Formal Systems & Languages

The Mathematical Foundation of Computation

How do we precisely define computational problems?

Learning Objectives

By the end of this session, you will be able to:

- **Define** alphabets, strings, and formal languages
- **Construct** examples of formal languages
- **Recognize** whether strings belong to specific languages
- **Explain** why formal languages are essential in computer science
- **Connect** formal languages to real-world computational problems

The Big Question

What exactly is a computational problem?

Answer: A **formal system** or **formal language**

Let's build up to this answer step by step...

Building Blocks: Alphabets

Definition

> An **alphabet** (denoted Σ) is a finite set of symbols

Examples:

Name	Alphabet Σ	Usage
Binary	{0, 1}	Computer memory
Decimal	{0, 1, 2, ..., 9}	Human numbers
DNA	{A, C, G, T}	Genetic sequences
English lowercase	{a, b, c, ..., z}	Text processing

🎯 Quick Check #1

Which of these are valid alphabets?

1. {0, 1, 2, ...}
2. {♠, ♣, ♥, ♦}
3. $\{\text{red}, \text{green}, \text{blue}\}$
4. All real numbers between 0 and 1
5. {+, -, *, /}

Think: What's the key requirement for an alphabet?

Quick Check #1 - Answers

Valid Alphabets ✓

2. {♠, ♣, ♥, ♦} - Finite set of symbols ✓
3. $\{\text{red}, \text{green}, \text{blue}\}$ - Finite set ✓
4. {+, -, *, /} - Finite set of operators ✓

Invalid Alphabets ✕

1. {0, 1, 2, ...} - **Infinite** set ✕
2. All real numbers - **Infinite** and uncountable ✕

Key: Alphabets must be **finite**!

Building Blocks: Strings

Definition

> A **string** (or word) is a finite sequence of symbols from an alphabet

Notation:

- Empty string: ε (epsilon) or λ (lambda)
- String length: |w| where w is a string
- Concatenation: xy (string x followed by string y)

String Examples

Over different alphabets:

Alphabet	Example Strings	Notes
{0, 1}	`1011`, `00`, `ε`	Binary strings
{a, ..., z}	`hello`, `xyz`, `aaa`	English-like
{0, ..., 9}	`42`, `007`, `999`	Numeric strings

Things to note:

- Strings are **finite**
- Order matters: `abc` ≠ `bca`
- Repetition allowed: `aaa` is valid

📝 Active Learning: String Construction

Your Turn: Given alphabet $\Sigma = \{a, b\}$

1. Write all strings of length 0
2. Write all strings of length 1
3. Write all strings of length 2
4. How many strings of length n exist?

String Construction - Solution

For alphabet $\Sigma = (\{a, b\})$:

Length	Strings	Count
0	`ε`	1
1	`a` , `b`	2
2	`aa` , `ab` , `ba` , `bb`	4
3	`aaa` , `aab` , `aba` , ...	8
n	All combinations	(2^n)

Pattern: For alphabet of size k, there are (k^n) strings of length n

The Main Concept: Formal Languages

Definition

> A **formal language** L is a set of strings over an alphabet

Key Points:

- Can be finite or **infinite**
- Subset of all possible strings ($L \subseteq \Sigma^*$)
- Σ^* denotes all possible strings over alphabet Σ

Examples:

- $L_1 = (\{00, 11\})$ - finite language
- $L_2 = (\{a^n b^n \mid n \geq 0\}) = (\{\varepsilon, ab, aabb, aaabbb, \ldots\})$ - infinite

Formal Language Examples

Language Description	In Language ✓	Not in Language ✕
English	`apple`, `racecar`, `computer`	`qptey`, `znwy`
Palindromes	`bob`, `racecar`, `noon`	`apple`, `banana`
Prime numbers	`2`, `3`, `5`, `7`, `11`	`1`, `4`, `6`, `8`
Valid emails	`user@domain.com`	`user@`, `@domain`
Balanced parentheses	`()`, `(())`, `(()())`	`(`, `())`, `(()`

More Complex Patterns

1. Even Binary Numbers

L = $\{w \in \{0,1\}^* | w \text{ ends with } 0\}$

- In: `10`, `100`, `1110`
- Not in: `11`, `101`, `1`

2. Equal Count Language

L = {a^n b^n | n ≥ 0}

- In: `ε`, `ab`, `aabb`, `aaabbb`
- Not in: `aab`, `abab`, `ba`

🤔 Active Learning: Language Detective

Challenge: Determine the pattern!

Language L contains: `a`, `aa`, `aaaa`, `aaaaaaaa`
Language L does NOT contain: `ε`, `aaa`, `aaaaa`, `aaaaaa`

Questions:

1. What's the pattern?
2. Write L in set notation
3. Is `aaaaaaaaaaaaaaaa` in L?

(5 minutes - discuss with partner)

Language Detective - Solution

The Pattern Revealed

Observation: String lengths are 1, 2, 4, 8...

Answer:

L = $\{a^{2^n} | n \geq 0\} = \{a^1, a^2, a^4, a^8, a^{16}, \ldots\}$

Strings with length equal to powers of 2!

