

Turing Machines

The Ultimate Computational Model

The Foundation of Computer Science

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ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO
THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

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The “computable” numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbersome technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

Alan Turing's 1936 paper: "On Computable Numbers, with an Application to the Entscheidungsproblem"

This single paper laid the foundation for modern computer science

Learning Objectives

By the end of this lecture, you will be able to:

- **Define** what a Turing Machine is and how it works
- **Trace** TM execution on sample inputs
- **Design** simple TMs for basic computations
- **Understand** why TMs are the theoretical model of computation
- **Implement** virtual TMs in Java

Recall: Finite Automata Limitations

DFAs and NFAs:

- Fixed, finite memory (states)
- Can only read input left-to-right once
- Cannot write or modify input

Cannot recognize: $0^n 1^n$ and other non-regular languages

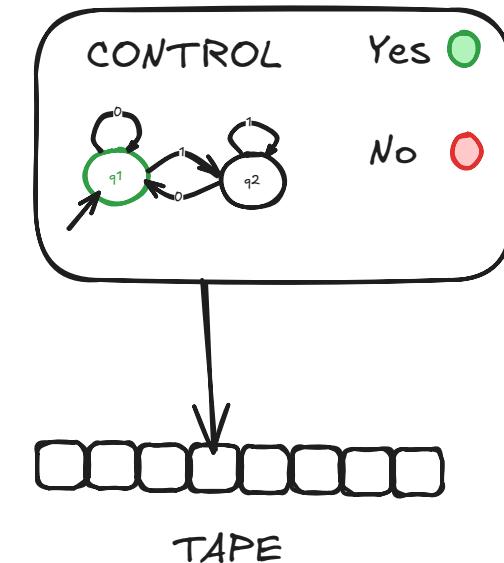
We need something more powerful...

The Turing Machine Model

Only slightly more complex than DFAs, but a lot more powerful!

Three key enhancements to DFAs:

1. Enhanced Tape
2. Reject States
3. Halting Behavior



Enhancement 1: The Tape

Capabilities:

- Move both left and right
- Read and write symbols
- Infinite in both directions
- Special blank symbol: \sqcup

State Transitions Include:

- Input symbol read
- Output symbol to write
- Direction: L or R

Example: $0:1,R$

- Read 0, write 1, move Right

Enhancement 2: Reject States

DFAs: Only accept states (implicit rejection)

TMs: Explicit accept AND reject states

- Needed because TMs can run indefinitely
- Must explicitly specify rejection

Enhancement 3: Halting

When a TM reaches an **accept** or **reject** state:

- It stops immediately
- No further processing

This defines the computational output

TM State Diagram Notation

Transition format: input:output,direction

Example: 0:0,R

- Read: 0
- Write: 0
- Move: Right

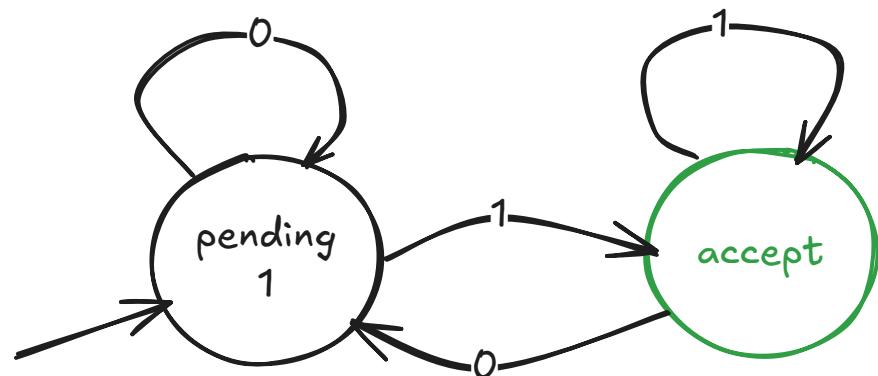
Shorthand:

- Unlabeled input → any other symbol e.g. :1,L
- Unlabeled output → same as input e.g. 1:L
- No transition defined → reject

