UNIT-II
Integral Calculus

Integration as inverse operation of differentiation. Simple integration by substitution, by parts and by partial fractions (for linear factors only). Introduction to definite integration. Use of formulae $\int_0^{\frac{\pi}{2}} \sin^n x dx$, $\int_0^{\frac{\pi}{2}} \cos^n x dx$, $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ for solving problems ,where m and n are positive integers. Applications of integration for (i). Simple problems on evaluation of area bounded by a curve and axes. (ii). calculation of volume of a solid formed by revolution of an area about axes. (Simple problems).

TOPICS

1. समाकलन की परिभाषा (Definition of Integration
2. समाकलन के प्रकार (Types of Integration)
3. समाकलन से संबंधित सूत्र (Formula related to Integration)
4. प्रतिस्थापन द्वारा समाकलन (Integration by Substitution)
5. खण्डश: समाकलन (Integration by Parts)
6. आंशिक भिन्नों द्वारा समाकलन (Integration by partial fractions)
7. गामा फलन द्वारा समाकलन (Integration Using Gama Function)
8. समाकलन के अनुप्रयोग (Applications of Integration)

1. समाकलन (Integration)

Definition :-

- **समाकलन, अवकलन की विपरीत क्रिया है**

(Integration is reverse process of differentiation) .

if F(x) is a function. $\frac{d\,F(x)}{dx} = f(x)$ **तो** $\int f(x).\,dx = F(x) + c$

c = समाकलन स्थिरांक (integration constant)

समाकल्य (Integrand) :- जिस फलन का समाकलन करते हैं उसे समाकल्य (Integrand) कहते हैं।
(The function which is integrated is called integral.)

समाकल (Integral) :- समाकलन की प्रक्रिया से प्राप्त फलन को समाकल (integral) कहते हैं।
(The function obtained from the process of integration is

समाकलन (Integration) :- समाकलन की प्रक्रिया से प्राप्त फलन को समाकल (integral) कहते हैं। जिस प्रक्रिया (process) द्वारा समाकल प्राप्त होता है उसे समाकलन (integration) कहते हैं।
The function obtained from the process of integration is called integral.
The process by which the integral is obtained is called integration.

2. समाकलन के प्रकार (Types of Integration)

समाकलन दो प्रकार के होते हैं :-

(i) अनिश्चित समाकलन (Indefinite integration)

$$\int f(x).\,dx = F(x) + c$$

(ii) निश्चित समाकलन (Definite integration)

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

a = Lower Limit (निम्न सीमा)

b = Upper Limit (उच्च सीमा)

जिसमें Limits का use करते हैं ⟶ निश्चित समाकलन (Definite integration)

जिसमें Limits का use नहीं करते ⟶ अनिश्चित समासन (Indefinite integration)

3. समाकलन से संबंधित सूत्र (Formula related to Integration)

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

2. $\int \frac{1}{x}\,dx = \log_e x + C$

3. $\frac{d}{dx}(e^x) = e^x$

4. $\frac{d}{dx}(a^x) = a^x \log_e a$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x$

10. $\frac{d}{dx}(\text{cosec}x = -\text{cosec}x \cot x$

3. $\int e^x \, dx = e^x + C$

4. $\int a^x \, dx = \frac{a^x}{\log_e a} + C$

5. $\int \sin x \, dx = -\cos x + C$

6. $\int \cos x \, dx = \sin x + C$

7. $\int \sec^2 x \, dx = \tan x + C$

8. $\int \text{cosec}^2 x \, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + C$

11. $\int \tan x \, dx = \log_e \sec x + C$

12. $\int \cot x \, dx = \log_e \sin x + C$

13. $\int \sec x \, dx = \log_e(\sec x + \tan x) + C$

14. $\int \text{cosec}\, x \, dx = \log_e(\text{cosec}\, x - \cot x) + C$

15. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

17. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

15. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C = -\cos^{-1} x + C$

16. $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C = -\cot^{-1} x + C$

17. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C = -\csc^{-1} x + C$

Note:

यदि $k = \text{constant}$

$$\int k \cdot f(x) \, dx = k \cdot \int f(x) \, dx + c$$

$$\int 1 \cdot dx = x + c \quad (1 = x^0)$$

$$\int x^0 \cdot dx = \frac{x^{0+1}}{0+1} = x$$

Q. 1. :- $\int (2x^3 + 5x + 6)\, dx$ का x के सापेक्ष समाकलन ज्ञात करो।

Find the integral of $\int (2x^3 + 5x + 6)\, dx$ with respect to x.

$$\int (2x^3 + 5x + 6)\, dx$$

$$= \int 2x^3\, dx + \int 5x\, dx + \int 6\, dx$$

$$= 2\int x^3\, dx + 5\int x\, dx + 6\int 1 \cdot dx$$

$$= 2\left(\frac{x^{3+1}}{3+1}\right) + 5\left(\frac{x^{1+1}}{1+1}\right) + 6(x) + C$$

$$= \frac{x^4}{2} + \frac{5x^2}{2} + 6x + C \quad \underline{\text{Ans}}$$

Q. 2 :- $\int \frac{(1 - x^2)^3}{x^2}$ dx का x के सापेक्ष समाकलन ज्ञात करो।

Find the integral of with respect to x.

$$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$$

$$\int \frac{(1-x^2)^3}{x^2}\, dx$$

$$= \int \frac{1 - x^6 - 3 \cdot x \cdot 1 \cdot x \cdot (1 - x^2)}{x^2}\, dx$$

$$= \int \left[\frac{1}{x^2} - x^4 - \frac{3x^2(1-x^2)}{x^2}\right] dx$$

$$= \int \left(\frac{1}{x^2} - x^4 - 3 + 3x^2\right) dx$$

$$= \int (x^{-2} - x^4 - 3 + 3x^2)\, dx$$

$$= \frac{x^{-2+1}}{-2+1} - \frac{x^{4+1}}{4+1} - 3x + 3 \cdot \frac{x^{2+1}}{3} + C$$

$$= -\frac{1}{x} - \frac{x^5}{5} - 3x + x^3 + C \quad \underline{\text{Ans}}$$

Q. 3:- $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$ **का x के सापेक्ष समाकलन ज्ञात करो।**
Find the integral of with respect to x.

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int \left[(\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 + 2\cdot\sqrt{x}\cdot\frac{1}{\sqrt{x}}\right] dx$$

$$= \int \left[x + \frac{1}{x} + 2\right] dx$$

$$= \frac{x^2}{2} + \log_e x + 2x + C \quad \underline{\text{Ans}}$$

Q. 4 :- $\int \frac{5}{\sin^2 x \cdot \cos^2 x} dx$ **का x के सापेक्ष समाकलन ज्ञात करो।**
Find the integral of with respect to x.

Method I

$$= \frac{4\times 5}{4\times \sin^2 x \cdot \cos^2 x} dx$$

$$= 20 \int \frac{1}{(2\sin x \cdot \cos x)^2} dx$$

$$= \int \frac{20}{(2\sin x \cdot \cos x)^2} dx$$

$$= 20 \int \frac{1}{\sin^2 2x} dx$$

$$= 20 \int \text{cosec}^2 2x \, dx$$

$$= 20\left(\frac{-\cot 2x}{2}\right) + C$$

$$= -10\cot 2x + C \quad \underline{\text{Ans}}$$

$$\int \text{cosec}^2 x \, dx = -\cot x$$

Note: समाकलन में x के गुणांक से भाग कर देते हैं।

Method II

$$\int \frac{5}{\sin^2 x \cdot \cos^2 x}\,dx$$

$$= 5\int \frac{1}{\sin^2 x \cdot \cos^2 x}\,dx$$

$$= 5\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx$$

$$= 5\int \left(\frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x}\right)dx$$

$$= 5\int (\sec^2 x + \csc^2 x)\,dx$$

$$= 5(\tan x - \cot x) + C \quad \underline{\text{Ans}}$$

Q. 5 :- $\int \frac{1-\tan x}{1+\tan x}\,dx$ **का x के सापेक्ष समाकलन ज्ञात करो।**
Find the integral of with respect to x.

$$= \int \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \cdot \tan x}\,dx \qquad \because \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B} = \tan(A-B)$$

$$= \int \tan(45^\circ - x)\,dx$$

$$\int \tan x\,dx = \log \sec x$$

$$= \frac{\log \sec(45^\circ - x)}{-1} + C$$

$$= -\log \sec(45^\circ - x) + C \quad \underline{\text{Ans}}$$

चरघातांकी फलनों पर आधारित प्रश्न
Questions based on exponential functions

Q. 6 :- $\int \frac{e^{7\log_e x} - e^{6\log_e x}}{e^{5\log_e x} - e^{4\log_e x}}\,dx$

Sol:-

$$= \int \frac{e^{\log_e x^7} - e^{\log_e x^6}}{e^{\log_e x^5} - e^{\log_e x^4}} \, dx$$

$$\log m^n = n \log m$$
$$e^{\log_e x} = x$$

$$\int \frac{x^7 - x^6}{x^5 - x^4} \, dx$$

$$\int \frac{x^6(x-1)}{x^4(x-1)} \, dx$$

$$\int x^2 \, dx = \frac{x^3}{3} + C \quad \underline{\text{Ans}}$$

Q. 7 :- यदि $f'(x) = 3x^2 - \dfrac{2}{x^3}$ तथा f(1) = 0 तो f (x) का मान ज्ञात करो।
If f(1) = 0 then find the value of f(x).

$f'(x)$ का अर्थ → $f(x)$ का अवकलन (differential)

$$\frac{df(x)}{dx} = f'(x) \Rightarrow \int f'(x)\, dx = f(x)$$

$$\int f'(x)\, dx = \int \left(3x^2 - \frac{2}{x^3}\right) dx$$

$$f(x) = 3 \cdot \frac{x^3}{3} - 2 \cdot \frac{x^{-3+1}}{-3+1} + C$$

$$f(x) = x^3 + \frac{1}{x^2} + C \qquad (1)$$

समीकरण (1) में x=1 रखने पर

$$f(1) = 1 + \frac{1}{1} + C = 2 + C$$

दिया है $f(1) = 0,$

$$0 = 2 + C \Rightarrow C = -2$$

अब $C = -2$ को (1) में रखें:

$$\boxed{f(x) = x^3 + \frac{1}{x^2} - 2}$$

Q. 8 :- यदि $f'(x) = 4x^3 - \dfrac{3}{x^4}$ **तथा f(2) = 0 तो f (x) का मान ज्ञात करो।**
If f(2) = 0 then find the value of f(x).

Integrate both sides

$$\int f'(x)\, dx = \int \left(4x^3 - \frac{3}{x^4}\right) dx$$

$$f(x) = 4 \cdot \frac{x^4}{4} - 3 \cdot \frac{x^{-3}}{-3} + C$$

$$f(x) = x^4 + \frac{1}{x^3} + C \quad (\text{Equation } ①)$$

$$f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$f(2) = 16 + \frac{1}{8} + C = 0$$

$$C = -\left(16 + \frac{1}{8}\right) = -\frac{129}{8}$$

$$\boxed{f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}} \quad \underline{\text{Ans}}$$

4. प्रतिस्थापन द्वारा समाकलन (Integration by Substitution)

प्रतिस्थापन (Substitution) :-

- **किसी एक चर (Variable) 'x' के स्थान पर कोई दूसरा चर (Variable) 't' रखने को प्रतिस्थापन कहते हैं।**

Hints ⟶ (t मानकर समाकलन करेंगे)

NOTE :-

(i) यदि कोई function करणी (√) में है तो इस function को t या t^2 मानकर Integration करते हैं।

(ii) यदि कोई function (ax ± b) के रूप में है तो (ax ± b) को t मानकर Integral करते हैं या फिर Direct Formula use करके x के गुणांक (coefficient) से भाग कर देते हैं।

(iii) जिस function का Differential (अवकलन) दिया होता है, उसे t मानकर Integral करते हैं।

Type - I ⟶ यदि function (ax ± b)के रूप में हो

Rule :-
- **(ax ± b) को t मानकर Integral करेंगे या फिर**
- **Direct formula use करके x के गुणांक से भाग करके**

Q.1 :- $\int (8x+5)^5 \, dx$ **को हल करें।**

Method I: Direct (Using Chain Rule)

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int (8x+5)^5\,dx = \frac{(8x+5)^6}{8\cdot 6} + C = \frac{(8x+5)^6}{48} + C$$

$$\boxed{\frac{(8x+5)^6}{48} + C} \quad \underline{\text{Ans}}$$

$$\int (8x+5)^5 dx$$

Method II: Substitution Method

Let:

$$t = 8x+5 \Rightarrow \frac{dt}{dx} = 8 \Rightarrow dx = \frac{dt}{8}$$

$$\int (8x+5)^5 dx = \int t^5 \cdot \frac{dt}{8} = \frac{1}{8}\int t^5\,dt$$

$$\frac{1}{8}\cdot\frac{t^6}{6} + C = \frac{t^6}{48} + C$$

Back-substitute $t = 8x+5$:

$$\boxed{\frac{(8x+5)^6}{48} + C} \quad \underline{\text{Ans}}$$

Q.2 :- $\displaystyle\int \frac{1}{7x-3}\,dx \quad \because \int \frac{1}{x}\,dx = \log_e x$

Direct →

$$\frac{\log_e(7x-3)}{7} + C \quad \text{Ans}$$

Substitution (प्रतिस्थापन) →

माना $7x - 3 = t$

Taking derivative w.r.t. x :

$$\frac{d}{dx}(7x-3) = \frac{dt}{dx} \Rightarrow 7 = \frac{dt}{dx} \Rightarrow 7\,dx = dt \Rightarrow dx = \frac{dt}{7}$$

$$\int \frac{1}{t}\cdot\frac{dt}{7} = \frac{1}{7}\int \frac{1}{t}\,dt = \frac{1}{7}\log_e t + C = \frac{1}{7}\log_e(7x-3) + C \quad \text{Ans}$$

Q.3 :- $\int a^{(3x+3)}\,dx$

$$\int a^{(3x+3)}\,dx = \frac{a^{(3x+3)}}{3\cdot\log_e a} + C \quad \text{Ans}$$

$$\text{(Formula used: } \int a^x dx = \frac{a^x}{\log_e a} + C\text{)}$$

Q.4 :- $\int \frac{2}{1+\cos 2x}\,dx$

$$= \int \frac{2}{1+(2\cos^2 x - 1)}\,dx \qquad \because \cos 2x = 2\cos^2 x - 1$$

$$= \int \frac{2}{2\cos^2 x}\,dx = \int \sec^2 x\,dx \qquad \because \sec^{2x} \text{ dx =tanx +c}$$

$$= \tan x + C \quad \text{Ans}$$

Q.5 :- $\int \sin^2(x^2+3)\cdot x\,dx$

माना $x^2 + 3 = t$

d.w.r.t. to x

$$2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$$\int \sin^2 t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}\int \sin^2 t\,dt$$

$$\therefore \cos 2t = 1 - 2\sin^2 t$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$= \frac{1}{2}\int \frac{1-\cos 2t}{2}\,dt$$

$$= \frac{1}{4}\int (1-\cos 2t)\,dt$$

$$= \frac{1}{4}\left[t - \frac{\sin 2t}{2}\right] + C$$

$$= \frac{1}{4}\left[(x^2+3) - \frac{\sin 2(x^2+3)}{2}\right] + C \quad \underline{\text{Ans}}$$

