FACULTY OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF APPLIED SCIENCE AND HUMANITIES

4th SEMESTER B.TECH PROGRAMME

PROBABILITY, STATISTICS AND NUMERICAL METHODS

(203191251)

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Finite Differences:

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, ...x_0 + nh$ giving $y = y_0, y_1, y_2, ...y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful:

(1) Forward Differences: The differences $y_1 - y_0, y_2 - y_1, ...y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, ..., \Delta y_{n-1}$ respectively are called the first forward differences where Δ is the forward difference operator. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$

Similarly, the second forward differences are defined by $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ defines the p th forward differences.

Forward difference table

Values of x	Values of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0	Δy_0	$\Delta^2 y_0$			
$x_0 + h$	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$	
$x_0 + 2h$	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_0$
$x_0 + 3h$	y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_2$		
$x_0 + 4h$	y_4	Δy_4				

x_0+5h	y_5

(2) Backward Differences: The differences $y_1-y_0, y_2-y_1, ...y_n-y_{n-1}$ when denoted by $\nabla y_0, \nabla y_1, \nabla y_2, ..., \nabla y_{n-1}$ respectively are called the first Backward differences where ∇ is the Backward difference operator. Similarly we defined higher order backward differences. Thus we have $\nabla y_r = y_r - y_{r-1}$.

In general, $\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1}$ defines the p th forward differences.

Backward difference table

Values of x	Values of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		∇y_1				
x_0+h	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_0+2h	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_0+3h	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_0+4h	y_4		$\nabla^2 y_5$			
		∇y_5				
x_0+5h	y_5					

(3) Central Differences:

Sometime it is convenient to employ another system of differences known as central differences. In this system, the central difference operator δ is defined by the relations:

$y_1-y_0=\delta y_{1/2}, y_2-y_1=\delta y_{3/2}, ..., y_n-y_{n-1}=\delta y_{n-1/2}$

Values of x	Values of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		$\delta y_{1/2}$				
x_0+h	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_0+2h	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_0+3h	y_3		$\delta^2 y_3$		$\delta^4 y_3$	

		$\delta y_{7/2}$		$\delta^3 y_{7/2}$
x_0+4h	y_4		$\delta^2 y_4$	
		$\delta y_{9/2}$		
x_0+5h	y_5			

Relationship between operators:

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(1) $\Delta\nabla = \Delta - \nabla = \nabla\Delta$

Solution:

$$\begin{aligned}\Delta\nabla f(x) &= \Delta[\nabla f(x)]\\ &= \Delta[f(x)-f(x-h)]\\ &= \Delta f(x) - \Delta f(x-h)\\ &= f(x+h)-f(x)-[f(x)-f(x-h)]\\ &= \Delta f(x) - \nabla f(x)\\ &= (\Delta-\nabla)f(x)\end{aligned}$$

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(2) $\Delta+\nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

Solution:

$$\begin{aligned}\frac{\Delta}{\nabla}-\frac{\nabla}{\Delta} &= \frac{\Delta^2-\nabla^2}{\nabla\Delta}\\ &= \frac{(\Delta-\nabla)(\Delta+\nabla)}{\Delta-\nabla}\\ &= \Delta-\nabla\end{aligned}$$

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(3) The shift operator E **:**

The operator E is defined as

$Ef(x) = f(x+h)$

$E^2 f(x) = E[Ef(x)] = f(x+2h)$

.

.

.

$E^n f(x) = f(x+nh)$

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(4) $E = 1+\Delta$

Solution:

$$\begin{aligned}Ef(x) &= f(x+h)\\ &= f(x+h)-f(x)+f(x)\\ &= \Delta f(x) + f(x)\\ &= (\Delta+1)f(x)\end{aligned}$$

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(5) $E\nabla = \Delta$

Solution:

$$\begin{aligned}E\nabla f(x) &= E(\nabla f(x))\\ &= E(f(x)-f(x-h))\\ &= Ef(x) - Ef(x-h)\\ &= f(x+h)-f(x)\\ &= \Delta f(x)\end{aligned}$$

