AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

UNIT - 1 : MATRICES

Lecture - 1

Today's target

- Inverse of a matrix by Elementary transformation
- PYQs
- DPP

By Gulshan sir

UNIT-1 : Matrices

- Inverse of a matrix by Elementary transformations (L-1)
- Rank of matrix
  - (i) By determinant method (L-2)
  - (ii) By echelon form method (L-3)
  - (iii) By normal form method (L-4)
  - (iv) By normal form method (PAQ Form) (L-5)
- Solution of system of linear equations
  - (i) Non Homogenous System of linear equations (L-6)
  - (ii) Homogenous System of linear equations (L-7)
- Linear Dependence and Independence of vectors (L-8)
- Complex Matrices (L-9)
  - (i) Hermitian
  - (ii) Skew-Hermitian
  - (iii) Unitary Matrices
- Eigen values and Eigen vectors (L-10 + L-11)
- Cayley Hamilton Theorem (L-12)
- Applications to Engineering problems (L-12)

Elementary Transformations / Elementary Operations

Elementary Row Transformation

(i) Interchanging of any two rows

$$A = \begin{array}{c} \begin{bmatrix} 3 & 1 & 4 \\ 2 & -3 & 4 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_1 \longleftrightarrow R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ 3 & 1 & 4 \end{bmatrix}$$

Elementary Column Transformation

(i) Interchanging of any two columns

$$A = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 3 & 1 & 4 \\ 2 & -3 & 4 \\ 1 & -1 & 1 \end{bmatrix} \end{array}$$

$$C_1 \longleftrightarrow C_2$$

$$A \sim \begin{bmatrix} 1 & 3 & 4 \\ -3 & 2 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

Elementary Row Transformation

(ii) Multiplication/Division of a row by a non-zero number

$$A = \begin{array}{c} \begin{bmatrix} 1 & 4 & 8 \\ 4 & -2 & 4 \\ 4 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_2 \longrightarrow -\frac{1}{2} R_2$$

$$A \sim \begin{bmatrix} 1 & 4 & 8 \\ -2 & 1 & -2 \\ 4 & -1 & 1 \end{bmatrix}$$

Elementary Column Transformation

(ii) Multiplication/Division of a column by a non-zero number

$$A = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1 & 5 & 6 \\ 3 & -3 & 4 \\ 5 & -1 & 2 \end{bmatrix} \end{array}$$

$$C_1 \longrightarrow 5C_1$$

$$A \sim \begin{bmatrix} 5 & 5 & 6 \\ 15 & -3 & 4 \\ 25 & -1 & 2 \end{bmatrix}$$

(iii) Addition of k times the element of a row to corresponding elements of another row, $k \neq 0$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 4 \\ 2 & -3 & 4 \\ 5 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_2 \longrightarrow R_2 + (-2R_1)$$

$$R_3 \longrightarrow R_3 + (-5R_1)$$

$$A \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -4 \\ 0 & -11 & -19 \end{bmatrix}$$

(iii) Addition of k times the element of column to corresponding elements of another column, $k \neq 0$

$$A = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 4 \\ 2 & -3 & 4 \\ 2 & -1 & 1 \end{bmatrix} \end{array}$$

$$C_2 \longrightarrow C_2 + (-2C_1)$$

$$A \sim \begin{bmatrix} 1 & 0 & 4 \\ 2 & -7 & 4 \\ 2 & -5 & 1 \end{bmatrix}$$

Using elementary transformation

(i) We can reduced a given matrix(square) in to Identity matrix

(ii) We can reduced a given matrix in to Echelon form

(iii) We can reduced a given matrix in to Normal form

Reduced a given matrix (square) in to Identity matrix

(i) Using elementary row transformation

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

(i) Using elementary column transformation

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Q.1 Transform $\begin{bmatrix}1 & 3 & 3\\ 2 & 4 & 10\\ 3 & 8 & 4\end{bmatrix}$ into a unit matrix by using elementary row transformation

$$A = \begin{bmatrix}\boxed{1} & 3 & 3\\ 2 & 4 & 10\\ 3 & 8 & 4\end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 + (-3R_1)$

$$\sim \begin{bmatrix}1 & 3 & 3\\ 0 & -2 & 4\\ 0 & -1 & -5\end{bmatrix}$$

$R_2 \longrightarrow -\frac{1}{2} R_2$

$$\sim \begin{bmatrix}1 & 3 & 3\\ 0 & \boxed{1} & -2\\ 0 & -1 & -5\end{bmatrix}$$

$R_1 \longrightarrow R_1 + (-3R_2)$

$R_3 \longrightarrow R_3 + R_2$

$$\sim \begin{bmatrix}1 & 0 & 9\\ 0 & 1 & -2\\ 0 & 0 & -7\end{bmatrix}$$

$R_3 \longrightarrow -\frac{1}{7} R_3$

$$\sim \begin{bmatrix}1 & 0 & 9\\ 0 & 1 & -2\\ 0 & 0 & \boxed{1}\end{bmatrix}$$

$R_1 \longrightarrow R_1 + (-9R_3)$

$R_2 \longrightarrow R_2 + 2R_3$

$$\sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Q.2 Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix by using elementary column transformation

$$A = \begin{bmatrix} \boxed{1} & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$$

$$C_2 \to C_2 + (-3C_1)$$
$$C_3 \to C_3 + (-3C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 4 \\ 3 & -1 & -5 \end{bmatrix}$$

$$C_2 \to -\frac{1}{2} C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & \boxed{1} & 4 \\ 3 & \frac{1}{2} & -5 \end{bmatrix}$$

$$C_1 \to C_1 + (-2C_2)$$
$$C_3 \to C_3 + (-4C_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \frac{1}{2} & -7 \end{bmatrix}$$

$$C_3 \to -\frac{1}{7} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \frac{1}{2} & \boxed{1} \end{bmatrix}$$

$$C_1 \to C_1 + (-2C_3)$$
$$C_2 \to C_2 + \left(-\frac{1}{2} C_3\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find inverse of a Matrix

(i) Apply Elementary row transformation only

$$\boxed{\begin{aligned} A &= IA \\ I &= A^{-1}A \end{aligned}}$$

(ii) Apply Elementary column transformation only

$$\boxed{\begin{aligned} A &= AI \\ I &= AA^{-1} \end{aligned}}$$

NOTE :

(1) Don't apply both transformation in the same question

(2) if on applying one or more elementary row operation, we obtain all zeros in one or more rows, then A^{-1} does not exist

Q.3 Find the inverse by employing elementary row transformation $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ **AKTU 2019**

$$A = IA$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \longrightarrow R_1 + (-R_2)$$

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2 + (-2R_1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \longleftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} A$$

$$R_2 \longrightarrow -1R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & 0 \end{bmatrix} A$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & -3 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$I = A^{-1} A$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}$$