Q.6 :- $\int \tan(3x+1)\,dx$

$$\int \tan(3x+1)\,dx$$

$$= \frac{\log \sec(3x+1)}{3} + C \quad \text{Ans}$$

Q.7 :- $\int e^{(5x-6)}\,dx$

$$= \frac{e^{(5x-6)}}{5} + C \quad \text{Ans}$$

Q.8 :- $\int e^{2x-3}\,dx$ का समाकलन ज्ञात करो।

Find the integral.

$$= \int e^{2x-3}\,dx$$

$$= \frac{e^{2x-3}}{2} + C \quad \underline{\text{Ans}}$$

Type - II

$\int \frac{f'(x)}{f(x)} . dx$ के रूप के समाकलन

जहाँ f(x) is a function (फलन)

& f'(x) ⟶ f(x) का अवकलन (differential)

Rule :- जिसका Differential दिया होता है, उसे t मानकर समाकलन करते हैं।

यहां पर, Let $f(x) = t$

Differentiating w.r.t. x

$$f'(x) = \frac{dt}{dx}$$

$$f'(x)\,dx = dt$$

अत: (Therefore) :-

$$\int \frac{1}{t}\,dt = \log_e t + C = \log_e f(x) + C \quad \underline{\text{Ans}}$$

Q.1 :- Prove that $\int \tan x \, dx = \log_e \sec x + C$

$$= \int \frac{\sin x}{\cos x} \, dx$$

Let $\cos x = t$

Differentiating w.r.t. x:

$$-\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$$

$$= \int \frac{1}{t} \cdot (-dt)$$

$$= -\log_e t + C$$

$$= -\log_e(\cos x) + C$$

$$= \log_e(\cos x)^{-1} + C$$

$$= \log_e\left(\frac{1}{\cos x}\right) + C$$

$$= \log_e(\sec x) + C \quad \boxed{\text{Proved!}}$$

NOTE :- **जब दो functions एक दूसरे के अवकल (Differential) हों और वह भिन्न में हों,तो हमेशा हर (denominator) वाले function को t मानते हैं।**

Q.2 :- Poove that $\int \cot x \, dx = \log_e \sin x + C$

$$= \int \frac{\cos x}{\sin x} \, dx$$

Let $\sin x = t$

Differentiating w.r.t. x:

$$\cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$$

$$= \int \frac{dt}{t}$$

$$= \log_e t + C = \log_e(\sin x) + C \quad \boxed{\text{Proved!}}$$

Q.3 :- $\int \frac{e^x}{1+e^x}\,dx$ का समाकलन ज्ञात करो।
Find the integral.

Method 1: Let $e^x = t$

Differentiating w.r.t. x:

$$\frac{dt}{dx} = e^x \Rightarrow e^x dx = dt$$

$$\int \frac{1}{1+t}\,dt = \log_e(1+t) + C$$

$$= \log_e(1+e^x) + C \quad \underline{\text{Ans}}$$

Method 2 Let $1 + e^x = t$

Differentiating w.r.t. x:

$$\frac{dt}{dx} = e^x \Rightarrow e^x dx = dt$$

$$\int \frac{1}{t}\,dt = \log_e t + C$$

$$= \log_e(1+e^x) + C \quad \underline{\text{Ans}}$$

Q.4 :- $\int \frac{\cos x}{(\sin^4 x)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $\sin x = t$

d. w. r. to x

$$\cos x\,dx = dt$$

$$= \int \frac{1}{t^4}\,dt$$

$$= \int t^{-4}\,dt$$

$$= \frac{t^{-4+1}}{-4+1} + C$$

$$= \frac{(\sin x)^{-3}}{-3} + C$$

$$= -\frac{1}{3\sin^3 x} + C \quad \textbf{Ans}$$

Q.5 :- $\int \frac{3x^2}{(x^3+4)^3}\,dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $(x^3+4) = t$

d. w. r. to x

$$3x^2 + 0 = \frac{dt}{dx}$$

$$3x^2 dx = dt$$

$$= \int \frac{1}{t^3}\,dt$$

$$= \int t^{-3}\,dt$$

$$= \frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{(x^3+4)^{-2}}{-2} + C$$

$$= -\frac{1}{2(x^3+4)^2} + C \quad \textbf{Ans}$$

Q.6 :- $\int \sin^3 x\,dx$ का समाकलन ज्ञात करो।
Find the integral.

Method–I

$$\because \sin 3x = 3\sin x - 4\sin^3 x$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\int \frac{3\sin x - \sin 3x}{4}\,dx$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\,dx$$

$$= \frac{1}{4}\left[3\cdot(-\cos x) - \left(-\frac{\cos 3x}{3}\right)\right] + C$$

$$= \frac{-3\cos x}{4} + \frac{\cos 3x}{12} + C \quad \textbf{Ans}$$

Method–II

$$\int \sin x \cdot \sin^2 x \, dx \qquad \because \ \sin^2 x = 1 - \cos^2 x$$

$$\int \sin x(1 - \cos^2 x) \, dx$$

माना $\cos x = t$

d. w. r. to x:

$$-\sin x \, dx = dt$$
$$\sin x \, dx = -dt$$

$$= \int (1 - t^2)(-dt)$$

$$= \int (t^2 - 1) \, dt$$

$$= \frac{t^3}{3} - t + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C \quad \textbf{Ans}$$

Q.7 :- $\displaystyle\int \frac{e^{m \tan^{-1} x}}{(1 + x^2)} dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $\tan^{-1} x = t$

d. w. r. to x:

$$\frac{1}{1 + x^2} = \frac{dt}{dx} \Rightarrow \frac{dx}{1 + x^2} = dt$$

$$= \int e^{m \cdot t} \, dt$$

$$= \frac{e^{mt}}{m} + C$$

$$= \frac{e^{m \cdot \tan^{-1} x}}{m} + C \quad \textbf{Ans}$$

Q.8 :- $\int \frac{(e^x - e^{-x})}{(e^x + e^{-x})} dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $e^x + e^{-x} = t$

d. w. r. to x:

$$e^x + (-e^{-x}) = \frac{dt}{dx} \Rightarrow (e^x - e^{-x})dx = dt$$

$$\int \frac{dt}{t} = \int \frac{1}{t} dt = \log_e t + C$$

$$= \log_e(e^x + e^{-x}) + C \quad \textbf{Ans}$$

Q.9 :- $\int \frac{\cot x}{\sqrt{\sin x}} dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\because \cot x = \frac{\cos x}{\sin x}$$

$$= \int \frac{\cos x}{\sin x \cdot (\sin x)^{1/2}} dx = \int \frac{\cos x}{(\sin x)^{3/2}} dx$$

माना $\sin x = t$

d. w. r. to x:

$$\cos x \, dx = dt$$

$$= \int \frac{1}{t^{3/2}} dt$$

$$= \int t^{-3/2} dt$$

$$= \frac{t^{-3/2+1}}{-3/2+1} + C$$

$$\frac{(\sin x)^{-1/2}}{-1/2} + C$$

$$= -2 \cdot \frac{1}{(\sin x)^{1/2}} + C$$

$$= -\frac{2}{\sqrt{\sin x}} + C \quad \underline{\text{Ans}}$$

Q.10 :- $\int \frac{\sec^2 x}{3 + 4\tan x} dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $3 + 4\tan x = t$

d. w. r. to x:

$$0 + 4 \cdot \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x\, dx = \frac{dt}{4}$$

$$\int \frac{1}{t} \cdot \frac{dt}{4} = \frac{1}{4} \int \frac{1}{t}\, dt = \frac{1}{4} \log_e t + C = \frac{1}{4} \log_e (3 + 4 \tan x) + C \quad \textbf{Ans}$$

Q.11 :- $\displaystyle\int \frac{e^x (1 + x)}{\cos^2 (x e^x)}\, dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $x e^x = t$

d. w. r. to x:

$$x \cdot e^x + e^x \cdot 1 = \frac{dt}{dx} \Rightarrow e^x (1 + x) = \frac{dt}{dx} \Rightarrow e^x (1 + x)\, dx = dt$$

$$\int \frac{1}{\cos^2 t}\, dt = \int \sec^2 t\, dt = \tan t + C = \tan(x e^x) + C \quad \textbf{Ans}$$

Q.12 :- $\displaystyle\int \frac{x \cdot \tan^{-1} x^2}{1 + x^4}\, dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $\tan^{-1} x^2 = t$

d. w. r. to x:

$$\frac{1}{1 + (x^2)^2} \cdot (2x) = \frac{dt}{dx} \Rightarrow \frac{2x}{1 + x^4} = \frac{dt}{dx} \Rightarrow \frac{x}{1 + x^4}\, dx = \frac{dt}{2}$$

$$= \int t \cdot \frac{dt}{2} = \frac{1}{2} \int t\, dt = \frac{1}{2} \cdot \frac{t^2}{2} + C = \frac{t^2}{4} + C = \frac{(\tan^{-1} x^2)^2}{4} + C \quad \textbf{Ans}$$

Q.13 :- $\displaystyle\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx$ का समाकलन ज्ञात करो।
Find the integral.

माना $\sqrt{x} = t$

d w. r. to x:

$$\frac{1}{2\sqrt{x}}\,dx = dt$$

$$\frac{dx}{\sqrt{x}} = 2\,dt$$

$$\int \cos t \cdot (2\,dt)$$

$$= 2\int \cos t\,dt$$

अब $t = \sqrt{x}$ रखने पर:

$$= 2\sin t + C$$

$$= 2\sin\sqrt{x} + C \quad \underline{\text{Ans}}$$

Q.14 :- $\displaystyle\int \frac{x^2+1}{(x+1)^2}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$\int \frac{x^2+1}{(1+x)^2}dx = \int \frac{x^2+1+2x-2x}{(1+x)^2}dx$$

$$= \int \frac{(x+1)^2-2x}{(1+x)^2}dx = \int \left[\frac{(x+1)^2}{(x+1)^2} - \frac{2x}{(1+x)^2}\right]dx$$

$$= \int 1 \cdot dx - 2\int \frac{x}{(1+x)^2}dx$$

अब $I_1 = \displaystyle\int \frac{x}{(1+x)^2}dx$ निकालते हैं।

Let $t = 1+x \Rightarrow dt = dx,\ x = t-1$

$$I_1 = \int \frac{t-1}{t^2}dt = \int \left(\frac{t}{t^2} - \frac{1}{t^2}\right)dt = \int \left(\frac{1}{t} - \frac{1}{t^2}\right)dt$$

$$I_1 = \int \frac{1}{t}dt - \int t^{-2}dt = \ln|t| + t^{-1} + C$$

$$I_1 = \ln|1+x| + \frac{1}{1+x} + C$$

अब मुख्य समाकलन में रखें:

$$\int \frac{x^2+1}{(1+x)^2}dx = \int 1\,dx - 2I_1$$

$$\int \frac{x^2+1}{(x+1)^2}\,dx = x - 2\left(\log_e(x+1) + \frac{1}{x+1}\right) + C.$$

$$\boxed{x - 2\,\log_e(x+1) - \frac{2}{(x+1)} + C}$$ **Ans.**

Q.15 :- $\int \frac{1+\sin 3x}{3x-\cos 3x}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $3x-\cos 3x = t$

d w.r.t. x:

$$3+\sin(3x)(3) = \frac{dt}{dx}$$

$$= 3+3\sin 3x = \frac{dt}{dx}$$

$$3(1+\sin 3x)\,dx = dt$$

$$(1+\sin 3x)\,dx = \frac{dt}{3}$$

$$= \int \frac{1}{t}\cdot\frac{dt}{3}$$

$$= \frac{1}{3}\log_e t + C$$

$$= \frac{1}{3}\log_e(3x-\cos 3x) + C \quad \underline{\text{Ans}}$$

$$\int \sqrt{\tan x}\,dx$$

Substitute $\tan x = t^2$

$$\Rightarrow \sec^2 x\,dx = 2t\,dt \Rightarrow (1+t^4)dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{1+t^4}$$

So the integral becomes:

$$\int \frac{t\cdot 2t\,dt}{1+t^4}$$

$$= \int \frac{2t^2\,dt}{1+t^4}$$

$$= \int \frac{t^2+1}{1+t^4}\,dt + \int \frac{t^2-1}{1+t^4}\,dt$$

Dividing both numerators and denominators by t^2, we get:

$$\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt + \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt + \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}\,dt$$

Substitute $t-\frac{1}{t}=u$ in the first integral and
$t+\frac{1}{t}=v$ in the second integral

$$\Rightarrow \left(1+\frac{1}{t^2}\right)dt=du \quad \text{and} \quad \left(1-\frac{1}{t^2}\right)dt=dv$$

Hence, the integral becomes:

$$\int \frac{du}{u^2+2}+\int \frac{dv}{v^2-2}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|+C$$

Here, $u=t-\frac{1}{t}$ and $v=t+\frac{1}{t}$
and $t=\sqrt{\tan x}$

Substituting back the values, we get the integral in the original variable x.

Type - III $\longrightarrow \int \frac{1}{x^{1/m}+x^{1/n}}\,dx$ **के रूप में समाकलन**

Rule :- इस प्रकार के Question में m तथा n का L.C.M. लेते हैं। (माना P है।)
$x = t^p$ मानकर समाकलन करते हैं।

Q.1 :- $\int \frac{1}{x^{1/2}+x^{1/3}}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

2 और 3 का L.C.M. = 6

माना $x = t^6$

d.w.r.t. x

$$dx = 6t^5\,dt$$

$$=\int \frac{1}{(t^6)^{1/2}+(t^6)^{1/3}}\cdot 6t^5\,dt$$

$$=6\int \frac{t^5}{t^3+t^2}\,dt$$

$$=6\int \frac{t^5}{t^2(t+1)}\,dt$$

$$= 6\int \frac{t^3}{t+1}\,dt$$

$$= 6\int \frac{t^3+1-1}{t+1}\,dt$$

$$= 6\int \left[\frac{t^3+1}{t+1} - \frac{1}{t+1}\right] dt$$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$= 6\int \left[(t+1)\frac{(t^2+1-t)}{(t+1)} - \frac{1}{(t+1)}\right] dt$$

$$= 6\int (t^2+1-t)\,dt - 6\int \frac{1}{(t+1)}dt$$

$$= 6\left(\frac{t^3}{3} + t - \frac{t^2}{2}\right) - 6\cdot\log_e(t+1) + C \qquad \text{अब } x = t^6 \qquad t = x^{1/6}$$

$$= 2\cdot(x^{1/6})^3 + 6\cdot(x^{1/6}) - 3\cdot(x^{1/6})^2 - 6\cdot\log_e(x^{1/6}+1) + C$$

$$\boxed{= 2x^{1/2} + 6x^{1/6} - 3x^{1/3} - 6\log_e(x^{1/6}+1) + C}$$ **Ans.**

Q.2 :- $\int \frac{dx}{(1+x)^{1/2} + (1+x)^{1/3}}$ **का समाकलन ज्ञात करो।**
Find the integral.