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(6) $(1+\Delta)(1-\nabla) = 1$

Solution:

$$\begin{aligned}(1+\Delta)(1-\nabla) &= 1-\nabla+\Delta-\Delta\nabla\\ &= 1-\nabla+\Delta-(\Delta-\nabla)\\ &= 1-\nabla+\Delta-\Delta+\nabla\\ &= 1\end{aligned}$$

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(7) $\nabla = 1-E^{-1}$

Solution:

$$\begin{aligned}1-E^{-1} &= 1-(1+\Delta)^{-1}\\ &= 1-\frac{1}{1+\Delta}\\ &= \frac{1+\Delta-1}{1+\Delta} = \frac{E\nabla}{E} = \nabla\end{aligned}$$

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(8) $E = e^{hD}$

Solution:

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	$Ef(x) = f(x+h)$ $= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + ...$ $= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + ...$ $= \left[1 + hD + \frac{h^2 D^2}{2!} + ...\right] f(x)$ $= e^{hD} f(x)$

- **Relations between the various operators:**

In terms of	E	Δ	$\boldsymbol{\nabla}$	$\boldsymbol{\delta}$	hD
E	------	$\Delta + \mathbf{1}$	$(\mathbf{1} - \boldsymbol{\nabla})^{-\mathbf{1}}$		
Δ		------	$(\mathbf{1} - \boldsymbol{\nabla})^{-\mathbf{1}} - \mathbf{1}$		
$\boldsymbol{\nabla}$		$\mathbf{1} - (\Delta + \mathbf{1})^{-\mathbf{1}}$	-----		
$\boldsymbol{\delta}$		$\Delta(\Delta + \mathbf{1})^{-\mathbf{1/2}}$		-----	
$\boldsymbol{\mu}$		$\left(\mathbf{1} + \frac{\Delta}{\mathbf{2}}\right)(\Delta + \mathbf{1})^{-\mathbf{1/2}}$			
hD					--------

Example: Prove with the usual notations, that

(1) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

(2) $\Delta - \nabla = \Delta\nabla = \delta^2$

Solution:(1) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})\, E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta$

(2)we know that $\Delta = E - 1, \nabla = 1 - E^{-1}$ and $\delta = E^{1/2} - E^{-1/2}$

$$\Delta - \nabla = \mathrm{E} - 2 + \mathrm{E}^{-1} = \left(\mathrm{E}^{\frac{1}{2}}\right)^2 - 2\left(\mathrm{E}^{\frac{1}{2}}\right)\left(\mathrm{E}^{-\frac{1}{2}}\right) + \left(\mathrm{E}^{-\frac{1}{2}}\right)^2$$

$$= \left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right)^2 = \delta^2 ----\text{(i)}$$

$$\Delta\nabla = (\mathrm{E} - 1)(1 - \mathrm{E}^{-1}) = \mathrm{E} + \mathrm{E}^{-1} - 2 = \left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right)^2 = \delta^2 ----\text{(ii)}$$

From (i) and (ii) we get, $\Delta - \nabla = \Delta\nabla = \delta^2$

EXAMPLE: Write forward difference table if

x:	10	20	30	40
y:	1.1	2.0	4.4	7.9

Solution:

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1			
		0.9		
20	2.0		1.5	
		2.4		-0.4
30	4.4		1.1	
		3.5		
40	7.9			

INTERPOLATION

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, y_3, \ldots\ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n$ of x. The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. Thus, interpolation is a technique of finding the value of a function for any intermediate value of the independent variable.

The process of computing the value of the function outside the range of given values of the variable is called extrapolation. The study of interpolation is based on the concept of finite differences which were discussed later.

Newton's forward interpolation formula:

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, y_3, \ldots\ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$y_r = f(x_0 + rh)$

$= E^r f(x_0)$

$= (1+\Delta)^r f(x_0)$

$= (1+\Delta)^r y_0$

$= \left[1 + r\Delta + \frac{r(r-1)}{2!}\Delta^2 + \frac{r(r-1)(r-2)}{3!}\Delta^3 + \ldots\ldots\ldots\right] y_0$

[Using Binomial theorem]

$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

The above formula is known as Newton's forward interpolation formula.

