Expropriation: A Mechanism Design Approach*

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Abstract

We consider a buyer that must purchase a fixed amount of units from multiple firms privately informed about their costs. While the cost minimizing mechanism can be computed through standard techniques, its practical implementation turns out to be difficult. Thus, we present and fully characterize two sequential mechanisms that are easily implementable: the optimal sequential posted prices scheme and a novel optimal non-linear sequential mechanism. Through extensive numerical simulations, we show that the latter closely approximates the expected cost of the optimal mechanism, while linear prices perform much worse. We finally show that a mechanism typically used in these settings, *pay-as-clear*, performs quite badly if costs are convex. Our results have natural applications to the problem of buying back pollution permits in order to meet international agreements and the problem of buying energy from generators, among others.

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1. Introduction

We consider a government who must buy back a fixed quantity of pollution permits (for example, to meet an international environmental agreement) from multiple firms that have private and convex costs of giving them up. We study the problem of cost minimization in this context. Classical mechanism design literature (e.g. Myerson (1981)) provides tools to answer this question in a straightforward way.

Nevertheless, the optimal mechanism lacks simplicity, as it entails the buyer offering to each firm a transfer-quantity menu that depends not only on its type but also on the types of all the other firms. Moreover, it can't be implemented through any standard pricing schedule[1]. Thus, the optimal mechanism doesn't seem very likely to be implemented in practice. Therefore, a natural question arises: are there simpler ways of conducting expropriations whose cost is close to the optimal one? The present article aims to answer this question by presenting alternative simpler schedules and characterizing their performance relative to the optimal one. In particular, we consider two sequential mechanisms: the natural benchmark of simplicity, sequential posted prices (i.e. announcing a price-per-unit menu to each firm in the given order) and an optimal sequential mechanism, which offers to each firm a transfer-quantity menu that depends only on its type and the total quantity of items already bought to previous firms (or, equivalently, the remaining quantity to be bought).

We are able to fully characterize both sequential mechanisms, taking advantage of their recursive structure. The optimal sequential mechanism features at each stage a simple menu of quantities and transfers as if there were only two sellers, the current firm and a fictitious player whose "cost parameter" encompasses the expected cost of subsequent stages (i.e. the cost of delaying the purchase). Moreover, these fictitious sellers can be computed and announced in advance, since they do not depend on the realization of types in previous stages. On the other hand, sequential posted prices entails prices that are linear functions of the quantity to be bought at each stage. These functions can also be computed ex-ante, as they only depend on the distribution of the cost of the firm faced at the corresponding stage and the order of subsequent firms. Furthermore, we establish through numerical simulations that while sequential posted prices performs quite badly, the optimal sequential mechanism has a near-optimal performance. Thus, it seems that it is possible to achieve simplicity at a low cost only if non-linear menus are considered.

Besides the pollution permits problem, our results have natural applications in several issues frequently faced by governments, as fisheries management through buying back previously issued licenses (for example, when a zone is running out of fishes as a result of a particularly strong warm or cool phase), retrieving mining permits (for example, in an attempt to transfer exploitation rights to public firms) and buying energy from generators. Crucially, in all these settings there are usually large incumbent conglomerates with several plants of varying efficiency. This naturally gives rise to a convex cost function, making the standard linear cost settings ill-suited for addressing these problems. As we show in section 3.1, this makes a crucial difference

[1] First-price auctions, pay-as-clear, pay-as-bid, dutch auctions, etc.

regarding the optimal mechanism.

1.1 Related Literature

This article is inscribed in the mechanism design literature, so two key methodological references are of course Myerson (1979) and Myerson (1981). While the first proves the revelation principle, lowering the complexity of these types of problems enormously by reducing the space of mechanism to consider, the second characterizes incentive compatibility in a very useful way that also holds for the problem we study, allowing thus the derivation of both the optimal mechanism and the optimal sequential one. Regarding the two alternative mechanisms explored, the consideration of an ordering or permutation of agents in an essentially static environment closely relates to the approach in Halac et al. (2019), though the problem they study is of a fundamentally different nature in both the methodological approach (contracting with externalities) and the particular question studied (investment with heterogeneous agents). Additionally, the sequential posted prices mechanism and a similar one (discriminatory posted prices) have been considered previously, though in the linear costs/valuations settings (see, for example, Chawla et al. (2010), Sandholm and Gilpin (2006) and Blumrosen and Holenstein (2008)) Interestingly, in these linear settings it turns out that they perform quite well and closely approximate the optimal mechanism. As far as we know, sequential mechanisms have not been studied in static environments before[2].

On the other hand, there is a recent and growing literature that tries to address the issue of the complexity of mechanism design in different settings. However, there is no agreement regarding what constitutes a "simple" mechanism, and most discussions focus on the demands of rationality (strategic thinking) and information-processing abilities that mechanisms require from the participants in order to be successfully implemented (see, for example Li (2017), Pycia and Troyan (2019) and Börgers and Li (2019)). In a similar vein, there is a body of work at the intersection of economics and computer science that, given some restriction on the design space (e.g. limited distributional knowledge), tries to find mechanisms that are sufficiently near the optimal "unconstrained one" (see the recent survey Roughgarden and Talgam-Cohen (2019); Garrett (2014), Hartline and Roughgarden (2009) and McAfee (1992) are examples of different approaches to this issue). More generally, this article is part of a large strand of literature that explores whether complex strategies can be approximated by using simpler alternatives. For example, Rogerson (2003) argues that in a standard principal-agent model, most of the gains to the principal from offering the optimal continuous menu of contracts can be captured by simpler alternatives. McAfee (2002) establishes similar findings in the context of nonlinear pricing, as does Neeman (2003) in the context of auctions.

More directly related to this work, Montero (2008) and Babaioff et al. (2015) present, in different settings, mechanisms that are simple in terms of having an intuitive interpretation and implementation (notions that depend, of course, on the particular environment and problem

[2]There is a strand of literature that explicitly studies sequential auctions and mechanism design (e.g., Skreta (2006), Cisternas and Figueroa (2015) and Jofre-Bonet and Pesendorfer (2014)) but it addresses problems in settings that are "fundamentally" sequential in nature and mostly deal with issues raised by the problem of non commitment, so it does not relate to this work in any relevant way.

studied) and prove they achieve the efficient allocation in the case of the first paper and perform very near the optimal (revenue maximizing) one in the case of the second. Finally, Chu et al. (2011) is similar to this work methodologically speaking, albeit the problem studied is the one of a multi product monopolist facing buyers with private information regarding their valuations. They propose a bundle strategy that is considerably simpler than the optimal (profit maximizing) one[3] and show through a carefully executed numerical analysis how it is very close to the optimal one in terms of profit attained.

The rest of the paper is organized as follows. In section 2, we present the model and the full information benchmark mechanism. In section 3, we characterize the optimal mechanism and show how does it compare to the optimal one in two closely related settings. In section 4, we present and discuss two simpler mechanisms: sequential posted prices and the optimal sequential one. In section 5, we show through numerical analysis how do our simple mechanisms perform relative to the optimal one for representative families of distributions of types in the common distribution setting. Finally, in section 6 we conclude and advance possible avenues for future research.

2. The Model

Consider an agent who needs to buy a fixed amount $\overline{Q}$ of an item. The set of potential sellers (firms) is $S = \{1, ..., k\}$, indexed by i. Each firm is risk-neutral and has a cost of selling a quantity Q given by $C_i(Q) = \theta_i Q^2/2$, where θ_i, hereafter the "type", is private information. This cost can be taken to be a standard production cost, but interpreted more generally, it also encompasses "alternative" costs like the expected cost of giving up permits in terms of future profits, as in our pollution permits application[4]). The type of the firm i, θ_i, is drawn from a publicly known distribution with cumulative distribution function F_i supported on the interval $[a_i, b_i]$, where $a_i > 0$ for every $i \in \{1, ..., k\}$. Moreover, for every $i \in \{1, ..., k\}$ we take $F_i(\cdot)$ to be absolutely continuous and thus a density function $f_i(\cdot)$ exists. We denote as $\boldsymbol{\theta} = (\theta_1, ..., \theta_k)$ the vector of types, as $\Theta \equiv \times_{i=1}^{k}[a_i, b_i]$ the joint product of type supports and as

$$f(\boldsymbol{\theta}) = \prod_{i=1}^{k} f_i(\theta_i)$$

the joint densities.

We assume that, for every $i \in \{1, ..., k\}$,

$$J_i(\theta_i) \equiv \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}$$

is strictly increasing in θ_i. This assumption —usually referred to as *regularity*— is standard in the mechanism design literature and does not compromise our findings (Myerson (1981) and more

[3]In this environment, simplicity is defined in terms of the number of prices that is the monopolist needs to set.

[4]Given this interpretation, it seems natural to assume increasing marginal costs as we do, as firms would likely close off their less profitable plants first and so, as the number of sold permits increases, they would need to close more efficient plants, thus giving up more future rents per marginal unit sold.

recently Toikka (2011) deal with standard mechanism design problems when this assumption does not hold). Moreover, we assume the buyer can fully commit to any mechanism before the realization of the types as it is also standard in the mechanism design literature. Consequently, the revelation principle trivially holds and we can without lost of generality restrict our search of the optimal mechanism to a particular class of them, namely, direct mechanisms.

Definition A **direct mechanism** Γ is an ordered pair

$$\Gamma = (q, t)$$

where $q : \Theta \to \mathbb{R}^k_+$ and $t : \Theta \to \mathbb{R}^k$.

The interpretation of the precedent class of mechanisms is simply that each potential seller reports a type $\theta_i \in \Theta_i$ rather than an arbitrary message. Given the vector of reports $\boldsymbol{\theta}$, the buyer buys $q(\boldsymbol{\theta}) = (q_1(\boldsymbol{\theta}), ..., q_k(\boldsymbol{\theta}))$ from each firm at a "price" $t(\boldsymbol{\theta}) = (t_1(\boldsymbol{\theta}), ..., t_k(\boldsymbol{\theta}))$, hereafter referred to as transfers (bear in mind that the buyer commits to an allocation rule $q(\cdot)$ and a transfer rule $t(\cdot)$ **before** firms report types). Note that these mechanisms are deterministic, as for a given a realization of $\boldsymbol{\theta}$, the bought quantities and transfers corresponding to each firm are deterministic (it is straightforward to see this is without loss of generality in our setting).

Any direct mechanism induces a game of incomplete information among the potential sellers. The sellers' strategies are mappings from their private types to reports. These strategies must maximize expected payoffs and exceed some exogenous payoff (which we will normalize to 0 for simplicity) in order to ensure participation.

2.1 Full Information Benchmark

When the types of the firms are publicly known (in particular, when they are known by the buyer), the solution of the problem of the buyer is straightforward, as the following proposition shows.