2 और 3 का L.C.M. = 6

माना $(1 + x) = t^6$

d.w.r.t. x

$$(0+1)\,dx = 6t^5\,dt$$

$$dx = 6t^5\,dt$$

$$= \int \frac{6t^5\,dt}{(t^6)^{1/2} + (t^6)^{1/3}}$$

$$= 6\int \frac{t^5}{t^3+t^2}\,dt$$

$$= 6\int \frac{t^5}{t^2(t+1)}\,dt$$

$$= 6\int \frac{t^3}{t+1}\,dt$$

$$= 6\int \frac{t^3+1-1}{t+1}\,dt$$

$$= 6\int \left[\frac{t^3+1}{t+1} - \frac{1}{t+1}\right] dt$$

$$\because\ a^3+b^3 = (a+b)(a^2+b^2-ab)$$

$$= 6\int\left[\frac{(t+1)(t^2+1-t)}{(t+1)} - \frac{1}{(t+1)}\right]dt$$

$$= 6\int(t^2+1-t)\,dt \;-\; 6\int\frac{1}{(t+1)}\,dt$$

$$= 6\left(\frac{t^3}{3}+t-\frac{t^2}{2}\right) - 6\log_e(t+1) + C$$

$$\because (1+x) = t^6 \quad\Rightarrow\quad (1+x)^{1/6} = t$$

$$= 2\cdot((1+x)^{1/6})^3 + 6\cdot(1+x)^{1/6} - 3\cdot((1+x)^{1/6})^2 - 6\log_e\left[(1+x)^{1}\right.$$

$$\boxed{= 2(1+x)^{1/2} + 6(1+x)^{1/6} - 3(1+x)^{1/3} - 6\log_e\left((1+x)^{1/6}+1\right) + C}$$ **Ans.**

Type - IV $\longrightarrow \displaystyle\int \frac{f(x)}{\sqrt{x+a}\pm\sqrt{x+b}}\,dx$ के रूप में समाकलन

Rule :- हर (Denominator) का परिमेयकरण करके समाकलन करते हैं।

Q.1 :- $\displaystyle\int\frac{1}{\sqrt{2x+3}-\sqrt{2x+1}}\,dx$ का समाकलन ज्ञात करो।
Find the integral.

हर (Denominator) का परिमेयकरण

$$= \int\frac{1}{\sqrt{2x+3}-\sqrt{2x+1}} \times \frac{(\sqrt{2x+3}+\sqrt{2x+1})}{(\sqrt{2x+3}+\sqrt{2x+1})}\,dx$$

$$\because (a-b)(a+b) = a^2 - b^2$$

$$= \int\frac{\sqrt{2x+3}+\sqrt{2x+1}}{(\sqrt{2x+3})^2-(\sqrt{2x+1})^2}\,dx$$

$$= \int\frac{\sqrt{2x+3}+\sqrt{2x+1}}{2x+3-(2x+1)}\,dx$$

$$= \int\frac{\sqrt{2x+3}+\sqrt{2x+1}}{2}\,dx$$

$$= \frac{1}{2}\int\left[(2x+3)^{1/2}+(2x+1)^{1/2}\right]dx$$

$$= \frac{1}{2}\left[\frac{(2x+3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times 2} + \frac{(2x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times 2}\right] + C$$

$$= \frac{1}{2}\left[\frac{(2x+3)^{3/2}}{\frac{3}{2}\times 2} + \frac{(2x+1)^{3/2}}{\frac{3}{2}\times 2}\right] + C$$

$$\boxed{= \frac{1}{6}\left[(2x+3)^{3/2} + (2x+1)^{3/2}\right] + C \quad \text{Ans}}$$

खण्डश: समाकलन (Integration by Parts):-

- इसके अन्तगत दो फलनों के गुणनफलन का समाकलन किया जाता है।

  Under this, the integration of the product of two functions is done.

- यदि दो फलन u तथा v हो, तब,

  If there are two functions u and v, then,

$$\int u \cdot v \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

या

$$\int I \cdot II \, dx = I \int II \, dx - \int \left(\frac{dI}{dx} \int II \, dx \right) dx$$

ILATE शब्द के आधार पर प्रथम तथा द्वितीय फलनों का चयन
(Selecting first and second functions on the basis of "ILATE")

	प्रकार	उदाहरण
I	**Inverse Trigonometric Function (प्रतिलोम त्रिकोणमिति अनुपात)**	$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ आदि
L	**Logarithmic Function (लघुगणकीय फलन)**	$\log_e x$
A	**Algebraic Function (बीजगणितीय फलन)**	x, x^2, x^3, x^4 आदि
T	**Trigonometric Function (त्रिकोणमितीय फलन)**	sinx, cosx, tanx, cotx आदि
E	**Exponential Function(घातांकात्मक फलन)**	e^x

Q.1 :- $\int x \sin x \, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\int I \cdot II \, dx = I \int II \, dx - \int \left(\frac{dI}{dx} \cdot \int II \, dx \right) dx$$

$$\int x \cdot \sin x \, dx = x \int \sin x \, dx - \int \left[\frac{dx}{dx} \cdot \int \sin x \, dx \right] dx$$

$$= x \cdot (-\cos x) - \int 1 \cdot (-\cos x) \, dx$$

$$= -x \cdot \cos x + \int \cos x \, dx$$

$$= -x \cdot \cos x + \sin x + C \quad \underline{\text{Ans}}$$

Q.2 :- $\int x \cos x \, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\int x \cdot \cos x \, dx = x \int \cos x \, dx - \int \left[\frac{dx}{dx} \cdot \int \cos x \, dx \right] dx$$

$$= x \cdot \sin x - \int 1 \cdot \sin x \, dx$$

$$= x \cdot \sin x + \cos x + C \quad \underline{\text{Ans}}$$

Q.3 :- $\int x \cdot e^{ax} \, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\int x \cdot e^{ax} \, dx = x \int e^{ax} \, dx - \int \left[\frac{dx}{dx} \cdot \int e^{ax} \, dx \right] dx$$

$$= x \cdot \frac{e^{ax}}{a} - \int 1 \cdot \frac{e^{ax}}{a} \, dx$$

$$= x \cdot \frac{e^{ax}}{a} - \frac{1}{a} \int e^{ax} \, dx$$

$$x \cdot \frac{e^{ax}}{a} - \frac{1}{a} \cdot \frac{e^{ax}}{a} + C = \frac{e^{ax}}{a} \left(x - \frac{1}{a} \right) + C \quad \underline{\text{Ans}}$$

Q.4 :- $\int x^2 \cdot e^x\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$\int x^2 \cdot e^x\, dx = x^2 \int e^x\, dx - \int \left[\frac{dx^2}{dx} \cdot \int e^x\, dx\right] dx$$

$$= x^2 \cdot e^x - \int 2x \cdot e^x\, dx$$

$$= x^2 \cdot e^x - 2 \int x \cdot e^x\, dx$$

$$= x^2 \cdot e^x - 2\left[x \cdot \int e^x\, dx - \int \left(\frac{dx}{dx} \cdot \int e^x\, dx\right) dx\right]$$

$$= x^2 \cdot e^x - 2\left[x \cdot e^x - \int e^x\, dx\right]$$

$$= x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C$$

$$= e^x\left(x^2 - 2x + 2\right) + C \quad \underline{\text{Ans}}$$

Q.5 :- $\int x^2 \sin x\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$\int x^2 \cdot \sin x\, dx = x^2 \int \sin x\, dx - \int \left(\frac{dx^2}{dx} \cdot \int \sin x\, dx\right) dx$$

$$= x^2 \cdot (-\cos x) - \int (2x \cdot (-\cos x))\, dx$$

$$= -x^2 \cos x + 2 \int x \cdot \cos x\, dx$$

$$= -x^2 \cos x + 2\left[x \cdot \int \cos x\, dx - \int \left(\frac{dx}{dx} \cdot \int \cos x\, dx\right) dx\right]$$

$$= -x^2 \cos x + 2\left[x \cdot \sin x - \int \sin x\, dx\right]$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

$$\boxed{= -x^2 \cos x + 2x \sin x + 2 \cos x + C} \quad \underline{\text{Ans}}$$

Q.5 :- $\int \log_e x\, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\int \log_e x \cdot 1\, dx = \log_e x \cdot \int 1\, dx - \left[\frac{d}{dx} \log_e x \cdot \int 1\, dx\right] dx$$

$$= \log_e x \cdot x - \int \frac{1}{x} \cdot x\, dx$$

$$= x \cdot \log_e x - x + C$$

$$= x\,(\log_e x - 1) + C \quad \underline{\text{Ans}}$$

Q.6 :- $\int \tan^{-1} x\, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$= \tan^{-1} x \cdot \int 1\, dx - \int \left[\frac{d}{dx}(\tan^{-1} x) \cdot \int 1\, dx\right] dx$$

$$= \tan^{-1} x \cdot x - \int \left[\frac{1}{1+x^2} \cdot x\right] dx$$

$$= x \cdot \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

माना $1 + x^2 = t$

d.w.r.t to x,

$$2x \cdot dx = dt \Rightarrow x \cdot dx = \frac{dt}{2}$$

$$x \cdot dx = \frac{dt}{2}$$

$$x \cdot \tan^{-1} x - \int \frac{1}{t} \cdot \frac{dt}{2}$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{1}{t}\, dt$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \log_e t + C$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \log_e (1 + x^2) + C$$ **Ans.**

Q.7 :- $\int (\log_e x)^2\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$\int (\log_e x)^2 \cdot 1\, dx \quad \text{(I)} \quad \text{(II)}$$

$$= (\log_e x)^2 \cdot \int 1\, dx - \left[\frac{d}{dx}(\log_e x)^2 \cdot \int 1\, dx\right] dx$$

$$= (\log_e x)^2 \cdot x - \int \left[2\log_e x \cdot \frac{1}{x} \cdot x\right] dx$$

$$= x(\log_e x)^2 - 2\int \log_e x\, dx$$

$$= x(\log_e x)^2 - 2 \cdot \int \log_e x \cdot 1\, dx$$

$$= x(\log_e x)^2 \; - \; 2\left[\log_e x \cdot \int 1\, dx \; - \; \int \left(\frac{d}{dx}(\log_e x) \cdot \int 1.dx\right) dx\right]$$

$$= x(\log_e x)^2 - 2\left[\log_e x \cdot x - \int x \cdot \frac{1}{x}\, dx\right]$$

$$= x(\log_e x)^2 - 2x \cdot \log_e x + 2x + C \quad \underline{\text{Ans}}$$

Q.8 :- $\int \frac{\sin(\tan^{-1} x)}{1+x^2}\, dx$

माना $\tan^{-1} x = t$

d.w.r. to x,

$$\frac{1}{1+x^2}\, dx = dt$$

$$= \int \sin t\, dt$$

$$= -\cos t + C$$

$$= -\cos(\tan^{-1} x) + C$$ **Ans.**

Q.9 :- $\int x \cdot \cos^2 2x\, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\because \cos 2x = 2\cos^2 x - 1$$
$$\cos 2x + 1 = 2\cos^2 x$$
$$\frac{\cos 2x + 1}{2} = \cos^2 x$$

$$= \int x \left(\frac{\cos 2x + 1}{2}\right) dx$$
$$= \frac{1}{2}\int (x\cos 2x + x)\, dx$$
$$= \frac{1}{2}\int x\cos 2x\, dx + \frac{1}{2}\int x\, dx$$
$$= \frac{1}{2}\left[x(\cos 2x)\, dx - \int \frac{d}{dx}(x)\,(\cos 2x\, dx)\right] + \frac{1}{2}\cdot\frac{x^2}{2}$$
$$= \frac{1}{2}\left[\frac{x\sin 2x}{2} - \int 1\cdot\frac{\sin 2x}{2}\, dx\right] + \frac{x^2}{4}$$
$$= \frac{x\sin 2x}{4} + \frac{1}{4}\cdot\frac{\cos 2x}{2} + \frac{x^2}{4} + C$$
$$\boxed{= \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^2}{4} + C} \text{ Ans}$$

Q.10 :- $\int x \cdot \sin 3x\, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\int x\cdot\sin 3x\, dx = x\int \sin 3x\, dx - \int\left(\frac{d}{dx}(x)\cdot\int \sin 3x\, dx\right) dx$$
$$= x\cdot\left(-\frac{\cos 3x}{3}\right) - \int 1\cdot\left(-\frac{\cos 3x}{3}\right) dx$$
$$= -\frac{x\cdot\cos 3x}{3} + \frac{1}{3}\int \cos 3x\, dx$$

$$= -\frac{x \cdot \cos 3x}{3} + \frac{1}{3} \cdot \frac{\sin 3x}{3} + C$$

$$= -\frac{x \cdot \cos 3x}{3} + \frac{\sin 3x}{9} + C$$

$$\boxed{\int x \cdot \sin 3x \, dx = -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + C}$$

Q.11 :- $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$ **का समाकलन ज्ञात करो।**
Find the integral.