Example 1: From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
Number of students	31	42	51	35	31

Solution:

Marks less than (x)	40	50	60	70	80
No. of students (f(x))	31	73	124	159	190

X	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We shall find y_{45} i.e. number of students with marks less than 45. Taking $x_0 = 40, x = 45,$ we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad (\because h = 10)$$

Newton's forward interpolation formula,

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \text{ x } 42 + \frac{0.5(-0.5)}{2} \text{ x } 9 + \frac{0.5(-0.5)(-1.5)}{6} \text{ x } (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \text{ x } 37$$

$$= 47.87$$

The number of students with marks less than 45 is 47.87 i.e. 48. But the number of students with marks less than 40 is 31. Hence the number of students getting marks between 40 and 45 = 48-31=17.

Example 2: The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

$x = height$	100	150	200	250	300	350	400
$y = disance$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

1. $\boldsymbol{x = 160ft}$
2. $\boldsymbol{x = 410}$

Solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	Δ	Δ^2	Δ^3	Δ^4
100	10.63	2.40	-0.39	0.15	-0.7
150	**13.03**	**2.01**	**-0.24**	**0.08**	**-0.05**
200	15.04	1.77	-0.16	0.03	-0.01
250	16.81	1.61	-0.13	**0.02**	
300	18.42	1.48	**-0.11**		
350	19.90	**1.37**			
400	**21.27**				

1. If we take $x_0 = 160 \; than \; y_0 = 13.03, \Delta y_0 = 2.01, \Delta^2 y_0 = -0.24, \Delta^3 = 0.08, \Delta^4 y_0 = -0.05$

Since $x = 160 \;\; and \; h = 150$

$$\therefore p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$$

∴ Using Newton’s forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots.$$

$$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46\ \mathit{bautical\ miles}$$

2. Since $x = 410$ is near the end of the table, we use Newton’s backward interpolation formula.

$$\therefore\ taking\ x_n = 400\ , p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$$

Using the line backward difference,

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02\ \ etc$$

$\therefore\ Newton'sbackward\ formula\ gives,$

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots.$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + .274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53\ \mathit{nautical\ miles}.$$

Example 3: Find the cubic polynomial which takes the following values:

X	0	1	2	3
f(x)	1	2	1	10

Hence or otherwise evaluate f(4).

Solution: The difference table is

X	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		2	
		-1		12
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \frac{x-0}{h} = x \qquad (\because h = 1)$

$$f(x) = f(0) + p\Delta f(0) + \frac{p(p-1)}{2!}\Delta^2 f(0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12) = 2x^3 - 7x^2 + 6x + 1$$ which is the required polynomial.

To compute f(4) we take $x_n = 3, x = 4$ so that $p = \frac{x - x_n}{h} = 1 \qquad (\because h = 1)$

($\because$ Newton's backward interpolation formula)

$$f(4) = f(3) + p\Delta f(3) + \frac{p(p+1)}{2!}\Delta^2 f(3) + \frac{p(p+1)(p+2)}{3!}\Delta^3 f(3)$$

= 10+9+10+12 = 41.

Example 4: Using Newton's forward interpolation formula, find the value of $f(1.6)$.

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

Solution:

$x=1.6,\ x_0=1\ ,h=0.4$

Let $r=\dfrac{x-x_0}{h}=\dfrac{1.6-1}{0.4}=1.5$

Difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

By Newton's forward interpolation formula,

$$y_r=y_0+r\Delta y_0+\frac{r(r-1)}{2!}\Delta^2 y_0+\frac{r(r-1)(r-2)}{3!}\Delta^3 y_0+..........$$

$$y_{1.6}=f(1.6)=3.49+(1.5)(1.33)+\frac{(1.5)(1.5-1)}{2!}(-0.19)+\frac{(1.5)(1.5-1)(1.5-2)}{3!}(-0.41)$$
$$=5.4393$$

