Operations on Regular Languages

- **Key Insight**: Regular languages are closed under fundamental operations
- **What this means**: Applying these operations to regular languages always produces regular languages
- **Today's Focus**: Three core operations and their constructions

The Three Core Operations

1. **Union** ($\cup$)
2. **Concatenation** (implicit or .)
3. **Closure** (*)

Each operation has:

- A formal definition
- Proof of closure (constructive proof using NFAs)
- Implementation strategy for virtual NFAs

1. Union Operation

Definition

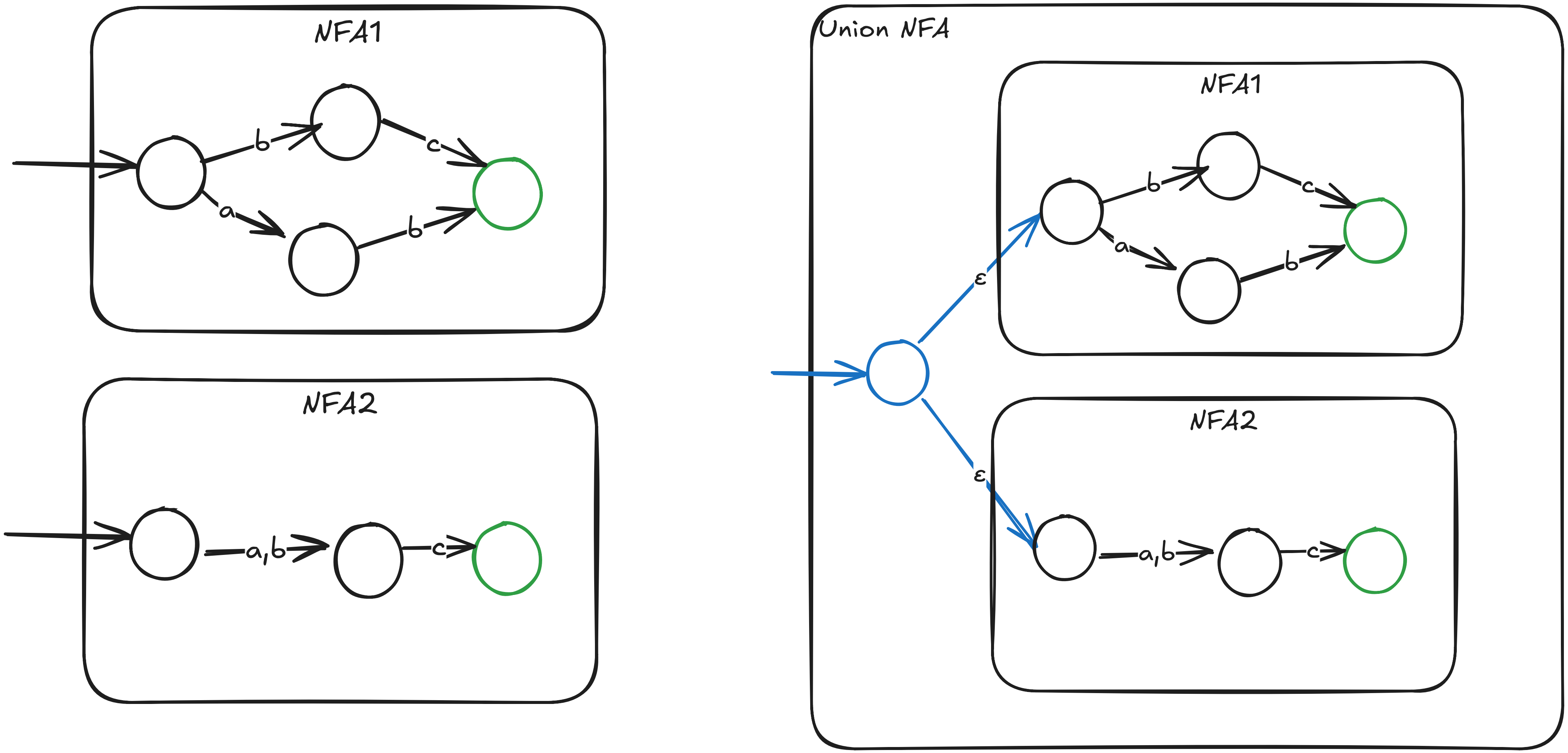
The **union** of two regular languages is a regular language of all strings that belong to at least one of the languages.