Q.4 Employing elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

AKTU 2018

$$A = IA$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \longleftrightarrow R_1$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 \longrightarrow R_3 + (-3R_1)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$$R_1 \longrightarrow R_1 + (-2R_2)$$

$$R_3 \longrightarrow R_3 + 5R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_3 \longrightarrow \frac{1}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$R_2 \to R_2 + (-2R_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$I = A^{-1} A$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}}$$

Q .5 Find the inverse of the following matrix using elementary row transformation $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

AKTU 2021

$$A = IA$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \longleftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2 + (-4R_1)$$

$$R_3 \longrightarrow R_3 + (-2R_1)$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$$R_2 \longleftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & -5 & -15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{bmatrix} A$$

$$R_2 \longrightarrow -1R_2$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & -5 & -15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 - 2R_2$

$R_3 \longrightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 0 & 2 \\ -5 & 1 & 6 \end{bmatrix} A$$

$R_3 \longrightarrow \frac{1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 0 & 2 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 + 4R_3$

$R_2 \longrightarrow R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$$I = A^{-1} A$$

$$A^{-1} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{5} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}}$$

Q .6 Find the inverse of the matrix by using elementary column transformation $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$A = AI$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_1 \to C_1 + C_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 0 \\ 3 & 1 & 3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 + C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 5 \\ 3 & 1 & 6 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$C_1 \to C_1 + (-5C_2)$$

$$C_3 \to C_3 + (-5C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ 1 & 0 & 2 \end{bmatrix}$$

$$C_1 \to C_1 + 2C_3$$

$$C_2 \to C_2 - C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$I = AA^{-1}$$

$$\boxed{A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}}$$

DPP-1 [Unit-1]

Inverse of a matrix by Elementary Transformation

Q.1 Transform $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ into a unit matrix by using elementary row transformation

Q .2 Find the inverse of the matrix by using elementary row transformation $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Ans $A^{-1} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$

Q.3 Employing elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Ans $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$

Q.4 Employing elementary row transformations, find the inverse of the following non-singular matrix

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Ans $A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

AKTU : B.Tech (First Sem)

GATEWAY CLASSES

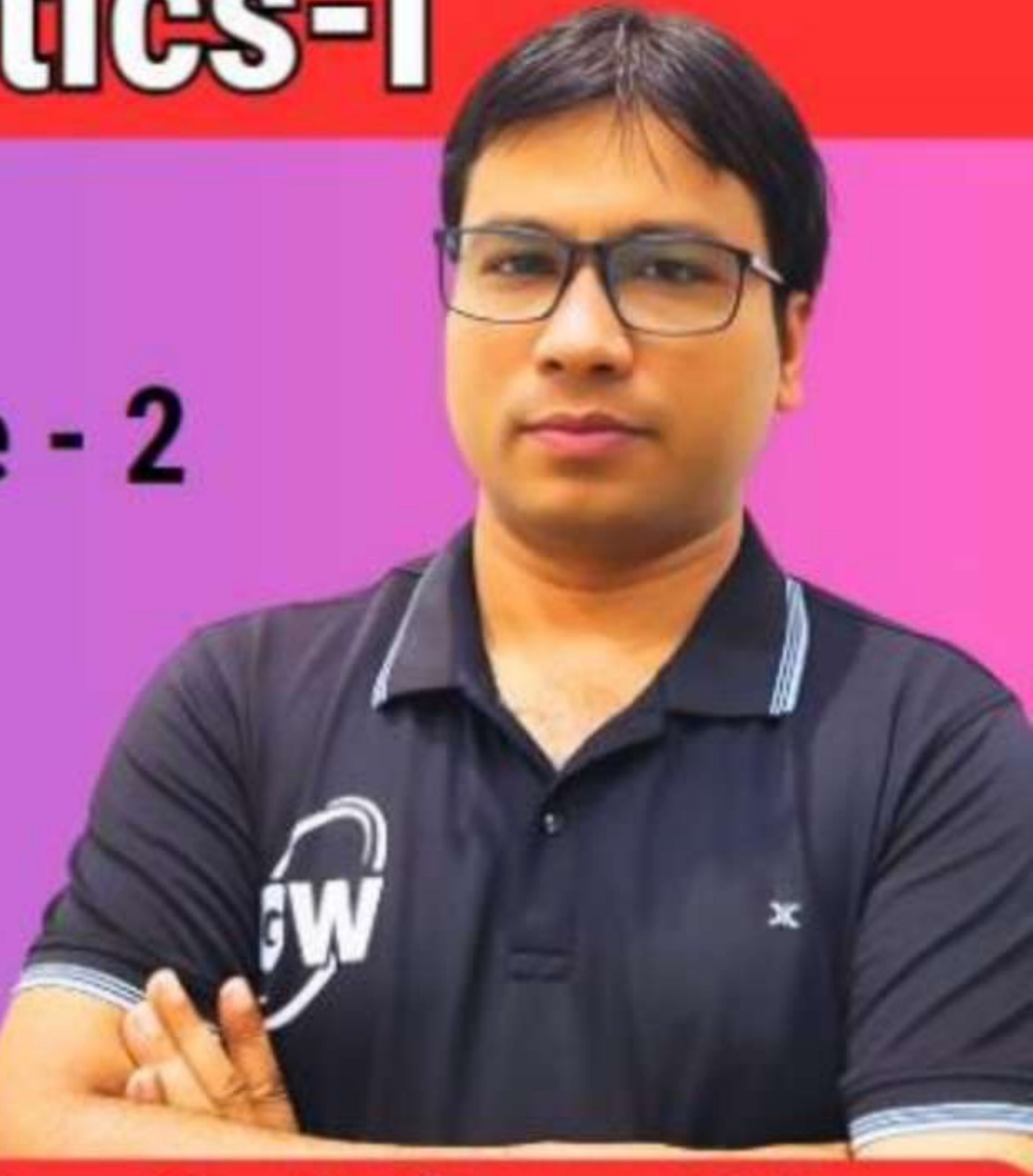
Engineering Mathematics-I

UNIT - 1 : Matrices

Lecture - 2

Today's target

- Rank of a matrix by Determinant Method
- DPP
- PYQ



By Gulshan Sir

UNIT-1 : Matrices

- Inverse of a matrix by Elementary transformations L-1
- Rank of matrix

  (i) By determinant method L-2

  (ii) By echelon form method L-3

  (iii) By normal form method L-4

  (iv) By normal form method (PAQ Form) L-5
- Solution of system of linear equations

  (i) Non Homogenous System of linear equations L-6

  (ii) Homogenous System of linear equations L-7
- Linear Dependence and Independence of vectors L-8
- Complex Matrices L-9