<u>Newton's Backward interpolation formula:</u>

Let the function $y=f(x)$ take the values $y_0,y_1,y_2,y_3,................,y_n$ corresponding to the values $x_0,x_1,x_2,x_3,................,x_n$ of x. Suppose it is required to evaluate $f(x)$ for $x=x_0+rh$, where r is any real number.

$$\begin{aligned}y_r&=f(x_n+rh)\\&=E^r f(x_n)\\&=(E^{-1})^{-r} f(x_n)\\&=(1-\nabla)^{-r} f(x_n)\\&=\left[1+r\nabla+\frac{r(r+1)}{2!}\nabla^2+\frac{r(r+1)(r+2)}{3!}\nabla^3+..........\right]y_n\end{aligned}$$

[Using Binomial theorem]

$$=y_n+r\nabla y_n+\frac{r(r+1)}{2!}\nabla^2 y_n+\frac{r(r+1)(r+2)}{3!}\nabla^3 y_n+..........$$

$$y_r=y_n+r\nabla y_n+\frac{r(r+1)}{2!}\nabla^2 y_n+\frac{r(r+1)(r+2)}{3!}\nabla^3 y_n+..........$$

The above formula is known as Newton's backward interpolation formula.

Example4: Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data: f(-0.75)=-0.0718125, f(-0.5)= -0.02475, f(-0.25)= 0.3349375, f(0)=1.10100.
Hence find f(-1/3).
Solution: The difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-0.75	-0.0718 125			
		0.0470625		
-0.5	-0.02475		0.312625	
		0.3596875		0.09375
-0.25	0.3349375		0.400375	
		0.7660625		
0	1.10100			

We use Newton's backward difference formula

$$y(x) = y_3 + p\Delta y_3 + \frac{p(p+1)}{2!}\Delta^2 y_3 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_3$$

$$x_3 = 0 \text{ and } p = \frac{x-0}{h} = \frac{x}{0.25} = 4x \qquad (\because h = 0.25)$$

$$y(x) = 1.10100 + 4x\,0.7660625 + \frac{4x(4x+1)}{2!}0.400375 + \frac{4x(4x+1)(4x+2)}{3!}0.09375$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101$$

Put $x = \frac{-1}{3}$

$$y(\frac{-1}{3}) = (\frac{1}{3})^3 + 4.001(\frac{-1}{3})^2 + 4.002(\frac{-1}{3}) + 1.101$$

$$= 0.1745$$

Example 5: Consider the following tabular values:

x	140	150	160	170	180
$y = f(x)$	3685	4845	6302	8076	10225

Determine $y(175)$ using Newton's backward interpolation formula.

Solution:

Let

$$x = 175,\ x_n = 180\ \ , h = 10$$

$$r = \frac{x - x_n}{h} = \frac{175-180}{10} = -0.5$$

Difference Table:

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
140	3685				
		1169			
150	4845		279		
		1448		47	
160	6302		326		2
		1774		49	
170	8076		375		
		2149			
180	10225				

By Newton's backward formula,

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \dots\dots\dots\dots$$

$$y_{175} = 10225 + (-0.5)(2149) + \frac{(-0.5)(-0.5+1)}{2!}(375) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(49)$$
$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(2)$$
$$= 9100.4844$$

<u>Gauss's forward interpolation formula:</u>

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

Where

$$r = \frac{x - x_0}{h}$$
$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2(1+\Delta)y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3(1+\Delta)y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 E y_{-1} = \Delta^4(1+\Delta)y_{-1} = \Delta^4 y_{-1} + \Delta^5 y_{-1}$$

Also,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.}$$

Substituting the values of $\Delta^2 y_0, \Delta^3 y_0, ...$in Eq.(1)

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + ...$$

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + ...$$

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \left[\frac{r(r-1)}{2!}\Delta^3 y_{-1} + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_{-1}\right]$$

$$+ \left[\frac{r(r-1)(r-2)}{3!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_{-1}\right] + ...$$

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + ...$$

....(1)

This is known as Gauss's forward interpolation formula.