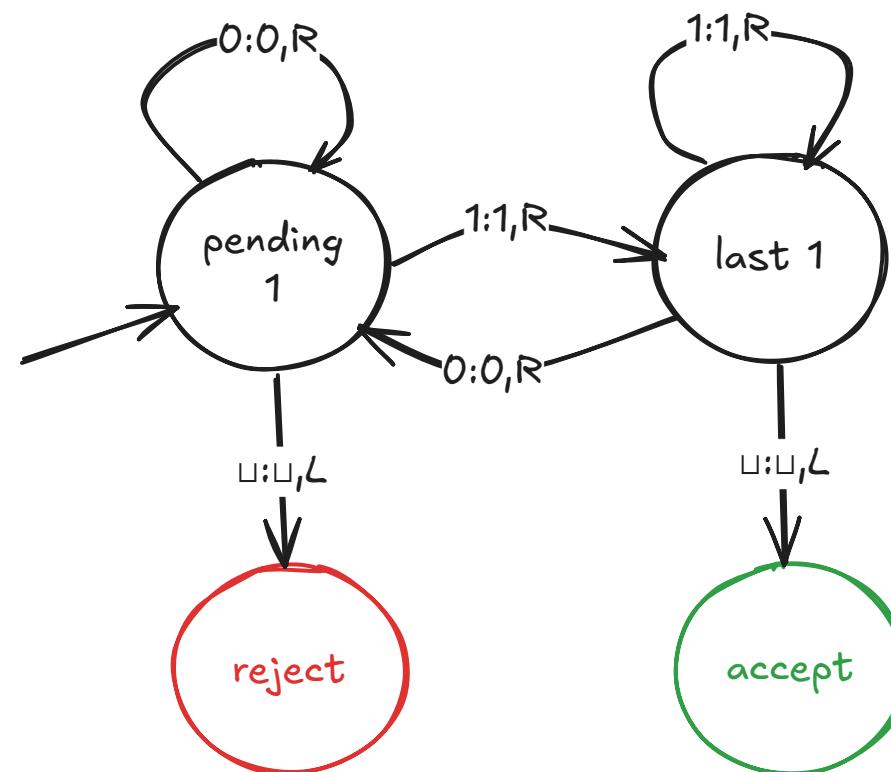
Example: DFA as TM

Language of binary strings ending in 1

DFA

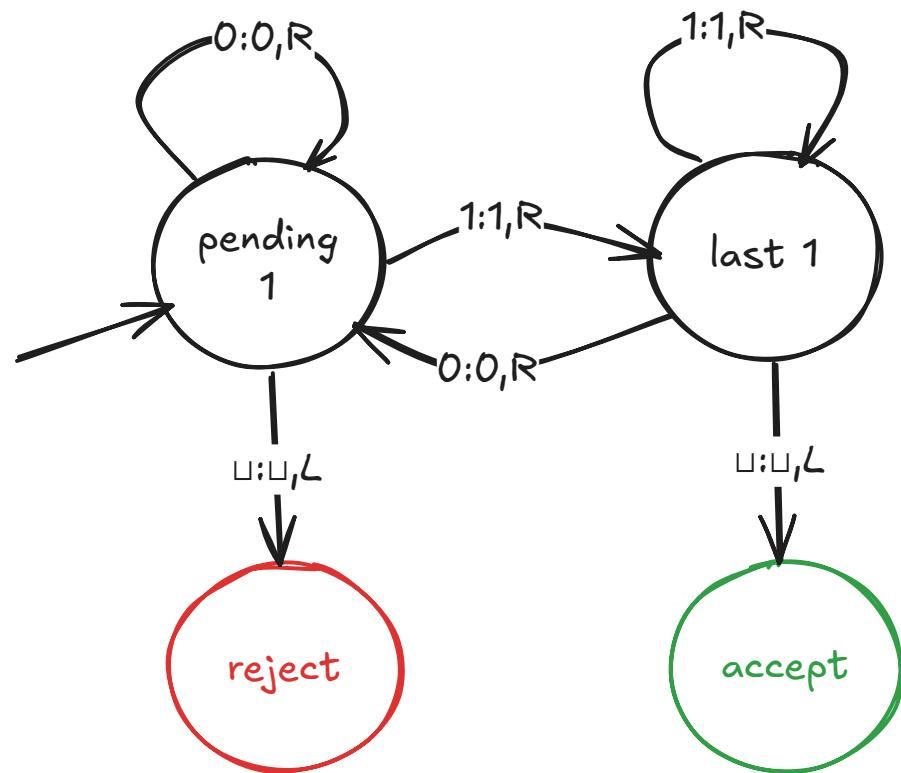


TM

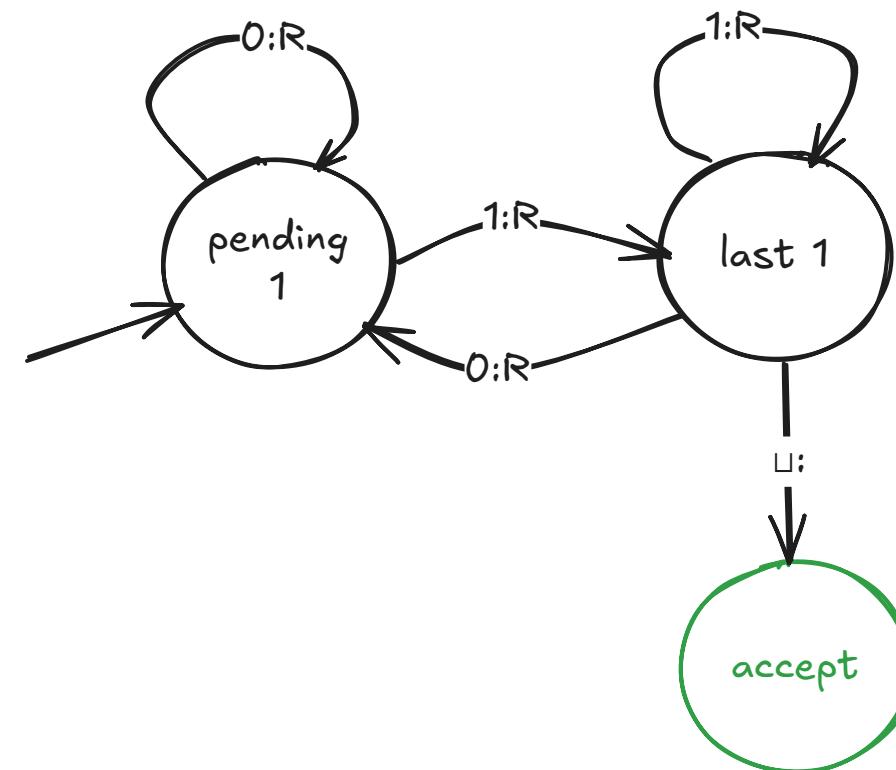


TM Shorthand

TM 1



TM 2



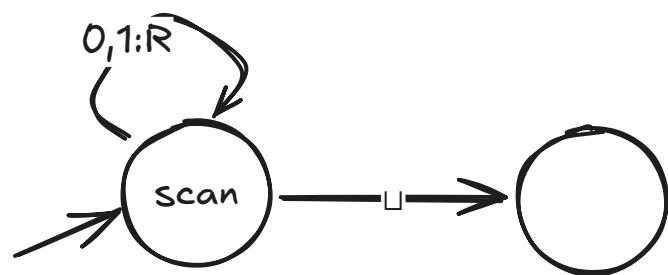
Example w/ output: Binary Incrementer

1. $\sqcup\sqcup 00\sqcup\sqcup$