  (i) Hermitian

  (ii) Skew-Hermitian

  (iii) Unitary Matrices
- Eigen values and Eigen vectors L-10, L-11
- Cayley Hamilton Theorem L-12
- Applications to Engineering problems. L-12

Rank of a matrix by Determinant Method

$\boxed{\text{Rank } (\rho) = \text{Order of highest non zero determinant obtained from the matrix.}}$

Note:

(i) If A is a null matrix, then

$\rho(A) = 0$

(ii) If A is NOT a null matrix, then

$\rho(A) \geq 1$

(iii) If A is $\boxed{\text{non singular matrix}}$ of order n x n , then

$\rightarrow |A| \neq 0$ $\rho(A) = n$

(iv) If A is m x n matrix then

$\rho(A) \leq$ Minimum value of m or n

3×3

$|A| \neq 0$, $\rho(A) = 3$

$|A| = 0$ → 2×2

$|A| \neq 0$, $\rho(A) = 2$

$|A| = 0$, $\rho(A) = 1$

Q.1 Find the rank of the matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix}$$

$$= 2 \times 5 - 4 \times 4$$

$$= 10 - 16$$

$$|A| = -6$$

$$\therefore |A| \neq 0$$

$$\boxed{\rho(A) = 2}$$

Q.2 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}_{3\times 3}$$

$$|D_1| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$|D_1| = 1\times 4 - 2\times 2$$

$$|D_1| = 0$$

$$|D_2| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$|D_2| = 2\times 5 - 4\times 3$$

$$|D_2| = 10 - 12$$

$$|D_2| = -2$$

$$|D_2| \neq 0$$

$$\boxed{\rho(A) = 2}$$

Q.3 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix}$$

$$|A| = 1(20-12) - 2(5-4) + 3(6-8)$$

$$= 8 - 2 - 6$$

$$= 8 - 8$$

$$\boxed{|A| = 0} \quad \rho(A) \neq 3$$

$$|D_1| = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 4 - 2$$

$$|D_1| = 2$$

$$|D_1| \neq 0$$

$$\boxed{\rho(A) = 2}$$

Q.4 Find the value of ‘b’ so that rank of $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$ is 2 .

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$$

$$\rho(A) = 2$$

$$\therefore |A| = 0$$

$$\begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{vmatrix} = 0$$

$$2(b-0) - 4(3b-2) + 2(0-1) = 0$$

$$2b - 12b + 8 - 2 = 0$$

$$-10b + 6 = 0$$

$$+10b = +6$$

$$b = \frac{\cancel{6}^{3}}{\cancel{10}_{5}}$$

$$\boxed{b = \frac{3}{5}}$$

Q.5 Find the value of 'P' so that rank of $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$ is 1 .

$$A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$$

$$\rho(A) = 1$$

$$\begin{vmatrix} 3 & P \\ P & 3 \end{vmatrix} = 0$$

$$9 - P^2 = 0$$

$$P^2 = 9$$

$$\boxed{P = \pm 3}$$

$$\begin{vmatrix} P & P \\ 3 & P \end{vmatrix} = 0$$

$$P^2 - 3P = 0$$

$$P(P-3) = 0$$

$$P = 0$$

$$P = 3$$

$$\boxed{P = 3}$$

Q.6 Under what condition the rank of the following matrix is 3. $A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$

$$\rho(A) = 3$$

$$|A| \neq 0$$

$$\begin{vmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0$$

$$2(x-0) - 4(2x-2) + 2(0-1) \neq 0$$

$$2x - 8x + 8 - 2 \neq 0$$

$$-6x + 6 \neq 0$$

$$+6x \neq +6$$

$$\boxed{x \neq 1}$$

Matrix : DPP-2

Topic : Rank of a matrix by Determinant Method

Q .1 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.

Ans.1 Rank = 3

Q .2 Find the value of 'K' so that rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & K \end{bmatrix}$ is 2 .

Ans.2 $K = 5$

GATEWAY CLASSES

AKTU : B.Tech (Semester - 1)

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 3

Today's Target

- Rank of a matrix by Echelon Form method
- PYQs
- DPP

GATEWAY CLASSES

Rank of a matrix by Echelon Form method

STEP-1 : Convert the given matrix in to the Echelon form

STEP-2 : Rank (ρ) = Number of non-zero rows in the Echelon form

$$A = \begin{bmatrix} \boxed{1} & 3 & 2 & 4 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{non zero row} \\ \text{non-zero row} \\ \text{non-zero row} \\ \text{zero row} \end{array}$$

$$\boxed{\rho(A) = 3}$$

Echelon Form

Any matrix A is said to be in Echelon form or row Echelon form if

(i) The number of zeros before the first non-zero element increases from top to bottom

Note : First non zero element in each row is called Pivot element or Leading element

(ii) If the column contain pivot element then all the entries below pivot element are zero.

Note : (a) Convert Pivot element in to 1, if require for easy calculation

(b) Pivot element shifted toward right

(iii) All zero rows of the matrix lie below non zero row.

Note : The Echelon Form of a square matrix is an upper triangular matrix

Rank Nullity Theorem : Let A be a square matrix of order n and if the rank of A is r, then $n-r$ is called the nullity of the matrix A and is usually denoted by $N(A)$

$$\boxed{N(A) = n-r}$$

Q.1 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$

$$R_3 \to R_3 + (-2R_1)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is an Echelon form

$\rho(A) =$ Number of non-zero rows

$$\boxed{\rho(A) = 2}$$

Q.2 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ by Echelon Form

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$R_2 \to R_2 + (-2R_1)$

$R_3 \to R_3 + (-3R_1)$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is an Echelon Form

$\rho(A) =$ Number of non-zero row

$$\boxed{\rho(A) = 1}$$

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Q.3 Find the rank of the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

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$$A = \begin{bmatrix} \boxed{2} & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is an Echelon Form

$\rho(A)$ = Number of non-zero rows

$$\boxed{\rho(A) = 1}$$

Q.4 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ Also find nullity

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_2$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 & 2 \\ 0 & \boxed{-1} & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) =$ Number of non-zero row

$$\boxed{\rho(A) = 2}$$

Nullity is not defined

Q.5 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 10 & 13 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 10 & 13 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 + (-3R_1)$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$\rho(A)$ = Number of non-zero rows

$$\boxed{\rho(A) = 3}$$

Q.6 Use elementary transformation to reduce the following matrix A to triangular form and hence,

Find the rank of $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 + (-3R_1)$