Proposition 1 *In the full information setting, the optimal quantities bought from each firm are given by*

$$q_i(\boldsymbol{\theta}) = \left(\frac{\theta_i^{-1}}{\sum_{j=1}^k \theta_j^{-1}} \right) \overline{Q}, \tag{1}$$

while the transfers are given by

$$t_i(\boldsymbol{\theta}) = \frac{\theta_i}{2} \left(\frac{\theta_i^{-1}}{\sum_{j=1}^k \theta_j^{-1}} \right)^2 \overline{Q}^2. \tag{2}$$

Moreover, the expected total cost for the buyer is given by

$$C_k^{fi}(\overline{Q}) = \left(\frac{1}{\sum_{j=1}^k \theta_j^{-1}} \right) \frac{\overline{Q}^2}{2}. \tag{3}$$

Proof

Note first that in this case the buyer will pay the firms exactly their costs as they are observable and the buyer wants to buy at minimum cost. Hence, the problem the buyer solves involves choosing only the optimal quantities to buy and is given by

$$\min_{q:\Theta\to\mathbb{R}^k} \quad \sum_{i=1}^{k} \theta_i \frac{q_i^2(\boldsymbol{\theta})}{2}$$

$$\text{subject to} \qquad \sum_{i=1}^{k} q_i(\boldsymbol{\theta}) \geq \overline{Q},$$

$$q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } i \in \{1, ..., k\} \text{ and all } \boldsymbol{\theta} \in \Theta.$$

Note that both the objective function and the constraints are convex, so we can solve the problem using Lagrange multipliers (see, for example, Theorem 5.6 in Carter (2001)).[5] Denoting by λ the Lagrange multiplier of the first constraint, the first order conditions are given by

$$\theta_i q_i = \lambda^{6}, \tag{4}$$

for every $i \in \{1, ..., k\}$, and

$$\sum_{j=1}^{k} q_j - \overline{Q} \geq 0. \tag{5}$$

Clearly (4) holds without slackness at the optima. Thus, substituting q_i from (3) and solving for λ, we get

$$\lambda = \frac{\overline{Q}}{\sum_{j=1}^{k} \theta_j^{-1}}. \tag{6}$$

Substituting (5) in (3) and solving for q_i we get the desired expression for the quantity bought from each firm i,

$$q_i(\boldsymbol{\theta}) = \left(\frac{\theta_i^{-1}}{\sum_{j=1}^{k} \theta_j^{-1}} \right) \overline{Q}. \tag{7}$$

The expressions for the transfers and the total cost follow directly from substituting (6) into the cost function of each firm and the objective function of the firm respectively. ■

Intuitively, the convexity of the cost function makes it optimal for the buyer to buy a positive amount from each firm, no matter how inefficient. In particular, the proportion of $\overline{Q}$ each firm sells is given by the ration of the inverse of its cost parameter —which can be naturally be taken to be an "efficiency" parameter— over the sum of the inverses of the cost parameters of all firms. Thus, naturally, the more productive a firm is relative to the rest, the more it sells. This contrasts starkly with the linear cost setting, where it's optimal to buy all the quantity from the lowest cost type.

Note that the full information optimal allocation is efficient, as it minimizes the social cost of producing the needed amount $\overline{Q}$ given the realization of $\boldsymbol{\theta}$. This coincidence stems

[5] This applies to all the remaining optimization problems in this paper.

[6] It is straightforward to see that the case with non interior solution leads to a contradiction so we only need to consider the interior one.

naturally from the fact that the buyer internalizes exactly the costs of production of the firms and doesn't have to incur in additional expenses (as opposed to the case in the incomplete information setting as we will see next).

3. Incomplete Information: Optimal Mechanism

Having established the solution of the buyer's problem in the full information benchmark setting, we move on to the —more relevant— incomplete information environment. In this setup, the buyer faces the problem of not directly observing the types of the firms, and thus, she is not able to simply buy the efficient amount from each of them and pay them their cost. First, we study the optimal (cost-minimizing) mechanism in our main setting, as characterized by the following proposition.

Proposition 2 *If regularity holds, the optimal mechanism entails the following allocation rule*

$$q_i(\boldsymbol{\theta}) = \left(\frac{J_i^{-1}(\theta_i)}{\sum_{j=1}^k J_j^{-1}(\theta_j)}\right)\overline{Q}. \tag{8}$$

The transfers (in their point-wise form) are, on the other hand, given by

$$t_i(\boldsymbol{\theta}) = \frac{\theta_i}{2}q_i^2(\boldsymbol{\theta}) + \frac{1}{2}\int_{\theta_i}^{b_i} q_i^2(s_i,\theta_{-i})ds_i.$$

Additionally, the expected total cost of this mechanism is given by

$$C_k^{om}(\overline{Q}) = \mathbb{E}_{\boldsymbol{\theta}}\left(\frac{1}{\sum_{j=1}^k J_j^{-1}(\theta_j)}\right)\frac{\overline{Q}^2}{2}. \tag{9}$$

Proof
As the revelation principle holds, the buyer needs only to consider direct mechanisms where it is optimal for every seller to report their types truthfully (see, for example, Proposition 2.1 in Börgers (2015)). Hence, the problem the buyer solves is given by

$$\min_{\substack{q:\Theta\to\mathbb{R}^k\\ t:\Theta\to\mathbb{R}^k}} \int_\Theta \left[\sum_{i=1}^k t_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$\begin{aligned}
\text{subject to}\qquad & \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i,\theta_{-i})}{2}\right] \geq \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i',\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right] \\
& \text{for all } \theta_i,\theta_i' \in \Theta_i \text{ and all } i \in \{1,...,k\} \\
& \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i,\theta_{-i})}{2}\right] \geq 0 \text{ for all } \theta_i \in \Theta_i \text{ and all } i \in \{1,...,k\}, \\
& \sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta} \in \Theta, \\
& q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } i \in \{1,...,k\} \text{ and all } \boldsymbol{\theta} \in \Theta.
\end{aligned}$$

The first constraint is the standard incentive compatibility one that imposes that it is optimal for

every seller to reveal their true type given the mechanism (q, t). The second one is usually called individual rationality or voluntary participation and simply imposes that every seller wants to participate, given their reservation utility which is normalized to zero for simplicity.
The following argument regarding a characterization of the incentive compatibility constraint is based on the one in Krishna (2010). Note that IC can alternatively be written as

$$U_i(\theta_i) = \max_{\theta_i'} \left\{ T_i(\theta_i') - \mathbb{E}_{\theta_{-i}} \left[\theta_i \frac{q_i^2(\theta_i', \theta_{-i})}{2} \right] \right\}, \tag{10}$$

where $T(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i})$ is the expected transfer of the seller i given a mechanism (q, t) and that she announced her type was θ_i and the function $U_i : \Theta_i \to \mathbb{R}$ is defined as the expected utility of seller i given her type and the fact that she truthfully revealed it. In other words, this conditions says that given the mechanism (q, t) the highest expected utility a seller can attain is the one she gets revealing her true type. Note that U_i is here the maximum of a family of affine functions (of θ_i), hence it is convex. Every convex function is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Hence, by the Envelope Theorem, at every point where it is differentiable, we have

$$U_i'(\theta_i) = -\frac{1}{2} \mathbb{E}_{\theta_{-i}} \{ q_i^2(\theta_i, \theta_{-i}) \}.$$

As U_i is convex, the previous expression implies that $\mathbb{E}_{\theta_{-i}} \{ q_i^2(\theta_i, \theta_{-i}) \}$ must be a weakly decreasing function of θ_i. Moreover, by Corollary 1 in Milgrom and Segal (2002), at every point where U_i is differentiable and for every IC allocation rule $q_i(\theta_i, \theta_{-i})$, we have

$$U_i(\theta_i) = U_i(b_i) + \frac{1}{2} \int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}} \{ q_i^2(s_i, \theta_{-i}) \} ds_i. \tag{11}$$

Additionally, for any IC allocation rule $q_i(\theta_i, \theta_{-i})$,

$$U_i(\theta_i) = T_i(\theta_i) - \frac{\theta_i}{2} \mathbb{E}_{\theta_{-i}} \{ q_i^2(\theta_i, \theta_{-i}) \}. \tag{12}$$

Putting (10) and (11) together, we can write the expected transfer as

$$T_i(\theta_i) = U_i(b_i) + \frac{\theta_i}{2} \mathbb{E}_{\theta_{-i}} \{ q_i^2(\theta_i, \theta_{-i}) \} + \frac{1}{2} \int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}} \{ q_i^2(s_i, \theta_{-i}) \} ds_i. \tag{13}$$

Given this, we can rewrite the buyer's objective function as follows

$$\begin{aligned} \int_{\Theta} \left[\sum_{i=1}^{k} t_i(\boldsymbol{\theta}) \right] f(\boldsymbol{\theta}) d\boldsymbol{\theta} &= \sum_{i=1}^{k} \int_{\Theta} t_i(\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \sum_{i=1}^{k} \int_{a_i}^{b_i} T_i(\theta_i) f_i(\theta_i) d\theta_i \\ &= \sum_{i=1}^{k} \left\{ \int_{a_i}^{b_i} \left\{ U_i(b_i) + \frac{\theta_i}{2} \mathbb{E}_{\theta_{-i}} \{ q_i^2(\theta_i, \theta_{-i}) \} + \frac{1}{2} \int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}} \{ q_i^2(s_i, \theta_{-i}) \} ds_i \right\} f_i(\theta_i) d\theta_i \right\}. \end{aligned}$$

By Fubini's Theorem, we can interchange the order of integration in the last integral, thus getting

$$\begin{aligned}\int_{a_i}^{b_i}\int_{\theta_i}^{b_i}\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}f_i(\theta_i)ds_i d\theta_i &= \int_{a_i}^{b_i}\int_{a_i}^{s_i}\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}f_i(\theta_i)d\theta_i ds_i\\ &= \int_{a_i}^{b_i}\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}\int_{a_i}^{s_i}f_i(\theta_i)d\theta_i ds_i\\ &= \int_{a_i}^{b_i}F(s_i)\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}ds_i\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_\Theta\left[\sum_{i=1}^k t_i(\boldsymbol{\theta})\right]f(\theta)d\theta =& \frac{1}{2}\sum_{i=1}^k\int_{a_i}^{b_i}\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\mathbb{E}_{\theta_{-i}}\{q_i^2(\theta_i,\theta_{-i})\}f_i(\theta_i)d\theta_i+\sum_{i=1}^n U_i(b_i)\\ =&\frac{1}{2}\sum_{i=1}^k\mathbb{E}_{\boldsymbol{\theta}}\left\{\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)q_i^2(\boldsymbol{\theta})\right\}+\sum_{i=1}^n U_i(b_i)\\ =&\frac{1}{2}\sum_{i=1}^k\int_\Theta\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)q_i^2(\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}+\sum_{i=1}^n U_i(b_i)\\ =&\frac{1}{2}\int_\Theta\sum_{i=1}^k\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)q_i^2(\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}+\sum_{i=1}^n U_i(b_i).\end{aligned}$$

Moreover, we established that an incentive compatible mechanism is such that $\mathbb{E}_{\theta_i}\{q_i^2(\theta_i,\theta_{-i})\}$ is a decreasing function of θ_i and (11) holds. To see that the converse is also true, note first that for any $\theta_i,\theta_i'\in\Theta_i$, we have

$$\begin{aligned}T_i(\theta_i')-\mathbb{E}_{\theta_{-i}}\left[\theta_i\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right] =& T_i(\theta_i')-\mathbb{E}_{\theta_{-i}}\left[\theta_i'\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right]-\mathbb{E}_{\theta_{-i}}\left[\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right](\theta_i-\theta_i')\\ =&U_i(\theta_i')-\mathbb{E}_{\theta_{-i}}\left[\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right](\theta_i-\theta_i')\end{aligned}$$

So, incentive compatibility can be written as

$$U_i(\theta_i)\geq U_i(\theta_i')-\mathbb{E}_{\theta_{-i}}\left[\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right](\theta_i-\theta_i'),$$

for all $\theta_i,\theta_i'\in\Theta_i$. Using (11) and rearranging, the previous inequality can be rewritten as

$$\int_{\theta_i}^{\theta_i'}\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}ds_i\geq\mathbb{E}_{\theta_{-i}}\{q_i^2(\theta_i',\theta_{-i})\}(\theta_i'-\theta_i),$$

which clearly holds if $\mathbb{E}_{\theta_{-i}}\{q_i^2(\theta_i,\theta_{-i})\}$ is decreasing. Hence a mechanism is incentive compatible if and only if $\mathbb{E}_{\theta_i}\{q_i^2(\theta_i,\theta_{-i})\}$ is decreasing in θ_i and (11) holds. By the previous results, the original problem can be rewritten only in terms of the allocation rule as follows

$$\min_{q:\Theta\to\mathbb{R}^k} \int_{\Theta}\left[\sum_{i=1}^{k}\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\frac{q_i^2(\boldsymbol{\theta})}{2}\right]f(\boldsymbol{\theta})d\boldsymbol{\theta}+\sum_{i=1}^{k}U_i(b_i)$$

subject to

$$\mathbb{E}_{\theta_{-i}}[q_i^2(\cdot,\theta_{-i})] \text{ is decreasing for all } i\in\{1,...,k\},$$

$$\sum_{i=1}^{k} q_i(\boldsymbol{\theta})\geq \overline{Q} \text{ for all } \boldsymbol{\theta}\in\Theta,$$

$$q_i(\boldsymbol{\theta})\geq 0 \text{ for all } \boldsymbol{\theta}\in\Theta \text{ and all } i\in\{1,...,k\}.$$