2. $\sqcup\sqcup 01\sqcup\sqcup$
3. $\sqcup\sqcup 10\sqcup\sqcup$
4. $\sqcup\sqcup 11\sqcup\sqcup$
5. $\sqcup 100\sqcup\sqcup$

Strategy:

1. Scan right to end of input
2. Move left, flipping 1s to 0s
3. When you hit a 0 or \sqcup , flip to 1 and stop

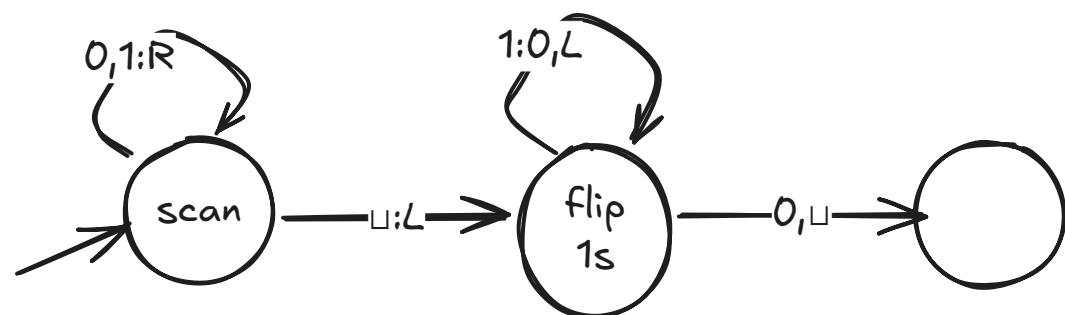
Example w/ output: Binary Incrementer



Strategy:

