On the compressors in quasi-static gas transmission network

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Outline

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Gas Flow Equations, Steady-State and Transient Cases

$$0 = p_t + Q_x$$
$$0 = (p^2)_x + Q|Q|$$

For some network given by a graph $G = (V, E)$

Steady-State Case: $p_t = 0$

$$\begin{aligned} p_j^2 - p_i^2 &= Q_{ij}|Q_{ij}|,\ (i, j) \in E \\ A^T \boldsymbol{Q} &= \boldsymbol{q},\ A - \text{incidence matrix of} G \end{aligned} \qquad (1)$$

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Energy Function Minimization in Steady-Stae Case

Gas Flow Problem without bounds (1) can be formulated in the following forms (and vice-versa):

$$\begin{aligned} &\min \sum_{(i,j)\in E} |Q_{ij}|^3 \\ &s.t.\ A^T \boldsymbol{Q} = \boldsymbol{q} \end{aligned} \tag{2}$$

$$\min \frac{2}{3} \sum_{(m,n)\in E} \frac{|\psi_m - \psi_n|^{3/2}}{\sqrt{a_{mn}}} - \boldsymbol{q}^T \boldsymbol{\psi} \tag{3}$$

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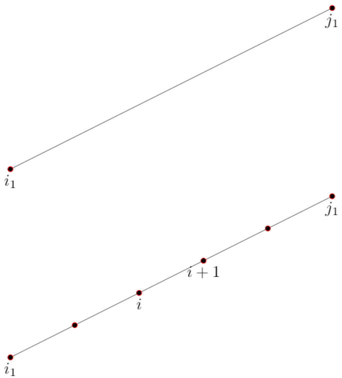
Transient Case: Adiabatics



Multi-pipe discretization

- The whole network is approximated by another one. The physics is governed by

$$\begin{cases} 0 = \dfrac{p_i^{m+1} - p_i^m}{\varepsilon_t} + \dfrac{Q_{i,i+1}^m - Q_{i-1,i}^m}{\varepsilon_x} \\ 0 = |Q_{i,i+1}^m| Q_{i,i+1}^m + \dfrac{(p_{i+1}^m)^2 - (p_i^m)^2}{\varepsilon_x} \end{cases}$$

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Transient Case: successive d^m

$$\min \sum_{m=1}^{\mathcal{M}} \left[\frac{\varepsilon_x}{3} \langle |\boldsymbol{Q}^m|^3, \mathbf{1} \rangle - \left\langle A^T \boldsymbol{d}^m, \boldsymbol{Q}^m \right\rangle \right] \tag{4a}$$

$$\text{s. t.} \frac{p_i^{m+1} - p_i^m}{\varepsilon_t} + \frac{Q_{i,i+1}^m - Q_{i-1,i}^m}{\varepsilon_x} = 0 \tag{4b}$$

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Transient Case: successive d^m. Motivation and Plan

Motivation:

- Successive solve optimization problems in order to make d^m equal to the squared pressures. **Why?**

$$|\boldsymbol{Q}^m|\boldsymbol{Q}^m - A^T\boldsymbol{d}^m = 0$$

$$\frac{p_i^{m+1} - p_i^m}{\varepsilon_t} + \frac{Q_{i,i+1}^m - Q_{i-1,i}^m}{\varepsilon_x} = 0$$

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Transient Case: successive d^m. Motivation and Plan

Action Plan:

- We solve a series of 4. For each of them we provide index k indicates the number of optimization problem. Suppose, for some k the optimization problem is solved with $\boldsymbol{d}_k$.
  - Choose next $\boldsymbol{d}_{k+1}$ to make it closer to $\boldsymbol{p}_k^2$.
  - Solve $(k+1)^{\text{th}}$ optimization problem and get ${\boldsymbol{p}^2}_{k+1}$.
  - Repeat, until the process converges.

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Transient Case: successive d^m. Notations clarifications

$$\min \sum_{m=1}^{\mathcal{M}} \left[\frac{\varepsilon_x}{3} \left\langle \left| \left(\boldsymbol{Q}^m \right)_k \right|^3, \mathbf{1} \right\rangle - \left\langle A^T \left(\left(\boldsymbol{d}^m \right)_k \right), \left(\boldsymbol{Q}^m \right)_k \right\rangle \right] \tag{5a}$$

$$\text{s. t.} \frac{\left(p_i^{m+1})\right)_k - \left(p_i^m)\right)_k}{\varepsilon_t} + \frac{\left(Q^m_{i,i+1}\right)_k - \left(Q^m_{i-1,i}\right)_k}{\varepsilon_x} = 0 \tag{5b}$$

- k – indicates number of optimization problem
- Bold symbols – vectors that contains all spatial elements. E.g., $\boldsymbol{d}^m = \left(d_1^m, \ldots, d_{\mathcal{I}}^m\right)^T$

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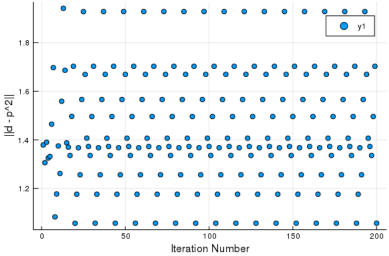
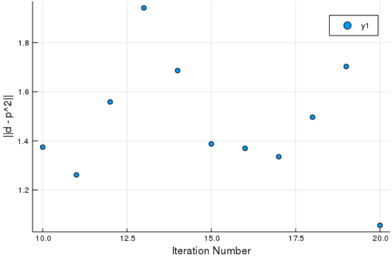
Transient Case: successive d^m. Strategies for $\boldsymbol{d}$ updates

$$\boldsymbol{d}_{k+1} = \boldsymbol{p}_k^2$$





Convergence (no) to the squared pressures. With close up

Transient Case: successive d^m. Strategies for $\boldsymbol{d}$ updates

- Minimize $\|\boldsymbol{d}_k - \boldsymbol{p}_k^2\|^2$ with gradient iteration
  - $\boldsymbol{d}_{k+1} = \boldsymbol{d}_k - \frac{1}{\gamma}\left(\boldsymbol{d}_k - \boldsymbol{p}_k^2\right)$
  - $\boldsymbol{p}_{k+1}$ – solution of 4 with new $\boldsymbol{d}_{k+1}$.