$$R_1 \cup R_2 = R_\cup = \{w \mid w \in R_1 \text{ or } w \in R_2\}$$

Example

- $R_1 = \{ab, bc\}$
- $R_2 = \{bc, ac\}$
- $R_1 \cup R_2 = \{ab, bc, ac\}$

Union: Proof by Construction



2. Concatenation Operation

Definition

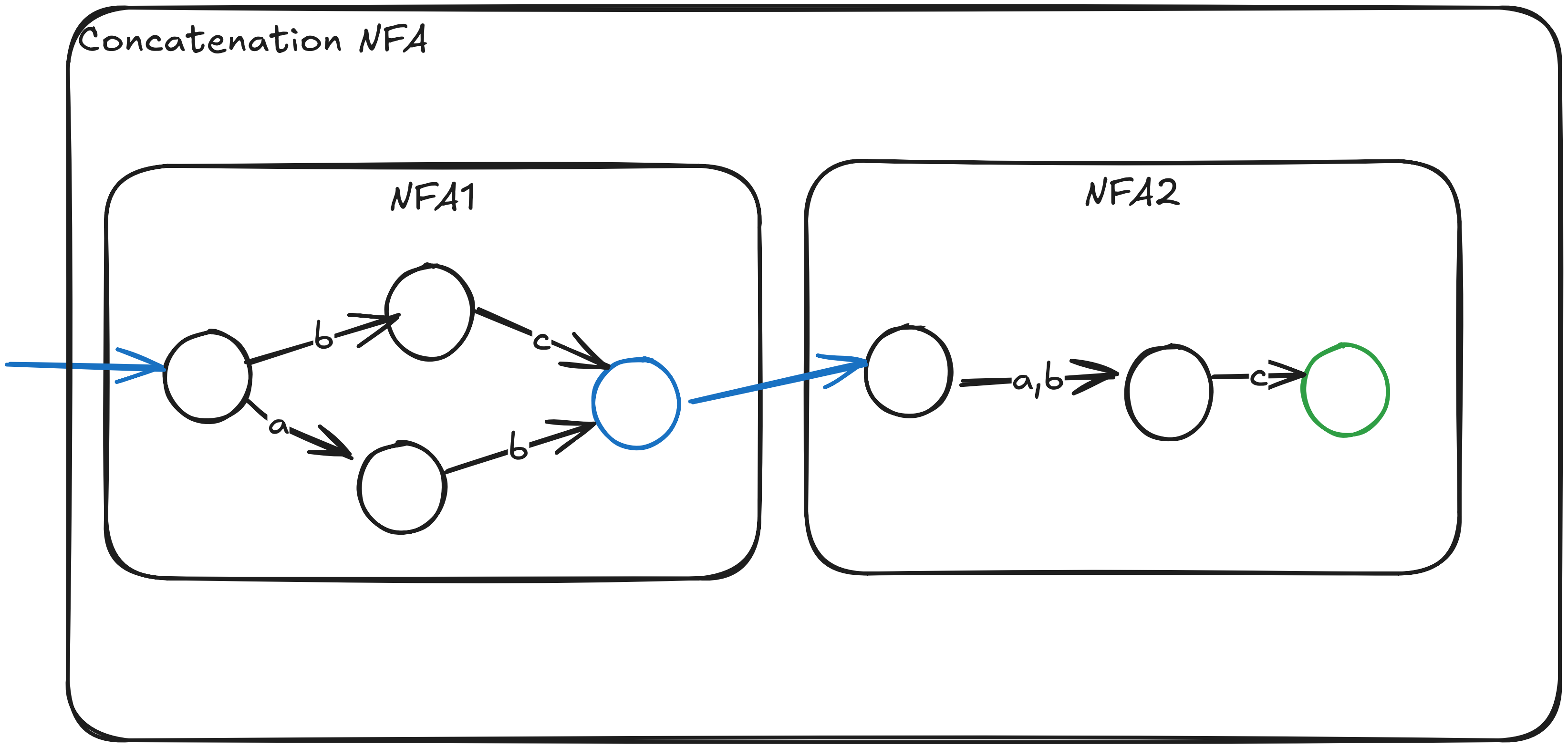
The **concatenation** of two regular languages R_1 and R_2 is the regular language of all strings formed by concatenating a string from R_1 with a string from R_2.

$$R_1 R_2 = R_{\text{concat}} = \{xy \mid x \in R_1 \text{ and } y \in R_2\}$$

Example

- $R_1 = \{ab, bc\}$
- $R_2 = \{bc, ac\}$
- $R_1 R_2 = \{abbc, abac, bcbc, bcac\}$

Concatenation: Proof by Construction



3. Closure Operation

Definition

The **closure** (or Kleene star) of a regular language R is a regular language of all strings formed by concatenating zero or more strings from R.

