Complexity

Which Solvable Problems Are Tractable?

Course Journey So Far

Universality - Turing machines can compute anything computable

Computability - Some problems can't be solved by any computer

Today: Complexity - Among solvable problems, which are practical?

Today's Journey

1. Tractable vs intractable
2. Measuring complexity (Big O)
3. The class P
4. The class NP
5. P vs NP
6. NP completeness

Extended Church-Turing Thesis

Universality plus efficiency:

A Turing machine can **efficiently** perform any computation that can be efficiently computed by any physically realizable computer

What this means:

- Any algorithm running in time $T(n)$ on a physical computer
- Can be simulated by a TM in time $(T(n))^k$ for some constant k
- TM may be slower, but not **exponentially** slower

Tractable vs Intractable

Tractable problems: Solvable in **polynomial time**

- Time proportional to n^k (for some constant k)
- Examples: $n^{\frac{1}{2}}, n, n^2, n^3, n^{10}$

Intractable problems: Require **exponential time**

- Time proportional to 2^{n^k}
- Examples: $2^{\sqrt{n}}, 2^n, 2^{n^2}$

The divide: Polynomial vs Exponential determines practicality

Polynomial vs Exponential: The Reality

A supercomputer can execute ~10^{18} instructions per second (exaflop)

Algorithm	O(n)	n	Supercomputer Run Time
Insertion sort	n^2 (polynomial)	10^9	~1 second
Subsets of n things	2^n (exponential)	100	> 10,000 years
Permutations of n things	$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (exponential)	52	> 10^{42} years > age of the universe (14×10^9 years)

Bottom line: Exponential = effectively impossible for modest inputs

Section 2: Measuring Complexity

What is Complexity?

Understanding Complexity

Complexity ≠ Difficulty of Solution

- Remove human ability from equation
- Assume optimal solver who can solve any solvable problem

Complexity = Tractability of Finding Solution

- **Time:** How long does it take?
- **Memory:** How much space is needed?

Practical reality: A program that's too slow or uses too much memory is useless

Two Analysis Strategies

A posteriori (after the fact):

- Run the program and measure time/memory
- Real-world performance data

A priori (before the fact):

- Analyze the program without running it
- Predict performance characteristics

Best practice: Use **both** approaches!

- Before coding: analyze algorithms for feasibility
- After coding: measure to verify requirements

Runtime Analysis Framework

Donald Knuth: Turing Award winner, father of algorithm analysis

Total run time = $\Sigma(c_i \times f_i)$

- c_i = cost of executing statement i (system-dependent)
- f_i = frequency of executing statement i (algorithm-dependent)

Problem: This calculation is tedious!

Solution: Use **order of growth** model

Order of Growth

Key concepts:

- Most programs have a **problem size** (n) that characterizes difficulty
- As problem size increases, runtime increases
- Order of growth expresses **how runtime scales** with problem size

Expressed using Big O notation: $O(f(n))$

- n is the problem size
- $f(n)$ is a function describing growth rate

Big O Simplification Rules

Rule 1: Keep only the leading term

- $n^2 + 2n + 3 \to \mathbf{n^2}$
- Highest power dominates for large n

Rule 2: Set coefficient to 1

- $5n^2 \to \mathbf{n^2}$
- $100n \to \mathbf{n}$

Why these rules?

- Focus on growth rate, not constants
- Constants matter less as n grows large
- Simplifies comparison between algorithms

Quick Poll

Which has the fastest growth rate?