1. Scan right to end of input
2. Move left, flipping 1s to 0s
3. When you hit a 0 or u , flip to 1 and stop

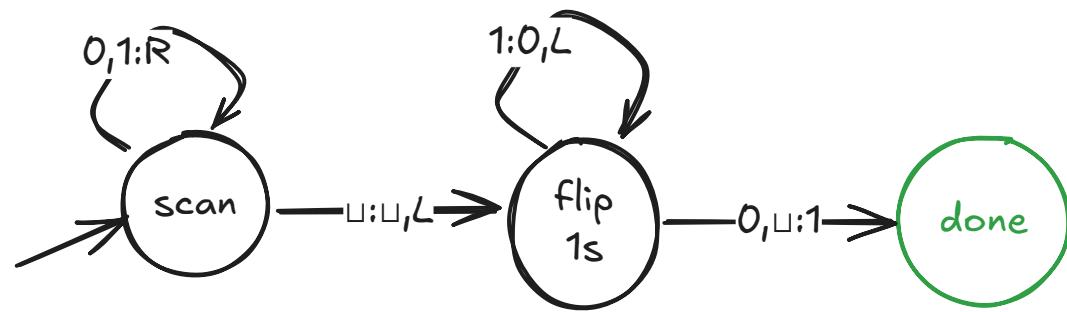
Example w/ output: Binary Incrementer



Strategy:

1. Scan right to end of input
2. Move left, flipping 1s to 0s
3. When you hit a 0 or □, flip to 1 and stop

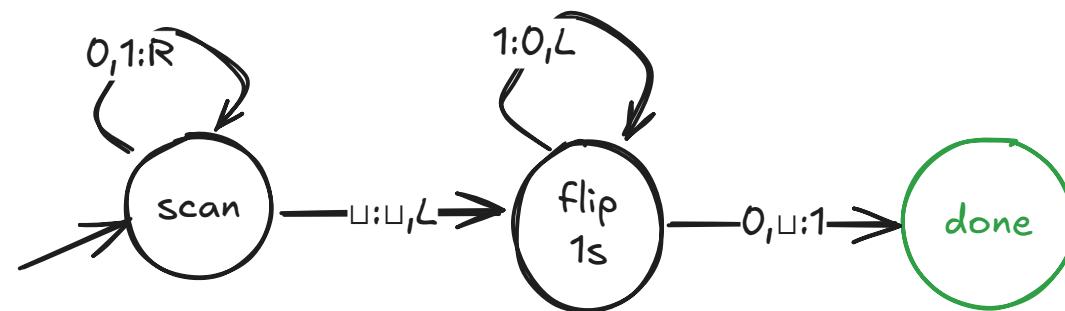
Example w/ output: Binary Incrementer



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Example w/ output: Binary Incrementer



Strategy:

1. Scan right to end of input
2. Move left, flipping 1s to 0s
3. When you hit a 0 or \sqcup , flip to 1 and stop

Example trace for $101 (5_{10})$:

State	Tape Position	Action
scan	1 0 1 \sqcup	R
scan	1 0 1 \sqcup	R
scan	1 0 1 \sqcup	R
scan	1 0 1 \sqcup	L, flip 1s
flip 1s	1 0 1 \sqcup	0, L
flip 1s	1 0 0 \sqcup	1, done
done	110	halt



Active Learning: Trace the Incrementer

Given input: 111

State	Tape Position	Action
scan	□ 1 1 1 □	R
...