मान लें $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$= \int \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) dx$$
$$= \int \cos^{-1}(\cos 2\theta) dx$$
$$= \int 2\theta \, dx$$
$$= \int 2\tan^{-1} x \, dx$$

$$= 2\int \tan^{-1} x \cdot 1 \, dx \quad \text{(I)}$$

$$= 2\left[\tan^{-1} x \int 1\, dx - \int \left(\frac{d}{dx}\tan^{-1} x \cdot \int 1\, dx\right) dx\right]$$

$$= 2\left[\tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx\right]$$
$$= 2x \cdot \tan^{-1} x - 2\int \frac{x}{1+x^2} dx$$

मान लें $1 + x^2 = t$

Differentiate w.r.t. x:

$$2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$$

$$= 2x \cdot \tan^{-1} x - 2\int \frac{1}{t} \cdot \frac{dt}{2}$$

$$= 2x \cdot \tan^{-1} x - \frac{2}{2} \int \frac{1}{t}\, dt$$

$$= 2x \cdot \tan^{-1} x - \log_e t + C$$

$$= 2x \cdot \tan^{-1} x - \log_e (1 + x^2) + C \quad \boxed{\text{Ans}}$$

Q.12 :- $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}}\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $x = \tan\theta \Rightarrow dx = \sec^2\theta \cdot d\theta$

$$= \int \frac{\tan\theta \cdot \tan^{-1}(\tan\theta)}{(1+\tan^2\theta)^{3/2}} \cdot \sec^2\theta \cdot d\theta$$

$$= \int \frac{\theta \cdot \tan\theta}{(\sec^2\theta)^{3/2}} \cdot \sec^2\theta \cdot d\theta$$

$$= \int \frac{\theta \cdot \tan\theta}{\sec^3\theta} \cdot \sec^2\theta\, d\theta$$

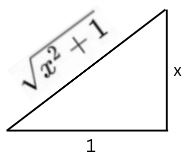
$$= \int \frac{\theta \cdot \tan\theta}{\sec\theta}\, d\theta$$

$$= \int \theta \cdot \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\, d\theta$$

$$= \int \theta \cdot \sin\theta\, d\theta$$

$$= \theta \int \sin\theta\, d\theta - \int \left[\frac{d\theta}{d\theta} \cdot \int \sin\theta\, d\theta\right] d\theta$$

$$= \theta \cdot (-\cos\theta) - \int (-\cos\theta) \cdot d\theta$$



$$= -\theta \cdot \cos\theta + \int \cos\theta \cdot d\theta = -\theta \cdot \cos\theta + \sin\theta + C$$

$\because x = \tan\theta$

$\tan\theta = \frac{x}{1} = \frac{L}{A}$

$\sin\theta = \frac{x}{\sqrt{x^2+1}}$,

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$$= -\tan^{-1} x \cdot \frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$$

$$= \frac{-\tan^{-1} x + x}{\sqrt{1+x^2}} + C \quad \text{Ans}$$

Q.13 :- $\displaystyle\int \frac{x^2 \tan^{-1} x}{(1+x^2)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $\tan^{-1} x = t \;\Rightarrow\; x = \tan t$

d w.r.t x

$$\frac{1}{1+x^2}\,dx = dt$$

$$= \int x^2 \cdot t\,dt$$

$$\because 1 + \tan^2 t = \sec^2 t$$

$$\tan^2 t = \sec^2 t - 1$$

$$= \int \tan^2 t \cdot t \cdot dt$$

$$= \int (\sec^2 t - 1)\cdot t \cdot dt = \int (t\cdot \sec^2 t - t)dt$$

$$= \int t\cdot \sec^2 t\,dt - \int t\,dt$$

$$= t\cdot \int \sec^2 t\,dt - \int \left(\frac{d}{dt}t \cdot \int \sec^2 t\,dt\right)dt - \frac{t^2}{2}$$

$$= t\cdot \tan t - \int \tan t\,dt - \frac{t^2}{2}$$

$$= t\cdot \tan t - \log \sec t - \frac{t^2}{2} + C$$

$$\because \tan^{-1} x = t \quad \& \quad x = \tan t$$

$$\because 1 + \tan^2 t = \sec^2 t$$

$$1 + x^2 = \sec^2 t$$

$$\sqrt{1+x^2} = \sec t$$

$$\Rightarrow \tan^{-1} x \cdot x - \log\left(\sqrt{1+x^2}\right) - \frac{(\tan^{-1} x)^2}{2} + C$$

$$= x\cdot \tan^{-1} x - \log\left(\sqrt{1+x^2}\right) - \frac{(\tan^{-1} x)^2}{2} + C \quad \text{Ans}$$

Q.14 :- $\displaystyle\int \frac{xe^x}{(1+x)^2}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$= \int \frac{(x+1-1)e^x}{(1+x)^2}\,dx$$

$$= \int \frac{(x+1)e^x - e^x}{(1+x)^2}\,dx$$

$$= \int \left[\frac{(x+1)e^x}{(1+x)^2} - \frac{e^x}{(1+x)^2} \right] dx$$

$$= \int \frac{1}{(1+x)} \cdot e^x \, dx - \int \frac{e^x}{(1+x)^2} \, dx$$

$$= \frac{1}{(1+x)} \int e^x \, dx - \int \left[\frac{d}{dx} \left(\frac{1}{1+x} \right) \cdot \int e^x dx \right] dx - \int \frac{e^x}{(1+x)^2} \, dx$$

$$= \frac{1}{1+x} \cdot e^x - \int \left(\frac{-1}{(1+x)^2} \cdot e^x \right) dx - \int \frac{e^x}{(1+x)^2} dx$$

$$= \frac{e^x}{1+x} + \int \frac{e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx$$

$$\boxed{= \frac{e^x}{1+x} + C \quad \text{Ans}}$$

Q.15 :- $\int \frac{e^x(x^2+1)}{(1+x)^2} \, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$= \int \frac{e^x(x^2+1+2x-2x)}{(1+x)^2} dx$$

$$= \int e^x \left[\frac{(x+1)^2 - 2x}{(1+x)^2} \right] dx$$

$$= \int \left[\frac{e^x(x+1)^2}{(1+x)^2} - \frac{2x \cdot e^x}{(1+x)^2} \right] dx$$

$$= \int e^x dx - \int \frac{2x \cdot e^x}{(1+x)^2} dx$$

$$= e^x - 2 \int \frac{e^x(x+1-1)}{(1+x)^2} dx$$

$$= e^x - 2 \int \frac{e^x(x+1) - e^x}{(x+1)^2} \, dx$$

$$= e^x - 2 \left[\frac{e^x(x+1)}{(x+1)^2} - \frac{e^x}{(x+1)^2} \right] dx$$

$$= e^x - 2 \int \frac{1}{(x+1)} \cdot e^x \, dx \; - \; 2 \int \frac{e^x}{(x+1)^2} \, dx$$

$$= e^x - 2 \left[\frac{1}{(x+1)} \int e^x dx - \int \frac{d}{dx} \left(\frac{1}{x+1} \right) \cdot e^x dx \right] - 2 \int \frac{e^x}{(x+1)^2} dx$$

$$= e^x - \frac{2}{(x+1)} e^x - 2 \int \left(-\frac{1}{(x+1)^2} \right) e^x dx - 2 \int \frac{e^x}{(x+1)^2} dx + C$$

$$\boxed{= e^x - \frac{2e^x}{x+1} + C \quad \text{Ans}}$$

Q.16 :- $\int e^x \sin x\, dx$ का समाकलन ज्ञात करो।
Find the integral.

मानें $\int e^x \sin x\, dx = I$

$$= e^x \int \sin x\, dx - \int \left[\frac{d}{dx} e^x \cdot \int \sin x\, dx\right] dx$$

$$I = -e^x \cos x + \int e^x \cos x\, dx$$

$$I = -e^x \cos x + e^x \int \cos x\, dx - \int \left[\frac{d}{dx} e^x \cdot \int \cos x\, dx\right] dx$$

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$I = -e^x \cos x + e^x \sin x - I$$

$$I + I = e^x(\sin x - \cos x) \Rightarrow 2I = e^x(\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^x}{2}(\sin x - \cos x) + C$$

$$\boxed{I = \frac{e^x}{2}(\sin x - \cos x) + C \quad \text{Ans}}$$

Q.17 :- $\int e^x(\sin x + \cos x)\, dx$ का समाकलन ज्ञात करो।
Find the integral.

$$\Rightarrow \int (e^x \cdot \sin x + e^x \cdot \cos x)\, dx$$

$$\Rightarrow \int e^x \cdot \sin x\, dx + \int e^x \cdot \cos x\, dx$$

$$\Rightarrow \int e^x \cdot \sin x\, dx + e^x \int \cos x\, dx - \int \left[\frac{de^x}{dx} \cdot \left(\int \cos x\, dx\right)\right] dx$$

$$\Rightarrow \int e^x \cdot \sin x\, dx + e^x \cdot \sin x - \int e^x \cdot \sin x\, dx$$

$$\boxed{= e^x \cdot \sin x + C \qquad \text{Ans}}$$

आंशिक भिन्नों द्वारा समाकलन (Integration by Partial Fractions)

- जब फलन भिन्नों के रूप में होते हैं, तो उन्हें दो या दो से अधिक आंशिक भिन्नों में तोड़कर समाकलन किया जाता है।
  When functions are in the form of fractions, integration is done by breaking them into two or more partial fractions.

Note:-

(i) यदि अंश (ऊपर का फलन) की घात, हर (नीचे का फलन) की घात के बराबर या उससे अधिक हो तो सर्वप्रथम अश को हर से भाग देगें। तत्पश्चात समाकलन करेंगे।

If the degree of the numerator (upper function) is equal to or greater than the degree of the denominator (lower function), then first divide the numerator by the denominator. Then integrate.

(ii) यदि हर के गुणनखण्ड करना सम्भव हो तो सर्वप्रथम गुणनखण्ड करके फिर आंशिक भिन्न बनाकर समाकलन करेंगे

If it is possible to factor the denominator, then first factor it and then make partial fractions and integrate.

क्र० सं०	प्रश्नों में मूल भिन्न का रूप	आंशिक भिन्न में परिवर्तित रूप
1.	$\frac{f(x)}{(x-a)(x-b)\ldots}$	$\Rightarrow \frac{A}{(x-a)}+\frac{B}{(x-b)}+\ldots$
2.	$\frac{f(x)}{(x-a)^2}$	$\Rightarrow \frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
3.	$\frac{f(x)}{(x-a)^2(x-b)}$	$\Rightarrow \frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
4.	$\frac{f(x)}{(x-a)(\alpha x^2+\beta x+\gamma)}$	$\Rightarrow \frac{A}{(x-a)}+\frac{Bx+C}{(\alpha x^2+\beta x+\gamma)}$

आचार्य श्रीधर के सूत्र अनुसार (According to Acharya Sridhar's sutra),

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Q.1 :- $\int \frac{x^2+1}{x^2-5x+6}\,dx$ **का समाकलन ज्ञात करो। Find the integral.**

$$\text{Divide: } \frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6}$$

$$\int \left[1 + \frac{5x-5}{x^2-5x+6}\right] dx$$

Hint:
भागफल + शेष / हर

$$\int 1\,dx + 5\int \frac{x-1}{x^2-3x-2x+6}\,dx$$

$$x + 5\int \frac{x-1}{(x-2)(x-3)}\,dx \qquad (1)$$

आंशिक भिन्न (Partial Fraction)

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$x-1 = A(x-3) + B(x-2) \qquad (2)$$

x = 2 **रखने पर:**

$$2-1 = A(2-3) + B(2-2)$$

$$1 = A(-1) + 0$$

$$\boxed{A = -1}$$

x = 3 **रखने पर:**

$$3-1 = A(3-3) + B(3-2)$$

$$2 = 0 + B(1)$$

$$\boxed{B = 2}$$

अब,

$$\frac{x-1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$$

इसको Eq.(1) में रखते हैं:

$$x + 5\int \left[\frac{-1}{x-2} + \frac{2}{x-3}\right] dx$$

$$= x - 5\int \frac{1}{x-2}dx + 10\int \frac{1}{x-3}dx$$

$$\boxed{x - 5\log(x-2) + 10\log(x-3) + C \qquad \text{Ans}}$$

Q.2 :- $\int \frac{x^2}{(x-1)(3x-1)(3x-2)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

आंशिक भिन्न द्वारा (by Partial fraction):

$$\frac{x^2}{(x-1)(3x-1)(3x-2)} = \frac{A}{(x-1)} + \frac{B}{(3x-1)} + \frac{C}{(3x-2)}$$

$$\frac{x^2}{(x-1)(3x-1)(3x-2)} = \frac{A(3x-1)(3x-2)+B(x-1)(3x-2)+C(x-1)(3x-1)}{(x-1)(3x-1)(3x-2)}$$

$$x^2 = A(3x-1)(3x-2)+B(x-1)(3x-2)+C(x-1)(3x-1) \qquad (1)$$

समीकरण (1) में $x=1$ रखने पर:

$$1^2 = A(3\times 1-1)(3\times 1-2)+B(1-1)(3\times 1-2)+C(1-1)(3\times 1-1)$$

$$1 = A(2)(1)+0+0 \Rightarrow A = \frac{1}{2}$$

समीकरण (1) में $x=\frac{1}{3}$ रखने पर:

$$\left(\frac{1}{3}\right)^2 = A(3\times\frac{1}{3}-1)(3\times\frac{1}{3}-2)+B(\frac{1}{3}-1)(3\times\frac{1}{3}-2)+C(\frac{1}{3}-1)(3\times\frac{1}{3}-1)$$

$$\frac{1}{9} = 0+B\left(\frac{-2}{3}\right)(-1)+0 \Rightarrow \frac{1}{9} = \frac{2}{3}B$$

$$\frac{1}{9}\div\frac{2}{3} = B \Rightarrow B = \frac{1}{6}$$

समी (1) में $x=\frac{2}{3}$ रखने पर:

$$\left(\frac{2}{3}\right)^2 = A\left(3\times\frac{2}{3}-1\right)\left(3\times\frac{2}{3}-2\right)+B\left(\frac{2}{3}-1\right)\left(3\times\frac{2}{3}-2\right)+C\left(\frac{2}{3}-1\right)\left(3\times\frac{2}{3}-1\right)$$

$$\frac{4}{9} = 0+0+C\cdot\left(-\frac{1}{3}\cdot 1\right)$$

$$\frac{4}{9} = -\frac{1}{3}\cdot C \Rightarrow \frac{4}{9}\times\left(\frac{3}{1}\right) = -C \Rightarrow C = -\frac{4}{3}$$

$$\frac{x^2}{(x-1)(3x-1)(3x-2)} = \frac{1}{2(x-1)}+\frac{1}{6(3x-1)}+\frac{-4}{3(3x-2)}$$

Integration (समाकलन):

$$= \tfrac{1}{2}\int\frac{1}{(x-1)}\,dx + \tfrac{1}{6}\int\frac{1}{(3x-1)}\,dx - \tfrac{4}{3}\int\frac{1}{(3x-2)}\,dx$$

$$= \tfrac{1}{2}\log_e|x-1| + \tfrac{1}{6}\cdot\frac{\log_e|3x-1|}{3} - \tfrac{4}{3}\cdot\frac{\log_e|3x-2|}{3} + C$$

$$\boxed{\frac{1}{2}\log_e(x-1)+\frac{1}{18}\log_e(3x-1)-\frac{4}{9}\log_e(3x-2)+C\quad \text{Ans}}$$