$R_4 \longrightarrow R_4 + (-6R_1)$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_3$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + (-4R_2)$$

$$R_4 \rightarrow R_4 + (-9R_2)$$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + (-2R_3)$$

$$A = \begin{bmatrix} \boxed{1} & -1 & -2 & -1 \\ 0 & \boxed{1} & -6 & -3 \\ 0 & 0 & \boxed{33} & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = Number of non-zero row

$$\boxed{\rho(A) = 3}$$

Matrix : DPP-3

Topic : Rank of a matrix by Echelon Form method

Q 1 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Ans 1 Rank = 2

Q 2 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

Ans 2 Rank = 3

Q 3 Use elementary transformation to reduce the following matrix A to triangular form and hence

find the rank of $A = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

Ans 3 Rank = 2

GATEWAY CLASSES

AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

UNIT - 1 : MATRICES

Lecture - 4

Today's target

- Rank of a matrix by Normal Form method
- PYQs
- DPP

By Gulshan sir

Rank of a matrix by Normal Form or Canonical Form method

- Convert given matrix in to Normal form using both elementary row and column transformation
- Standard Normal form are given below

(I) I_r

(II) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

(III) $\begin{bmatrix} I_r & 0 \end{bmatrix}$ ✓

(IV) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{3\times 4} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Q.1 Reduce the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ in to normal form and find its rank

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + (-3R_1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\sim [$

$$\sim [I_2 \quad 0]$$

It is a normal form

$$\boxed{\rho(A) = 2}$$

Q.2 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$ AKTU 2016

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$$

$C_1 \to C_1 - C_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$$

$R_2 \to R_2 + (-2R_1)$

$R_3 \to R_3 + (-3R_1)$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 0 & -2 & 8 \end{bmatrix}$$

$C_2 \to C_2 + (-2C_1)$

$C_3 \to C_3 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 8 \\ 0 & -2 & 8 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & -2 & 8 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 4C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ Normal Form

$$\boxed{\rho(A) = 2}$$

Q. 3 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

AKTU 2015

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

$C_2 \to C_2 + (-2C_1)$

$C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 8 & 5 & 0 \end{bmatrix}$$

$C_2 \to -\frac{1}{2} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

$R_3 \to \frac{1}{9} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

Normal Form

$$\boxed{\rho(A) = 3}$$

Q 4 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

AKTU 2018

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 + (-2C_1)$

$C_3 \longrightarrow C_3 + 3C_1$

$C_4 \longrightarrow C_4 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + 2R_2$

$R_4 \longrightarrow R_4 - R_2$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&-1&-1\\0&0&2&0\\0&0&2&0\end{bmatrix}$$

$C_3 \to C_3 + C_2$

$C_4 \to C_4 + C_2$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&2&0\end{bmatrix}$$

$C_3 \to \frac{1}{2} C_3$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\\hline 0&0&0&0\end{array}\right]$$

$$\sim\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$

$$\boxed{\rho(A) = 3}$$

Matrix : DPP-4

Topic : Rank of a matrix by Normal Form method

Q 1 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Ans.1 Rank = 3

Q.2 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \end{bmatrix}$ AKTU 2014

Ans.1 Rank = 3

Q 3 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ AKTU 2017

Ans.3 $\text{Rank} = 4$

Q 4 Reduce the matrix A to its normal form and hence find its rank where $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Ans.4 $\text{Rank} = 4$

AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 5

Today's target

- Rank of a matrix by Normal Form method (PAQ Form)
- PYQs
- DPP

By Gulshan sir

Rank of a matrix by Normal Form (PAQ Form)

Both row and column elementary transformation are allowed

$$\boxed{A_{m \times n} = I_m A I_n}$$

$$\boxed{\text{Normal Form} = PAQ}$$

(i) Reduce the matrix A on LHS in normal form by using row and column transformation

(ii) Row transformation is also applied on I_m (Pre factor)

(iii) Column transformation is also applied on I_n (Post factor)

NOTE : P and Q are not unique

$AA^{-1} = I$

Inverse a matrix by Normal Form (PAQ Form)

$$I = PAQ$$

$$P^{-1}I = P^{-1}PAQ$$

$$P^{-1} = IAQ$$

$$P^{-1}P = AQP$$

$$I = AQP$$

$$A^{-1}I = A^{-1}AQP$$

$$\boxed{A^{-1} = QP}$$

Q.1 If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Determine two non − singular matrices P and Q such that $PAQ = I$.

Hence find A^{-1}

$$A_{3\times3} = I_3 A I_3$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \longrightarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$C_3 \longrightarrow C_3 + (-4C_2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$I_3 = PAQ$$

$$\boxed{\rho(A) = 3}$$

$$P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = QP$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}$$

Q.2 Find non-singular matrices P and Q such that PAQ is in the normal form for the matrix hence find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

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$$A_{3\times 4} = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$ $\quad R_3 \longrightarrow R_3 + (-3R_1)$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 + (-2C_1)$

$C_3 \longrightarrow C_3 + (-3C_1)$

$C_4 \longrightarrow C_4 + 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 + C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 1 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + 5C_2 \qquad C_4 \to C_4 + (-7C_2)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & -6 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$\boxed{\rho(A) = 2}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \checkmark$$

$$Q = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & -6 \end{bmatrix} \checkmark$$

Matrix : DPP-5

Topic : Rank of a matrix by Normal Form method (PAQ Form)

Q 1 For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ Find non-singular matrices P and Q such that PAQ is a normal form.

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Ans.1 $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Q .2 Find non-singular matrices P and Q such that PAQ is a normal form for the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Ans.2 $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

GW
GATEWAY CLASSES

UNIT -1 : MATRICES

Lecture - 6

Today's target

- ➢ A system of Non-Homogenous Linear Equations
- ➢ PYQs
- ➢ DPP

By Gulshan sir

UNIT-1 : Matrices

- Inverse of a matrix by Elementary transformations (L-1)
- Rank of matrix
  - (i) By determinant method (L-2)
  - (ii) By echelon form method (L-3)
  - (iii) By normal form method (L-4)
  - (iv) By normal form method (PAQ Form) (L-5)
- Solution of system of linear equations
  - (i) Non Homogenous System of linear equations (L-6)
  - (ii) Homogenous System of linear equations (L-7)
- Linear Dependence and Independence of vectors (L-8)
- Complex Matrices (L-9)
  - (i) Hermitian
  - (ii) Skew-Hermitian
  - (iii) Unitary Matrices
- Eigen values and Eigen vectors (L-10 + L-11)
- Cayley Hamilton Theorem (L-12)
- Applications to Engineering problems (L-12)

System of Linear Equations

Non-Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

Next Lecture

Non-Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

In Matrix Form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

$$\cancel{A^{-1} = \frac{1}{|A|}\,\text{adj}\,A}$$

Augmented Matrix

$$[A : B] = \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

A System of non-Homogenous Linear Equations
$AX = B$

Find $\rho(A)$ and $\rho[A:B]$

$\rho(A) = \rho[A:B]$ → **Solution exists, system is consistent**

- $\rho(A) = \rho[A:B] = n$ (no. of unknowns) → **Unique Solution**
- $\rho(A) = \rho[A:B] < n$ (no. of unknowns) → **Infinite Solution** → consider arbitrary constants $= n - r$