A) $O(n \log n)$

B) $O(n^2)$

C) $O(2^n)$

D) $O(n!)$

Common Growth Rates

Big O	Name	Example
$O(1)$	Constant	Array access
$O(\log n)$	Logarithmic	Binary search
$O(n)$	Linear	Linear search
$O(n \log n)$	Linearithmic	Merge sort
$O(n^2)$	Quadratic	Bubble sort
$O(n^3)$	Cubic	Matrix multiplication
O(2^n)	Exponential	Power set generation
$O(n!)$	Factorial	Traveling salesman (brute force)

Analysis Focus: Worst Case

Three cases for any algorithm:

Best case: Fastest possible runtime

- Example: Linear search finds element first → $O(1)$

Average case: Expected runtime on random inputs

- Example: Linear search on average → $O(n/2) = O(n)$

Worst case: Slowest possible runtime

- Example: Linear search finds element last → $O(n)$

For complexity theory: Focus on **worst case**

- Provides guarantees and simplifies analysis

Section 3: The Class P

Polynomial Time Algorithms

Extended Church-Turing Thesis Revisited

All physically realizable models are polynomially equivalent

What this means:

- k-tape TM running in $T(n)$ time
- Can be simulated by single-tape TM in $T^2(n)$ time
- Polynomial slowdown, not exponential!

Important exception:

- Nondeterministic TMs are NOT physically realizable
- Single-tape nondeterministic TM in $T(n)$ time
- Requires $2^{O(T(n))}$ time on deterministic TM

Defining Class P

The class P consists of languages decidable in polynomial time on a deterministic single-tape Turing machine

Equivalently: P = problems solvable in polynomial time on any physically realizable computer

Key insight:

- Polynomial time = tractable
- No computational breakthrough needed
- Differences within polynomial (n^2 vs n^3) matter practically
- But theoretically, all polynomial time is "efficient"

Regular Languages in P

Theorem: Every regular language is a member of P

Proof:

1. Every regular language can be decided by a DFA
2. DFA scans input string w left to right once
3. Runtime is linear in length of w
4. Linear is polynomial → Regular languages $\in$ P

Takeaway: Pattern matching with regular expressions is efficient!

Sorting Problem

Problem: Given a list of comparable items, sort them in ascending order.

Brute-force approach: Compute each permutation and check whether sorted

- Exponential!

Efficient approach:

- $O(n^2)$ insertion sort etc.
- $O(n \lg n)$ quicksort, mergsesort, etc.

