. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^n x^{n+1}}{2n-1}$

Solution:

$$u_n = \frac{(-1)^{n+1}x^{n+1}}{2n-1} \quad u_{n+1} = \frac{(-1)^{n+2}x^{n+2}}{2n+1}$$

$$|u_n| = \frac{x^{n+1}}{2n-1} \quad |u_{n+1}| = \frac{x^{n+2}}{2n+1}$$

1)

$$|u_n| - |u_{n+1}| = \frac{x^{n+1}}{2n-1} - \frac{x^{n+2}}{2n+1}$$

$$= \frac{(2n+1)x^{n+1} - x^{n+2}(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{x^{n+1}[(2n+1)-(2n-1)x]}{(4n^2-1)} \succ 0$$

$$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Now

2)

$$n \xrightarrow{\lim} \infty |u_n| = n \xrightarrow{\lim} \infty \frac{x^{n+1}}{2n-1}$$

$$= 0 \qquad if \quad x < 1$$

Thus by Leibnitz's test the alternating series is convergent.

Que : Determine absolute or conditional convergence of the series $\sum_{n=1}^{\infty}(-1)^n . \frac{n^2}{n^3+1}$

Solution:

$$\text{Let} \quad u_n = (-1)^n . \frac{n^2}{n^3+1}$$

$$|u_n| = \frac{n^2}{n^3+1}$$

$$= \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{1^3}} = 1 \; [\,finite\ and\ nonzero]$$

And $\sum v_n = \sum \frac{1}{n}$ is divergent as p = 1.

By comparison test $\sum|u_n|$ is also divergent.

Hence, $\sum u_n$ is not absolutely convergent. To check conditional convergence applying Leibnitz's test.

(i)

$$|u_n| - |u_{n+1}| = \frac{n^2}{n^3+1} - \frac{(n+1)^2}{(n+1)^3+1}$$

$$= \frac{n^2(n^3+3n^2+3n+2) - (n^3+1)(n^2+2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+2n^3+n^2-2n-1}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4 + n^2(2n+1) - 1(2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$= \frac{n^4+(2n+1)(n^2-1)}{(n^3+1)[(n+1)^3+1]} > 0$ for all nϵ N.

$|u_n| > |u_{n+1}|$

(ii)

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n^3}\right)} = 0.$$

By Leibnitz's test, $\sum u_n$ is convergent. The series $\sum u_n$ is convergent and test $\sum |u_n|$ is divergent. Hence, the series is conditionally convergent.

Que.. Test the convergence of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ ...

Sol: $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}, a_{n+1} = \frac{(-1)^n}{\sqrt{n+1}},$

$$|a_n| = \frac{1}{\sqrt{n}}, |a_{n+1}| = \frac{1}{\sqrt{n+1}}$$

i) $|a_n| - |a_{n+1}| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} > 0$

$$|a_n| > |a_{n+1}|$$

ii) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

By Leibnitz's test, $\sum u_n$ is convergent. The series $\sum u_n$ is convergent

$$|a_n| = \frac{1}{\sqrt{n}}$$

By p-series p=$\frac{1}{2} < 1$, $\sum |a_n|$ is divergent.

Hence, the series is conditionally convergent.

❖ <u>Power Series:</u>

A power series about$x = a$ is given as

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

If $a = 0$, then the power series in powers of x is given by

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$wherec_0, c_1, c_2, \ldots arerealnumbers.$$

Note: For a power series in powers of $(x-a)$

(1) Apply D'Alembert Ratio Test or Cauchy's n[th] root test

$$i.e.\ \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = l, then\ R = \frac{1}{l} is \text{ called radius of convergence}$$
$$and\ |x-a| < R \text{ gives interval of convergence.}$$

$$i.e.\ \lim_{n\to\infty}(|c_n|)^{\frac{1}{n}} = l, then\ R = \frac{1}{l}\text{ is called radius of convergence}$$
$$and\ |x-a| < R \text{ gives interval of convergence.}$$

(2) $l = \infty \Longrightarrow R = 0 \Longrightarrow$series converges only at $x = a$.

(3) $l = 0 \Longrightarrow R = \infty \Longrightarrow$series converges for all x.

(4) l is finite and non-zero $\Rightarrow r = \frac{1}{l}$ and the interval of convergence is $(a - R, a + r)$.