$r = \text{rank}$

$\rho(A) \neq \rho[A:B]$ → **No Solution, system is inconsistent**

Q.1 Solve, with the help of matrices, the simultaneous equations:

$$x + y + z = 3,\ x + 2y + 3z = 4,\ x + 4y + 9z = 6.$$

$x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + 9z = 6$

In Matrix Form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$AX = B$$

Augmented Matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 1 & 4 & 9 & : & 6 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-R_1)$

$R_3 \longrightarrow R_3 + (-R_1)$

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 8 & : & 3 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + (-3R_2)$

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 2 & : & 0 \end{bmatrix}$$

$\rho(A) = 3$

$\rho(A : B) = 3$

$n = 3$

$\therefore \rho(A) = \rho(A:B) = n(\text{Unknown})$

The given system is consistent and have unique solution

$$x + y + z = 3 \quad \text{——} \ (1)$$

$$y + 2z = 1 \quad \text{——} \ (2)$$

$$2z = 0 \quad \text{——} \ (3)$$

$$\boxed{\begin{aligned} z &= 0 \\ y &= 1 \\ x &= 2 \end{aligned}}$$

Q.2 Test the consistency for the following system of equations and if system is consistent, solve them :

$$x + y + z = 6, \qquad x + 2y + 3z = 14, \qquad x + 4y + 7z = 30$$

AKTU 2023

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

Augmented Matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$R_3 \to R_3 + (-3R_2)$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = 2$$
$$\rho(A:B) = 2$$
$$n = 3$$
$$\rho(A) = \rho(A:B) < n$$

The given system is consist and have infinite solutions

$$x+y+z=6 \quad \text{——} \quad ①$$

$$y+2z=8 \quad \text{——} \quad ②$$

Let $z=k$

$$y+2k=8$$

$$y=8-2k$$

$$x+8-2k+k=6$$

$$x+8-k=6$$

$$x=k-2$$

$$\boxed{\begin{aligned} z &= k \\ y &= 8-2k \\ x &= k-2 \end{aligned}}$$

Q.3 Show that the system of equations $x + y + z = -3,\ 3x + y - 2z = -2,\ \ 2x + 4y + 7z = 7$ is not consistent.

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$

Augmented Matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_2 \longrightarrow R_2 + (-3R_1)$

$R_3 \longrightarrow R_3 + (-2R_1)$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \longrightarrow R_3 + R_2$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$\rho(A) = 2$

$\rho(A:B) = 3$

$\rho(A) \neq \rho(A:B)$

The given system is inconsistent and has no solution

Q.4 Investigate, for what values of λ and μ do the system of equations

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu$$

have (i) no solution (ii) unique solution (iii) infinite solutions?

AKTU 2016, 2018

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

Augmented Matrix

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) For no solution

$\lambda - 3 = 0$

$\boxed{\lambda = 3}$

$\mu - 10 \neq 0$

$\boxed{\mu \neq 10}$

(II) Unique solution

$$\lambda - 3 \neq 0$$

$$\boxed{\lambda \neq 3}$$

μ can have any value

(III) Infinite solution

$$\lambda - 3 = 0$$

$$\boxed{\lambda = 3}$$

$$\mu - 10 = 0$$

$$\boxed{\mu = 10}$$

Q.5 Apply the matrix method to solve the system of equations

$$x_1 - 2x_2 + x_3 - x_4 + 1 = 0 \qquad 3x_1 - 2x_3 + 3x_4 + 4 = 0 \qquad 5x_1 - 4x_2 + x_4 + 3 = 0$$

$$x_1 - 2x_2 + x_3 - x_4 = -1$$

$$3x_1 + 0x_2 - 2x_3 + 3x_4 = -4$$

$$5x_1 - 4x_2 + 0x_3 + x_4 = -3$$

Augmented Matrix

$$[A:B] = \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{array}\right]$$

$R_2 \longrightarrow R_2 + (-3R_1)$

$R_3 \longrightarrow R_3 + (-5R_1)$

$$[A:B] = \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 6 & -5 & 6 & 2 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_2$

$$[A:B] = \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$\rho(A) = 2$

$\rho(A:B) = 3$

$\rho(A) \neq \rho(A:B)$

Given system is inconsistent and has no solution

Q.6 For what value of k, the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case.

$$x + y + z = 1$$

$$2x + y + 4z = k$$

$$4x + y + 10z = k^2$$

Augmented Matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix}$$

$$R_2 \to R_2 + (-2R_1)$$

$$R_3 \to R_3 + (-4R_1)$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2-4 \end{bmatrix}$$

$$R_3 \to R_3 + (-3R_2)$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix}$$

The given system has solution

$$K^2 - 3K + 2 = 0$$

$$(K-1)(K-2) = 0$$

$$K = 1, 2$$

CASE-1 $(K=1)$

$$x + y + z = 1 \quad \text{--- (1)}$$

$$-y + 2z = -1 \quad \text{--- (2)}$$

Let $\boxed{z = a}$

$$-y + 2a = -1$$

$$-y = -2a - 1$$

$$\boxed{y = 2a + 1}$$

From (1)

$$x + 2a + 1 + a = 1$$

$$x + 3a + 1 = 1$$

$$\boxed{x = -3a}$$

CASE-2 $(K=2)$

$$x + y + z = 1 \quad \text{--- (3)}$$

$$-y + 2z = 0 \quad \text{--- (4)}$$

Let $\boxed{z = b}$

$$-y + 2b = 0$$

$$\boxed{y = 2b}$$

$$x + 2b + b = 1$$

$$\boxed{x = -3b + 1}$$

Engineering Mathematics -1 by Gulshan Sir

Matrix : DPP-6

Topic : Solution of Non-Homogenous System of Linear Equations

Q.1 Solve, the system of equations using matrix method:

$2x_1 + x_2 + 2x_3 + x_4 = 6,$ $6x_1 - 6x_2 + 6x_3 + 12x_4 = 36,$ Ans $x_1 = 2,\ x_2 = 1, x_3 = -1\ x_4 = 3$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1,$ $2x_1 + 2x_2 - x_3 + x_4 = 10.$

Q.2 Test for consistency and hence, solve the following set of equations :

$10y + 3z = 0$
$3x + 3y + z = 1$
$2x - 3y - z = 5$
$x + 2y = 4$

Ans Inconsistent, no solution exists

Q.3 Determine the values of ʎ and μ such that the system

$2x - 5y + 2z = 8$
$2x + 4y + 6z = 5$
$x + 2y + \lambda z = \mu$

have (i) no solution (ii) a unique solution (iii) infinite solutions

Ans (i) ʎ=3 and μ≠5/2 (ii) ʎ ≠ 3 and μ may have any value (iii) ʎ=3 and μ=5/2

Q .4 For what values of λ and μ, the system of linear equations **AKTU 2020**

$$x + y + z = 6$$
$$x + 2y + 5z = 10$$
$$2x + 3y + \lambda z = \mu$$

has (i) a unique solution (ii) no solution (iii) infinite solutions also find the solution for $\lambda = 2$ and $\mu = 8$.