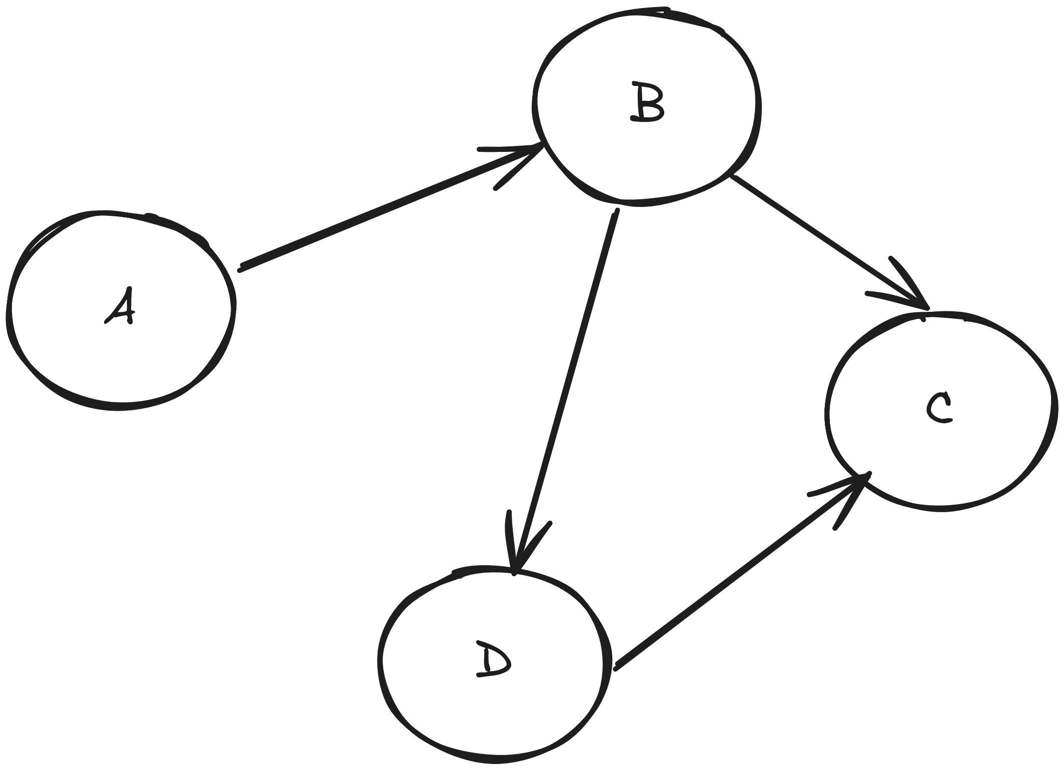
Graph Background

Graph: Represents relationships between pairs of entities

Components:

- **Vertices** (nodes): Entities
- **Edges**: Connections between vertices
- **Directed** vs **Undirected** edges

Applications: Web, social networks, transportation, circuits, state machines



Graph Representation in Code

Adjacency set representation:

```
// A → {B}, B → {C, D}, C → {}, D → {C}
Map<V, Set<V>> vertexToNeighbors = new HashMap<>();
```

Minimal Graph API:

```
public interface Graph<V> {
    void addVertex(V v);
    void addEdge(V v, V w);
    Iterable<V> getVertices();
    Iterable<V> getNeighbors(V v);
    boolean hasVertex(V v);
    boolean hasEdge(V v, V w);
}
```

The PATH Problem

Intuitive: Given a directed graph G and two nodes s and t, determine if there's a path from s to t

Formal language:
PATH = $\{\langle G, s, t\rangle \mid G$ is a directed graph with a path from s to t$\}$

Question: Is PATH in P?

PATH: Failed Approach

Attempt 1: Brute-force search

Examine all potential paths in G (which has m nodes):

- Length 1: 1 path (direct edge s → t)
- Length 2: (m-2) paths
    - s → u → t, where u can be chosen (m-2) ways
- Length 3: (m-2)(m-3) paths
- Length m-1: (m-2)(m-3)...1 paths

Total: O(m^{m-2}) paths → **Exponential!**

Result: Intractable approach

PATH: Successful Approach

Attempt 2: Breadth-First Search (BFS)

Algorithm:

- Maintain set R of reachable nodes from s, initially R = {s}
- Maintain queue Q of nodes to process, initially Q = [s]
- While Q is not empty:
  - Dequeue node u
  - For each neighbor n of u:
    - If n == t, return true
    - If n not in R: add n to R, enqueue n to Q
- Return false

Complexity: Each node processed at most once → O(m) → **Polynomial!**

Relatively Prime Problem

Problem: Given two natural numbers x and y, determine if they are relatively prime (share no common factors except 1)

Brute-force approach: Find all factors of both numbers

- For k-digit numbers, factors can be as large as 10^(k/2)
- Exponential in input size!

Efficient approach: Euclidean algorithm

- Repeatedly apply: gcd(x,y) = gcd(y, x mod y)
- Runs in $O(k^2)$ time
- Polynomial! → Relatively Prime ∈ P

Active Learning

What other problems can you think of that are in P?

Section 4: The Class NP

Verification vs Solution

Beyond P: Harder Problems

We've seen: Some problems have polynomial time solutions (Class P)

Reality: Many useful problems have no known polynomial solution

Open question:

- Have we just not discovered the right mathematical principles?
- Or are these problems intrinsically intractable?