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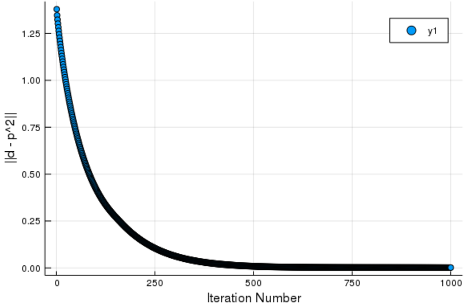
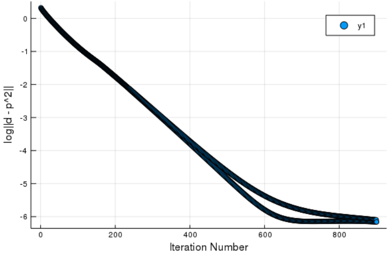
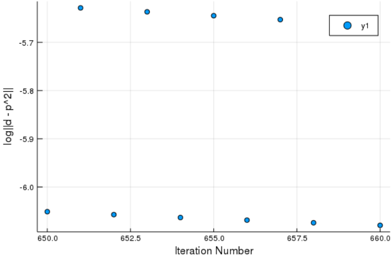
Transient Case: successive d^m. Strategies for d updates



Convergence to the squared pressures

Transient Case: successive d^m. Strategies for d updates





Convergence to the squared pressures (semi-log scale)

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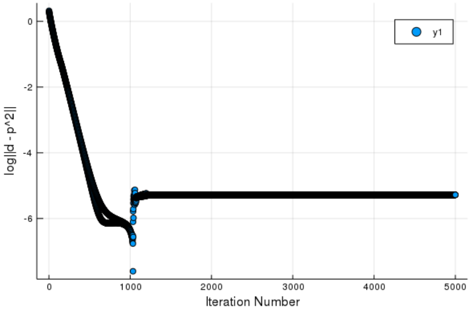
Transient Case: successive d^m. Strategies for d updates



Convergence to the squared pressures (semi-log scale)

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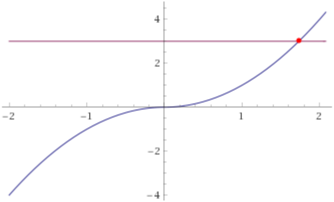
Transient Case: successive d^m. Convergence proof ideas

Theorem. Let x^* be a fixed point of $x^{k+1} = g(x^k)$, $g(x)$ is differentiable at x^* and spectral radius of Jacobian $g'(x^*)$ satisfies $\rho = \rho\,(g'(x^*)) < 1$. Then $x^{k+1} = g(x^k)$ converges to x^* locally. Particularly,
$\forall\ \varepsilon \in (0, 1-\rho)\ \exists\ \delta > 0,\ \ c \in \mathbb{R} : \forall k\ \ \geq 0\ \|x^k - x^*\| \leq c\,(\rho + \varepsilon)^k$ if $\|x^0 - x^*\| \leq \delta$.

Transient Case: successive d^m. Convergence proof ideas

$$\left|\left(Q^m_{i,i+1}\right)_k\right|\left(Q^m_{i,i+1}\right)_k - \left(\left(d^m_{i+1}\right)_k - \left(d^m_i\right)_k\right) = 0 \Longrightarrow$$

$$\Longrightarrow \left(Q^m_{i,i+1}\right)_k = \sqrt{\left|\left(d^m_{i+1}\right)_k - \left(d^m_i\right)_k\right|} sign\left(\left(d^m_{i+1}\right)_k - \left(d^m_i\right)_k\right)$$



Graphical representation of the dissipative equation

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Transient Case: successive d^m. Convergence proof ideas

$$\frac{\left(p_i^{m+1}\right)_k - \left(p_i^m\right)_k}{\varepsilon_t} + \frac{\left(Q_{i,i+1}^m\right)_k - \left(Q_{i-1,i}^m\right)_k}{\varepsilon_x} = 0 \Longrightarrow$$

$$\left(p_i^{m+1}\right)_k = \left(p_i^1\right)_k - \frac{\varepsilon_t}{\varepsilon_x} \sum_{m'=1}^{m-1} \left(Q_{i,i+1}^{m'}\right)_k - \left(Q_{i-1,i}^{m'}\right)_k,$$

$$\left(Q_{i,i+1}^m\right)_k = \sqrt{\left|\left(d_{i+1}^m\right)_k - \left(d_i^m\right)_k\right|} sign\left(\left(d_{i+1}^m\right)_k - \left(d_i^m\right)_k\right)$$

$$\boldsymbol{d}_k = \boldsymbol{d}_{k-1} - \frac{1}{\gamma}\left(\boldsymbol{d}_{k-1} - \boldsymbol{p}_k^2\right)$$

This is $g(\cdot)$! Magenta colored expressions allow to get $\boldsymbol{d}_{k+1}$ as a function of $\boldsymbol{p}_k$

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