Ans.4 (i) $\lambda \neq 6$, μ is arbitrary (ii) $\lambda = 6$, $\mu \neq 16$ (iii) $\lambda = 6$, $\mu = 16$

when $\lambda = 2$, $\mu = 8$, solution is $x = 8$, $y = -4$ and $z = 2$.

Q.5 Show that the equations

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

Have no solution unless a + b + c = 0. In which case they have infinitely many solutions? Find these solution when a = 1, b = 1, c = -2

Ans

Q .6 Solve by calculating the inverse by elementary row operations: **AKTU 2015**

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 - x_4 = 4$$
$$x_1 + x_2 - x_3 + x_4 = -4$$
$$x_1 - x_2 + x_3 + x_4 = 2$$

$X = A^{-1}B$

Ans.5 $x_1 = 1,\quad x_2 = -1,\quad x_3 = 2,\quad x_4 = -2$

GATEWAY CLASSES

AKTU : B.Tech (Semester – 1)

GATEWAY CLASSES

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 7

Today's target

- A system of Homogenous Linear Equations
- PYQs
- DPP

By Gulshan sir

System of Linear Equations

Non-Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

Homogenous System of Linear Equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

In Matrix Form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

A System of Homogenous Linear Equations
$AX = 0$

Always has a Solution
(or system is consistent)

Find $\rho(A)$

$\rho(A) = n$(no. of unknowns)

Unique or trivial Solution

Zero Solution

$\rho(A) < n$(no. of unknowns)

Infinite no. of non trivial Solution

Non-Zero Solution

consider Arbitrary constants $= n - r$

Q.1 Test whether the following system of equations possess a non-trivial solution:

$$x_1 + x_2 + 2x_3 + 3x_4 = 0 \qquad 3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$$

$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0 \qquad 6x_1 + 8x_2 + 13x_3 + 16x_4 = 0.$$

$$x_1 + x_2 + 2x_3 + 3x_4 = 0$$
$$3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$$
$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$$
$$6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$$

In Matrix Form

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \boxed{1} & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + (-3R_1) \\ R_3 \to R_3 + (-5R_1) \\ R_4 \to R_4 + (-6R_1) \end{array}$$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + (-2R_2) \\ R_4 \to R_4 + (-2R_2) \end{array}$$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$R_4 \longrightarrow R_4 + (-R_3)$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Echelon Form

$\rho(A) = 4 =$ No. of Unknown

No, The given system has unique solution or trivial solution or zero solution

$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}}$$

Q.2 Solve the system of homogenous equations :

$x_1 + x_2 + x_3 + x_4 = 0 \qquad x_1 + 3x_2 + 2x_3 + 4x_4 = 0 \qquad 2x_1 + x_3 - x_4 = 0$

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$x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + 3x_2 + 2x_3 + 4x_4 = 0$

$2x_1 + 0x_2 + x_3 - x_4 = 0$

In Matrix Form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-R_1)$

$R_3 \longrightarrow R_3 + (-2R_1)$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \boxed{2} & 1 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ No. of Unknown

The given system of equation has Infinite non-trivial solution

$x_1 + x_2 + x_3 + x_4 = 0 \quad — (1)$

$2x_2 + x_3 + 3x_4 = 0 \quad — (2)$

Let $\boxed{x_3 = K_1}$

$\boxed{x_4 = K_2}$

Put x_3 and x_4 in (2)

$2x_2 + K_1 + 3K_2 = 0$

$$\boxed{x_2 = -\frac{1}{2}(K_1 + 3K_2)}$$

Put x_2, x_3 and x_4 in (1)

$$\boxed{x_1 = \frac{1}{2}(K_1 - K_2)}$$

Q.3 Determine k such that the system of homogenous equations

$$2x + y + 2z = 0,\ \ x + y + 3z = 0,\ \ 4x + 3y + kz = 0$$

has (i) trivial solution (ii) non trivial solutions. Find the non trivial solutions using matrix method.

$2x + y + 2z = 0$

$x + y + 3z = 0$

$4x + 3y + kz = 0$

In Matrix Form

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$A \sim \begin{bmatrix} \boxed{1} & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & k \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 + (-4R_1)$

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & k-12 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + (-R_2)$

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & k-8 \end{bmatrix}$$

Echelon Form

(i) For trivial solution

$$K-8 \neq 0$$

$$\boxed{K \neq 8}$$

(ii) Non trivial solution

$$K-8 = 0$$

$$\boxed{K = 8}$$

Put $K=8$ in Echelon Form

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + y + 3z = 0 \quad \text{——} \; (1)$$

$$-y - 4z = 0 \quad \text{——} \; (2)$$

Let
$$\boxed{\begin{aligned} z &= K \\ y &= -4K \\ x &= K \end{aligned}}$$

Q .4 Find the values of λ for which the equations

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0, \qquad (\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$

$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

have a non-trivial solution. Also find the solution in each case.

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$

$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$

$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

In Matrix Form

$$\begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A=\begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{bmatrix}$$

For Non trial solution

$$|A|=0$$

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix}=0$$

$C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 6\lambda & 3\lambda+1 & 2\lambda \\ 6\lambda & 4\lambda-2 & \lambda+3 \\ 6\lambda & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

Taking 6λ common from C_1

$$6\lambda \begin{vmatrix} 1 & 3\lambda+1 & 2\lambda \\ 1 & 4\lambda-2 & \lambda+3 \\ 1 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

$R_2 \to R_2 + (-R_1)$

$R_3 \to R_3 + (-R_1)$

$$6\lambda \begin{vmatrix} 1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & -\lambda+3 \\ 0 & 0 & \lambda-3 \end{vmatrix} = 0$$

Expand along C_1

$$6\lambda\left[(\lambda-3)(\lambda-3) - 0\right] = 0$$

$$6\lambda(\lambda-3)(\lambda-3) = 0$$

$$\boxed{\lambda = 0, 3}$$

CASE-1 When $\lambda = 0$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

$$A \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

$\rho(A) = 2 <$ No. of unknown

Non-trivial solution

$-x + y = 0$ —— (1)

$-3y + 3z = 0$ —— (2)

Let $z = K$

From (1) and (2)

$y = K$

$x = K$

$$\boxed{\begin{aligned} x &= K \\ y &= K \\ z &= K \end{aligned}}$$

CASE-3 When $\lambda = 3$

$$A = \begin{bmatrix} \boxed{2} & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-R_1)$