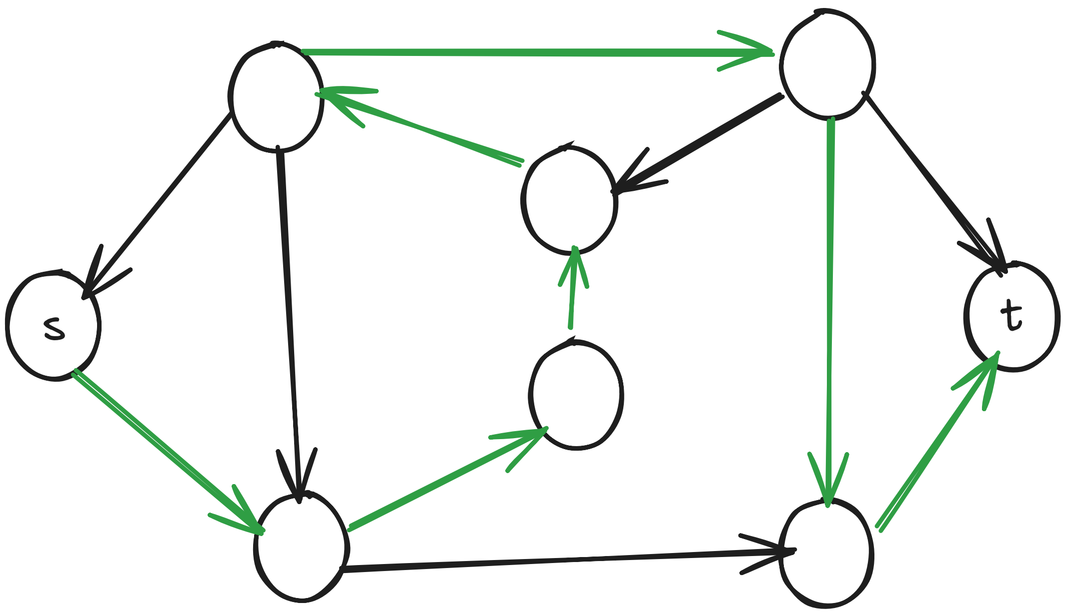
The Hamiltonian Path Problem

Definition: A Hamiltonian path in directed graph G goes through each node exactly once

Problem: Given graph G and vertices s, t, does G have a Hamiltonian path from s to t?

Formal language:
HAMPATH = {⟨G, s, t⟩ | G has a Hamiltonian path from s to t}

Also known as: The Traveling Salesman Problem



HAMPATH: Exponential Solution

Brute-force approach:

- Consider all potential paths from s to t
- Check if any visit every node exactly once
- **Complexity:** O(m^(m-2)) paths to check

Why exponential?

- Must consider all permutations of nodes
- No known way to prune search space efficiently

Result: Appears intractable for large graphs

HAMPATH: Polynomial Verification

Key insight: Can't efficiently **find** a Hamiltonian path, but can efficiently **verify** one!

Verification algorithm:

- Given a proposed path p from s to t
- Check that each edge in p exists in G
- Check that p visits every node exactly once
- **Complexity:** O(m) → Polynomial!

This asymmetry is fundamental to NP

Verifier Definition

A verifier for language L is algorithm V where:

$L = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some certificate } c\}$

For HAMPATH:

- $w = \langle G, s, t\rangle$ (the problem instance)
- $c = p$ (the Hamiltonian path, if it exists)
- V verifies in polynomial time whether p is valid

Certificate (or proof): The "hint" that makes verification easy

Defining NP (Version 1)

Definition 1: NP is the class of languages that have polynomial time verifiers

In other words:

- Problems where you can verify a solution in polynomial time
- Even if you can't find the solution efficiently

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Think-Pair-Share

Question: Can you think of everyday problems that are hard to solve but easy to verify once you have the answer?

Hint: Think about puzzles, games, or other challenges

Nondeterministic Solution to HAMPATH

Nondeterministic TM approach:

- For each permutation of nodes (starting with s, ending with t)
- Use polynomial time verifier to check if it's Hamiltonian
- Since permutations checked in parallel (nondeterministically)
- Total time is polynomial!

The catch: Nondeterministic TMs aren't physically realizable

- Simulating on deterministic TM takes exponential time

Defining NP (Version 2)

Definition 2: NP is the class of languages decided by some nondeterministic polynomial time Turing machine

Two equivalent definitions:

1. Has polynomial time verifier (deterministic, with certificate)
2. Decidable in polynomial time nondeterministically

NP stands for: Nondeterministic Polynomial

- **NOT** "Non-Polynomial" (common misconception!)