Q.3 :- $\int \frac{1}{1+x-x^2-x^3}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$\int \frac{1}{1+x-x^2-x^3}\,dx = \int \frac{1}{(1-x)(1+x)^2}\,dx$$

$$= \int \frac{1}{(1-x)(1+x)^2}\,dx$$

$$= \int \frac{1}{(1-x)(1+x)^2}\,dx$$

(By Partial fraction)

$$\frac{1}{(1-x)(1+x)^2} = \frac{A}{(1-x)} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2} \quad \ldots\ldots (1)$$

$$\Rightarrow \frac{1}{(1-x)(1+x)^2} = \frac{A(1+x)^2 + B(1-x)(1+x) + C(1-x)}{(1-x)(1+x)^2}$$

$$1 = A(1+x)^2 + B(1-x)(1+x) + C(1-x) \quad \ldots\ldots (2)$$

Putting $x=1$ in equation (1)

$$1 = A(1+1)^2 + B(1-1)(1+1) + C(1-1)$$

$$1 = A(2)^2 + 0 + 0 \Rightarrow A = \frac{1}{4}$$

Putting $x=-1$ in equation (1)

$$1 = A(1-1)^2 + B(1-(-1))(1-1) + C(1-(-1))$$

$$1 = 0 + 0 + 2C \Rightarrow C = \frac{1}{2}$$

Putting $x=0$ in equation (1)

$$1 = A(1+0)^2 + B(1-0)(1+0) + C(1-0)$$

$$1 = A + B + C \Rightarrow 1 = \frac{1}{4} + B + \frac{1}{2}$$

$$1 = \frac{3}{4} + B \Rightarrow B = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\frac{1}{(1-x)(1+x)^2} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x)^2}$$

Integrating:

$$= \frac{1}{4}\int \frac{1}{(1-x)}dx + \frac{1}{4}\int \frac{1}{(1+x)}dx + \frac{1}{2}\int \frac{1}{(1+x)^2}dx$$

$$= \frac{1}{4}\cdot\frac{\log(1-x)}{-1} + \frac{1}{4}\log(1+x) + \frac{1}{2}\int (1+x)^{-2}dx$$

$$= -\frac{1}{4}\log(1-x) + \frac{1}{4}\log(1+x) + \frac{(1+x)^{-1}}{-1}\cdot\frac{1}{2} + C$$

$$= -\frac{1}{4}\log(1-x) + \frac{1}{4}\log(1+x) - \frac{1}{2(1+x)} + C \quad \text{Ans}$$

Q.3 :- $\int \frac{1+4x}{x(x^2-4)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$= \int \frac{1+4x}{x((x+2)(x-2))}\,dx$$

$$= \int \frac{1+4x}{x(x+2)(x-2)}\,dx$$

By Partial Fraction:

$$\frac{1+4x}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

Multiply both sides by $x(x+2)(x-2)$:

$$1+4x = A(x^2-4) + Bx(x-2) + Cx(x+2) \qquad (1)$$

समीकरण (1) में $x=0$ रखने पर:

$$1+0 = A(0-4) + B\cdot 0\cdot(-2) + C\cdot 0\cdot 2$$

$$1 = -4A + 0 + 0 \Rightarrow A = -\frac{1}{4}$$

समीकरण (1) में $x=-2$ रखने पर:

$$1+4(-2)=A(4-4)+B(-2)(-2-2)+C(-2)(-2+2)$$

$$1-8=0+8B+0 \Rightarrow -7=8B \Rightarrow B=\frac{-7}{8}$$

समीकरण (1) में $x=2$ रखने पर:

$$1+4(2)=A(4-4)+B(2)(2-2)+C(2)(2+2)$$

$$1+8=0+0+8C \Rightarrow 9=8C \Rightarrow C=\frac{9}{8}$$

A, B और C का मान रखने पर

$$\frac{1+4x}{x(x+2)(x-2)}=\frac{-1}{4x}+\frac{-7}{8(x+2)}+\frac{9}{8(x-2)}$$

Integrating (समाकलन करने पर)

$$=-\frac{1}{4}\int\frac{1}{x}dx-\frac{7}{8}\int\frac{1}{x+2}dx+\frac{9}{8}\int\frac{1}{x-2}dx$$

$$=-\frac{1}{4}\log x-\frac{7}{8}\log(x+2)+\frac{9}{8}\log(x-2)+C \quad \text{Ans}$$

Q.4 :- $\int\frac{1}{x^3+1}dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$x^3+1=(x+1)(x^2-x+1)$$

$$=\int\frac{1}{(x+1)(x^2-x+1)}dx$$

By Partial Fraction:

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

Multiply both sides by $(x+1)(x^2-x+1)$:

$$1=A(x^2-x+1)+(Bx+C)(x+1) \qquad (1)$$

समीकरण (1) में $x=-1$ रखने पर:

$$1 = A((-1)^2+1+1)+(B(-1)+C)(0)$$

$$1 = A(1+1+1) = 3A \Rightarrow A = \frac{1}{3}$$

अब समीकरण (1) को पूरी तरह expand करें:

$$1 = A(x^2-x+1)+(Bx+C)(x+1)$$

$$= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (-A+B+C)x + (A+C)$$

Compare with RHS 1:

From (1):

$$A+B=0 \Rightarrow B=-A=-\frac{1}{3}$$

$$-A+B+C=0 \Rightarrow -\frac{1}{3}+(-\frac{1}{3})+C=0 \Rightarrow -\frac{2}{3}+C=0 \Rightarrow C=\frac{2}{3}$$

A B C का मान रखने पर

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x+\frac{2}{3}}{(x^2-x+1)}$$

Integrating (समाकलन)

$$= \frac{1}{3}\int \frac{1}{(x+1)}dx - \int \frac{-x+2}{3(x^2-x+1)}dx$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{3}\int \frac{2-x}{x^2-x+1}dx$$

$$\int \frac{x-2}{x^2-x+1}\,dx$$

Let:

$$x-2 = \frac{1}{2}(2x-1)+\frac{1}{2}-2$$

$$\int \frac{x-2}{x^2-x+1}dx = \int \frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1}dx$$

$$= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx - \frac{3}{2}\int \frac{1}{x^2-x+1}dx$$

Let: $t = x^2 - x + 1 \Rightarrow dt = (2x-1)dx$

$$= \frac{1}{2}\int \frac{1}{t}\,dt - \frac{3}{2}\int \frac{1}{x^2 - 2x\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{1}{2}\log t - \frac{3}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$= \frac{1}{2}\log(x^2-x+1) - \frac{3}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2}\log(x^2-x+1) - \frac{3}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\boxed{\frac{1}{2}\log(x^2-x+1) - \sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C}$$ (Ans)

Q.5 :- $\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $x^2 = t$ by Partial Fraction

$$\frac{t}{(t+1)(t+4)} = \frac{A}{(t+1)} + \frac{B}{(t+4)}$$

$$t = A(t+4) + B(t+1) \quad (1)$$

समीकरण (1) **में** $t = -1$ **रखने पर**

$$-1 = A(-1+4) + B(-1+1)$$
$$-1 = 3A + 0 \Rightarrow A = -\frac{1}{3}$$

समीकरण (1) **में** $t = -4$ **रखने पर**

$$-4 = A(-4+4) + B(-4+1)$$
$$-4 = 0 + B(-3) \Rightarrow B = \frac{4}{3}$$

✅ A, B का मान रखते हुए:

$$\frac{t}{(t+1)(t+4)} = \frac{-\frac{1}{3}}{(t+1)} + \frac{4}{3(t+4)}$$

अब $t = x^2$ रखने पर:

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-\frac{1}{3}}{(x^2+1)} + \frac{4}{3(x^2+4)}$$

Integrating:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{4}{3}\int \frac{1}{x^2+4}\,dx$$

$$= -\frac{1}{3}\tan^{-1}(x) + \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{3}\tan^{-1}(x) + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + C \quad \boxed{\text{Ans}}$$

Q.6 :- $\displaystyle\int \frac{x^4}{x^2+1}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$x^2 = t \Rightarrow dx$$

$$= \int \frac{t^2}{t+1}\,dx = \int \frac{t^2-1+1}{t+1}\,dx = \int \frac{(t^2-1)}{t+1}\,dx + \int \frac{1}{t+1}\,dx$$

$$= \int \frac{(t+1)(t-1)}{t+1}\,dx + \int \frac{1}{t+1}\,dx = \int (t-1)\,dx + \int \frac{1}{t+1}\,dx$$

$t = x^2$:

$$= \int (x^2 - 1)\, dx + \int \frac{1}{x^2+1}\, dx$$

$$\boxed{\frac{x^3}{3} - x + \tan^{-1}(x) + C}$$ **Ans.**

Q.7 :- $\int \frac{\sec^2 x}{(1+\tan x)(2+\tan x)}\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

Let:

$$\tan x = t \Rightarrow \sec^2 x\, dx = dt$$

$$\Rightarrow \int \frac{1}{(1+t)(2+t)}\, dt$$

By Partial Fraction:

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$$

Multiply both sides:

$$1 = A(2+t) + B(1+t)$$

To find A and B:

- Put $t = -1$:

$$1 = A(2-1) + B(0) \Rightarrow A = 1$$

- Put $t = -2$:

$$1 = A(0) + B(1-2) \Rightarrow B = -1$$

A, B का मान रखने पर

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} - \frac{1}{2+t}$$

Integrating:

$$\int \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt = \log(1+t) - \log(2+t) + C$$ $\because$ t = tanx

$$\boxed{\log(1+\tan x) - \log(2+\tan x) + C}$$ **Ans.**

Q.8 :- $\int \frac{1-x}{1+x}\, dx$ **का समाकलन ज्ञात करो।**
Find the integral.

$$= \int \frac{1}{1+x}\,dx - \int \frac{x}{1+x}\,dx$$

$$= \log(1+x) - \int \frac{x+1-1}{1+x}\,dx$$

$$= \log(1+x) - \int \frac{1+x}{1+x}\,dx + \int \frac{1}{1+x}\,dx$$

$$= \log(1+x) - \int 1\,dx + \int \frac{1}{1+x}\,dx$$

$$= \log(1+x) - x + \log(1+x) + C$$

$$\boxed{2\log(1+x) - x + C}$$ **Ans.**

Q.9 :- $\int \frac{1}{x(x^4+1)}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

अंश व हर में x^3 से गुणा करने पर

$$= \int \frac{x^3}{x^4(x^4+1)}\,dx$$

$$x^4 = t \Rightarrow 4x^3dx = dt \Rightarrow x^3dx = \frac{dt}{4}$$

$$= \int \frac{1}{t(t+1)} \cdot \frac{dt}{4} = \frac{1}{4}\int \frac{1}{t(t+1)}\,dt$$

By Partial Fractions:

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

Multiply both sides:

$$1 = A(t+1) + B(t)$$

To find A and B:

- Let $t = 0$:

$$1 = A(0+1) \Rightarrow A = 1$$

- Let $t = -1$:

$$1 = B(-1) \Rightarrow B = -1$$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

Integration:

$$\frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \frac{1}{4}\log t - \frac{1}{4}\log(t+1) + C$$

Now substitute $t = x^4$:

$$\boxed{\frac{1}{4}\log(x^4) - \frac{1}{4}\log(x^4+1) + C}$$ **Ans.**

Q.9 :- $\int \frac{\sin x}{\cos^2 x - 5\cos x + 4}\,dx$ **का समाकलन ज्ञात करो।**
Find the integral.

माना $\cos x = t$

$$\text{d w.r.t. to } x \Rightarrow -\sin x\,dx = dt$$

$$\sin x\,dx = -dt$$

$$= \int \frac{-dt}{t^2 - 5t + 4}$$

$$= -\int \frac{1}{t^2 - 4t - t + 4}\,dt$$

$$= -\int \frac{1}{(t-1)(t-4)}\,dt$$

By Partial Fraction :

$$\frac{1}{(t-1)(t-4)} = \frac{A}{(t-1)} + \frac{B}{(t-4)}$$

$$1 = A(t-4) + B(t-1) \quad ----(1)$$

समी (1) में $t = 1$ रखने पर :

$$1 = A(1-4) + B(1-1)$$

$$1 = -3A + 0 \Rightarrow A = -\frac{1}{3}$$

समी (1) में $t = 4$ रखने पर :

$$1 = A(4-4) + B(4-1)$$

$$1 = 0 + 3B \Rightarrow B = \frac{1}{3}$$

A, B का मान रखने पर :

$$-\frac{1}{(t-1)(t-4)} = -\left[\frac{-1/3}{(t-1)} + \frac{1/3}{(t-4)}\right]$$

Integrating :

$$= \frac{1}{3}\int \frac{1}{(t-1)}\,dt - \frac{1}{3}\int \frac{1}{(t-4)}\,dt$$

$$= \frac{1}{3}\log(t-1) - \frac{1}{3}\log(t-4) + C$$

$$= \frac{1}{3}\log(\cos x - 1) - \frac{1}{3}\log(\cos x - 4) + C$$

$$= \frac{1}{3}\log\left(\frac{\cos x - 1}{\cos x - 4}\right) + C \quad \textbf{Ans}$$

निश्चित समाकलन (Definite Integration)

• जब किसी फलन का समाकलन दिए गए किन्हीं दो निश्चित सीमाओं के लिए किया जाता है तो उसे निश्चित समाकलन कहते हैं।
(When the integration of a function is done for any two given fixed limits then it is called definite integral.)