Clearly it is optimal for the buyer to set $U_i(b_i)=0$ for all $i\in\{1,...,k\}$. Thus, we can finally rewrite the problem as

$$\min_{q:\Theta\to\mathbb{R}^k} \int_{\Theta}\left[\sum_{i=1}^{k}\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\frac{q_i^2(\boldsymbol{\theta})}{2}\right]f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

subject to

$$\mathbb{E}_{\theta_{-i}}[q_i^2(\cdot,\theta_{-i})] \text{ is decreasing for all } i\in\{1,...,k\},$$

$$\sum_{i=1}^{k} q_i(\boldsymbol{\theta})\geq \overline{Q} \text{ for all } \boldsymbol{\theta},$$

$$q_i(\boldsymbol{\theta})\geq 0 \text{ for all } \boldsymbol{\theta}\in\Theta \text{ and all } i\in\{1,...,n\}.$$

Leaving aside the first restriction, we can solve the unrestricted version of the preceding problem point-wise. Noting that this problem is completely analogous to the full information one substituting θ_i by $J_i(\theta_i)=\theta_i+\dfrac{F_i(\theta_i)}{f_i(\theta_i)}$, we find the optimal allocation rule

$$q_i(\boldsymbol{\theta})=\left(\frac{J_i^{-1}(\theta_i)}{\sum_{j=1}^{k}J_j^{-1}(\theta_j)}\right)\overline{Q}. \tag{14}$$

It is straightforward to see that given our regularity assumption, this function is decreasing on θ_i. Thus, it is the solution to the original restricted problem. On the other hand, the expression for the transfers is really only one of the possible ones, as both agents are risk-neutral and, as we showed, what is fixed given the allocation rule is $T_i(\theta_i)$, the expected value of the transfer taken with respect to the types of the other sellers. Finally, the expression for the costs follows directly by substituting the allocation rule in the expression we proved to be equal to the expected sum of transfers, namely

$$\frac{1}{2}\sum_{i=1}^{k}\mathbb{E}_{\boldsymbol{\theta}}\left\{\left(\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right)q_i^2(\boldsymbol{\theta})\right\}.$$

■

A few remarks regarding the optimal mechanism. First, note that the allocation rule differs from the full information one and thus, from the efficient one. This departure from efficiency is pervasive in incomplete information settings. It follows from the fact that inducing truth-telling on the sellers is costly and this cost depends on the distribution of types of each firm. In particular, the optimal allocation rule assigns production through virtual costs parameters rather than actual costs, where the former reflect the cost of inducing truth telling given each distribution and realization of type. Second, note that with this mechanism, all sellers —except the ones with the highest realization of cost types— have positive profits. These are known as informational rents and, as the inefficiency of the optimal mechanism, follow from the need to induce truth telling. Finally, note that this mechanism, however optimal, is quite difficult to implement in any practical real-world setting, as 1) The expression for the transfers is complex and doesn't have closed-form (thus, the mechanism can't be implemented by using posted prices, two-part tariffs nor any other known pricing technique) and 2) It entails offering transfer-quantity menus to each firm that depend in a non-trivial way on the types' realizations of all firms, not only on its own.

Given this last two features of the optimal mechanism, there is clearly a need for a simpler and more intuitive way of buying. A natural and widely studied mechanism that is used in multiple settings because of its simplicity and ease of implementation is simply announcing a price-per-unit and buying all the quantity that is offered. In the following section we will characterize a sequential version of this well-known mechanism (hereafter referred to as "posted prices"). However, before that, we will briefly look at two related problems and compare their optimal mechanisms to ours in order to highlight the particular character of our problem.

3.1 Linear Costs and Capacity Constraints

Consider the same problem than before but with costs functions given by

$$C_i(Q) = \begin{cases} \theta_i Q & \text{if } Q \leq c \\ \infty & \text{if not,} \end{cases}$$

where $c > 0$ for every $i \in \{1, ..., k\}$ and a common distribution of types F. This corresponds to the case where every firm has linear costs and a hard capacity constraint c such that it isn't able to produce more than that. We assume $kc > \overline{Q}$ as the alternative case, having to buy a quantity that can't be produced, doesn't make much sense. Additionally, we assume $(k-1)c > \overline{Q}$ (that is, it is always possible to attain the quantity needed without having to rely on any particular firm) just in order to avoid pathological features that complicate the characterization of the optimal mechanism without adding any relevant insight. This setting, which can be seen as closely related to ours, turns out to have an optimal mechanism that is quite simple and implementable, as the following proposition shows.

Proposition 3 *If regularity holds, the optimal mechanism of the previously described problem is given by the following allocation rule*

$$q_i(\boldsymbol{\theta}) = \begin{cases} c & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) \leq \overline{Q} \\ \overline{Q} - c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) > \overline{Q} \text{ and } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| < \overline{Q} \\ 0 & \text{if } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| \geq \overline{Q}, \end{cases}$$

and the following transfers (in their point-wise form)

$$t_i(\boldsymbol{\theta}) = \begin{cases} \theta_l \cdot c & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) \leq \overline{Q} \\ \theta_l \cdot (\overline{Q} - c \cdot |\mathcal{J}_i(\boldsymbol{\theta})|) & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) > \overline{Q} \text{ and } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| < \overline{Q} \\ 0 & \text{if } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| \geq \overline{Q}, \end{cases}$$

where

$$\mathcal{J}_i(\boldsymbol{\theta}) \equiv \{j \in \{1, ..., k\} \setminus \{i\} \mid \theta_j \leq \theta_i\},$$

and $\theta_l = \theta_l(\boldsymbol{\theta}) \in \{\theta_n\}_{n=1}^k$ *is such that*

$$c \cdot |\mathcal{J}_i(\theta_l, \theta_{-i})| < \overline{Q}$$

$$c \cdot (|\mathcal{J}_i(\theta_l, \theta_{-i})| + 1) \geq \overline{Q}.$$

Proof See Appendix.

Note that even though this mechanism does condition the amount bought to each firm in the whole vector $\boldsymbol{\theta}$, it can be implemented in a very natural and simple way (by announcing a unique price per-unit and buying all the quantity offered at that price), which is not the case in our quadratic costs setting. Thus, we can regard this setup as being one where the optimal mechanism is simple enough to be implementable without much trouble.

Additionally, note that the optimal price set is almost the price that clears the market given the aggregate supply generated by $\boldsymbol{\theta}$ (a step function with the size of each step equal to the capacity c and the value of each step given by the costs of the firms, ordered from lowest to highest) and an inelastic demand, $Q^d = \overline{Q}$[7]. Hence, this mechanism has a natural analog in our quadratic cost setting, namely, posting a price-per-unit that clears the market given $\boldsymbol{\theta}$ and buying all that firms offer at that price. This mechanism, however, is not incentive compatible, as the following proposition shows.

Proposition 4 *In our quadratic cost setup, the mechanism that sets a uniform price that clears the market is not truth-telling.*

Proof See Appendix.

This results shows that the convexity of the costs plays a key role in shaping the complexity of the optimal mechanism and the low performance of natural and simple alternatives. Now, let's look at another set up where the optimal mechanism is a relatively simple one.

3.2 Open Procurement

Consider the problem of a government that has a constant valuation $b > 0$ per unit of an item produced by k different firms in the incomplete information setting described in Section 2. The following proposition characterizes the optimal mechanism for this problem.

Proposition 5 *If regularity holds, the optimal mechanism of the previously described problem is*

[7] Actually, it is equal to the type that is just above the one of the last firm that sells a positive quantity. Thus, graphically, it is just above the natural "market clearing price" (note that this has a second-price auction flavor as a consequence of the linearity of the costs and the common distribution of types assumption).

given by the following allocation rule

$$q_i(\boldsymbol{\theta}) = \frac{b}{J_i(\theta_i)}$$

The transfers (in their point-wise form) are, on the other hand, given by

$$t_i(\boldsymbol{\theta}) = \frac{\theta_i}{2} q_i^2(\boldsymbol{\theta}) + \frac{1}{2}\int_{\theta_i}^{b_i} q_i^2(s_i, \theta_{-i}) ds_i.$$

Proof See Appendix.

Note that for this problem, the optimal transfer-quantity menus offered to each firm depend only on their individual types. This is a consequence of the fact that the problem is "separable", as the relevant variable for the buyer is the cost of each firm —adjusted by the distribution in order to induce incentive compatibility— relative to the valuation $b > 0$, not relative to the cost of others (on the other hand, as the problem is still convex, it remains true that every firm is optimally mandated to sell a positive, no matter how inefficient). This makes the optimal mechanism in this setting quite simpler, implementation-wise, than the optimal one in our main setup. This highlights the particular nature of our setting and how the convexity of the costs combined with the buyer's need for a fixed quantity of items is the source of the complex interdependency of the optimal menus and thus, of its lack of implementation's simplicity.

In the next section, we will characterize two mechanisms that share some of the features of these two simple and optimal mechanisms that are specially tailored to the fixed quantities and convex costs setting.

4. Two Simple Mechanisms

4.1 Sequential Posted Prices

What if the buyer could face the firms not all at once but in a given order and approach them one by and one offering a price-per-unit at each "selling stage"? The buyer, anticipating the choice of the firm given each price, would choose the price optimally at each stage, accounting for the fact that the quantity that is not bought in each period (regarding the remaining quantity to be bought) will have to be bought to the subsequent firms possibly at a higher cost (in fact, in the last stage, the remaining quantity is bought at the cost of the most expensive firm). Given that at each stage the price would be announced without knowing the realization of the the cost type, it should also account for the whole range of possible quantities offered. Formulating the problem recursively, we can characterize the optimal prices to be charged and the correspondent expected cost —provided a condition over the distributions of types holds— through the following proposition

Proposition 6 *Assume without lost of generality that firms are ordered by their index i (i.e. firm 1 trades first, firm 2 second, etc...). Define, for $i \in \{1, ..., k\}$, $\mu_{i,1} \equiv \mathbb{E}_{\theta_i}(1/\theta_i)$ and $\mu_{i,2} \equiv$*

$\mathbb{E}_{\theta_i}(1/\theta_i^2)$. *Also define recursively*

$$B_k^k = b_k$$

and

$$B_j^k = B_{j+1}^k - \frac{(B_{j+1}^k)^2 \mu_{j,1}^2}{2\mu_{j,1} + B_{j+1}^k \mu_{j,2}}$$

for $j < k$. *If for every* $j < k$

$$\frac{B_{j+1}^k \mu_{j,1}}{2\mu_{j,1} + B_{j+1}^k \mu_{j,2}} \leq a_j,$$

then the total expected cost for the buyer given she uses an optimal sequential posted prices mechanism is given by

$$C_k^{pp}(\overline{Q}) = B_1^k \frac{\overline{Q}^2}{2},$$

Moreover, the optimal price the buyer chooses when faced with firm $j < k$ *is given by*

$$P_j^k(\overline{Q}_j) = \frac{B_{j+1}^k \mu_{j,2}}{2\mu_{j,2} + B_{j+1}^k \mu_{j,2}} \overline{Q}_j,$$

where $\overline{Q}_1 = \overline{Q}$ *and* $\overline{Q}_j = \overline{Q} - \sum_{i=1}^{j-1} q_i(\theta_i)$ *for* $j > 1$. *Finally, the quantity sold by firm* j *given a realization of their type* θ *is*

$$q_j(\theta) = \frac{P_j^k(\overline{Q}_j)}{\theta}.$$

Proof

Naturally, we will use a backward induction argument. Note first that for any given P chose by the seller, a firm with type θ solves

$$\max_{Q^s} \;\; \Pi(Q_s) = PQ^s - \theta \frac{Q^{s\,2}}{2}$$

The solution to the previous problem is clearly given by

$$Q^s(\theta, P) = \frac{P}{\theta}. \tag{15}$$

Now lets proceed by induction. First, for $k = 1$, the buyer simply buys $\overline{Q}$ units to the firm paying as if it were the highest possible type (this is natural as in this case this mechanism is quite simple and it would be more expensive to use posted prices[8]). Thus, the total cost is given by

$$C_1^{pp} = b_1 \frac{\overline{Q}^2}{2} = b_k \frac{\overline{Q}^2}{2} = B_k^k \frac{\overline{Q}^2}{2} = B_1^k \frac{\overline{Q}^2}{2}.$$