Que. $\boldsymbol{Find\ the\ radius\ of\ convergence\ of}\ \sum_{n=0}^{\infty} n!\, x^n$

$a_n = n!\, x^n$

$a_{n+1} = (n+1)!\, x^{(n+1)}$

By Ratio Test,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!\, x^{(n+1)}}{n!\, x^n}$$
$$= \lim_{n\to\infty}(n+1)x \quad = \infty$$
$$Hence\ ,\ l = \infty \to R = 0$$
$$Radius\ of\ convegence\ is\ 0$$
$$series\ converges\ at\ x = 0$$

Que. Determine the interval of convergence for the following series and also their behaviour at each end points.

$$\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$$

Solution:

$$u_n = \frac{2^n x^n}{n!}$$
$$u_{n+1} = \frac{2^{n+1}x^{n+1}}{(n+1)!}$$
$$\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!}\frac{2^{n+1}x^{n+1}}{2^n x^n}$$
$$= \frac{2x}{n+1}$$
$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| \qquad = 0$$

Hence, the series is convergent for all values of x i.e., $-\infty < x < \infty$ and interval of convergence is $(-\infty, \infty)$

Que. Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$, $x > 0$.

Sol:
$$u_n = \frac{x^n}{2^n}$$
$$u_{n+1} = \frac{x^{n+1}}{2^{n+1}}$$
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{2^{n+1}} \frac{2^n}{x^n} = \frac{x}{2}$$

By D'Alembert's ratio test,the series is

convergent if $\frac{x}{2} < 1$ or $x < 2$.

divergent if $\frac{x}{2} > 1$ or $x > 2$.

The test fails if $\frac{x}{2} = 1$ or $x = 2$

Then $u_n = \frac{x^n}{2^n} = 1$
$$\sum_{n=1}^{\infty} u_n = 1 + 1 + 1 + \cdots \infty$$

Which is a divergent series.

Hence, series is convergent for $0 < x < 2$ and the range of convergence is $0 < x < 2$.

Que. Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n . n}$.

SOLUTION:

Let $u_n = \frac{(x+1)^n}{3^n . n}$
$$u_{n+1} = \frac{(x+1)^{n+1}}{3^{n+1}.(n+1)}$$
$$\frac{u_{n+1}}{u_n} = \frac{(x+1)^{n+1}}{3^{n+1}(n+1)} . \frac{3^n(n)}{(x+1)^n}$$
$$= \frac{x+1}{3\left(1+\frac{1}{n}\right)}$$
$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x+1}{3\left(1+\frac{1}{n}\right)}\right|$$
$$= \left|\frac{x+1}{3\left(1+\frac{1}{n}\right)}\right| = \left|\frac{x+1}{3}\right|$$

The series is convergence if
$$\left|\frac{x+1}{3}\right| < 1$$
$$|x+1| < 3$$
$$-3 < (x+1) < 3$$
$$-4 < x < 2$$

At x=2 $u_n = \frac{1}{n}$

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as p=1.

At x=-4 $u_n = \frac{(-1)^n}{n}$

$|u_n| = \frac{1}{n}$

The given series is an alternating series.

$(i)|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1} > 0$

$$|u_n| > |u_{n+1}|$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus,By Leibnitz test , the series is convergent at x = -4

Hence , the series is convergent for -4<x<2 and the range of convergence is [-4,2).

Taylor's Series: If *f(x+a)* is a differential function of *x* up to n[th] order and *a* is a constant, then *f(x)* can be expanded into a power series of *x* as follows

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + + \frac{x^3}{3!}f'''(0) \ldots \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$

Maclaurin's Series: If the fuction *f(x)* is differentiable n times at *x=0*, then it can be expanded into finite series in ascending power of x as follows

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(2)$$

Case 1: Put a=0 in (1), we get Macluarin's series

Case2: Take *a=x* and *x=h* in (1)

$$f(x+h) = f(x) + hf'(h) + \frac{h^2}{2!}f''(x) + \cdots \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(3)$$

Case3: take a for *x* and $x - a$ for *h* in (3)

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \quad \text{\_\_\_\_\_\_\_\_\_\_}(4)$$

Some special series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$a^x = 1 + xloga + \frac{x^2}{2!}(loga)^2 + \frac{x^3}{3!}(loga)^2 + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right]$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

Que. Find the Maclaurin's series for the function $f(x) = \frac{1}{\sqrt{4-x}}$

Solution:

$$f(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-1/2} \qquad \Rightarrow f(0) = 1/2$$

$$f'(x) = \frac{1}{2}(4-x)^{-3/2} \qquad \Rightarrow f'(0) = \frac{1}{2^4}$$

$$f''(x) = \frac{1}{2}\times\frac{3}{2}(4-x)^{-5/2} \qquad \Rightarrow f'(0) = \frac{1}{2^5}\times\frac{3}{4}$$

$$f'''(x) = \frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}(4-x)^{-7/2} \qquad \Rightarrow f'(0) = \frac{1}{2^7}\times\frac{15}{8}$$