Examples of NP Problems

1. **Puzzles**
   - Sudoku, KenKen, Crossword, Jigsaw
   - Hard to solve, easy to check solution
2. **Prime Factorization**
   - Given 100-digit number, find prime factors
   - Would take > 10^{24} years on supercomputer!
   - But verifying factors is easy: just multiply
3. **CLIQUE**
   - Find k nodes in graph where all are connected to each other
   - Verification: check k(k-1)/2 edges exist
4. **SUBSET-SUM**
   - Given set of numbers and target t, find subset summing to t
   - Verification: check if provided subset sums to t

Application Exercise

Problem: Show that SUBSET-SUM is in NP

SUBSET-SUM={⟨S,t⟩ ∣ S is a finite multiset of integers and $\exists T \subseteq S : \sum_{x \in T} x = t$}

Examples:

- ⟨{3, 5, 2, 8}, 10⟩ ∈ SUBSET-SUM (because 2 + 8 = 10 or 3 + 5 + 2 = 10)
- ⟨{1, 4, 7}, 6⟩ ∉ SUBSET-SUM (no subset sums to 6)

Your task:

1. Method 1: Describe polynomial time verifier (what is the certificate?)
2. Method 2: Describe nondeterministic TM deciding SUBSET-SUM

Complement Class: coNP

Definition: coNP contains complements of NP languages

Examples:

- NOT-CLIQUE = {⟨G, k⟩ | G has NO k-clique}
- NOT-SUBSET-SUM = {⟨S, t⟩ | No subset of S sums to t}

Challenge: Hard to verify something doesn't exist!

- No certificate proves absence
- Would need to check all possibilities

Open question: Is coNP = NP?

Section 5: P vs NP

The Million Dollar Question

The Relationship Question

What we know:

- **P:** Problems solvable in polynomial time
- **NP:** Problems verifiable in polynomial time

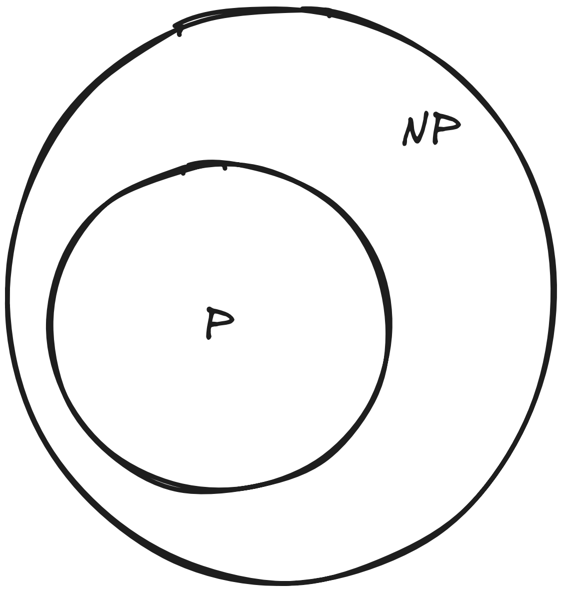
Key observation:
Every problem in P is also in NP

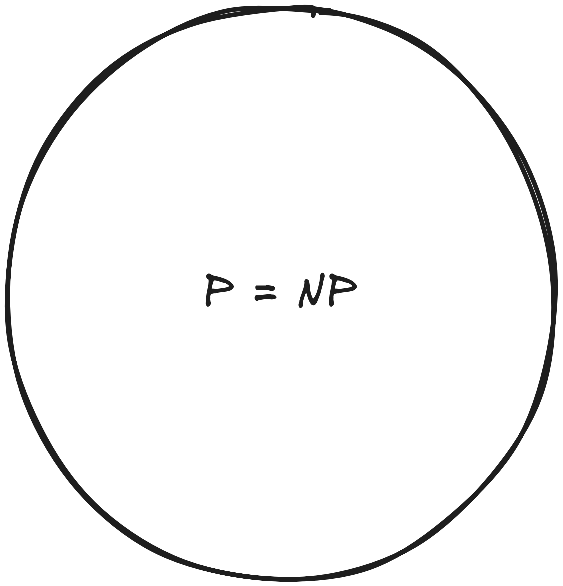
- Why? If you can solve it in polynomial time, you can verify in polynomial time (just solve it!)