Example: Use Gauss's forward formula to evaluate y_{30} given that y_{21} = 18.4708, y_{25} =17.8144, y_{29} =17.1070, y_{33} =16.3432 and y_{37} =15.5154.

Solution: Taking $x_0 = 29$**,** h=4**,** we require the value of y for x=30

$$p = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25$$

X	P	Yp	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
21	-2	18.4708				
			-0.6564			
25	-1	17.8144		-0.0510		
			-0.7074		-0.0074	
29	0	17.1070		-0.0564		-0.0022
			-0.7638		-0.0076	
33	1	16.3432		-0.0640		
			-0.8278			
37	2	15.5154				

Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-2} + ...$$

$y_{30} = 16.9216$approx

Example 6: Find $y(32)$ from the following table:

x	25	30	35	40
y	0.2707	0.3027	0.3386	0.3794

Solution:

Let

$$x = 32,\ x_0 = 30\ ,h = 5$$

$$r = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4$$

Central Difference Table:

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	-1	0.2707			
			0.0320		
30	0	0.3027		0.0039	
			0.0359		0.0010
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

By Gauss's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + ...$$

$$y(32) = 0.3027 + (0.4)(0.0359) + \frac{(0.4)(0.4-1)}{2!}(0.0039) + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(0.0010)$$

$$= 0.3165$$

<u>Gauss's backward interpolation formula:</u>

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + ...$$

....(1)

Where

$r = \dfrac{x - x_0}{h}$

$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1}$

$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$\Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3 (1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

$\Delta^4 y_0 = \Delta^4 E y_{-1} = \Delta^4 (1+\Delta) y_{-1} = \Delta^4 y_{-1} + \Delta^5 y_{-1}$

Also,

$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$, etc.

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, ...$in Eq.(1)

$$\begin{aligned} y_r &= y_0 + r(\Delta y_{-1} + \Delta^2 y_{-1}) y_0 + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + ... \\ &= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + ... \\ &= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + ... \end{aligned}$$

....(1)

This is known as Gauss's backward interpolation formula.

Example 7: From the following table, find y when $x = 38$.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Solution:

Let

$$x = 38,\ x_0 = 40\ \ , h = 5$$

$$r = \frac{x - x_0}{h} = \frac{38-40}{5} = -0.4$$

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	-2	15.9				
			-1			
35	-1	14.9		0.2		

			-0.8		-0.2	
40	0	14.1		0		0.2
			-0.8		0	
45	1	13.3		0		
			-0.8			
50	2	12.5				

By Gauss's backward interpolation formula,

$$y_r = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \dots\dots\dots$$

$$y(38) = 14.1 + (-0.4)(-0.8) + \frac{(-0.4+1)(-0.4)}{2!}(0) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(-0.2)$$
$$+ \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(0.2)$$

$$= 14.4133$$

Example 8: Interpolate by means of Gauss's backward formula, the population of a town for the year1974, given that:

Year	**1939**	**1949**	**1959**	**1969**	**1979**	**1989**
Population	**12**	**15**	**20**	**27**	**39**	**52**

Solution:

Taking $x_0 = 1969, h = 10$ the population of the town is to be found for $p = \frac{1974-1969}{10} = 0.5$

The central difference table is,

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	-3	12	3	2	0	3	
1949	-2	15	5	2			
1959	-1	20	7		3		-10
1969	0	27		5			
1979	1	39	12			-7	
1989	2	52	13	1	-4		

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_3$$

$$\therefore ie\ y_{0.5} = 27 + (0.5)(7) + \frac{(1.5)(0.5)}{2}(5) + \frac{(1.5)(0.5)(-0.5)}{6}(3) + \frac{(2.5)(1.5)(-0.5)}{24}(-7) + \frac{(2.5)(1.5)(0.5)(-0.5)(1.5)}{120}(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$

$$= 32.532\ thousands\ approx$$

Example: Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that:

Year:	1939	1949	1959	1969	1979	1989
Population:	12	15	20	27	39	52

<u>Stirling's interpolation formula:</u>

By Gauss's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \dots\dots\dots$$

By Gauss's backward interpolation formula,

$$y_r = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \dots\dots\dots\dots$$

Adding both equations and then dividing by 2.