Now, assume that for $k = n$ the expected total cost is given by

$$C_n^{pp}(\overline{Q}) = B_1^n \frac{\overline{Q}^2}{2}.$$

Given this, when $k = n+1$, the problem the buyer solves when facing the first firm is the following

[8]As in that case the price chosen would be $P = \overline{Q} b_1$ in order to make sure that for every possible type realization the firm offers at least $\overline{Q}$. This would imply a cost of $\overline{Q}^2 b_1 > \overline{Q}^2 b_1/2$

[9]

$$
\begin{aligned}
\min_{P_1\geq 0} \ & \int_{a_1}^{b_1} \left[P_1 Q^s(\theta_1, P_1) + \frac{B_1^n}{2} \max\{\overline{Q} - Q^s(\theta_1, P_1), 0\}^2 \right] f_1(\theta_1) d\theta_1 \\
&= \int_{a_1}^{b_1} \left[\frac{P_1^2}{\theta_1} + \frac{B_1^n}{2} \max\{\overline{Q} - \frac{P_1}{\theta_1}, 0\}^2 \right] f_1(\theta_1) d\theta_1,
\end{aligned}
$$

where the max operator on the second term of the objective function accounts for the fact that the first seller could potentially offer to sell all the remaining quantity to-buy (or even more), in which case the buyer would naturally stop buying from subsequent firms. The second equality, on the other hand, follows from the fact that the buyer anticipates the seller will offer (15) given price P. Equivalently, this can be written as

$$
\min_{P_1\geq 0} \ F(P_1) \tag{P}
$$

where

$$
F(P_1) = \begin{cases} P_1^2 \displaystyle\int_{a_1}^{b_1} \frac{f_1(\theta_1)}{\theta_1} d\theta_1 + \frac{B_1^n}{2} \int_{P_1/\overline{Q}}^{b_1} \left(\overline{Q} - \frac{P_1}{\theta_1} \right)^2 f_1(\theta_1) d\theta_1 & \text{if } P_1 > a_1 \overline{Q} \\ P_1^2 \displaystyle\int_{a_1}^{b_1} \frac{f_1(\theta_1)}{\theta_1} d\theta_1 + \frac{B_1^n}{2} \int_{a_1}^{b_1} \left(\overline{Q} - \frac{P_1}{\theta_1} \right)^2 f_1(\theta_1) d\theta_1 & \text{if } P_1 \leq a_1 \overline{Q}. \end{cases}
$$

If we solve the problem as if $P_1 \leq a_1 \overline{Q}$, we get from the FOC

$$
P_1^* = \left(\frac{B_1^n \mu_{1,1}}{2\mu_{1,1} + B_1^n \mu_{1,2}} \right) \overline{Q}.
$$

Which satisfies $P_1 \leq a_1 \overline{Q}$ given our assumption. Hence, the solution to problem (P) is indeed

$$
P_1^* = \left(\frac{B_1^n \mu_{1,1}}{2\mu_{1,1} + B_1^n \mu_{1,2}} \right) \overline{Q} = \left(\frac{B_2^{n+1} \mu_{1,1}}{2\mu_{1,1} + B_2^{n+1} \mu_{1,2}} \right) \overline{Q}. \tag{16}
$$

Substituting (16) in the objective function of the previous minimization problem, we get our desired expression

$$
\begin{aligned}
C_{n+1}^{pp}(\overline{Q}) &= \left(\frac{B_2^{n+1} \mu_{1,1} \overline{Q}}{2\mu_{1,1} + B_2^{n+1} \mu_{1,2}} \right)^2 \mu_{1,1} + \frac{B_2^{n+1}}{2} \mathbb{E}_{\theta_1} \left\{ \left(\overline{Q} - \left(\frac{B_2^{n+1} \mu_{1,1} \overline{Q}}{2\mu_{1,1} + B_2^{n+1} \mu_{1,2}} \right) \frac{1}{\theta_1} \right)^2 \right\} \\
&= \left(B_2^{n+1} - \frac{(B_2^{n+1})^2 \mu_{1,1}^2}{2\mu_{1,1} + B_2^{n+1} \mu_{1,2}} \right) \frac{\overline{Q}^2}{2} \\
&= B_1^{n+1} \frac{\overline{Q}^2}{2},
\end{aligned}
$$

where the second equality follows simply from rearranging and using the definition of $\mu_{1,1}$ and $\mu_{1,2}$. ∎

In the common distribution scenario, we can provide a more intuitive and very easy to check condition that ensures the analytical characterization found in the previous theorem holds.

[9] We assume for expositional purposes that the new firm is added "first in the line" (for any other assumption regarding the "place" of the new firm in the ordering we can find analogous expressions for the optimal mechanism and its cost)

Corollary 1 *If all firms have the same type distributions (i.e. $a_i = a$, $b_i = b$ and $F_i = F$ for all $i \in \{1, ..., k\}$), the previous recursive characterization of the sequential posted prices mechanism holds if*

$$b\left(1 - \frac{\mu_1}{\mu_2}a\right) \leq 2a \tag{17}$$

where $\mu_1 = \mu_{i,1}$ and $\mu_2 = \mu_{i,2}$ for all $i \in \{1, ..., k\}$.

Proof As $a_j = a$ for all $j \in \{1, ..., k\}$, the condition in Proposition 6 can be written as

$$\frac{B^k_{j+1}\mu_1}{2\mu_1 + B^k_{j+1}\mu_2} \leq a.$$

Rearranging, we get

$$B^k_{j+1}\left(1 - \frac{\mu_1}{\mu_2}a\right) \leq 2a$$

It's straightforward to see that $B^k_j < B^k_{j+1}$ for all $j \in \{1, ..., k-1\}$, so it's enough to show the previous condition holds for $j = k-1$. Using that $B^k_k = b$, the previous condition becomes

$$b\left(1 - \frac{\mu_1}{\mu_2}a\right) \leq 2a,$$

as desired. ∎

Note that the recursive nature of the problem makes the optimal prices and expected costs very easy to compute (both amount to iterating an operator that involves only a single variable integral of a simple expression) and to implement (at each period, the buyer just looks at their remaining to-buy quantity and the number/distributions of the subsequent firms and announces a per unit price so that she will buy all the quantity offered by the correspondent firm at that price). Hence, sequential posted prices appears to be a very good alternative to the optimal mechanism when faced with implementability constraints (the buyer could simply arrange the firms in order instead of facing them all at once and proceed to implement this mechanism[10]).

Nevertheless, the essential features of a posted price menu seem problematic: (1) Its linearity prevents tailoring the price-per-unit to different quantities, which as it is well known, is crucial for optimality in convex costs environments and (2) Its ex-ante character prevents the possibility of conditioning on the realization of the cost parameter, thus having to rely on expected values which can be extremely inefficient, particularly when the heterogeneity in costs is high (i.e. the range of the support is high). These serious shortcomings of the sequential posted prices mechanism suggest its performance can easily be much worse than the optimal mechanism's, thus giving rise to the question: can we find a mechanism that offers a better balance on the trade-off between ease of implementation and performance? (putting it differently, can we find a less costly mechanism if we can actually handle more complexity but not as much as the optimal mechanism requires?) In the following section, we will study a natural mechanism that address this issue.

[10] In this setting, a natural question regarding the optimal order for the buyer arises (in the heterogenous distributions setup, naturally, otherwise all firms are ex ante identical and thus the order doesn't matter). This question exceeds the scope of this article and so we will deal only with exogenously given orderings.

4.2 An Optimal Sequential Mechanism

After studying the sequential posted prices mechanism and noting its simplicity and lack of "tailoring" stressed at the end of the previous section, a natural question arises: what is the optimal way of buying $\overline{Q}$ if firms are faced sequentially but the buyer is not restricted to use linear prices? In other words, what is the optimal "sequential mechanism"? The following proposition recursively characterizes this mechanism.

Proposition 7 *Suppose without lost of generality that firms are ordered by their index i (i.e. firm 1 trades first, firm 2 second, etc...). The total expected cost for the buyer of the optimal sequential mechanism is*

$$C_k^{sm}(\overline{Q}) = A_1^k \frac{\overline{Q}^2}{2},$$

where $A_n^n = b_n$ *and* $A_j^n = \mathbb{E}_{\theta_j}\left(\frac{1}{J_j^{-1}(\theta_j) + (A_{j+1}^n)^{-1}}\right)$ *for every* $j < n$*. Moreover, the quantity sold by firm* $i < n$ *is given by*

$$q_i(\theta_i) = \left(\frac{J_i^{-1}(\theta_i)}{J_i^{-1}(\theta_i) + (A_{i+1}^n)^{-1}}\right)\overline{Q}_i,$$

where $\overline{Q}_i$ *is the remaining quantity to be bought after the first* $i-1$ *firms have sold and thus given by* $\overline{Q}_1 = \overline{Q}$ *and*

$$\overline{Q}_i = \overline{Q} - \sum_{j=1}^{i-1} q_j(\theta_j),$$

for $i > 1$*.*

Proof
We will also prove this by induction. First, it is straightforward to note that for $k = 1$, the cost of buying $\overline{Q}$ units is given by

$$C_1(\overline{Q}) = b_1 \frac{\overline{Q}^2}{2} = A_1^1 \frac{\overline{Q}^2}{2},$$

while the quantity bought to the only selling firm is, of course, $q_1 = \overline{Q}$. Now, assume that for $k = n$ the total expected cost of the sequential mechanism is

$$C_n(\overline{Q}) = A_1^n \frac{\overline{Q}^2}{2},$$

and the quantity bought to each firm $i < n$ is given by

$$q_i(\theta_i) = \left(\frac{J_i^{-1}(\theta_i)}{J_i^{-1}(\theta_i) + (A_{i+1}^n)^{-1}}\right)\overline{Q}_i.$$