This means: $P \subseteq NP$

The question: Is $P = NP$, or is $P \subset NP$?

Two Possible Universes





Universe 1: P ⊂ NP (P is strictly smaller)

- Verifying is easier than solving
- Some problems are intractable

Universe 2: P = NP (P equals NP)

- Every verifiable problem is efficiently solvable
- Finding = Checking

Why This Matters

If P = NP:

- Many "hard" problems become tractable
- Cryptography would collapse
- Optimization problems easily solved
- Fundamental shift in computer science

If P ≠ NP:

- Confirms our intuition about hard problems
- Cryptography remains secure
- Some problems inherently intractable
- Limits of efficient computation established

The Million Dollar Question

Current status:

- Millennium Prize Problem (one of seven)
- $1,000,000 prize from Clay Mathematics Institute
- Announced in 2000, still unsolved
- Most computer scientists believe $P \neq NP$

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Quick Poll

Which do you think is more likely?

A) P = NP (verification and solution are equally hard)

B) P ≠ NP (verification is easier than solution)

C) We'll never know

D) The question is somehow ill-posed

Section 6: NP-Completeness

The Hardest Problems in NP

1970s Breakthrough

Key discovery: Not all NP problems are equally hard

Central insight:

- ALL NP problems can reduce to a **subset** of NP problems
- These hardest problems are called **NP-complete**

Why this matters:

- Focus effort on NP-complete problems
- If ANY NP-complete problem has polynomial solution → P = NP
- If ANY NP-complete problem proven intractable → P ≠ NP

Polynomial Time Reducibility

Reduction concept (recall):

Convert problem A to problem B such that B's solution solves A

Polynomial time reduction:

The conversion itself must run in polynomial time

Implication:

If A reduces to B in polynomial time, and $B \in P$, then $A \in P$

Key idea: Reductions transfer tractability

The Cook-Levin Theorem (1971)

Theorem: Every nondeterministic Turing Machine (every problem in NP) can be reduced to Boolean Satisfiability (SAT) in polynomial time

What this means:

- SAT is in NP
- Every NP problem reduces to SAT
- Therefore, SAT is **NP-complete**

First NP-complete problem discovered!

Boolean Satisfiability (SAT)

Problem: Given boolean expression, does a satisfying assignment exist?

Example expression:

$(x_1 \lor \neg x_2) \land (x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3)$

Satisfying assignment:

x_1 = true, x_2 = true, x_3 = false makes expression true

Why NP-complete:

- Can verify assignment in polynomial time
- Can encode any NP problem as SAT instance

Cook-Levin Proof Sketch (1/3)

Step 1: Represent TM computation as table

Example: TM deciding if binary string ends in 1, on input "101"

Time	State	Tape Head	cell 0	cell 1	cell 2	cell 3
0	pending1	cell 0	**1**	0	1	⊔
1	last1	cell 1	1	**0**	1	⊔
2	pending1	cell 2	1	0	**1**	⊔
3	last1	cell 3	1	0	1	**⊔**
4	accept	cell 2	1	0	**1**	⊔

Cook-Levin Proof Sketch (2/3)

Step 2: Encode table as boolean variables

Variable types:

- $Q_{t,q} = 1$ iff at time t, TM in state q
- $H_{t,c} = 1$ iff at time t, tape head at cell c
- $S_{t,c,s} = 1$ iff at time t, symbol at cell c is s

Example:

- $Q_{0,\text{pending1}} = 1$, $Q_{0,\text{last1}} = 0$, $Q_{0,\text{accept}} = 0$
- $H_{0,0} = 1$, $H_{0,1} = 0$, $H_{0,2} = 0$, $H_{0,3} = 0$
- $S_{0,0,1} = 1$, $S_{0,0,0} = 0$, $S_{0,0,\sqcup} = 0$

Cook-Levin Proof Sketch (3/3)

Step 3: Write boolean expressions for constraints

Constraint types:

1. Initial configuration: Q_{0,start} ∧ H_{0,0} ∧ S_{0,0,input[0]} ∧ ...
2. Accepting configuration: Q_{T,accept}
3. Valid transitions: If configuration C_1 then configuration C_2
4. Uniqueness: One state, one head position, one symbol per cell

Final expression: e = e_start ∧ e_move ∧ e_accept ∧ e_cell

Key insight: e is satisfiable ⟺ TM accepts input!