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \dots\dots\dots\dots$$

This is known as Stirling's formula.

Example 8: Using Stirling's formula, find $y(25)$ from the following table:

x	20	24	28	32
y	0.01427	0.01581	0.01772	0.01996

Solution:

Let

$$x = 25,\ x_0 = 24\ \ , h = 4$$
$$r = \frac{x - x_0}{h} = \frac{25-24}{4} = 0.25$$

Central Difference Table:

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	0.01427			
			0.00154		
24	0	0.01581		0.00037	
			0.00191		-0.00004
28	1	0.01772		0.00033	
			0.00224		
32	2	0.01996			

By Stirling's formula,

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \dots\dots\dots\dots$$

$$y(25) = 0.01581 + (0.25)\left(\frac{0.00154 + 0.00191}{2}\right) + \frac{(0.25)^2}{2!}(0.00037) + \frac{(0.25)((0.25)^2 - 1)}{3!}\left(\frac{-0.00004}{2}\right)$$
$$= 0.01625$$

Example: Given

θ^0 : 0 5 10 15 20 25 30

$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula estimate the value of $\tan 16^0$.

INTERPOLATION WITH UNEQUAL INTERVALS:

If the values of x are unequally spaced then interpolation formula for equally spaced points cannot be used. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x . There are two such formulae for unequally spaced values of x .

(i) Lagrange's interpolation formula

(ii) Newton's interpolation formula with divided difference

(i) Lagrange's interpolation formula:

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, y_3, ..., y_n$ corresponding to the values $x_0, x_1, x_2, x_3, ..., x_n$ of x . Since there are $(n+1)$ values of x and y, $f(x)$ can be represented by a polynomial in x of degree n .

$$\begin{aligned} y = f(x) = {} & a_0(x-x_1)(x-x_2)...........(x-x_n) \\ & + a_1(x-x_0)(x-x_2)...........(x-x_n) \\ & + a_2(x-x_0)(x-x_1)...........(x-x_n) \\ & + ... \\ & + a_n(x-x_0)(x-x_1)...........(x-x_{n-1}) \\ & \qquad ....(1) \end{aligned}$$

Where $a_0, a_1, a_2, ..., a_n$ are constants.

Putting $x = x_0, y = y_0$ in Eq.(1),

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2).......(x_0 - x_n)$$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2).......(x_0 - x_n)}$$

Similarly,putting $x = x_1, y = y_1$ in Eq.(1)

Proceeding in the same way,

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1).......(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, ..., a_n$

$$f(x) = \frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2)........(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).......(x-x_n)}{(x_1-x_0)(x_1-x_2)........(x_1-x_n)} y_1 +$$

$$.................+ \frac{(x-x_0)(x-x_1).......(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)........(x_n-x_{n-1})} y_n$$

This is known as Lagrange's interpolation formula.

Example 9: Find the value of y when $x=10$ from the following table:

x	5	6	9	11
y	12	13	14	16

Solution:

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2)........(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2).......(x-x_n)}{(x_1-x_0)(x_1-x_2)........(x_1-x_n)}y_1+$$
$$..................+\frac{(x-x_0)(x-x_1).......(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)........(x_n-x_{n-1})}y_n$$

$$\begin{aligned}y(10)&=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)\\&\quad+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(5-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)\\&=2-4.3333+11.6666+5.3333\\&=14.6666\end{aligned}$$

Example 10: Find the Lagrange interpolating polynomial from the following data:

x	0	1	4	5
$y=f(x)$	1	3	24	39

Solution:

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2)........(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2).......(x-x_n)}{(x_1-x_0)(x_1-x_2)........(x_1-x_n)}y_1+$$
$$..................+\frac{(x-x_0)(x-x_1).......(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)........(x_n-x_{n-1})}y_n$$

$$f(x)=\frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(1)+\frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$
$$+\frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24)+\frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$
$$=\frac{1}{20}(3x^3+10x^2+27x+20)$$

Example 9: Using Lagrange's formula, express the function $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions.