Active Learning: Trace the Incrementer (soln)

State	Tape Position	Action
scan	□ 1 1 1 □	R
scan	...	R
scan	□ 1 1 1 □	L, flip 1s
flip 1s	□ 1 1 1 □	0,L
flip 1s	□ 1 1 0 □	0,L
flip 1s	□ 1 0 0 □	0,L
flip 1s	□ 0 0 0 □	1,done
done	1 0 0 0 □	halt

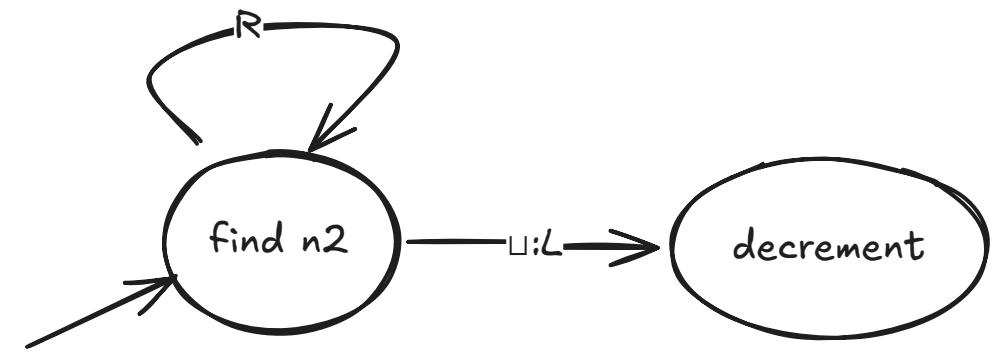
Example: Binary Adder

Input format: $\sqcup 101+10\sqcup$, **Output format:** $\sqcup 111\sqcup$ ($5 + 2 = 7$)

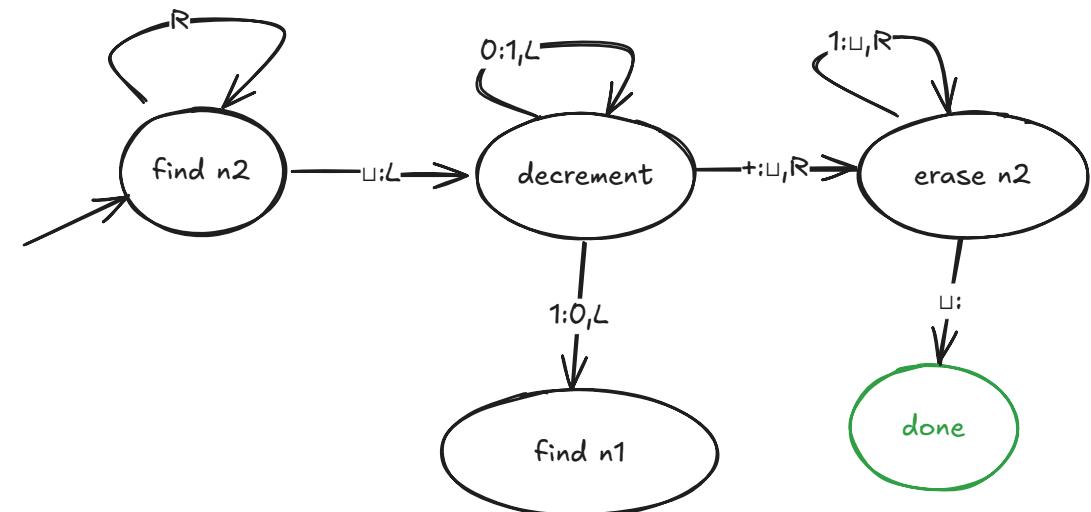
Strategy:

1. Scan right to the end of n_2
2. Decrement n_2
3. If n_2 was all 0s before the decrement (resulting in all 1s after the decrement):
 - i. Replace $+111\dots\sqcup$ with $\sqcup\sqcup\sqcup\dots$
 - ii. Accept
4. Scan left to the end of n_1
5. Increment n_1
6. Repeat from step 1