Karp's 21 NP-Complete Problems (1972)

Extended the list through polynomial time reductions

Selection of famous NP-complete problems:

1. SAT (Boolean Satisfiability) - Starting point
2. 3-SAT (Satisfiability with 3 literals per clause)
3. CLIQUE (Finding densely connected subgraphs)
4. Vertex Cover (Selecting nodes to cover edges)
5. Hamiltonian Path (Visiting all nodes once)
6. Traveling Salesman (Shortest Hamiltonian cycle)
7. Graph Coloring (Coloring with k colors)
8. Knapsack (Selecting items for maximum value)

More NP-Complete Problems

9. Partition (Splitting set into equal sums)
10. Subset Sum
11. Integer Programming
12. Set Packing
13. Set Cover
14. Feedback Arc Set
15. Exact Cover
16. Hitting Set
17. Steiner Tree
18. 3-Dimensional Matching

Common thread: All practical, important problems!

Reduction Chains

How new NP-complete problems are found:

SAT → 3-SAT → CLIQUE → Vertex Cover → Hamiltonian Path → TSP

Process:

1. Start with known NP-complete problem
2. Show polynomial reduction to new problem
3. Show new problem is in NP
4. Conclude new problem is NP-complete

Result: Thousands of NP-complete problems identified

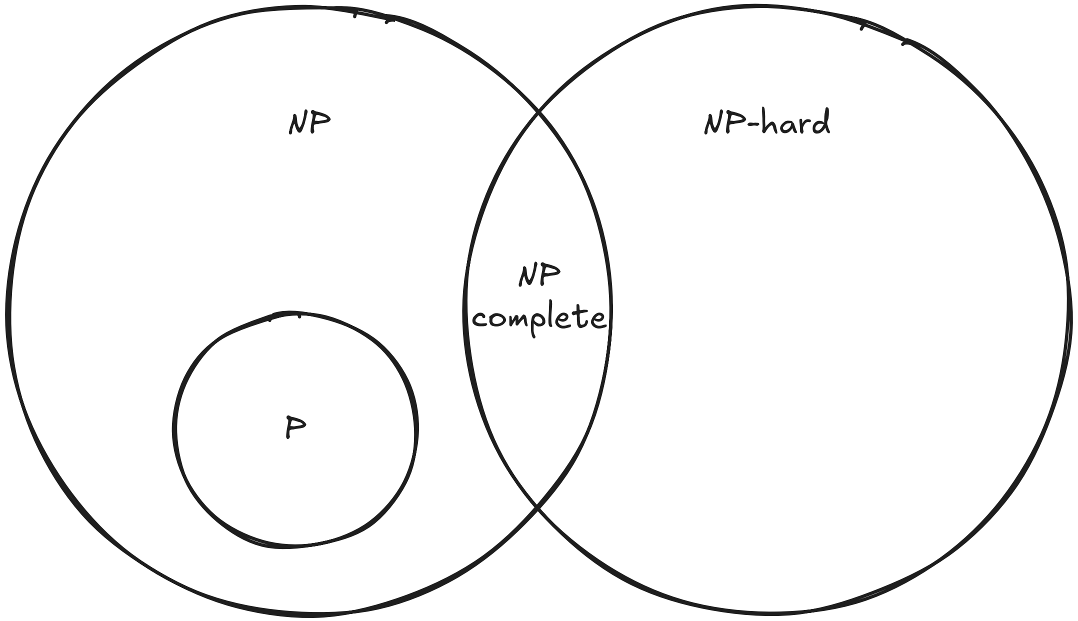
Beyond NP-Complete: NP-Hard

NP-hard: At least as hard as NP-complete

Key difference: NP-hard problems not necessarily in NP!

