Homework assignment 4

September 17, 2019

To be handed in by 23:59 on Sunday September 15th. For all your coding exercises, I recommend that you use a Jupyter notebook which you can later export as PDF.

1. Finish implementing the fully implicit solver from Homework 2 by now including the iterative process to determine $\Delta\mathbf{x}$ outlined in the book.

2. (i) Demonstrate that a second order approximation of the derivative of $f(x) = \sin(kx)$ is $f'(x) \approx \sin(k\Delta x)\cos(kx)/\Delta x$.

(ii) The exact derivative of $f(x)$ is $f'(x) = k\cos(kx)$. When deriving using a second order finite difference scheme, we end up with a different pre-factor multiplying the cosine. This is called k', the *modified* wavenumber. Plot how the relationship $k'/k = \sin(k\Delta x)/(k\Delta x)$ changes as a function of $k\Delta x$.[1]

(iii) To show numerically the effect of wavenumber modification, start off with $f(x) = \sin x$ in the interval $[0, 2\pi)$. Take the Fourier transform and show that it has all the energy in a single component. Represent the function using a number of points from 2 to 10^5 (this is like changing $k\Delta x$). For a few selected cases, calculate the derivative using a first, or second order scheme and show the magnitude of the Fourier component (which is k'). These are *dispersion* errors.

3. Write a code that solves the Poisson equation $\nabla^2\phi = f$, with the source term:

$$\begin{aligned} f = & 8\pi^2\cos(4\pi y)\left[\cos(4\pi x) - \sin(4\pi x)\right] - \\ & 16\pi^2\left[\sin(4\pi x)\cos(2\pi y)^2 + \sin(2\pi x)^2\cos(4\pi y)\right] \end{aligned} \tag{1}$$

in the two-dimensional torus[2] bounded by $[0, 2\pi) \times [0, 2\pi)$.

The analytic solution is given by:

$$\phi = \sin(2\pi x)^2\cos(4\pi y) + \sin(4\pi x)\cos(2\pi y)^2 \tag{2}$$

Make the code fully spectral, i.e. substitute the derivatives by multiplications. What is the accuracy of the scheme?

4. Repeat the exercise above, but with now take the derivatives following the finite difference approximation/modified wave number approach outlined in Equation 9.16.

[1] It only makes sense to represent $k\Delta x$ in the range $[0, \pi)$, as $k = \pi/\Delta x$ is the Nyquist frequency

[2] Two-dimensional torus is shorthand for the domain is doubly-periodic, i.e. periodic in both the x and y directions.