As in Proposition 6, we assume for expositional purposes that when $k = n + 1$ the "additional" firm regarding the previous scenario of $k = n$ gets to sell first. Thus, given that we can restrict our mechanisms' space to direct and truth-telling ones as before, when facing the new firm the buyer solves the following problem

$$\min_{\substack{q_1:\Theta_1\to\mathbb{R}\\ t_1:\Theta_1\to\mathbb{R}}} \int_{\Theta_1} \left[t_1(\theta_1) + A_1^n \frac{(\overline{Q} - q_1(\theta_1))^2}{2} \right] f_1(\theta_1) d\theta_1$$

subject to

$$t_1(\theta_1) - \theta_1 \frac{q_1^2(\theta_1)}{2} \geq t_1(\theta_1') - \theta_1 \frac{q_1^2(\theta_1')}{2} \text{ for all } \theta_1, \theta_1' \in \Theta_1$$

$$t_1(\theta_1) - \theta_1 \frac{q_1^2(\theta_1)}{2} \geq 0 \text{ for all } \theta_1 \in \Theta_1,$$

$$\overline{Q} \geq q_1(\theta) \geq 0 \text{ for all } \theta_1 \in \Theta_1.$$

Using an argument analogous to the one in Proposition 2's proof regarding the IC constraint (only now there's no need to take expectations over θ_i in the utility, the transfer and the quantities), it can be shown that the problem can be reduced to

$$\min_{q_1:\Theta_1\to\mathbb{R}} \int_{\Theta_1} \left[\left(\theta_1 + \frac{F_1(\theta_1)}{f_1(\theta_1)} \right) \frac{q_1^m(\theta_1)}{m} + A_1^n \frac{(\overline{Q} - q_1(\theta_1))^2}{2} \right] f_1(\theta_1) d\theta_1$$

subject to

$$q_1^2(\cdot) \text{ is decreasing,}$$

$$\overline{Q} \geq q_1(\theta) \geq 0 \text{ for all } \theta_1 \in \Theta_1.$$

Leaving aside the first restriction, we can solve the unrestricted version of the preceding problem point-wise. Noting that this problem is completely analogous to the full information one with $k = 2$, substituting θ_1 by $J_1(\theta_1)$ and θ_2 by A_1^n, we find the optimal allocation rule,

$$q_1(\theta_1) = \left(\frac{J_1^{-1}(\theta_1)}{J_1^{-1}(\theta_1) + (A_1^n)^{-1}} \right) \overline{Q} = \left(\frac{J_1^{-1}(\theta_1)}{J_1^{-1}(\theta_1) + (A_2^{n+1})^{-1}} \right) \overline{Q}_1,$$

where the second equality follows simply by noting that adding a firm "first in the line" to the k firms scenario makes all the $A's$ to change index for the subsequent one. It is straightforward to see that given our regularity assumption, this function is decreasing on θ_1. Thus, it is the solution to the original restricted problem. Note also that, for $i > 1$,

$$q_i(\theta_i) = \left(\frac{J_i^{-1}(\theta_i)}{J_i^{-1}(\theta_i) + (A_{i+1}^n)^{-1}} \right) \overline{Q_i} = \left(\frac{J_i^{-1}(\theta_i)}{J_i^{-1}(\theta_i) + (A_{i+1}^{n+1})^{-1}} \right) \overline{Q}_i,$$

where now the index i of every firm in the n firm setting is substituted by the subsequent number (firm 2 is now 3, 3 is now 4, etc...). Finally, to find the expected costs, we need only to substitute the allocation rule we found in the expression we proved to be equal to the expected sum of transfers, namely

$$\frac{1}{2} \mathbb{E}_{\theta_1} \left\{ J_1(\theta_1) q_1^2(\theta_1) + A_2^{k+1} (\overline{Q} - q_1(\theta_1))^2 \right\}.$$

After substituting and a bit of algebraic manipulation, the previous expression becomes

$$\frac{1}{2} \mathbb{E}_{\theta_1} \left(\frac{1}{J_1^{-1}(\theta_1) + (A_2^{n+1})^{-1}} \right) \frac{\overline{Q}^2}{2} = A_1^{n+1} \frac{\overline{Q}^2}{2}.$$

■

Note that the allocation rule of this mechanism has a structure that is very similar to the optimal one's, the only difference being that it considers "two sellers at the time" at each stage, the actual "contemporaneous firm" and a virtual competitor with a cost parameter given by the expected cost of delaying the purchase of a unit (thus incorporating the number of remaining firms and the expected future realizations of their types in a sequential fashion). The computation of this mechanisms' costs and allocation rules turns out to be almost as straightforward as posted prices (it too amounts to the iteration of a simple operator, albeit a different, slightly more complex, one). Moreover, even though it does not have an interpretation as natural as posted prices, it is considerably simpler than the optimal mechanism, as the transfer-quantity menus offered to the potential sellers rely only on two key variables, a backward looking one (the remaining to-buy quantity to meet the original goal) and a forward looking one (the already mentioned "type" of the virtual competitor, summarizing all future expectation of costs faced by the buyer), which are constant at each selling stage. Thus, like sequential posted prices and in contrast to the optimal mechanism, this sequential mechanism avoids the complexity of the transfer-quantity menus in terms of the simultaneous dependence on all cost parameters (though note that the transfers' complexity and consequent "unimplementability" through familiar pricing strategies is still an issue with this mechanism, not so with posted prices). To sum up, the optimal sequential mechanism turns out to be simpler to implement than he optimal one without sacrificing some key features of it.

Clearly this mechanism performs at least weakly better than the sequential prices one, as the latter is included in the feasible space of the former (i.e. the mechanism designer can choose to use sequential posted prices). Moreover, as this mechanism takes advantage of the convex nature of the costs by using non linear pricing and offers menus contingent on the realization of the cost parameters, it seems natural to think it performs strictly better, at least for some distributions. In the next section we will use numerical analysis in order to asses whether this is the case and compare both mechanisms' performance to the optimal one's for different parametrizations and distributions of cost types.

5. Comparing Mechanisms: Numerical Analysis

Take $\overline{Q} = 1$[11] and consider the common distribution scenario. We simulated the total cost of the previous mechanisms for a varying number of firms, different supports and distributions using the Monte Carlo method for computing the expectations (we used a sample of $m = 500000$ draws). In particular, we used supports of the form $[i, i+1]$ with $i \in \{1, ..., 100\}$ number of firms ranging from 1 to 10 and the following distributions for θ [12]: (a) Triangular with three different modes, $c \in \{i, (2i+1)/2, i+1\}$; (b) Power Function with $\beta \in \{0.5, 1, 2, 5\}$, (c) Parabolic; (d) Inverse Parabolic; (e) Truncated Normal with $\mu = (2i+1)/2$ (average) and $\sigma \in \{0.5, 1, 10, 100\}$. The following tables and Figure 1 illustrate the results of the simulations (all these consider

[11]This particular assumption is harmless regarding the generality of the numerical analysis, as the cost of all the considered mechanisms are a monomials of order two with respect to $\overline{Q}$ and so the percentual difference of the costs are constant in $\overline{Q}$.

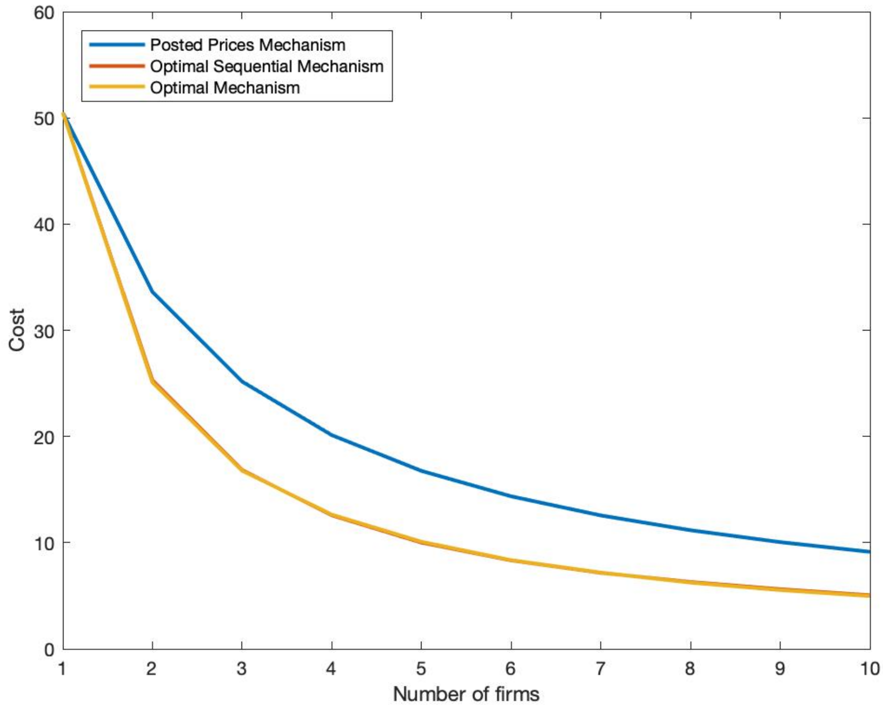
[12]It was verified these distributions satisfy the condition in Corollary 1 and so our analytical characterization holds.

supp = $[100, 101]$, the results for different supports are quite similar)[13]:

Table 1: Proportional excess cost over the optimal mechanism: Triangular dist. (average c)

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	0.16	32.99
3	0.04	49.59
4	0.95	60.38
5	0.73	67.15
6	0.53	71.52
7	0.77	76.02
8	1.31	79.52
9	1.70	82.82
10	1.76	85.08

Figure 1: Costs of Mechanisms: Triangular distribution (average c)



[13]Results in tables are shown up to 2 decimals and negative differences of order up to 10^{-3} are shown as 0 as they are a result of the approximation error.

Table 2: Proportional excess cost over the optimal mechanism: Power Function distr. ($\beta = 1$)

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	0.00	33.11
3	0.00	49.63
4	0.00	59.52
5	0.00	66.12
6	0.00	70.82
7	0.00	74.35
8	0.00	77.10
9	0.00	79.29
10	0.00	81.11

Table 3: Proportional excess cost over the optimal mechanism: Parabolic distribution

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	0.68	33.60
3	0.78	50.71
4	0.74	59.82
5	1.07	67.37
6	0.91	72.34
7	3.34	79.02
8	0.00	73.40
9	0.00	76.66
10	0.00	73.76

Table 4: Proportional excess cost over the optimal mechanism: Inverse Parabolic distribution

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	1.36	34.42
3	1.14	51.10
4	1.03	60.96
5	1.29	67.69
6	1.83	73.49
7	1.51	77.22
8	1.63	80.13
9	1.63	82.26
10	1.78	84.61

Table 5: Proportional excess cost over the optimal mechanism: Truncated Normal distribution ($(\mu, \sigma) = (100.5, 10)$))

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	0.00	33.12
3	0.00	49.63
4	0.00	59.53
5	0.00	66.12
6	0.00	70.83
7	0.00	74.35
8	0.00	77.10
9	0.00	79.29
10	0.00	81.09

Additionally, we performed the numerical exercise for supports of the form $[11 - i, 11 + i]$ with $i \in \{1, ..., 10\}$[14], and the qualitative result regarding the relative performances of the optimal sequential and the posted prices mechanism stands, though in a less pronounced way[15]. It should be noted, nevertheless, that the condition of Corollary 1 holds only up to a certain $i < 10$ (which one depends on the and parametrization considered), so our assessment of the performance of the posted prices mechanism is impaired for large supports. As an example, see Table 6 and Figure 2 (the results for the remaining distributions and parametrizations are available upon request).

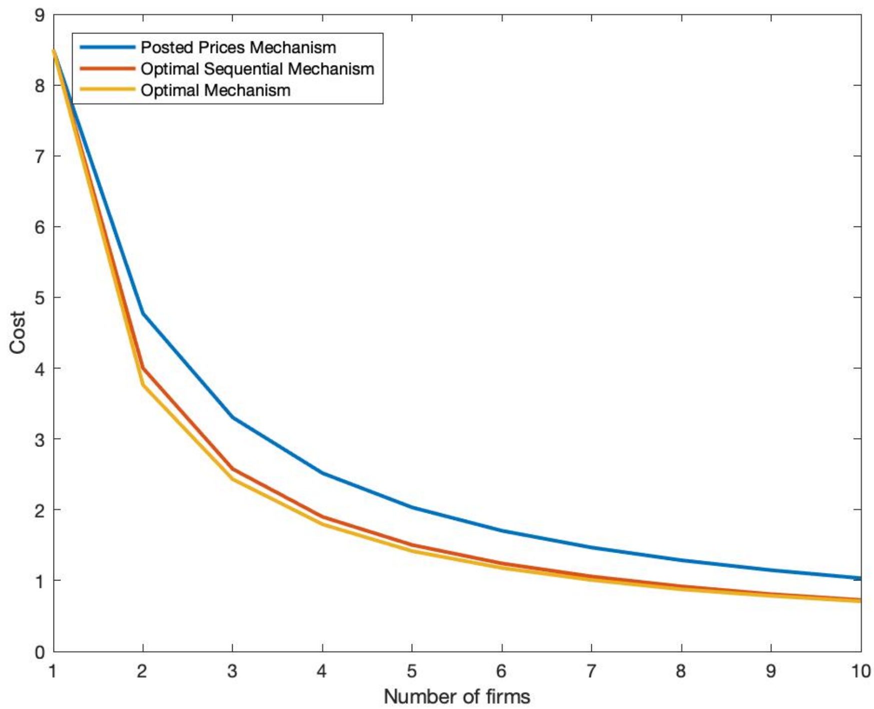
Table 6: Proportional excess cost over the optimal mechanism: Triangular dist. (average c), $supp(F) = [5, 17]$

Mechanism / No. of firms	Optimal Sequential (%)	Sequential Posted Prices (%)
1	0.00	0.00
2	6.31	26.81
3	5.95	35.90
4	5.89	40.26
5	6.12	43.54
6	5.45	44.63
7	4.95	45.50
8	4.74	46.84
9	3.09	46.33
10	2.65	46.12

[14]For most applications, it does not seem reasonable to assume the less efficient firm is more than 20 times more costly than the most efficient one.