$R_3 \longrightarrow R_3 + (-R_1)$

$$A \sim \begin{bmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 1 <$ No. of Unknown

Non-trivial solution

$2x + 10y + 6z = 0$ ——— (1)

Let $\boxed{y = k_1}$

$\boxed{z = k_2}$

Put y and z in (1)

$2x + 10k_1 + 6k_2$

$$\boxed{x = 5k_1 + 3k_2}$$

Matrix : DPP-7

Topic : A system of Homogenous Linear Equations

Q.1 Solve the equations using matrix method:

$x_1 + 3x_2 + 2x_3 = 0,\quad 2x_1 - x_2 + 3x_3 = 0,\quad 3x_1 - 5x_2 + 4x_3 = 0,\quad x_1 + 17x_2 + 4x_3 = 0.$ Ans non trivial

Q.2 Find the value of k so that the equations : $x + y + 3z = 0,\ 4x + 3y + kz = 0,\ 2x + y + 2z = 0$ have a non trivial solutions. Ans k = 8

Q.3 Find the values of k for which the system of equations: AKTU 2022

$$(3k-8)\,x + 3y + 3z = 0$$

$$3x + (3k-8)y + 3z = 0$$

$$3x + 3y + (3k-8)z = 0$$

has a non-trivial solution. Ans K = 2/3 , 11/3 , 11/3

Q.4 Find the values of λ for which the equations AKTU 2015

$$x + (\lambda+4)y + (4\lambda+2)z = 0$$

$$x + 2(\lambda+1)y + (3\lambda+4)z = 0$$

$$2x + 3\lambda y + (3\lambda+4)z = 0$$

have a non-trivial solution. Also find the solution in each case. Ans λ = 2 , -2

AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 8

Today's target

- Linear Dependence and Linear Independence of Vectors
- PYQs
- DPP

By Gulshan sir

AKTU : Engineering Mathematics-I

UNIT-1 : Matrices

- Inverse of a matrix by Elementary transformations (L-1)
- Rank of matrix
  - (i) By determinant method (L-2)
  - (ii) By echelon form method (L-3)
  - (iii) By normal form method (L-4)
  - (iv) By normal form method (PAQ Form) (L-5)
- Solution of system of linear equations
  - (i) Non Homogenous System of linear equations (L-6)
  - (ii) Homogenous System of linear equations (L-7)
- Linear Dependence and Independence of vectors (L-8)
- Complex Matrices (L-9)
  - (i) Hermitian
  - (ii) Skew-Hermitian
  - (iii) Unitary Matrices
- Eigen values and Eigen vectors (L-10 + L-11)
- Cayley Hamilton Theorem (L-12)
- Applications to Engineering problems (L-12)

Linear Dependence and Linear Independence of Vectors

WORKING RULE

Step – 1 : Construct coefficient matrix A with elements of given vectors as columns.

Step – 2 : Find $\rho(A)$

Step – 3 : (i) If $\rho(A)$ < no. of vectors

Then the given set of vectors is linearly dependent.

A set of n Vectors(Matrices) $X_1, X_2 \ldots\ldots X_n$ is said to be linearly dependent if there exist n scalars $k_1, k_2, k_3 \ldots\ldots k_n$, not at all zero, such that

$$\boxed{k_1X_1 + k_2X_2 + \ldots\ldots k_nX_n = 0}$$

$\boxed{\text{(ii) If } \rho(A) = \text{no. of vectors}}$

Then the given set of vectors is linearly independent.

A set of n Vectors(Matrices) $X_1, X_2 \ldots\ldots X_n$ is said to be linearly independent if $k_1 = k_2 = k_3 = \ldots\ldots k_n = 0$

Q.1 Find whether or not the following vectors are linearly dependent or independent $(1, 1, 1, 1)$, $(0, 1, 1, 1)$, $(0, 0, 1, 1)$, $(0, 0, 0, 1)$.

Let

✓ $X_1 = (1, 1, 1, 1)$

✓ $X_2 = (0, 1, 1, 1)$

✓ $X_3 = (0, 0, 1, 1)$

✓ $X_4 = (0, 0, 0, 1)$

coefficient Matrix

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$R_4 \rightarrow R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_4 \longrightarrow R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is an Echelon Form

$\rho(A) = 4$ (= No. of vectors)

Hence, the given set of vectors is linearly independent

Here

$K_1 = K_2 = K_3 = K_4 = 0$

Q.2 Show that the column vectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ are linearly independent.

OR

Show that the vectors $X_1 = (1, 6, 4)$, $X_2 = (0, 2, 3)$, $X_3 = (0, 1, 2)$ are linearly dependent. **AKTU 2020**

Given

$X_1 = (1, 6, 4)$

$X_2 = (0, 2, 3)$

$X_3 = (0, 1, 2)$

Coefficient Matrix

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2 + (-6R_1)$

$R_3 \to R_3 + (-4R_1)$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{2} & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + (-3R_2)$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$ Echelon Form

$\rho(A) = 3$ (= Number of vectors)

Hence,

The given set of vectors is linearly independent

Q.3 Show that the vectors $X_1 = (1, -1, 1)$, $X_2 = (2, 1, 1)$, $X_3 = (3, 0, 2)$ are linearly dependent. Find the relation between them.

AKTU 2013

$X_1 = (1, -1, 1)$

$X_2 = (2, 1, 1)$

$X_3 = (3, 0, 2)$

Coefficient Matrix

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{3} & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{3} R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

$\rho(A) = 2 <$ Number of vectors

Hence

The given set of vectors is linearly dependent

Let the relation between X_1, X_2 and X_3 is

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0 \qquad \text{---} (1)$$

K_1, K_2 and K_3 are calculated from Echelon Form

$$K_1 + 2K_2 + 3K_3 = 0 \quad \text{--- (2)}$$

$$3K_2 + 3K_3 = 0 \quad \text{--- (3)}$$

Let $\boxed{K_2 = k}$

Put K_2 in (3)

$$3k + 3K_3 = 0$$

$$\cancel{3}K_3 = -\cancel{3}k$$

$$\boxed{K_3 = -k}$$

Put K_2 and K_3 in (1)

$$K_1 + 2k - 3k = 0$$

$$K_1 - k = 0$$

$$\boxed{K_1 = k}$$

Put K_1, K_2 and K_3 in (1)

$$kX_1 + kX_2 - kX_3 = 0$$

$$kX_1 + kX_2 = kX_3$$

$$\cancel{k}(X_1 + X_2) = \cancel{k}X_3$$

$$\boxed{X_1 + X_2 = X_3}$$

Q.4 Find the value of λ for which the vectors $(1, -2, \lambda)$, $(2, -1, 5)$, and $(3, -5, 7\lambda)$ are linearly dependent.