$$R^* = \{w_1 w_2 \cdots w_k \mid k \geq 0, w_i \in R\}$$

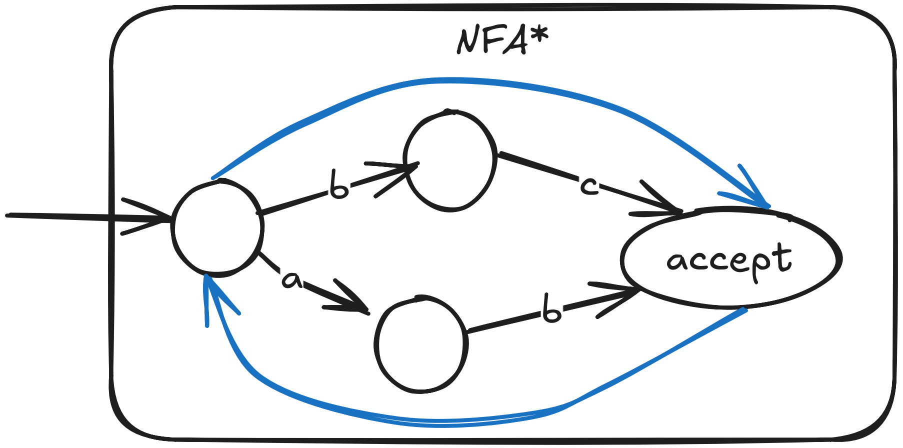
Note: $\varepsilon \in R^*$ (concatenating zero strings)

Example

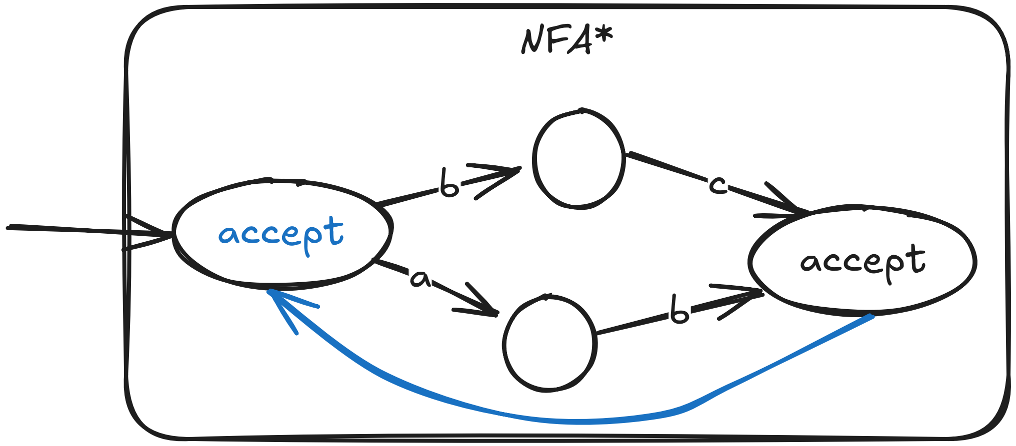
- $R = \{ab, bc\}$
- $R^* = \{\varepsilon, ab, bc, abab, abbc, bcab, bcbc, \ldots\}$

Closure: Proof by Construction

Construction 1



Construction 2



Active Learning: Identify the Operation

$L_1 = \{0, 1\}$ i.e. the language of 1 digit binary numbers

1. $L_1 \cup L_1$ = ?
2. $L_1 L_1$ = ?
3. $L_1 *$ = ?

Implementing Union in Virtual NFA

```
public static NFA union(NFA n1, NFA n2) {
    NFA union = new NFA();

    // Create a new start state.
    ...

    // Add ε-transition from the new start state to each NFA's start state
    ...

    // Add each NFA's accept states to the union accept states
    ...

    return union;
}
```

Implementing Concatenation in Virtual NFA

```
public static NFA concatenation(NFA n1, NFA n2) {
    NFA concat = new NFA();

    // Start state is n1's start state
    ...

    // Add ε-transitions from n1's accept states to n2's start
    ...

    // Accept states are n2's accept states
    ...

    return concat;
}
```

Implementing Closure in Virtual NFA

```
public static NFA closure(NFA n) {
    NFA star = new NFA();

    // Create a new start state.
    ...

    // New start is an accept state (accept ε)
    ...

    // ε-transition to original start
    ...

    // ε-transitions from accept states back to original start
    ...

    return star;
}
```

Key Takeaways

Operation	Symbol	Meaning	Property
Union	$\cup$	Strings in either language	Closed
Concatenation	. (implicit)	Prefix from R_1, suffix from R_2	Closed
Closure	*	Zero or more repetitions	Closed

Main Point: These constructions show that regular languages form a **closed system** under these operations.

Application: Building Complex Languages

You can combine these operations to build sophisticated regular languages:

- $(a \cup b)^*$: Any string of a's and b's
- $ab(cd)^*$: "ab" followed by zero or more "cd"
- $(a^*ba^*)^*$: Complex patterns with a's and b's

Understanding these operations is crucial for:

- Regular expression design
- Compiler lexical analysis
- String pattern matching