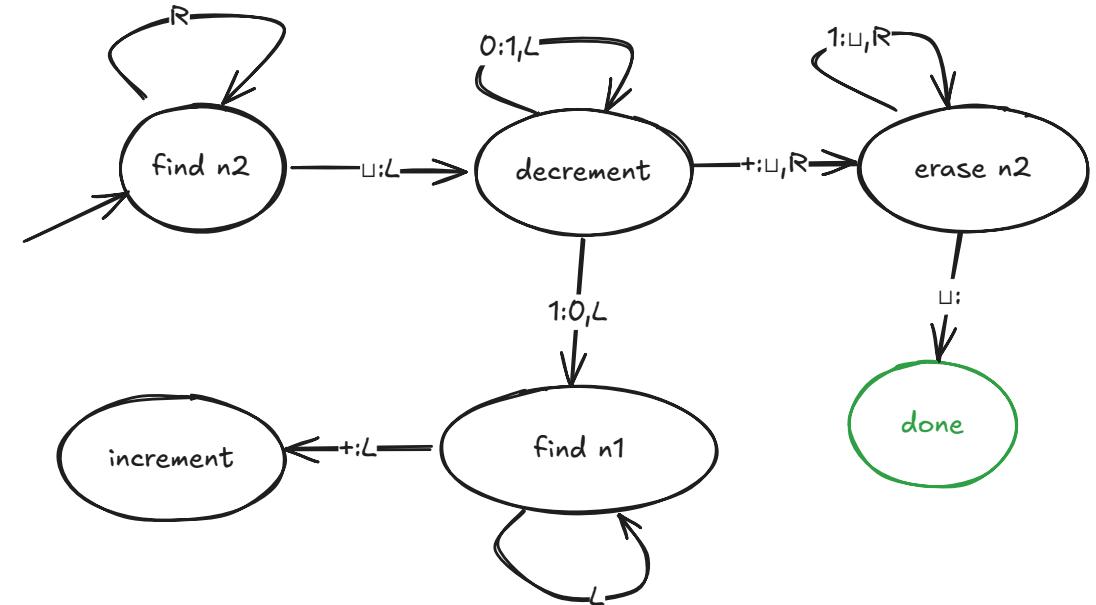
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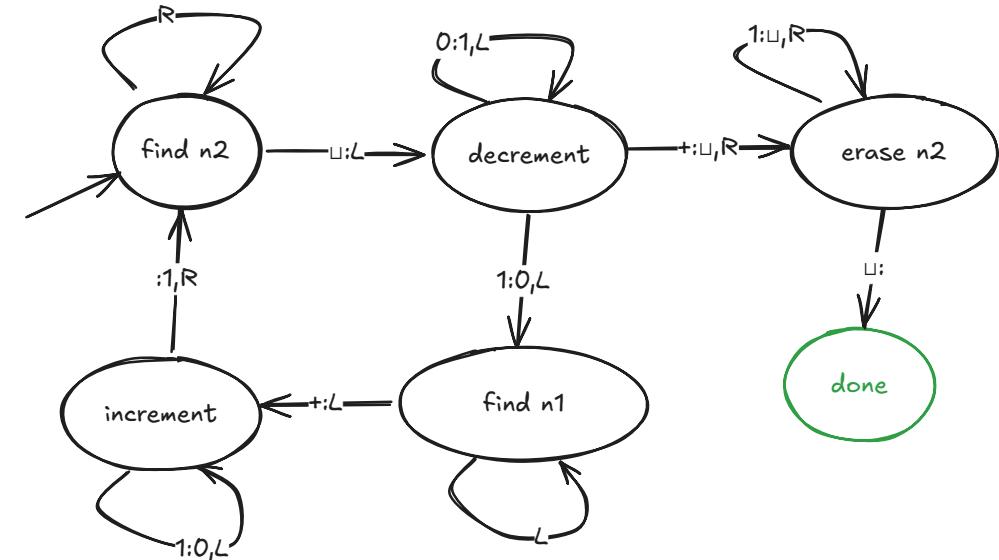
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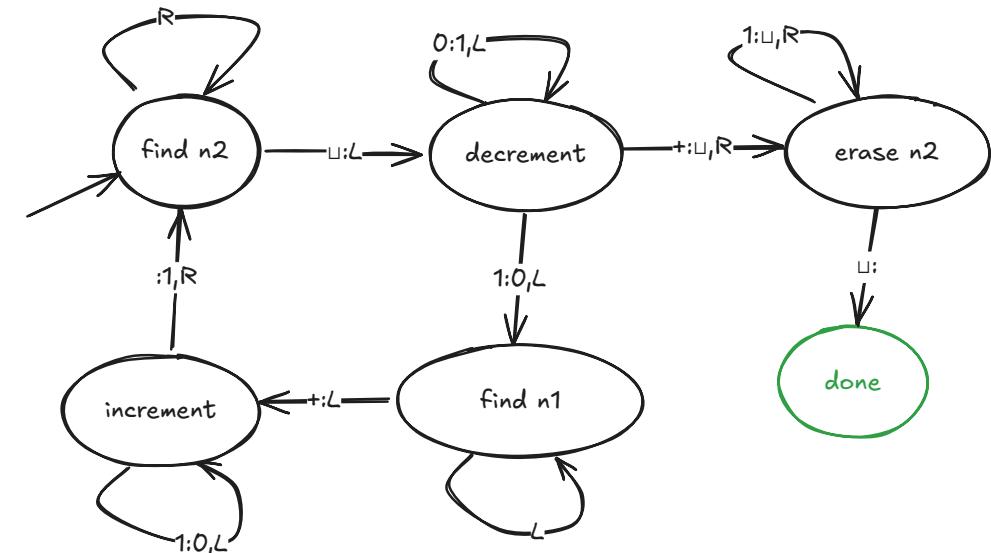
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6. Repeat from step 1