Solution:

Let us evaluate $y=3x^2+x+1$ for $x=1, x=2, x=3$

x	1	2	3
y	5	15	31

Lagrange's formula is,

$$y=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)y_0+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)y_1+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)y_2$$

Substituting the above values, we get

$$y=\left(\frac{x-2}{1-2}\right)\left(\frac{x-3}{1-3}\right)(5)+\left(\frac{x-1}{2-1}\right)\left(\frac{x-3}{2-3}\right)(15)+\left(\frac{x-1}{3-1}\right)\left(\frac{x-2}{3-2}\right)(31)$$
$$y=2.5(x-2)(x-3)-15(x-1)(x-3)+15.5(x-1)(x-2)$$

Thus,

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}=\frac{2.5(x-2)(x-3)-15(x-1)(x-3)+15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$

Example: Given the values

x:	5	7	11	13	17
f(x):	150	392	1452	2366	5202

Evaluate f(9), using Lagrange's formula.

Example:A curve passes through the points (0,18), (1,10), (3,-18) and (6, 90). Find the slope of the curve at x=2.

DIVIDED DIFFERENCES:

In Lagrange's interpolation formula, if another interpolation value is added then the interpolation coefficients are required to be recalculated. To avoid this recalculation, Newton's general interpolation formula is used.

If $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$ be given points then the first divided difference for x_0, x_1 is defined by the relation,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$, etc.

The second divided difference for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

And so on.

NEWTON'S DIVIDED DIFFERENCE FORMULA:

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, y_3, ..., y_n$ corresponding to the values $x_0, x_1, x_2, x_3, ..., x_n$ respectively. According to the definition of divided differences,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$y = y_0 + (x - x_0)[x, x_1] ....(1)$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting in Eq.(1),

$$\begin{aligned} y &= y_0 + (x - x_0)\{[x_0, x_1] + (x - x_1)[x, x_0, x_1]\} \\ &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \\ &\qquad ....(2) \end{aligned}$$

Also, $[x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting in Eq.(2)

$$\begin{aligned} y &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \\ &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)\{[x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]\} \\ &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &\qquad + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] \end{aligned}$$

and so on.

Finally, we have

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ... \\ &+ (x - x_0)(x - x_1).....(x - x_{n-1})[x, x_0, x_1, x_2, ..., x_n] ....(3) \end{aligned}$$

Eq.(3) is known as Newton's general interpolation formula for divided differences

Example 11: Using Newton's divided difference interpolation, compute the value of $f(6)$ from the table given below:

x	1	2	7	8
$y = f(x)$	1	5	5	4

Solution:

Divided Difference Table:

x	$f(x)$	1stDD	2ndDD	3rdDD

1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		-1		
8	4			

By Newton's divided difference formula,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots$$
$$+ (x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})[x, x_0, x_1, x_2, \ldots, x_n]$$

$$f(6) = 1 + (6-1)(4) + (6-1)(6-2)\left(-\frac{2}{3}\right) + (6-1)(6-2)(6-7)\left(\frac{1}{14}\right)$$
$$= 6.2381$$

Example 12: Using Newton's divided difference interpolation formula, find a polynomial.

x	1	2	4	7
$y = f(x)$	10	15	67	430

Solution:

Divided Difference Table:

x	$f(x)$	1stDD	2ndDD	3rdDD
1	10			
		5		
2	15		7	
		26		2
4	67		19	
		121		
7	430			

By Newton's divided difference formula,

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \ldots\ldots$$
$$+ (x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})[x,x_0,x_1,x_2,\ldots,x_n]$$

$$f(x) = 10 + (x-1)(5) + (x-1)(x-2)(7) + (x-1)(x-2)(x-4)(2)$$
$$= 2x^3 - 7x^2 + 12x + 3$$

Example:

Given the values

x:	5	7	11	13	17
f(x):	150	392	1452	2366	5202

Evaluate f(9), using Newton's divided difference formula.

Example: Using Newton's divided difference formula, find the missing value from the table:

x:	1	2	4	5	6
y:	14	15	5	___	9