[15]Unfortunately, we can't assert this for the truncated normal distribution, as, for the time being, we lack computational power to compute the costs of the optimal and sequential mechanism under this distribution of costs, even for the simplest parametrizations.

Figure 2: Costs of Mechanisms: Triangular distribution (average c, Support $[5, 17]$)



Clearly, the results above suggest that, at least in the common distribution setting, the standard optimal mechanism and the optimal sequential one have a very similar performance. On the other hand, the sequential posted prices mechanism performs a lot worse than its optimal counterpart. It should be noted that even though this is only a conjecture for the time being, the performance in the chosen distributions constitutes a strong argument in favor of it, as they include cases where most mass is (a) in the middle of the distribution (parabolic, triangular with average mode and truncated normal); (b) in either the right or left section of the support (triangular with right/left mode respectively and power function with $\beta > 1$ and $\beta < 1$ respectively); (c) in both extremes of the support (inverse parabolic). Moreover, we consider a reasonable number of firms (it's not difficult to get to $k = 20$, which is more than enough for most potential applications), different degrees of variance (truncated normal distribution), different "sizes" of costs (i.e. different scales of the support) and different degrees of heterogeneity among firms (as measured by the range of the support).

6. Concluding Remarks

In this article, we tackled the problem of buying a fixed quantity of items in a convex cost environment with incomplete information. We characterized the optimal mechanism and proposed two alternative ones that are simpler to implement and thus potentially more relevant for actual procurement design. Moreover, we showed through numerical analysis that the expected costs of one of our alternative mechanisms closely approximates the cost of the optimal one, while posted prices, performs much worse.

There are several possible avenues for further research. The most immediate one is, without much doubt, tackling analytically the problem we only addressed numerically. That is, finding an analytical bound on the difference in the expected costs of the optimal and the optimal sequential mechanism. Our numerical results suggest that this bound should be quite tight (and, moreover, should depend on the range of the support of the types distribution), though this might not be the case for pathological distributions that fail to be qualitatively similar to the ones we simulated in our numerical exercises.

Another natural and potentially fruitful direction ahead regards the conceptual issue of "implementation simplicity". It is clear that our notion of simple mechanisms is vague and, to the best of our knowledge, it does not exist any formal definition that precisely captures what we regard as simple (as we mentioned in the introduction, there is a large literature concerning simplicity in mechanism design, but it mostly focuses on different —yet related— issues). It would be quite valuable to elaborate on this in order to 1) Clarify what features of a mechanism are really of practical importance in environments such as the one we study and 2) Determine whether there are other mechanisms that perform better than our proposed sequential mechanism in the simplicity-cost trade off.

Finally, in a more practical vein, a natural next step would be to explore the impact of the firms' orderings in the performance of the optimal sequential mechanism, both from the perspective of the buyer and of individual firms. In the first case, it would be valuable to understand how does the expected cost of the buyer varies when the firm's ordering changes, eventually characterizing the optimal ordering for different families of distributions. In the second, it would be interesting to study the firms' incentives if the order is to be determined endogenously through a bayesian game[16] (either from an ex-ante and an interim perspective). Both of these approaches would deliver insights to inform actual buying policies (for example, if firms are all willing to go first, there could be room for further cost minimization by extracting rents through auctioning the positions in the ordering).

[16]Clearly there would be a trade off from the point of view of the firm, as the buyer is willing to pay more per quantity sold when there are few firms left, but chances are the quantity left to be bought would be slim.

7. Appendix

7.1 General Monomial Costs

Consider the same problem than before but with the firms' cost functions given by

$$C(Q;\theta) = \theta \frac{Q^m}{m},$$

where $m \geq 2$. The following propositions show that we can characterize the optimal and the optimal sequential mechanism in a way analogous to the quadratic environment considered in the main body of the article (the sequential posted prices mechanism does not have a closed form expression for its costs or optimal prices for $m > 2$).

Proposition 8 *If regularity holds, the optimal mechanism given the previous monomial cost function entails the following allocation rule*

$$q_i(\boldsymbol{\theta}) = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{\sum_{j=1}^{k} J_j(\theta_j)^{-1/(m-1)}} \right) \overline{Q}, \tag{18}$$

where, as before, $J_i(\theta_i) = \theta_i + \dfrac{F_i(\theta_i)}{f_i(\theta_i)}$.
The transfers (in their point-wise form) are, on the other hand, given by

$$t_i(\boldsymbol{\theta}) = \frac{\theta_i}{m} q_i^m(\boldsymbol{\theta}) + \frac{1}{2} \int_{\theta_i}^{b_i} q_i^m(s_i, \theta_{-i}) ds_i.$$

Additionally, the expected total cost of this mechanism is given by

$$C_k^{om}(\overline{Q}) = \mathbb{E}_{\boldsymbol{\theta}} \left(\frac{1}{\sum_{j=1}^{k} J_j(\theta_j)^{-1/(m-1)}} \right)^{m-1} \frac{\overline{Q}^m}{m}. \tag{19}$$

Proof
The proof is analogous to the one of proposition 2. Again, as the revelation principle holds, the buyer needs only to consider direct mechanisms where it is optimal for every seller to report their types truthfully.

Hence, the problem the buyer solves is given by

$$\min_{\substack{q:\Theta\to\mathbb{R}^k \\ t:\Theta\to\mathbb{R}^k}} \int_\Theta \left[\sum_{i=1}^k t_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

subject to

$$\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i \frac{q_i^m(\theta_i,\theta_{-i})}{m}\right] \geq \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i',\theta_{-i}) - \theta_i \frac{q_i^m(\theta_i',\theta_{-i})}{m}\right]$$

for all $\theta_i, \theta_i' \in \Theta_i$ and all $i \in \{1, ..., k\}$

$$\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i \frac{q_i^m(\theta_i,\theta_{-i})}{m}\right] \geq 0 \text{ for all } \theta_i \in \Theta_i \text{ and all } i \in \{1, ..., k\},$$

$$\sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta} \in \Theta,$$

$$q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } i \in \{1, ..., k\} \text{ and all } \boldsymbol{\theta} \in \Theta.$$

The first constraint is the standard incentive compatibility one that imposes that it is optimal for every seller to reveal their true type given the mechanism (q, t). The second one is usually called individual rationality or voluntary participation and simply imposes that every seller wants to participate given their reservation utility which is normalized to zero here for simplicity.
Note that the Incentive Compatibility constraint can alternatively be written as

$$U_i(\theta_i) = \max_{\theta_i'} \left\{ T_i(\theta_i') - \mathbb{E}_{\theta_{-i}}\left[\theta_i \frac{q_i^m(\theta_i',\theta_{-i})}{m}\right]\right\}, \tag{20}$$

where $T(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i})$ is the expected transfer of the seller i given a mechanism (q, t) and that she announced her type was θ_i and the function $U_i : \Theta_i \to \mathbb{R}$ is defined as the expected utility of seller i given her type and the fact that she truthfully revealed it. In other words, this conditions says that given the mechanism (q, t) the highest expected utility a seller can attain is the one she gets revealing her true type. Note that U_i is here the maximum of a family of affine functions (of θ_i), hence it is convex. Every convex function is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Hence, by the Envelope Theorem, at every point where it is differentiable, we have

$$U_i'(\theta_i) = -\frac{1}{m}\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i,\theta_{-i})\}.$$

As U_i is convex, the previous expression implies that $\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i,\theta_{-i})\}$ must be a weakly decreasing function of θ_i. Moreover, by Corollary 1 in Milgrom and Segal (2002), at every point where U_i is differentiable and for every IC allocation rule $q_i(\theta_i, \theta_{-i})$, we have

$$U_i(\theta_i) = U_i(b_i) + \frac{1}{m}\int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i,\theta_{-i})\}ds_i. \tag{21}$$

Additionally, at every IC allocation rule $q_i(\theta_i, \theta_{-i})$,

$$U_i(\theta_i) = T_i(\theta_i) - \frac{\theta_i}{m}\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i,\theta_{-i})\}. \tag{22}$$

Putting (20) and (21) together, we can express the expected transfer as

$$T_i(\theta_i) = U_i(b_i) + \frac{\theta_i}{m}\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i, \theta_{-i})\} + \frac{1}{m}\int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\}ds_i. \tag{23}$$

Given this, we can rewrite the buyer's objective function as follows

$$\begin{aligned}
\int_\Theta \left[\sum_{i=1}^{k} t_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta} =& \sum_{i=1}^{k} \int_\Theta t_i(\boldsymbol{\theta}) f(\boldsymbol{\theta})d\boldsymbol{\theta}\\
=& \sum_{i=1}^{k} \int_{a_i}^{b_i} T_i(\theta_i) f_i(\theta_i)d\theta_i\\
=& \sum_{i=1}^{k} \left\{ \int_{a_i}^{b_i} \left\{ U_i(b_i) + \frac{\theta_i}{m}\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i, \theta_{-i})\} + \frac{1}{m}\int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\}ds_i \right\} f_i(\theta_i)d\theta_i \right\}.
\end{aligned}$$

By Fubini's Theorem, we can interchange the order of integration in the last integral, thus getting

$$\begin{aligned}
\int_{a_i}^{b_i}\int_{\theta_i}^{b_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\} f_i(\theta_i) ds_i d\theta_i &= \int_{a_i}^{b_i}\int_{a_i}^{s_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\} f_i(\theta_i) d\theta_i ds_i\\
&= \int_{a_i}^{b_i} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\} \int_{a_i}^{s_i} f_i(\theta_i) d\theta_i ds_i\\
&= \int_{a_i}^{b_i} F(s_i)\mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\} ds_i
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\int_\Theta \left[\sum_{i=1}^{k} t_i(\boldsymbol{\theta})\right] f(\theta)d\theta =& \frac{1}{m}\sum_{i=1}^{k} \int_{a_i}^{b_i} \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right) \mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i, \theta_{-i})\} f_i(\theta_i)d\theta_i + \sum_{i=1}^{n} U_i(b_i)\\
=& \frac{1}{m}\sum_{i=1}^{k} \mathbb{E}_{\boldsymbol{\theta}}\left\{ \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right) q_i^m(\boldsymbol{\theta}) \right\} + \sum_{i=1}^{n} U_i(b_i)\\
=& \frac{1}{m}\sum_{i=1}^{k} \int_\Theta \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right) q_i^m(\boldsymbol{\theta}) f(\boldsymbol{\theta})d\boldsymbol{\theta} + \sum_{i=1}^{n} U_i(b_i)\\
=& \frac{1}{m}\int_\Theta \sum_{i=1}^{k} \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right) q_i^m(\boldsymbol{\theta}) f(\boldsymbol{\theta})d\boldsymbol{\theta} + \sum_{i=1}^{n} U_i(b_i).
\end{aligned}$$

Moreover, we established that a direct incentive compatible mechanism is such that $\mathbb{E}_{\theta_i}\{q_i^m(\theta_i, \theta_{-i})\}$ is a decreasing function of θ_i and (22) holds. To see that the converse is also true, note first that for any $\theta_i, \theta_i' \in \Theta_i$, we have

$$\begin{aligned}
T_i(\theta_i') - \mathbb{E}_{\theta_{-i}}\left[\theta_i \frac{q_i^m(\theta_i', \theta_{-i})}{m}\right] =& T_i(\theta_i') - \mathbb{E}_{\theta_{-i}}\left[\theta_i' \frac{q_i^m(\theta_i', \theta_{-i})}{m}\right] - \mathbb{E}_{\theta_{-i}}\left[\frac{q_i^m(\theta_i', \theta_{-i})}{m}\right](\theta_i - \theta_i')\\
=& U_i(\theta_i') - \mathbb{E}_{\theta_{-i}}\left[\frac{q_i^m(\theta_i', \theta_{-i})}{m}\right](\theta_i - \theta_i')
\end{aligned}$$