Adder: Trace Example

Input: $101 + 10$

State	Tape	Action
find n2	$\sqcup 1 0 1 + 1 0 \sqcup$	R
...
find n2	$\sqcup 1 0 1 + 1 0 \sqcup$	L, decrement
decrement	$\sqcup 1 0 1 + 1 0 \sqcup$	1, L
decrement	$\sqcup 1 0 1 + 1 1 \sqcup$	0, L, find n1
find n1	$\sqcup 1 0 1 + 0 1 \sqcup$	L, increment
increment	$\sqcup 1 0 1 + 0 1 \sqcup$	0, L
increment	$\sqcup 1 0 0 + 0 1 \sqcup$	1, R, find n2
find n2	$\sqcup 1 1 0 + 0 1 \sqcup$	R
...



Implementing TMs in Java

Just as we implemented virtual DFAs and NFAs, we can implement virtual TMs!

Key differences:

- The tape structure
 - writes
 - bidirectional
 - infinite
- accept/reject states
- halting

Java Implementation: Tape Class

```
public class Tape {  
  
    public char read() {...}  
    public void write(char symbol) {...}  
    public void moveLeft() {...}  
    public void moveRight() {...}  
}
```

Tape: Two-Stack Strategy

Problem: Tape is infinite

Solution: Use two stacks

- **Left stack:** Symbols to the left
- **Right stack:** Symbols to the right
- **Current symbol:** Between them

Example: For input 101

Left: []
Current: 1
Right: [0, 1]

After moveRight():
Left: [1]
Current: 0
Right: [1]

Java Implementation: Transition Class

```
public class Transition {  
    public enum Direction { L, R }  
  
    public Transition(State nextState,  
                      Character writeSymbol,  
                      Direction direction) {  
        this.nextState = nextState;  
        this.writeSymbol = writeSymbol;  
        this.direction = direction;  
    }  
  
    private final State nextState;  
    private final Character writeSymbol;  
    private final Direction direction;  
}
```

Java Implementation: State Class

```
public class State {  
    public void addTransition(Character inputSymbol,  
                             Transition transition) {  
        transitions.put(inputSymbol, transition);  
    }  
  
    public Transition getTransition(Character inputSymbol) {  
        return transitions.get(inputSymbol);  
    }  
  
    private final Map<Character, Transition> transitions = new HashMap<>();  
}
```

Java Implementation: TM Class

```
public class TM {  
    public TM() {...}  
  
    public void setStartState(State state) {...}  
    public void addAcceptState(State state) {...}  
    public void addRejectState(State state) {...}  
  
    /** Returns final decision (accept/reject) and updates tape contents. */  
    public boolean run(Tape tape) {...}  
}
```

Why Turing Machines Matter

Three Fundamental Reasons:

1. Theoretical Foundation

- Precise model of computation
- Enables mathematical proofs

2. Universal Model

- Church-Turing Thesis (next lecture)
- As powerful as any physical computer

3. Practical Impact

- Inspired von Neumann architecture
- Foundation for compiler theory
- Basis for computability theory

Key Takeaways

- ✓ TMs add tape read/write and bidirectional movement to DFAs
- ✓ TMs can recognize non-regular languages like $0^n 1^n$
- ✓ TMs can be implemented in Java using two stacks for the tape
- ✓ TMs are the theoretical model for all computation

Looking Ahead

Next Topics:

1. **Church-Turing Thesis** - TMs = maximal computational power
2. **Decidability** - What can TMs compute?
3. **The Halting Problem** - What CAN'T TMs compute?
4. **Complexity Theory** - What's practical vs. impractical?