अनिश्चित समाकलन (Indefinite Integration) $\int f(x)\,dx = F(x) + C$

निश्चित समाकलन (Definite Integration) $\int_a^b f(x)\,dx = [F(x)]_a^b$

Note :- निश्चित समाकलन में 'C' का use नहीं करते हैं।

Find the value of Definite Integration(निश्चित समाकलन का मान ज्ञात करना) :-

$$\int_a^b f(x)\,dx = ?$$ **(f(x) का समाकलन F(x) है।)**

a = Lower Limit (निम्न सीमा)

b = Upper Limit (उच्च सीमा)

- सबसे पहले दिये गये function का समाकलन (Integration) करते हैं।
- इसके बाद Uppers Limit & lower Limit को बड़े कोष्ठक (Bracket) के ऊपर व नीचे लिखते हैं।
- अब पहले upper Limit & बाद में Lower limit को x वे स्थान पर लिखकर घटाते हैं।

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Que. 1:- $\int_a^b \frac{1}{x}\,dx$

$$= [\log_e x]_a^b$$

$$= \log_e b - \log_e a$$

$$= \log_e\left(\frac{b}{a}\right) \quad \textbf{Ans}$$

Que.2 :- $\int_0^{\pi/4} \tan^2 x\, dx$

Sol:-

$$= \int_0^{\pi/4} (\sec^2 x - 1)\, dx$$

$$= [\tan x - x]_0^{\pi/4}$$

$$= (\tan\frac{\pi}{4} - \frac{\pi}{4}) - (\tan 0 - 0)$$

$$= (1 - \frac{\pi}{4}) - (0 - 0)$$

$$= 1 - \frac{\pi}{4} \quad \textbf{Ans}$$

Que.3 :- $\int_0^{1} e^x \cos(e^x)\, dx$

माना $e^x = t$

$$\text{d w.r.t. to } x \Rightarrow e^x dx = dt$$

Limits change :

जब $x = 0$, तब $t = e^0 = 1$

जब $x = 1$, तब $t = e^1 = e$

$$\int_1^e \cos t\, dt$$

$$= [\sin t]_1^e$$

$$= \sin(e) - \sin(1) \quad \textbf{Ans}$$

Que.4 :- $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\, dx$

माना $\tan^{-1} x = t$

$$\text{d w.r.t. to } x \Rightarrow \frac{1}{1+x^2} dx = dt$$

Limit change :

जब $x = 0$, तब $t = \tan^{-1}(0) = 0$

जब $x = 1$, तब $t = \tan^{-1}(1) = \frac{\pi}{4}$

$$\int_0^{\pi/4} t^2\, dt$$

$$= \left[\frac{t^3}{3}\right]_0^{\pi/4}$$

$$= \frac{(\pi/4)^3}{3} - 0$$

$$= \frac{\pi^3}{192} \quad \textbf{Ans}$$

गामा फलन (Gamma Function)

- निश्चित समाकलन (Definite integration) $\int_0^{\infty} e^{-x}\, x^{n-1}\, dx$ को n का Gamma function कहते हैं।
- n के Gramma function को Γn से प्रदर्शित करते हैं।

$$\Gamma(n) = \int_0^{\infty} e^{-x}\, x^{n-1}\, dx \quad \text{where } n > o$$

गामा फलन के गुण (Properties of Gamma Function)

1. Γ(n) = (n – 1)!

जैसे :- Γ(5) = 4! = 4 × 3 × 2 × 1 = 24

जब n Positive integer (धन पूर्णक) हो।

2. Γ(n) = (n – 1) Γ(n – 1)

जैसे :- (i) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$$= \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{4}\sqrt{\pi}$$

(ii) $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$

$$= \frac{105}{16}\sqrt{\pi}$$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. $\Gamma 1 = 1$

Factorial :

1. $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$

   $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

2. $1! = 1$
3. $0! = 1$

Que.1 :- $\Gamma(7)$ तथा $\Gamma\left(\frac{9}{2}\right)$ का मान निकालों।
(Find the value.)

(i) $\Gamma(7) = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$

(ii) $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

$= \frac{105}{16}\sqrt{\pi}$ Ans.

Que.2 :- $\Gamma\left(-\frac{1}{2}\right)$ तथा $\Gamma\left(-\frac{3}{2}\right)$ का मान निकालों।
(Find the value.)

(i) $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$

$= \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}$

$$\boxed{\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}}$$

$\Gamma(n) = (n-1)\Gamma(n-1)$

या $\Gamma(n+1) = n\,\Gamma(n)$

या $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

(ii) $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$

$= \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}$

$$\boxed{\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}}$$ Ans.

Que.3 :- $\int_0^\infty x^5 e^{-x}\, dx$ **का मान निकालों।**
(Find the the value.)

$\because$ **Gamma function** $\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma(n)$

$$\int_0^\infty e^{-x} x^5\, dx = \Gamma(6)$$ $\because$ यहाँ तुलना करने पर $n-1=5 \Rightarrow n=6$

$$\Gamma(6) = 5! = 5\times4\times3\times2\times1 = 120$$

$$\boxed{\text{Ans} = 120}$$

Que.4 :- $\int_0^\infty 4\sqrt{x}\, e^{-\sqrt{x}}\, dx$ **का मान निकालों।**
(Find the value.)

$$\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$$

माना $\sqrt{x} = t \quad \Rightarrow x = t^2,$

d.w.r.t to $x \Rightarrow dx = 2t\, dt$

$$= \int_0^\infty (t^2)^{1/4} e^{-t}\, (2t)\, dt$$

$$= 2\int_0^\infty t^{1/2} e^{-t}\, t\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

From definition of Gamma function

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma(n)$$

$$= 2\int_0^\infty e^{-t}\, t^{\frac{5}{2}-1}\, dt$$

$$= 2\Gamma\left(\frac{5}{2}\right)$$

$$= 2\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}$$

$$= \frac{3\sqrt{\pi}}{2} \quad \textbf{Ans}$$

Que.5 :- $\int_0^\infty x^{1/2} e^{-3\sqrt{x}}\, dx$ **का मान निकालों।**
(Find the value.)

$$\int_0^\infty x^{1/2} e^{-(x)^{1/3}}\, dx$$

माना $\quad x^{1/3} = t$ या $x = t^3$

d.w.r.t to $x \Rightarrow dx = 3t^2\, dt$

$$= \int_0^\infty (t^3)^{1/2} e^{-t}\, 3t^2\, dt$$

$$= 3\int_0^\infty t^{7/2} e^{-t}\, dt$$

From definition of Gamma function

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma(n)$$

$$= 3\int_0^\infty e^{-t} t^{\frac{9}{2}-1}\, dt$$

$$= 3\times\Gamma\left(\frac{9}{2}\right)$$

$$= 3\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}$$

$$= \frac{315\sqrt{\pi}}{16} \quad \textbf{Ans}$$

Q:- $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ **के रूप में समाकल का मान:**
(The value of the integral in the form).

$$\boxed{\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}$$

जैसे- $$\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx = \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{3+5+2}{2}\right)}$$

$$= \frac{\Gamma(2)\,\Gamma(3)}{2\,\Gamma(5)}$$

$$= \frac{1! \times 2!}{2 \times 4!} = \frac{1 \times 2 \times 1}{2 \times 4 \times 3 \times 2 \times 1} = \frac{1}{24} \quad \textbf{Ans}$$

1. $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

2. $$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^n x \,(\cos x)^0 \, dx$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{n+2}{2}\right)}$$

3. $$\int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} (\sin x)^0 \cos^n x \, dx$$

$$= \frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+2}{2}\right)}$$

Que.1 :- $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$ **का मान निकालों।**
(Find the value.)

यह $\int_0^{\pi/2}$ $\cos^m x \sin^n x dx$ के रूप में है।

Formula :

$$\boxed{\int_0^{\pi/2} \cos^m x \sin^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}$$

$$\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx = \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{5+4+2}{2}\right)} = \frac{\Gamma(3)\,\Gamma\left(\frac{5}{2}\right)}{2\,\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{2! \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{2 \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}$$

$$= \frac{1}{\frac{315}{8}}$$

$$= \frac{8}{315} \quad \textbf{Ans}$$

Que.2 :- दिखाइए कि $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx = \pi$

$$= \int_0^{\pi/2} (\sin x)^{-1/2}\, dx \times \int_0^{\pi/2} (\sin x)^{1/2}\, dx$$

$$= \int_0^{\pi/2} (\sin x)^{-1/2} (\cos x)^0 dx \times \int_0^{\pi/2} (\sin x)^{1/2} (\cos x)^0 dx$$

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{1/2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{1/2+0+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)} \times \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\sqrt{\frac{1}{4}}\, \sqrt{\pi} \times \sqrt{\pi}}{4 \times \frac{1}{4} \times \sqrt{\frac{1}{4}}}$$

$$= \pi \quad \textbf{Ans}$$

Que.3 :- $\displaystyle\int_0^{\pi/2} \cos^8 x\, dx$

$$= \int_0^{\pi/2} (\cos x)^8 (\sin x)^0\, dx \quad \text{(Gamma function form)}$$

$$= \frac{\Gamma\left(\frac{8+1}{2}\right)\, \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{8+0+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{9}{2}\right)\, \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(5)}$$

$$= \frac{\left(\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right) \times \sqrt{\pi}}{2 \times 4!}$$

$$= \frac{\left(\frac{105}{16}\pi\right)}{2 \times 24}$$

$$= \frac{105\pi}{768} \quad \textbf{Ans}$$

Que.4 :- $\displaystyle\int_0^a \frac{x^4}{\sqrt{a^2 - x^2}}\, dx$

माना: $x = a\sin\theta$

$$\text{d.w.r.t. to } x \Rightarrow dx = a\cos\theta\, d\theta$$

Limit change:

जब $x = 0 \Rightarrow \theta = 0$

जब $x = a \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^{\pi/2} \frac{a^4 \sin^4\theta}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{a^4 \sin^4\theta}{a\cos\theta}\, a\cos\theta\, d\theta$$

$$= a^4 \int_0^{\pi/2} \sin^4\theta (\cos\theta)^0\, d\theta$$

$$= a^4 \cdot \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{4+0+2}{2}\right)}$$

$$= a^4 \cdot \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(3)}$$

$$= a^4 \cdot \frac{\left(\frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right) \times \sqrt{\pi}}{2 \times 2!}$$

$$= a^4 \cdot \frac{\frac{3}{4}\pi}{4}$$

$$= \frac{3\pi a^4}{16} \quad \textbf{Ans}$$

Que.5 :- $\displaystyle\int_0^{\pi/2} \sin^6 x \cos^3 x\, dx$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$\int_0^{\pi/2} \sin^6 x \cos^3 x\, dx = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{6+3+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma(2)}{2\,\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\left(\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right) \times 1!}{2 \times \left(\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right)}$$

$$= \frac{1}{\frac{63}{2}}$$

$$= \frac{2}{63} \quad \textbf{Ans}$$

Que.6 :- $\int_0^{\pi/2} \sin^8 x \cos^2 x \, dx$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{8+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{8+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(6)}$$

$$= \frac{\left(\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right) \times \left(\frac{1}{2}\sqrt{\pi}\right)}{2 \times 5!}$$

$$= \frac{\left(\frac{105}{32}\pi\right)}{2 \times 120}$$

$$= \frac{7\pi}{512} \quad \textbf{Ans}$$

समाकलन के अनुप्रयोग (Applications of Integration)

- निश्चित समाकलन का प्रयोग, निम्नलिखित क्षेत्रों में किया जा सकता है -
  Definite integral can be used in the following areas -

1. वक्रों के चापों की लम्बाई निकालने में (In determining the length of arcs of curves)
2. समाकलन द्वारा क्षेत्रफल निकालने में (In determining areas by integration)
3. ठोसों के पृष्ठ तथा आयतन निकालने में (In determining surfaces and volumes of solid), इत्यादि।

वक्रों के चापों की लम्बाईयाँ (Lengths of the Arcs of Curves)

- प्राय: इस लम्बाई को 'S' से प्रदर्शित किया जाता है।
  Usually this length is represented by 'S'.
- किसी वक्र y = f(x) के लिये समाकलन द्वारा वक्र के कुछ भाग या पूरे वक्र की लम्बाई निकालने। विधि/सूत्र इस प्रकार है -
  For any curve y = f(x), find the length of some part of the curve or the whole curve by integration. The method/formula is as follows -

1. वक्र y = f (x) के बिन्दुओं x = a तथा x = b के बीच चाप की लम्बाई
   The length of the arc between the points x = a and x = b of the curve y = f (x)

$$S = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

2. वक्र x = f (y) के बिन्दुओं y = a तथा y = b के बीच चाप की लम्बाई,
The length of the arc between the points y = a and y = b of the curve x = f (y),

$$S = \int_{y=a}^{y=b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Note :-

- उपरोक्त सभी में ध्यान देने वाली बात यह है कि जिसके सापेक्ष समाकलन होगा सीमायें भी उसी के लिये रखी जायेगी।
  The point to be noted in all the above is that the limits will be set for the person with respect to whom the integration will be done.

Que.1 :- वक्र $y^2 = x^3$ के x = 0 से x = $\frac{5}{9}$ तक के चाप की लम्बाई ज्ञात कीजिये।
Find the length of the arc.

$$y^2 = x^3 \Rightarrow y = x^{3/2}$$

differentiating w.r.t. x

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\frac{dy}{dx} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2} \quad \ldots. (1)$$

Limits:

$x = 0$ और $x = \frac{5}{9}$

Length of Arc:

$$S = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^{5/9} \sqrt{1 + \left(\frac{3\sqrt{x}}{2}\right)^2}\, dx$$

$$S = \int_0^{5/9} \sqrt{1 + \frac{9x}{4}}\, dx$$

$$S = \int_0^{5/9} \frac{\sqrt{4+9x}}{4}\, dx$$

$$S = \frac{1}{2}\int_0^{5/9} (4+9x)^{1/2}\, dx$$

$$S = \frac{1}{2}\left[\frac{(4+9x)^{\frac{1}{2}+1}}{(\frac{1}{2}+1)\times 9}\right]_0^{5/9}$$

$$S = \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{\frac{27}{2}}\right]_0^{5/9}$$

$$S = \frac{1}{27}\left[(4+9x)^{3/2}\right]_0^{5/9}$$

$$S = \frac{1}{27}\left[(4+9\times\frac{5}{9})^{3/2} - (4+9\times 0)^{3/2}\right]$$

$$S = \frac{1}{27}\left[9^{3/2} - 4^{3/2}\right]$$

$$S = \frac{1}{27}\left[3^3 - 2^3\right]$$

$$S = \frac{1}{27}(27 - 8)$$

$$\boxed{S = \frac{19}{27}\text{ units Ans}}$$

Que.2 :- वक्र $y = x^{3/2}$ की $x = 0$ से $x = 5$ के बीच चाप की लम्बाई ज्ञात करों।
Find the length of the arc.

$$y = x^{3/2}$$

differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2} \quad (1)$$

Limits: $x = 0$ and $x = 5$

Length of Arc:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^5 \sqrt{1 + \left(\frac{3\sqrt{x}}{2}\right)^2}\, dx$$

$$S = \int_0^5 \sqrt{1 + \frac{9x}{4}}\, dx$$

$$S = \int_0^5 \frac{\sqrt{4 + 9x}}{4}\, dx$$

$$S = \frac{1}{2}\int_0^5 (4 + 9x)^{1/2}\, dx$$

$$S = \frac{1}{2}\left[\frac{(4+9x)^{\frac{1}{2}+1}}{(\frac{1}{2}+1)\times 9}\right]_0^5$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{\frac{3}{2}\times 9}\right]_0^5$$

$$= \frac{1}{2}\times\frac{2}{27}\left[(4+9x)^{3/2}\right]_0^5$$

$$= \frac{1}{27}\left[(4+9\times 5)^{3/2} - (4+9\times 0)^{3/2}\right]$$

$$= \frac{1}{27}\left[(49)^{3/2} - (4)^{3/2}\right]$$

$$= \frac{1}{27}\left[(49)^{3/2} - (4)^{3/2}\right]$$

$$= \frac{1}{27}\left[(7^2)^{3/2} - (2^2)^{3/2}\right]$$

$$= \frac{1}{27}(343 - 8)$$

$$= \frac{335}{27} \text{ unit Ans}$$

Que.3 :- वक्र $y^2 = x^3$ की बिन्दुओं (0,0) तथा (1, 2) के बीच चाप की लम्बाई ज्ञात करों।
Find the length of the arc.