So, incentive compatibility can be written as

$$U_i(\theta_i) \geq U_i(\theta_i') - \mathbb{E}_{\theta_{-i}}\left[\frac{q_i^m(\theta_i', \theta_{-i})}{m}\right](\theta_i - \theta_i'),$$

for all $\theta_i, \theta_i' \in \Theta_i$. Using (22) and rearranging, the previous inequality can be rewritten as

$$\int_{\theta_i}^{\theta_i'} \mathbb{E}_{\theta_{-i}}\{q_i^m(s_i, \theta_{-i})\}ds_i \geq \mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i', \theta_{-i})\}(\theta_i' - \theta_i),$$

which clearly holds if $\mathbb{E}_{\theta_{-i}}\{q_i^m(\theta_i, \theta_{-i})\}$ is decreasing. Hence a direct mechanism (q, t) is incentive compatible if and only if $\mathbb{E}_{\theta_i}\{q_i^m(\theta_i, \theta_{-i})\}$ is decreasing in θ_i and (22) holds. By the previous results, the original problem can be rewritten only in terms of the allocation rule as follows

$$\min_{q:\Theta\to\mathbb{R}^k} \int_\Theta \left[\sum_{i=1}^k \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\frac{q_i^m(\boldsymbol{\theta})}{m}\right] f(\boldsymbol{\theta})d\boldsymbol{\theta} + \sum_{i=1}^k U_i(b_i)$$

subject to

$$\mathbb{E}_{\theta_{-i}}[q_i^m(\cdot, \theta_{-i})] \text{ is decreasing for all } i \in \{1, ..., k\},$$
$$\sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta} \in \Theta,$$
$$q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } \boldsymbol{\theta} \in \Theta \text{ and all } i \in \{1, ..., k\}.$$

Clearly it is optimal for the buyer to set $U_i(b_i) = 0$ for all $i \in \{1, ..., k\}$. Thus, we can finally rewrite the problem as

$$\min_{q:\Theta\to\mathbb{R}^k} \int_\Theta \left[\sum_{i=1}^k \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\frac{q_i^m(\boldsymbol{\theta})}{m}\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

subject to

$$\mathbb{E}_{\theta_{-i}}[q_i^m(\cdot, \theta_{-i})] \text{ is decreasing for all } i \in \{1, ..., k\},$$
$$\sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta},$$
$$q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } \boldsymbol{\theta} \in \Theta \text{ and all } i \in \{1, ..., n\}.$$

Forgetting about the first restriction, we can solve the unrestricted version of the preceding problem point-wise, finding thus the optimal allocation rule

$$q_i(\boldsymbol{\theta}) = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{\sum_{j=1}^k J_j(\theta_j)^{-1/(m-1)}}\right)\overline{Q}. \tag{24}$$

It is straightforward to see that given our regularity assumption, this function is decreasing on θ_i. Thus, it is the solution to the original restricted problem. On the other hand, the expression for the transfers is really only one of the possible ones, as both agents are risk-neutral and, as we showed, what is fixed given the allocation rule is $T_i(\theta_i)$, the expected value of the transfer taken with respect to the types of the other sellers. Finally, the expression for the costs follows directly by substituting the allocation rule in the expression we proved to be equal to the expected sum of transfers, namely

$$\frac{1}{2}\sum_{i=1}^k \mathbb{E}_{\boldsymbol{\theta}}\left\{\left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right) q_i^2(\boldsymbol{\theta})\right\}.$$

$\blacksquare$

Proposition 9 *Suppose without lost of generality that firms are ordered by their index i (i.e. firm 1 trades first, firm 2 second, etc...). The total expected cost for the buyer of the optimal sequential mechanism is*

$$C_k^{sm}(\overline{Q}) = A_1^k \frac{\overline{Q}^m}{m},$$

where $A_n^n = b_n$ and $A_j^n = \mathbb{E}_{\theta_j}\left(\frac{1}{J_j(\theta_j)^{-1/(m-1)} + (A_{j+1}^n)^{-1/(m-1)}}\right)$ for every $j < n$. Moreover, the quantity sold by firm $i < n$ is given by

$$q_i(\theta_i) = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{J_i(\theta_i)^{-1/(m-1)} + (A_{i+1}^n)^{-1/(m-1)}}\right)\overline{Q}_i,$$

where $\overline{Q}_i$ is the remaining quantity to be bought after the first $i-1$ firms have sold and thus given by $\overline{Q}_1 = \overline{Q}$ and

$$\overline{Q}_i = \overline{Q} - \sum_{j=1}^{i-1} q_j(\theta_j),$$

for $i > 1$.

Proof

As all the other propositions regarding the sequential setting, we will prove this by induction. First, it is straightforward to note that for $k = 1$, the cost of buying $\overline{Q}$ units is given by

$$C_1(\overline{Q}) = b_1 \frac{\overline{Q}^m}{m} = A_1^1 \frac{\overline{Q}^m}{m},$$

while the quantity bought to the only selling firm is, of course, $q_1 = \overline{Q}$. Now, assume that for $k = n$ the total expected cost of the sequential mechanism is

$$C_n(\overline{Q}) = A_1^n \frac{\overline{Q}^m}{m},$$

and the quantity bought to each firm $i < n$ is given by

$$q_i(\theta_i) = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{J_i(\theta_i)^{-1/(m-1)} + (A_{i+1}^n)^{-1/(m-1)}}\right)\overline{Q}_i.$$

Given this, when $k = n + 1$, the problem the buyer solves when facing the first firm is the following[17]

$$\min_{\substack{q_1:\Theta_1\to\mathbb{R}\\ t_1:\Theta_1\to\mathbb{R}}} \int_{\Theta_1} \left[t_1(\theta_1) + A_1^n \frac{(\overline{Q}-q_1(\theta_1))^m}{m} \right] f_1(\theta_1) d\theta_1$$

subject to

$$t_1(\theta_1) - \theta_1 \frac{q_1^m(\theta_1)}{m} \geq t_1(\theta_1') - \theta_1 \frac{q_1^m(\theta_1')}{m} \text{ for all } \theta_1, \theta_1' \in \Theta_1$$
$$t_1(\theta_1) - \theta_1 \frac{q_1^m(\theta_1)}{m} \geq 0 \text{ for all } \theta_1 \in \Theta_1,$$
$$\overline{Q} \geq q_1(\theta) \geq 0 \text{ for all } \theta_1 \in \Theta_1.$$

Using an argument analogous to the one in the proof of Proposition 2 regarding the IC constraint (only now there's no need to take expectations over θ_i in the utility, the transfer and the quantities), it can be shown that the problem can be reduced to

$$\min_{q_1:\Theta_1\to\mathbb{R}} \int_{\Theta_1} \left[\left(\theta_1 + \frac{F_1(\theta_1)}{f_1(\theta_1)}\right) \frac{q_1^m(\theta_1)}{m} + A_1^n \frac{(\overline{Q}-q_1(\theta_1))^m}{m} \right] f_1(\theta_1) d\theta_1$$

subject to

$$q_1^m(\cdot) \text{ is decreasing,}$$
$$\overline{Q} \geq q_1(\theta_1) \geq 0 \text{ for all } \theta_1 \in \Theta_1.$$

Leaving aside the first restriction, we can solve the unrestricted version of the preceding problem point-wise, finding thus the optimal allocation rule:

$$q_1(\theta_1) = \left(\frac{J_1(\theta_1)^{-1/(m-1)}}{J_1(\theta_1)^{-1/(m-1)} + (A_1^n)^{-1/(m-1)}} \right) \overline{Q} = \left(\frac{J_1(\theta_1)^{-1/(m-1)}}{J_1(\theta_1)^{-1/(m-1)} + (A_2^{n+1})^{-1/(m-1)}} \right) \overline{Q}_1,$$

where the second equality follows simply by noting that adding a firm "first in the line" to the k firms setting makes all the $A's$ to change index for the subsequent one. It is straightforward to see that given our regularity assumption, this function is decreasing on θ_1. Thus, it is the solution to the original restricted problem. Note also that, for $i > 1$,

$$q_i(\theta_i) = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{J_i(\theta_i)^{-1/(m-1)} + (A_{i+1}^n)^{-1/(m-1)}} \right) \overline{Q}_i = \left(\frac{J_i(\theta_i)^{-1/(m-1)}}{J_i(\theta_i)^{-1/(m-1)} + (A_{i+1}^{n+1})^{-1/(m-1)}} \right) \overline{Q}_i,$$

where now the index i of every firm in the n firm setting is substituted by the subsequent number (firm 2 is now 3, 3 is now 4, etc...). Finally, to find the expected costs, we need only to substitute the allocation rule we found in the expression we proved to be equal to the expected sum of transfers, namely

$$\frac{1}{m} \mathbb{E}_{\theta_1} \left\{ J_1(\theta_1) q_1^m(\theta_1) + A_2^{n+1} (\overline{Q} - q_1(\theta_1))^m \right\}.$$

After substituting and a bit of algebraic manipulation, the previous expression becomes

$$\mathbb{E}_{\theta_1} \left(\frac{1}{J_1(\theta_1)^{-1/(m-1)} + (A_2^{n+1})^{-1/(m-1)}} \right) \frac{\overline{Q}^m}{m} = A_1^{n+1} \frac{\overline{Q}^m}{m}.$$

■

[17] Again, we assume without lost of generality that the new firm is added "first in the line" for expositional purposes.

7.2 Omitted Proofs

Proof of Proposition 3

As in all previous settings, the revelation principle holds and thus the buyer needs only to consider direct mechanisms where it is optimal for every seller to report their types truthfully. Hence, the problem the buyer solves is given by

$$\min_{\substack{q:\Theta\to\mathbb{R}^k\\ t:\Theta\to\mathbb{R}^k}} \int_\Theta \left[\sum_{i=1}^k t_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$\text{subject to} \quad \begin{aligned} &\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i q_i(\theta_i,\theta_{-i})\right] \geq \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i',\theta_{-i}) - \theta_i q_i(\theta_i',\theta_{-i})\right] \\ &\text{ for all } \theta_i,\theta_i' \in \Theta_i \text{ and all } i\in\{1,...,k\}, \\ &\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i q_i(\theta_i,\theta_{-i})\right] \geq 0 \text{ for all } \theta_i\in\Theta_i \text{ and all } i\in\{1,...,k\}, \\ &\sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta}\in\Theta, \\ &c \geq q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } i\in\{1,...,k\} \text{ and all } \boldsymbol{\theta}\in\Theta. \end{aligned}$$

By a procedure analogous to the one in the proof of Proposition 2 regarding the characterization of incentive compatibility and the use's of Fubini's Theorem to write the expected sum of transfers conveniently, we can both write the expected transfer of any incentive compatible mechanism as

$$T_i(\theta_i) = \theta_i \cdot \mathbb{E}_{\theta_{-i}}\{q_i(\theta_i,\theta_{-i})\} + \int_{\theta_i}^b \mathbb{E}_{\theta_{-i}}\{q_i(s_i,\theta_{-i})\}ds_i. \tag{25}$$

and write the buyer's problem only in terms of the allocation rule as follows

$$\min_{q:\Theta\to\mathbb{R}^k} \int_\Theta \left[\sum_{i=1}^k \left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right) q_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$\text{subject to} \quad \begin{aligned} &\mathbb{E}_{\theta_{-i}}[q_i(\cdot,\theta_{-i})] \text{ is decreasing for all } i\in\{1,...,k\}, \\ &\sum_{i=1}^k q_i(\boldsymbol{\theta}) \geq \overline{Q} \text{ for all } \boldsymbol{\theta}, \\ &c \geq q_i(\boldsymbol{\theta}) \geq 0 \text{ for all } \boldsymbol{\theta}\in\Theta \text{ and all } i\in\{1,...,n\}. \end{aligned}$$