Is `aaaaaaaaaaaaaaaa` (16 a's) in L?

Yes! 16 = 2^4, so $a^{16} \in$ L

Programming Languages as Formal Languages

Connecting Theory to Practice

Formal Language	Valid Strings	Invalid Strings
Java Identifiers	`x`, `myVar`, `_test`	`123abc`, `my-var`, `class`
Java Programs	Complete compilable code	Syntax errors
SQL Queries	`SELECT * FROM users`	`SELECTING FROM users`

Key Insight:

Compilers are language recognizers - they check if your code belongs to the formal language of valid programs!

Special Mathematical Example

Fermat's Last Theorem as a Language

Language Definition:

L = $\{n \in \mathbb{Z} | x^n + y^n = z^n \text{ for some integers } x, y, z > 0 \text{ and } n > 2\}$

Amazing Fact:

- **L = ∅** (empty set)
- Proved by Andrew Wiles in 1995
- Took 350+ years to prove!

This shows how formal languages can encode deep mathematical problems

Why Formal Languages Matter

1. Precise Problem Definition

Instead of asking vaguely "Is x prime?", we ask:

> "Is string x in the language PRIMES = {2, 3, 5, 7, 11, ...}?"

2. Computability Analysis

We can formally prove:

- What problems are solvable
- What problems are unsolvable
- Complexity of solutions

Graph Problems as Languages

Encoding Complex Objects

Problem: Is graph G connected?

Formal Language Approach:

1. Encode graph as string: `<G>` = `"[n1,n2,n3],[(n1,n2),(n2,n3)]"`
2. Define: CONNECTED = $\{< G > | G \text{ is a connected graph}\}$
3. Question becomes: Is `<G>` ∈ CONNECTED?

This transformation enables mathematical analysis!

🎮 Active Learning: Language Building Game

Group Activity: Create your own formal language!

1. Choose an alphabet (2-3 symbols)
2. Define a rule for your language
3. Give 3 strings IN your language
4. Give 3 strings NOT in your language
5. Other groups guess your rule!

(7 minutes total - 4 to create, 3 to share)

The Power of Formalization

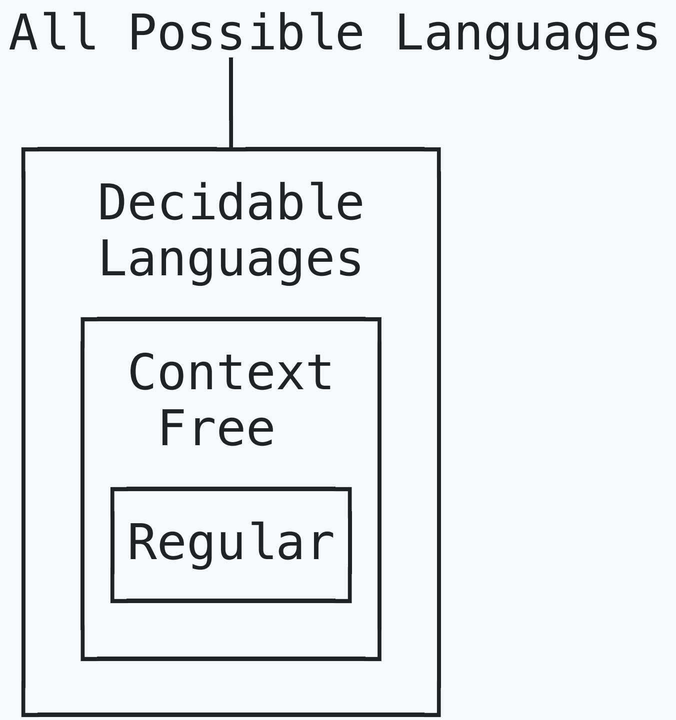
From Informal to Formal

Informal Question	Formal Language
"Is this password strong?"	L = {passwords with 8+ chars, number, special}
"Is this chemical formula valid?"	L = {valid molecular formulas}
"Can I win this game?"	L = {game states with winning strategy}

Key Benefit: Mathematical tools can now be applied!

Hierarchy of Language Classes

Preview of Coming Attractions



Different computational models can recognize different language classes!

Key Takeaways

Essential Concepts

1. **Alphabet** → Finite set of symbols
2. **String** → Finite sequence from alphabet
3. **Language** → Set of strings (possibly infinite)

Why This Matters:

- **Precision**: Exact problem specification
- **Analysis**: Mathematical proofs of possibility/impossibility
- **Applications**: Compilers, pattern matching, AI, cryptography

Practical Applications

Immediate Applications:

- **Regular Expressions** - Pattern matching in code
- **Parsing** - Understanding programming languages
- **Validation** - Input checking

Advanced Applications:

- **Machine Learning** - Language models (GPT, BERT)
- **Bioinformatics** - DNA sequence analysis
- **Cryptography** - Security protocols
- **Quantum Computing** - Quantum languages

Go Deeper?

Read my notes on the PQ formal language invented Invented by Douglas Hofstadter in his 1979 book "Gödel, Escher, Bach"

- A useful illustration of a formal system and its relevance to math and computation