Limits: $x = 0$ तथा $x = 1$

$$y^2 = x^3$$

differentiating w.r.t. x

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\frac{dy}{dx} = \frac{3x^2}{2x^{3/2}} = \frac{3\sqrt{x}}{2}$$

$$\frac{dy}{dx} = \frac{3x^{1/2}}{2}$$

Length of Arc:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^1 \sqrt{1 + \left(\frac{3\sqrt{x}}{2}\right)^2}\, dx$$

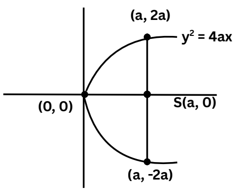
$$S = \int_0^1 \sqrt{1 + \frac{9x}{4}}\, dx$$

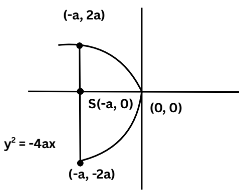
$$S = \frac{1}{27}\left[(4+9x)^{3/2}\right]_0^1$$

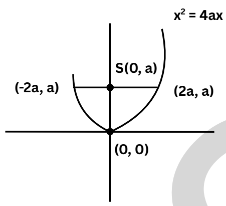
$$S = \frac{1}{27}\left[(13)^{3/2} - 8\right]$$

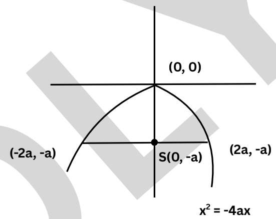
$$\boxed{S = \frac{13\sqrt{13} - 8}{27} \text{ Ans}}$$

Parabola
(परवलय)









Que.1 :- परवलय $y^2 = 4ax$ तथा उसके नाभिलम्ब द्वारा कटी लम्बाई ज्ञात करें।

Find the length cut off by the parabola $y^2 = 4ax$ and its latus rectum.

$$y^2 = 4ax$$

Differentiate with respect to y:

$$2y = 4a\frac{dx}{dy}$$

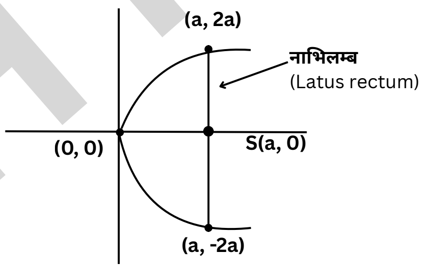
$$\frac{dx}{dy} = \frac{y}{2a} \quad ...(1)$$

$$y = 0 \text{ to } y = 2a$$

$$\text{Length} = 2\int_0^{2a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$S = 2\int_0^{2a}\sqrt{1+\left(\frac{y}{2a}\right)^2}\,dy$$

$$S = 2\int_0^{2a}\sqrt{\frac{4a^2+y^2}{4a^2}}\,dy$$



$$= 2\int_0^{2a}\frac{\sqrt{4a^2+y^2}}{2a}\,dy$$

$$= \frac{2}{2a}\int_0^{2a}\sqrt{4a^2+y^2}\,dy$$

formula: $\boxed{\int\sqrt{a^2+x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2\ln\left(x+\sqrt{a^2+x^2}\right)\right]}$

$$S = \frac{1}{a}\times\frac{1}{2}\left[y\sqrt{4a^2+y^2} + 4a^2\log\left(y+\sqrt{4a^2+y^2}\right)\right]_0^{2a}$$

$$S = \frac{1}{2a}\left[2a\sqrt{4a^2+(2a)^2} + 4a^2\log(2a+\sqrt{4a^2+(2a)^2}) - 4a^2\log(0+2a)\right]$$

$$S = \frac{1}{2a}\left[2a\sqrt{8a^2} + 4a^2\log\left(2a+\sqrt{8a^2}\right) - 4a^2\log(2a)\right].$$

$$2a\sqrt{8a^2} = 2a \cdot 2\sqrt{2}\,a = 4\sqrt{2}\,a^2.$$

$$S = 2\sqrt{2}\,a \;+\; 2a\log\left(2a + 2\sqrt{2}\,a\right) \;-\; 2a\log(2a).$$

$$2a\log(2a + 2\sqrt{2}a) - 2a\log(2a) = 2a\left[\log(2a(1+\sqrt{2})) - \log(2a)\right].$$

$$2a\left[\log(2a(1+\sqrt{2})) - \log(2a)\right] = 2a\log\left(1+\sqrt{2}\right).$$

$$\boxed{S = 2a\left(\sqrt{2} + \log(1+\sqrt{2})\right)}$$ **Ans.**

Que.2 :- समाकलन विधि से वृत्त $x^2 + y^2 = a^2$ की परिधि ज्ञात करें।

Find the circumference of the circle $x^2 + y^2 = a^2$ by integration method.

Given:

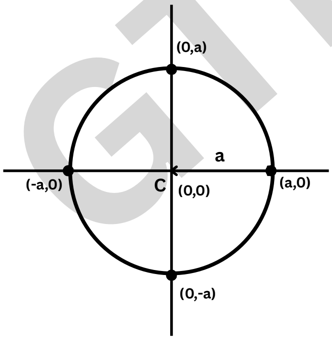
$$x^2 + y^2 = a^2$$

Differentiate w.r.t x:

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



Limits: $x = 0$ तथा $x = a$

Length of arc:

$$S = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 4\int_0^a \sqrt{1 + \left(-\frac{x}{y}\right)^2}\,dx$$

$$S = 4\int_0^a \sqrt{1 + \frac{x^2}{y^2}}\,dx$$

Since $y^2 = a^2 - x^2$,

$$S = 4\int_0^a \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,dx$$

$$S = 4\int_0^a \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\,dx$$

$$S = 4\int_0^a \frac{a}{\sqrt{a^2 - x^2}}\,dx$$

$$S = 4a\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$S = 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}\left(\tfrac{a}{a}\right) - \sin^{-1}(0)\right]$$

$$S = 4a\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$S = 4a\left[\frac{\pi}{2} - 0\right]$$

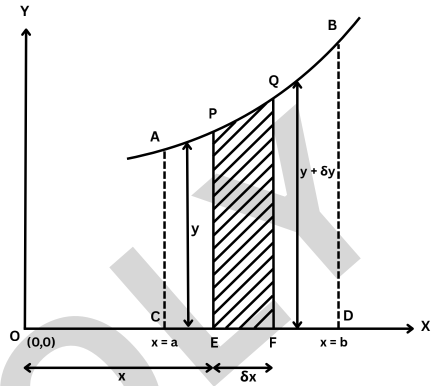
$$\boxed{S = 2\pi a}$$ **Ans.**

समाकलन द्वारा क्षेत्रफल ज्ञात करना (To find the Area by Integration)

- **किसी वक्र y = f(x) के लिये, सीमाओं x = a, x = b तथा X - अक्ष द्वारा घिरे क्षेत्रफल को ज्ञात करना**
  For a curve y = f(x), find the area bounded by the limits x = a, x = b and the X - axis

$$\text{क्षेत्रफल } ACDBA = A = \int_{x=a}^{x=b} f(x)\,dx = \int_{x=a}^{x=b} y\,dx$$

$$\text{क्षेत्रफल (A)} = A = \int_{x=a}^{x=b} y\,dx$$

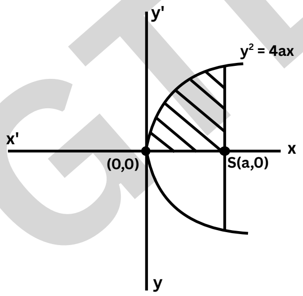


- **इसी प्रकार, वक्र x = f (y), (y), सीमाओं y = c, y = d तथा अक्ष द्वारा घिरे क्षेत्रफल को ज्ञात किया जाता है।**
  Similarly, the area bounded by the curve x = f (y), (y), the boundaries y = c, y = d and the axis is found.

$$\text{क्षेत्रफल 'A'} = \int_{y=c}^{y=d} f(y)\,dy = \int_{y=c}^{y=d} x\,dy$$

$$\text{A क्षेत्रफल (A)} = \int_{y=c}^{y=d} x\,dy$$

Que.1 :- परवलय $y^2 = 4ax$ तथा इसके नाभिलम्ब के बीच घिरे क्षेत्र का क्षेत्रफल ज्ञात करें।
Find the area of the region enclosed between the parabola $y^2 = 4ax$ and its foci.



Limits $x = 0$ तथा $x = a$

$$y^2 = 4ax$$

$$\text{या } y = \sqrt{4ax}$$

Area

$$\text{Area} = 2 \times \int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2 \times 2\sqrt{a}\int_0^a x^{1/2}dx$$

$$= 4\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a$$

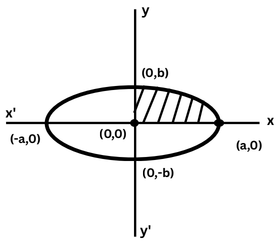
$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= 4(a)^{1/2} \times \frac{2}{3}\left[a^{3/2} - 0\right]$$

$$= \frac{8a^2}{3}\ \text{unit}^2 \quad \underline{\text{Ans.}}$$

Que.2 :- दीर्घवृत्त $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ का क्षेत्रफल निकालें।

Find the area of the ellipse.



Limits, $x=0$ तथा $x=a$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2(a^2-x^2)}{a^2}$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$=4\times\int_0^a y\,dx$$

$$=4\times\int_0^a \frac{b\sqrt{a^2-x^2}}{a}\,dx$$

$$=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$\text{We know that }\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \text{Area}=\frac{4b}{a}\times\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

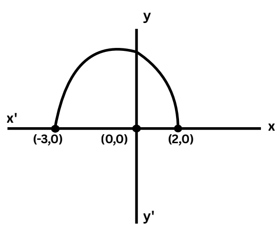
$$=\frac{2b}{a}\left[a\sqrt{a^2-a^2}+a^2\sin^{-1}\left(\frac{a}{a}\right)-(0+0)\right]$$

$$=\frac{2b}{a}\left[0+a^2\frac{\pi}{2}\right]$$

$$\boxed{A=\pi ab \text{ वर्ग इकाई (Ans)}}$$

Que.3 :- परवलय y = 6 - x - x^2 तथा x - अक्ष के बीच घिरे स्थान का क्षेत्रफल ज्ञात कीजिये।

Find the area of the space enclosed between the parabola y = 6 - x - x^2 and the x - axis.



$$y=6-x-x^2 \qquad (1)$$

x-अक्ष का समीकरण $y=0$। समीकरण (1) में रखने पर:

$$0=6-x-x^2$$

$$0=-(x^2+x-6)$$

$$x^2+x-6=0$$

$$x^2+3x-2x-6=0$$

$$x(x+3)-2(x+3)=0$$

$$(x+3)(x-2)=0$$

(Limits: $x = -3$ तथा $x = 2$)

$$\therefore \; x = 2, \; -3$$

$$\text{Area} = \int_a^b y\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(6(2) - \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(6(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3}\right)$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= (10 - \frac{8}{3}) - (-9 - \frac{9}{2})$$

$$= 30 - 11 - \frac{8}{3} + \frac{9}{2}$$

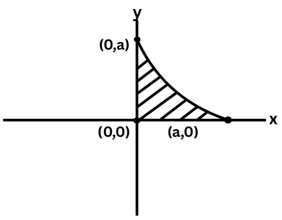
$$= \frac{19}{1} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{114 - 16 + 27}{6}$$

$$= \frac{141 - 16}{6} = \frac{125}{6} \text{ वर्ग इकाई (unit}^2\text{)}$$

Que.4 :- वक्र √x + √y = √a और अक्षों से घिरा क्षेत्रफल ज्ञात करो।

Find the area bounded by the curve √x + √y = √a and the axes.



$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad ----(1)$$

x-अक्ष का समीकरण $y = 0$

समीकरण (1) में रखने पर

Square on both sides

$$\sqrt{x} + 0 = \sqrt{a}$$

$$\boxed{x = a}$$

y-अक्ष का समीकरण $x = 0$

समीकरण (1) में रखने पर

$$0 + \sqrt{y} = \sqrt{a}$$

Square on both sides

$$\boxed{y = a}$$

Limits: $x = 0$ तथा $x = a$

समीकरण (1) से

$$\sqrt{y} = \sqrt{a} - \sqrt{x}$$

Square

$$y = (\sqrt{a} - \sqrt{x})^2 \quad ----(2)$$

Area (A) =

$$A = \int_a^b y\,dx$$

$$A = \int_0^a (\sqrt{a} - \sqrt{x})^2\,dx$$

$$= \int_0^a \left((\sqrt{a})^2 + (\sqrt{x})^2 - 2\sqrt{a}\sqrt{x}\right)dx$$

$$= \int_0^a \left(a + x - 2\sqrt{a}\,x^{1/2}\right)dx$$

$$= \left[a \cdot x + \frac{x^2}{2} - 2\sqrt{a} \cdot \frac{x^{3/2}}{3}\right]_0^a$$

$$A = \left[a^2 + \frac{a^2}{2} - \frac{4}{3}a^{3/2}\sqrt{a}\right]$$

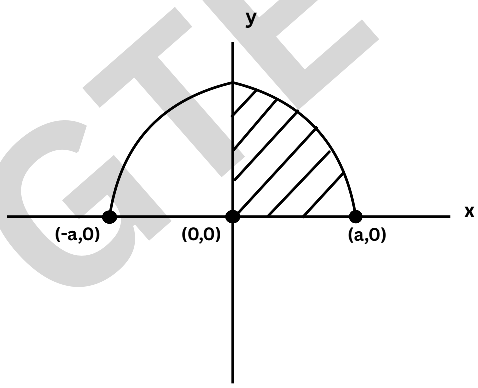
$$= \left[\frac{a^2}{1} + \frac{a^2}{2} - \frac{4a^2}{3}\right]$$

$$= \left[\frac{6a^2 + 3a^2 - 8a^2}{6}\right]$$

$$\boxed{A = \frac{a^2}{6} \text{ वर्ग इकाई } \text{Ans.}}$$

Que :- परवलय ay = 3 (a^2 - x^2) तथा x - अक्ष के बीच घिरे क्षेत्र का क्षेत्रफल ज्ञात कीजिये।

Find the area of the region enclosed between the parabola ay = 3 (a^2 - x^2) and the x - axis.



$$ay = 3(a^2 - x^2) \quad \text{———} (1)$$

x-**अक्ष का समीकरण:**

$$y = 0$$

समीकरण (1) में रखने पर,

$$0 = 3(a^2 - x^2)$$

$$0 = a^2 - x^2$$

$$x^2 = a^2$$

$$x = \pm a$$

Limits: $x = 0$ तथा $x = a$

$$y = \frac{3}{a}(a^2 - x^2) \quad \text{———} (2)$$

$$\text{Area} = 2\int_0^a y\,dx$$

$$= 2\int_0^a \frac{3}{a}(a^2 - x^2)\,dx$$

$$= \frac{6}{a}\int_0^a \left(a^2 - x^2\right) dx$$

$$= \frac{6}{a}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{6}{a}\left[a^3 - \frac{a^3}{3}\right]$$

$$\text{Area} = \frac{6}{a}\left[\frac{3a^3 - a^3}{3}\right]$$

$$= \frac{6}{a}\left[\frac{2a^3}{3}\right]$$

$= 4a^2$ वर्ग इकाई Ans.