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}f^{n}(0) + \ldots\ldots\ldots$$

$$= \frac{1}{2}. + x\left(\frac{1}{2^4}\right) + x^2\left(\frac{3}{2^5\times 4}\right) + x^3\left(\frac{15}{2^7\times 8}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Que. find the Taylor series expansion of $f(x) = x^3 - 2x + 4$ **about a=2.**

Solution:

$$f(x) = x^3 - 2x + 4 \qquad \Rightarrow f(2) = 8$$

$$f'(x) = 3x^2 - 2 \qquad \Rightarrow f'(2) = 10$$

$$f''(x) = 6x \qquad \Rightarrow f'(2) = 12$$

$$f'''(x) = 6 \qquad \Rightarrow f'(2) = 6$$

$$f(x)=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+\dots\dots\dots\dots$$

$$=8+10(x-2)+\frac{1}{2}(x-2)^2\times 12+\frac{1}{6}(x-2)^3\times 6$$

$$=8+10(x-2)+6(x-2)^2+(x-2)^3$$

Que. Expand $\sin\left(\frac{\pi}{4}+x\right)$ in power of x. hence find the value of $\sin 44^\circ$.

Solution: Suppose $f(x)=\sin x,\quad h=\frac{\pi}{4}$

Now by Taylor series expansion

$$f(x+h)=f(h)+x\,f'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+\dots\dots\dots\dots+\frac{x^n}{n!}f^n(h)+\dots$$

$$now\ f(x)=\sin x \qquad \Rightarrow f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}.=\frac{1}{\sqrt{2}}$$

$$f'(x)=\cos x \qquad \Rightarrow f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}.=\frac{1}{\sqrt{2}}$$

$$f''(x)=-\sin x \qquad \Rightarrow f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}.=\frac{-1}{\sqrt{2}}$$

$$f'''(x)=-\cos x \qquad \Rightarrow f'''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}.=\frac{-1}{\sqrt{2}}$$

$$f\left(x+\frac{\pi}{4}\right)=\sin\left(x+\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}\right)+x\,f'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)+\dots\dots\dots\dots$$

$$=\frac{1}{\sqrt{2}}+x\frac{1}{\sqrt{2}}+\frac{x^2}{2!}\left(\frac{-1}{\sqrt{2}}\right)+\frac{x^3}{3!}\left(\frac{-1}{\sqrt{2}}\right)+\dots\dots\dots\dots$$

$$=\frac{1}{\sqrt{2}}\left[1+x-\frac{x^2}{2}+\frac{x^3}{6}+\dots\dots\right]$$

$$now \qquad x=-1^\circ=\frac{-\pi}{180}=\frac{-3.14}{180}=-0.0175$$

$$\therefore \sin(45^\circ-1)=\frac{1}{\sqrt{2}}\left[1-0.0175-\frac{1}{2}(0.0175)^2+\dots..\right]\approx 0.6946$$

Que. Prove that $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots..$

Solution

Let $y=tanx$

$$\frac{dy}{dx}=\frac{1}{1+x^2}=(1+x^2)^{-1}=1-x^2+x^4-x^6+\cdots \qquad (1)$$

Integrating equation (1),

$$y=c+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

$$\tan^{-1}x=c+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$$

Putting $x = 0$

$$\tan^{-1} 0 = c$$
$$c = 0$$

Hence ,$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots ..$

Que. Prove that $\boldsymbol{\sin^{-1} \left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}$.

Solution :

let$y = \sin^{-1} \left(\frac{2x}{1+x^2}\right)$

putting $x = tan\theta$

$y = \sin^{-1} \left(\frac{2tan\theta}{1 + \tan^2 \theta}\right)$

$=\sin^{-1}(sin2\theta)$ $=2\theta$ $=2\tan^{-1} x$ $= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$

Hence $\boldsymbol{\sin^{-1} \left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}$

Que.Expand the function

$f(x) = x^4 - 11x^3 + 43x^2 - 60x + 14$ *in power of* $(x-3)$

Que..Find the Taylor series for $f(x) = \frac{1}{x}$ *at* $a = 2$.

Que. Find the Taylor series expansion of $f(x) = \tan x$ in power of $\left(x - \frac{\pi}{4}\right)$,showing at least four non zero terms. Hence find the value of $f(x) = \tan 46^\circ$.

Que. $5 + 4(x-1)^2 - 3(x-1)^3 + (x-1)^4$ **in ascending powers of x.**

Que. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2 -1}}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots\dots\right)$

Que. $\log(1+e^x)$ **in ascending power of x s far as term containing** x^4**.**

Que. Obtain Maclaurin's series of 1)$f(x) = sin^{-1} x$ 2) $y = e^{-x}$