Leaving aside the first restriction, we can solve the unrestricted version of the preceding problem point-wise. Noting that this problem is a linear one, the solution is apparent: ordering firms by virtual costs from lowest to highest and buying in ascending order until completing the quantity $\overline{Q}$. Formally, this is expressed as

$$q_i(\boldsymbol{\theta}) = \begin{cases} c & \text{if } c\cdot(|\mathcal{J}_i(\boldsymbol{\theta})|+1) \leq \overline{Q} \\ \overline{Q} - c\cdot|\mathcal{J}_i(\boldsymbol{\theta})| & \text{if } c\cdot(|\mathcal{J}_i(\boldsymbol{\theta})|+1) > \overline{Q} \text{ and } c\cdot|\mathcal{J}_i(\boldsymbol{\theta})| < \overline{Q} \\ 0 & \text{if } c\cdot|\mathcal{J}_i(\boldsymbol{\theta})| \geq \overline{Q}, \end{cases}$$

where

$$\mathcal{J}_i(\boldsymbol{\theta}) \equiv \{j \in \{1,...,k\} \setminus \{i\} \mid \theta_j \leq \theta_i\}.$$

Note that for any $\theta_i' > \theta_i$, $\mathcal{J}_i(\theta_i, \theta_{-i}) \subseteq \mathcal{J}_i(\theta_i', \theta_{-i})$ for every $\theta_{-i} \in \Theta_{-i}$ and every $i \in \{1,...,k\}$. Thus, this allocation rule is decreasing in θ_i for any $\theta_{-i} \in \Theta_{-i}$ and hence it is the solution to the original restricted problem. Now, substituting the allocation rule in the point-wise version of (25), we get

$$t_i(\boldsymbol{\theta}) = \begin{cases} \theta_i \cdot c + \displaystyle\int_{\theta_i}^{\theta_i^*(\boldsymbol{\theta})} c ds_i + \int_{\theta_i^*(\boldsymbol{\theta})}^{b_i} 0 ds_i & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) \leq \overline{Q} \\ \theta_i \cdot (\overline{Q} - c \cdot |\mathcal{J}_i(\boldsymbol{\theta})|) & \\ + \displaystyle\int_{\theta_i}^{\theta_i^*(\boldsymbol{\theta})} (\overline{Q} - c \cdot |\mathcal{J}_i(s_i, \theta_{-i})|) ds_i + \int_{\theta_i^*(\boldsymbol{\theta})}^{b_i} 0 ds_i & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) > \overline{Q} \text{ and } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| < \overline{Q} \\ 0 & \text{if } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| \geq \overline{Q}, \end{cases}$$

where $\theta_i^*(\boldsymbol{\theta})$ is the unique type such that any realization of type of firm i higher than it would have made optimal for the buyer not buying any positive quantity from firm i, keeping fixed the types of the other firms θ_{-i} (this always exists because of the assumption that it is always possible to buy all the required quantity from $k-1$ firms). Intuitively, note that $\theta_i^*(\boldsymbol{\theta})$ must correspond to the lowest type of the vector of type realizations $\boldsymbol{\theta}$ such that the firm with that type taken together with the firms with lower types are able to meet the demanded quantity $\overline{Q}$. Formally, $\theta_i^*(\boldsymbol{\theta}) \in \{\theta_n\}_{n=1}^k$ such that

$$c \cdot |\mathcal{J}_i(\theta_i^*, \theta_{-i})| < \overline{Q}$$

$$c \cdot (|\mathcal{J}_i(\theta_i^*, \theta_{-i})| + 1) \geq \overline{Q}.$$

Now, note that for every $m, n \in \{1, ...k\}$ such that $\theta_m, \theta_n \leq \theta_i^*(\boldsymbol{\theta})$, $|\mathcal{J}_m(\theta_i^*, \theta_{-i})| = |\mathcal{J}_n(\theta_i^*, \theta_{-i})|$ and thus, by construction, $\theta_m^* = \theta_n^*$. On the other hand, by construction of $\theta_i^*(\boldsymbol{\theta})$, if $q_i(\boldsymbol{\theta}) > 0$, every θ_m such that $q_m(\boldsymbol{\theta}) > 0$ is such that $\theta_m \leq \theta_i^*(\boldsymbol{\theta})$. Taken together, these two claim imply that $\theta_i^*(\boldsymbol{\theta})$ is the same for every $i \in \{1,...,k\}$ such that $q_i(\boldsymbol{\theta}) > 0$. Denote by $\theta_l(\boldsymbol{\theta})$ the common $\theta_i^*(\boldsymbol{\theta})$.

Finally, note that by definition of $\theta_l(\boldsymbol{\theta})$, $\mathcal{J}_i(\theta_i', \theta_{-i}) = \mathcal{J}_i(\theta_i, \theta_{-i})$ for every $\theta_i' \in [\theta_i, \theta_l(\boldsymbol{\theta})]$. Hence, it's straightforward to see that the previous expression for the transfers can be rearranged as

$$t_i(\boldsymbol{\theta}) = \begin{cases} \theta_l \cdot c & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) \leq \overline{Q} \\ \theta_l \cdot (\overline{Q} - c \cdot |\mathcal{J}_i(\boldsymbol{\theta})|) & \text{if } c \cdot (|\mathcal{J}_i(\boldsymbol{\theta})| + 1) > \overline{Q} \text{ and } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| < \overline{Q} \\ 0 & \text{if } c \cdot |\mathcal{J}_i(\boldsymbol{\theta})| \geq \overline{Q}, \end{cases}$$

thus completing the proof. ∎

Proof of Proposition 4

Given a price $P > 0$, a firm i with type θ_i has a "real" supply function given by

$$Q_i^s(\theta_i) = \frac{P}{\theta_i}.$$

Now, a firm can report any supply of the form $\widehat{Q}_i^s(\theta) = \dfrac{P}{\beta_i}$, where $\beta_i \in [a_i, b_i]$. Given the declared supplies of the k firms, the aggregate supply is given by

$$\widehat{Q}^s(\boldsymbol{\theta}) = \sum_{i=1}^{k} \widehat{Q}_i^s(\theta_i) = P\left(\sum_{i=1}^{k} \frac{1}{\beta_i}\right).$$

The price that clears the market is such that

$$\widehat{Q}^s(\boldsymbol{\theta}) = \overline{Q}.$$

Then,

$$P = \left(\sum_{i=1}^{k} \frac{1}{\beta_i}\right)^{-1} \overline{Q}.$$

To check whether truth-telling is an equilibrium of this bayesian game, we solve the problem of firm i taking as given that the other firms truthfully report their types, i.e.

$$\max_{\beta_i \in \Theta_i} \; \mathbb{E}_{\theta_{-i}}\{P(\beta_i, \theta_{-i})\widehat{Q}_i(P(\beta_i, \theta_{-i})) - \frac{\theta_i}{2}\widehat{Q}_i^2(P(\beta_i, \theta_{-i}))\}.$$

Plugging the expression for the reported supply, we can rewrite the problem as

$$\max_{\beta_i \in \Theta_i} \; \mathbb{E}_{\theta_{-i}} \left\{ \frac{P^2(\beta_i, \theta_{-i})}{\beta_i} - \frac{\theta_i}{2} \frac{P^2(\beta_i, \theta_{-i})}{\beta_i^2}) \right\}.$$

Substituting the expression for the price,

$$\max_{\beta_i \in \Theta_i} \; \mathbb{E}_{\theta_{-i}} \left\{ \left(\sum_{j \neq i} \frac{1}{\theta_j} + \frac{1}{\beta_i} \right)^{-2} \right\} \left(\frac{1}{\beta_i} - \frac{\theta_i}{2\beta_i^2} \right) \overline{Q}^2.$$

Taking the FOC and rearranging, we arrive at

$$\mathbb{E}_{\theta_{-i}} \left\{ \frac{\left(\prod\limits_{j\neq i} \theta_j\right)^2 \left[2\left(\beta_i \sum\limits_{\substack{l=1 \\ l\neq i}} \prod\limits_{\substack{j\neq l \\ j \neq i}} \theta_j + \prod\limits_{j \neq i} \theta_j \right) - 2 \sum\limits_{\substack{l=1 \\ l\neq i}} \prod\limits_{\substack{j\neq l \\ j \neq i}} \theta_j (2\beta_i - \theta_i) \right]}{\left(\beta_i \sum\limits_{\substack{l=1 \\ l\neq i}} \prod\limits_{\substack{j\neq l \\ j \neq i}} \theta_j + \prod\limits_{j \neq i} \theta_j \right)^4} \right\} = 0$$

It's easy to see that $\beta_i = \theta_i$ doesn't satisfy this. Thus, if all the other firms report their types truthfully, it's optimal for a firm to misreport its. Hence, the market clearing mechanism is not incentive compatible. ■

Proof of Proposition 5

We only provide the proof for completeness, as it's very similar to the one of Proposition 2. Again, as the revelation principle holds, the buyer needs only to consider direct mechanisms where it is optimal for every seller to report their types truthfully. Hence, the problem the buyer solves is given by

$$\max_{\substack{q:\Theta\to\mathbb{R}^k\\ t:\Theta\to\mathbb{R}^k}} \int_\Theta \left[\sum_{i=1}^k bq_i(\boldsymbol{\theta}) - t_i(\boldsymbol{\theta})\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

subject to

$$\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i,\theta_{-i})}{2}\right] \geq \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i',\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i',\theta_{-i})}{2}\right]$$

for all $\theta_i, \theta_i' \in \Theta_i$ and all $i \in \{1, ..., k\}$

$$\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i}) - \theta_i\frac{q_i^2(\theta_i,\theta_{-i})}{2}\right] \geq 0 \text{ for all } \theta_i \in \Theta_i \text{ and all } i \in \{1, ..., k\},$$

$q_i(\boldsymbol{\theta}) \geq 0$ for all $i \in \{1, ..., k\}$ and all $\boldsymbol{\theta} \in \Theta$.

By a procedure completely analogous to the one in Proposition 2 regarding the characterization of incentive compatibility and the use's of Fubini's Theorem to write the expected sum of transfers conveniently, the original problem can be rewritten only in terms of the allocation rule as follows

$$\max_{q:\Theta\to\mathbb{R}^k} \int_\Theta \left[\sum_{i=1}^k bq_i(\boldsymbol{\theta}) - \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right)\frac{q_i^2(\boldsymbol{\theta})}{2}\right] f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

subject to

$\mathbb{E}_{\theta_{-i}}[q_i^2(\cdot,\theta_{-i})]$ is decreasing for all $i \in \{1, ..., k\}$,

$q_i(\boldsymbol{\theta}) \geq 0$ for all $\boldsymbol{\theta} \in \Theta$ and all $i \in \{1, ..., n\}$.

Forgetting about the first restriction, we can solve the unrestricted version of the preceding problem point-wise. Simply taking first order conditions, we find the optimal allocation rule

$$q_i(\boldsymbol{\theta}) = \frac{b}{\theta_i + \dfrac{F_i(\theta_i)}{f_i(\theta_i)}} = \frac{b}{J_i(\theta_i)}$$

It is straightforward to see that given our regularity assumption, this function is decreasing on θ_i. Thus, it is the solution to the original restricted problem. On the other hand, the expression for the transfers is obtained by considering the point-wise "version" of $T_i(\theta_i)$, which can be shown exactly as in the proof of proposition 2 to be equal to

$$T_i(\theta_i) = \frac{\theta_i}{2}\mathbb{E}_{\theta_{-i}}\{q_i^2(\theta_i,\theta_{-i})\} + \frac{1}{2}\int_{\theta_i}^{b_i}\mathbb{E}_{\theta_{-i}}\{q_i^2(s_i,\theta_{-i})\}ds_i.$$

■

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