दो वक्रों के बीच घिरा क्षेत्रफल निकालना (To Find Area Between Two Curves)

- माना दो वक्रों के समीकरण $y = f_1(x)$ (वक्र ALB) तथा $y = f_2(x)$ (वक्र AMB) है। जो बिन्दुओं A तथा B या x = a तथा x = b पर काटते हैं। अत:

  Let the equations of two curves be $y = f_1(x)$ (curve ALB) and $y = f_2(x)$ (curve AMB). Which intersect at points A and B or x = a and x = b. So

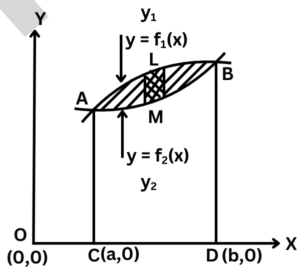
- दोनों वक्रों के बीच घिरा रेखांकित क्षेत्रफल ALBMA = A = क्षेत्रफल ALBDCA - क्षेत्रफल AMBDCA

  Area bounded between the two curves ALBMA = A = Area ALBDCA - Area AMBDCA

$$\boxed{A = \int_{x=a}^{x=b} f_1(x)\,dx - \int_{x=a}^{x=b} f_2(x)\,dx}$$

या

$$\boxed{Area = \int_a^b y_1\,dx - \int_a^b y_2\,dx}$$

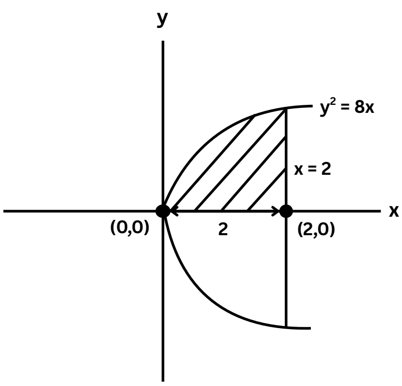


NOTE :-

- जब दो Carves के बीच Area ज्ञात करने के लिए Limits, Ques में न दी गई हो तो दोनों curves के प्रतिच्छेद बिन्दु (intersection Points) ही Limits होते हैं।
- Intersection Points, find करने के लिए दोनों curves के समी० को Solve करके x & y की value ज्ञात करते हैं।

Que.1 :- समाकलन विधि द्वारा उस क्षेत्र का क्षेत्रफल ज्ञात करें जो कि रेखा x = 2 और परवलय $y^2 = 8x$ द्वारा परिबद्ध है।

Find the area of the region bounded by the line x = 2 and the parabola $y^2 = 8x$ by the method of integration.



$$x = 2 \quad (1)$$

$$y^2 = 8x \quad (2)$$

$$\therefore y^2 = 4ax$$

Limits: $x = 0$ तथा $x = 2$

Area Formula: $\text{Area} = 2\int_0^2 y\,dx$

$$= 2\int_0^2 \sqrt{8x}\,dx$$

$$\text{Area} = 2 \times 2\sqrt{2}\int_0^2 x^{1/2}\,dx$$

$$= 4\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= 4\sqrt{2} \times \frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= \frac{8\sqrt{2}}{3}\left(2^{3/2} - 0\right)$$

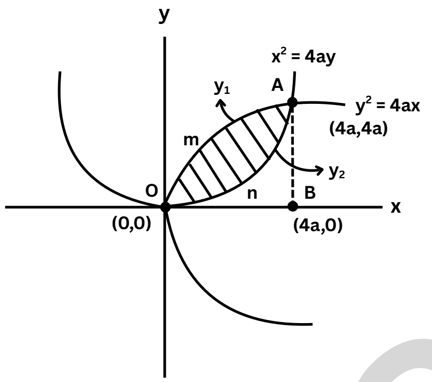
$$= \frac{8}{3} \times 2^2$$

$$= \frac{8}{3} \times 4 = \frac{32}{3}$$

$$\boxed{= \frac{32}{3}\ \text{वर्ग इकाई (Ans.)}}$$

Que.2 :- वक्रों $y^2 = 4ax$ तथा $x^2 = 4ay$ के बीच का क्षेत्रफल ज्ञात कीजिए।

Find the area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.



$$y^2 = 4ax \qquad (1)$$

$$x^2 = 4ay \qquad (2)$$

दोनों Curve का Intersection Point

समीकरण (1) से y का मान समीकरण (2) में रखने पर

$$x^2 = 4a\sqrt{4ax}$$

Square on both side

$x^4 = 16a^2 \times 4ax$

$x^3 = 64a^3$

$\boxed{x = 4a}$ समीकरण (1) में रखने पर

$y^2 = 4a \times 4a$

$y^2 = (4a)^2$

$y = 4a$

Intersection Point (4a, 4a)

Area (A) = Area of OmABO - Area of OnABO

$$A = \int_0^{4a} y_1\,dx - \int_0^{4a} y_2\,dx$$

$$A = \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$A = 2\sqrt{a}\int_0^{4a} x^{1/2}dx - \frac{1}{4a}\int_0^{4a} x^2 dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[(4a)^{3/2} - 0\right] - \frac{1}{12a}\left[(4a)^3 - 0\right]$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2}\right] - \frac{1}{12a}\left[64a^3\right]$$

$$= \frac{4\sqrt{a}}{3}\left[(2^2)^{3/2} \cdot a^{3/2}\right] - \frac{1}{12a}\left[64a^3\right]$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}\ \text{वर्ग इकाई Ans.}$$

समाकलन द्वारा ठोसों का आयतन (Volume of Solids by Integration)

- **परिक्रमित ठोसों के वक्रपृष्ठ तथा आयतन (Surface Areas and Volumes of Solids of Revolution)**

(1) वक्र y = f(x), X - अक्ष, सीमाओं x = a तथा x = b के बीच घिरे क्षेत्रफल को यदि X - अक्ष के परितः घुमाते हैं तो परिक्रमित ठोस का वक्र पृष्ठ तथा आयतन,

If the curve y = f(x), X - axis, and the area enclosed between the limits x = a and x = b are rotated around the

(i) Volume (आयतन)

$$V = \pi \int_a^b y^2 \, dx$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

(2) वक्र x = f(y), Y - अक्ष सीमाओं y = c तथा y = d के बीच घिरे क्षेत्रफल को यदि अक्ष के परितः घुमाते हैं तो परिक्रमित ठोस का वक्र पृष्ठ तथा आयतन,

Curve x = f(y), Y- axis If the area enclosed between the limits y = c and y = d is rotated around the Y - axis, then the curved surface and volume of the rotated solid,

(i) Volume (आपतन)

$$V = \pi \int_c^d x^2 \, dy$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$S = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Que.1 :- बेलन का वक्र पृष्ठ तथा आयतन (Surface Area and Volume of Cylinder) ज्ञात करें।

→ बेलन, एक आयत (Rectangle) को किसी अक्ष के परितः घुमाने से बनाता है।

→ माना चित्र के अनुसार आयत की AB भुजा को x - Axis के परितः घुमाने पर एक बेलन का निर्माण होता है।

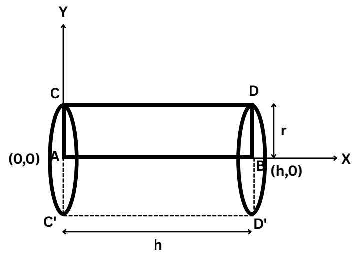
बेलन की त्रिज्या (radius of Cylinder) = r

बेलन की ऊंचाई (Height of Cylinder) = h

Limits x = 0 तथा x = h

रेखा CD का समीकरण (Equation of Line CD)

y = r —— (1)



(i) Volume (आयतन)

$$V = \pi \int_a^b y^2 \, dx$$

$$V = \pi \int_0^h r^2 \, dx$$

$$= \pi r^2 \int_0^h 1 \, dx$$

$$= \pi r^2 [x]_0^h$$

$$= \pi r^2 [h - 0]$$

$$= \pi r^2 h \quad \text{घन इकाई (unit}^3\text{)}$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 2\pi \int_0^h r\sqrt{1+0^2}\,dx$$

$$S = 2\pi r \int_0^h 1\,dx \quad (\text{क्योंकि } y = r, \frac{dy}{dx} = 0)$$

$$S = 2\pi r[x]_0^h$$

$$S = 2\pi r[h-0]$$

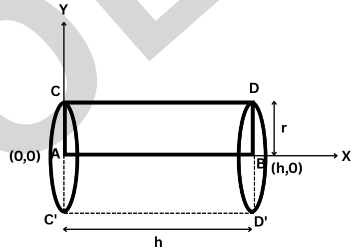
$$S = 2\pi rh \quad \text{वर्ग इकाई (unit}^2) \quad \underline{\text{Ans.}}$$

Que.2 :- किसी बेलन के आधार की त्रिज्या 3 सेमी० तथा ऊँचाई 8 सेमी० है। बेलन का वक्र पृष्ठ तथा आयतन ज्ञात कीजिये।

The radius of the base of a cylinder is 3 cm and its height is 8 cm. Find the curved surface area and volume of the cylinder.

Radius (त्रिज्या) r = 3cm

height (ऊँचाई) h = 8cm



चित्र के अनुसार एक आयत की AB भुजा को x - अक्ष के परितः घुमाने पर एक बेलन (Cylinder) का निर्माण होता है।

Limit x = 0 तथा x = h = 8

Line CD का समीकरण y = r

y = 3 —— (1)

(i) Volume (आयतन)

$$V = \pi \int_a^b y^2\,dx$$

$$V = \pi \int_0^8 3^2\,dx$$

$$V = 9\pi \int_0^8 1\,dx$$

$$V = 9\pi[x]_0^8$$

$$V = 9\pi[8-0]$$

$$V = 72\pi \quad \text{घन इकाई}$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 2\pi \int_0^8 3\sqrt{1+0^2}\,dx$$

$$S = 6\pi \int_0^8 1\,dx \quad (\text{क्योंकि } y = 3, \frac{dy}{dx} = 0)$$

$$S = 6\pi[x]_0^8$$

$$S = 6\pi[8-0]$$

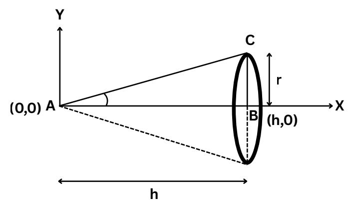
$$S = 48\pi \quad \text{वर्ग इकाई} \quad \underline{\text{Ans.}}$$

Que.3 :- लम्बवृत्तीय शंकु का वक्र पृष्ठ तथा आयतन ज्ञात कीजिए।

(Surface area and volume of a right circular cone)

→ शंकु, एक समकोण त्रिभुज (Right angle Triangle) को किसी अक्ष के परितः घुमाने से बनता है।

→ चित्र के अनुसार समकोण त्रिभुज की AB भुजा को x - अक्ष के परितः घुमाने पर शंकु का निर्माण होता है।



Limits: $x = 0$ तथा $x = h$

Line AC का समीकरण: $y = mx \quad (\because m = \tan\theta)$

$$y = \tan\theta \cdot x$$

$$y = \frac{r}{h}x \quad ...(1)$$

त्रिभुज ABC में $\tan\theta = \frac{\text{लंब}}{\text{आधार}} = \frac{r}{h}$

(i) Volume (आयतन)

$$\boxed{V = \pi\int_a^b y^2\,dx}$$

$$V = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx$$

$$= \pi\frac{r^2}{h^2}\int_0^h x^2 dx$$

$$= \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \pi\frac{r^2}{h^2}\left[\frac{h^3}{3} - 0\right]$$

$$= \frac{1}{3}\pi r^2 h \text{ घन इकाई}$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$\boxed{S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\left(\because y = \frac{r}{h}x,\ \frac{dy}{dx} = \frac{r}{h}\right)$$

$$S = 2\pi\int_0^h \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx$$

$$\because\ l = \sqrt{r^2+h^2}$$

$$S = \frac{2\pi r}{h}\int_0^h x\cdot\frac{l}{h}\,dx$$

$$= \frac{2\pi rl}{h^2}\left[\frac{x^2}{2}\right]_0^h$$

$$= \frac{2\pi rl}{h^2}\left[\frac{h^2}{2} - 0\right]$$

$$= \pi rl \text{ वर्ग इकाई}$$ Ans.

Que.4 :- गोले का वक्र पृष्ठ तथा आयतन ज्ञात कीजिए।
(Surface area and volume of a sphere)

एक अर्द्धवृत्त (Semicircle) जिसका केन्द्र मूलबिंदु (origin) पर है को x - अक्ष के परित: घुमाने पर गोले का निर्माण होता है।

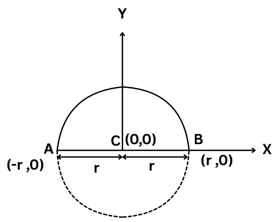
केन्द्र (Centre) = (0,0)

त्रिज्या (radius) = r

Limit $x = -r$ तथा $x = r$

अर्द्धवृत्त का समीकरण (Equation of Semi circle)

$x^2 + y^2 = r^2$ —— (1) $\Rightarrow y^2 = r^2 - x^2$ $\Rightarrow y = \sqrt{r^2 - x^2}$



(i) Volume (आयतन)

$$V = \pi \int_a^b y^2\, dx$$

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$= \pi \left[\frac{3r^3 - r^3 + 3r^3 - r^3}{3} \right]$$

$$= \pi \left[\frac{4r^3}{3} \right] \Rightarrow V = \frac{4}{3}\pi r^3 \text{ घन इकाई}$$

(ii) वक्रपृष्ठ (Curved surface Area)

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx$$

$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$y = \sqrt{r^2 - x^2}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{r^2 - x^2}} \cdot (-2x)$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}}$$

$$\left(\frac{dy}{dx} \right)^2 = \frac{x^2}{r^2 - x^2}$$

$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\, dx$$

$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$S = 2\pi r \int_{-r}^{r} 1\, dx$$

$$S = 2\pi r\, [x]_{-r}^{r}$$

$$S = 2\pi r\, [r - (-r)]$$

$$S = 2\pi r [2r] \Rightarrow S = 4\pi r^2 \text{ वर्ग इकाई}$$ Ans.