Examples:

- Halting Problem (undecidable!)
- Optimization versions of NP problems
  - Not "does solution exist?" but "what's the best solution?"



Practical Implications

If you encounter NP-complete problem:

Don't expect exact polynomial algorithm

- No one has found one yet
- If you do, you've solved P vs NP!

Instead, use practical approaches:

1. Approximation algorithms (near-optimal solutions)
2. Heuristics (work well in practice)
3. Restrict to special cases (polynomial subcases)
4. Accept exponential time for small inputs
5. Use probabilistic/randomized algorithms

Summary: The Complexity Hierarchy

P: Problems solvable in polynomial time

- Tractable, practical
- Examples: sorting, shortest path, pattern matching

NP: Problems verifiable in polynomial time

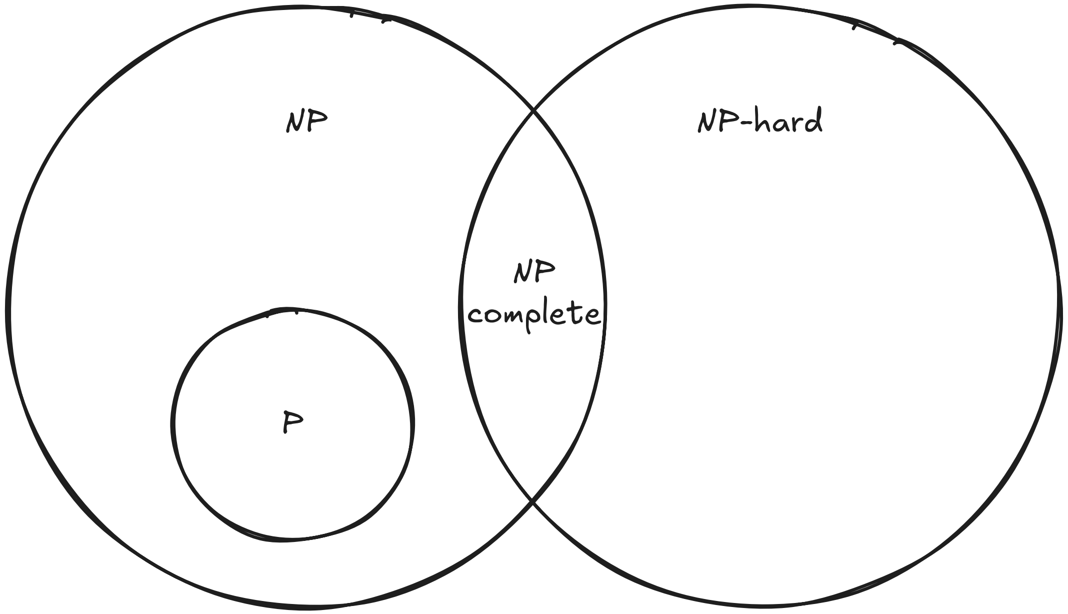
- Contains P
- Examples: Hamiltonian path, SAT, knapsack

NP-Complete: Hardest problems in NP

- All NP problems reduce to these
- Examples: SAT, 3-SAT, TSP, graph coloring

NP-Hard: At least as hard as NP-complete

- May not be in NP (may be undecidable)



The Big Picture

Hierarchy of problem difficulty:

1. **Regular** (DFA) - Always tractable, always polynomial
2. **Context-Free** (PDA) - Often tractable
3. **P** (Deterministic TM, polynomial) - Tractable by definition
4. **NP** (Nondeterministic TM, polynomial) - Verifiable, possibly tractable
5. **NP-Complete** - Hardest NP problems
6. **Decidable** - May require exponential time
7. **Recognizable** - May not halt on rejection
8. **Unrecognizable** - Impossible to compute