Let

$X_1 = (1, -2, \lambda)$

$X_2 = (2, -1, 5)$

$X_3 = (3, -5, 7\lambda)$

Coeffiuent Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{bmatrix}$$

$\rho(A) < 3$

$\Rightarrow |A| = 0$

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{vmatrix} = 0$$

Expand along R_1

$1(-7\lambda + 25) - 2(-14\lambda + 5\lambda) + 3(-10 + \lambda) = 0$

$-7\lambda + 25 + \underline{28\lambda} - 10\lambda - 30 + 3\lambda = 0$

$28\lambda - 14\lambda - 5 = 0$

$14\lambda = 5$

$$\boxed{\lambda = \frac{5}{14}}$$

Matrix : DPP-8

Topic : Linear Dependence and Linear Independence of Vectors

Q.1 Examine the following vectors for linear dependence and find the relation between them, if possible:

$X_1 = (1, 1, -1, 1)$, $X_2 = (1, -1, 2, -1)$, $X_3 = (3, 1, 0, 1)$

Ans Linearly dependent; $2X_1 + X_2 = X_3 = 8$

Q.2 Show that the vectors $X_1 = (1, 2, 4)$, $X_2 = (2, -1, 3)$, $X_3 = (0, 1, 2)$ and $X_4 = (-3, 7, 2)$ are linearly dependent and find the relation between them.

Ans Linearly dependent; $9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

Q.3 Find whether or not the following vectors are linearly dependent or independent (1, 1, 0), (1, 0, 1), (0, 1, 1).

Ans $\rho(A) = 2$ (<3, no of vectors) , Hence the given set of vectors is linearly dependent.

Q.4 Show that row vectors of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 4 & -2 & 1 \end{bmatrix}$ are linearly independent

AKTU : B.Tech (Semester – 1)

GATEWAY CLASSES

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 9

Today's target

- Complex Matrices

  (i) Hermitian Matrix (ii) Skew-Hermitian Matrix (iii) Unitary Matrix

- PYQs
- DPP

By Gulshan sir

Complex Matrix

If at least one element of a matrix is a complex number $Z = \overset{\text{Real Part}}{a} + \underset{\text{Imaginary Part}}{ib}$, where a, b are real *and* $i = \sqrt{-1}$, then the matrix is called complex matrix

$$\boxed{i = \sqrt{-1}}$$

Examples

(i) $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & i \\ 3 & 1 & 2 \end{bmatrix}$ ✓

(ii) $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ ✓

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ ✗

Conjugate of Complex Number ($\bar{Z}$)

$$Z = a + ib$$

$$\boxed{\bar{Z} = a - ib}$$

Conjugate of complex Matrix ($\bar{A}$)

$$A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

Note

(i) $\bar{\bar{A}} = A$

(ii) $\overline{AB} = \bar{A}\,\bar{B}$

(iii) $\overline{A+B} = \bar{A} + \bar{B}$

(iv) $\overline{\lambda A} = \bar{\lambda}\,\bar{A}$

Transpose of Conjugate of a Matrix (A^θ or A^*)

$$\boxed{A^\theta = (\bar{A})^T}$$

Example : if $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 5 \end{bmatrix}$ then find A^θ

$$\bar{A} = \begin{bmatrix} 2 & 2-i \\ 1+i & 5 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 2 & 1+i \\ 2-i & 5 \end{bmatrix}$$

$$\boxed{A^\theta = \begin{bmatrix} 2 & 1+i \\ 2-i & 5 \end{bmatrix}}$$

Note :

(i) $(A^\theta)^\theta = A$

(ii) $(AB)^\theta = B^\theta A^\theta$

(iii) $(A+B)^\theta = A^\theta + B^\theta$

(iv) $(\lambda A)^\theta = \bar{\lambda} A^\theta$ ✓

Types of Complex Matrices

(i) Hermitian Matrix

- A Square matrix A is said to be Hermitian if

$$\boxed{A^{\theta} = A}$$

- In Hermitian matrix $A = [a_{ij}]$

$$a_{ij} = \overline{a_{ji}} \quad \text{for all } i, j$$

- The diagonal elements of a Hermitian matrix are always real

Example : $A = \begin{bmatrix} 2 & 4-i \\ 4+i & 5 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2 & 4+i \\ 4-i & 5 \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 2 & 4-i \\ 4+i & 5 \end{bmatrix}$$

$$\boxed{A^{\theta} = A}$$

Hence, A is a Hermitian matrix

(ii) Skew-Hermitian Matrix

- A Square matrix A is said to be skew-Hermitian if

$$\boxed{A^{\theta} = -A}$$

- In skew Hermitian matrix $A = [a_{ij}]$

$$a_{ij} = -\overline{a_{ji}} \quad \text{for all } i, j$$

- The diagonal elements of a skew-Hermitian matrix are either zero or purely imaginary

Example : $A = \begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 0 & -1-i \\ 1-i & 0 \end{bmatrix}$$

$$A^{\theta} = -\begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$$

$$\boxed{A^{\theta} = -A}$$

Hence, A is Skew-Hermitian

Q.1 If A is a Hermitian matrix, then show that iA is a skew-Hermitian. AKTU 2023

A is a Hermitian Matrix

$$A^\theta = A \quad \text{——} \quad (1)$$

$$(iA)^\theta = \bar{i}\, A^\theta \quad \left\{ \because (\lambda A)^\theta = \bar{\lambda}\, A^\theta \right.$$

$$(iA)^\theta = -i\, A^\theta$$

$$\boxed{(iA)^\theta = -i\, A}$$

Henu, iA is a skew Hermitian matrix

Q.2 If A is a skew-Hermitian matrix, then show that iA is Hermitian.

Given

$$A^{\theta} = -A$$

$$(iA)^{\theta} = \bar{i}\, A^{\theta} \quad \left\{ (\lambda A)^{\theta} = \bar{\lambda}\, A^{\theta} \right.$$

$$(iA)^{\theta} = -i\, A^{\theta}$$

$$(iA)^{\theta} = (-i)(-A)$$

$$(iA)^{\theta} = iA$$

Hence, iA is a Hermitian matrix

Q.3 Show that the matrix A is Hermitian and iA is skew- Hermitian where $A = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$

To show that A is Hermitian

$$A = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$$

$$\boxed{A^{\theta} = A}$$

Hence, A is a Hermitian matrix

To show that iA is skew-Hermitian

$$iA = i\begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$$

$$iA = \begin{bmatrix} 2i & 3i-4i^2 \\ 3i+4i^2 & 2i \end{bmatrix}$$

$$iA = \begin{bmatrix} 2i & 4+3i \\ -4+3i & 2i \end{bmatrix} \quad — (1)$$

$$\overline{iA} = \begin{bmatrix} -2i & 4-3i \\ -4-3i & -2i \end{bmatrix}$$

$$(\overline{iA})' = \begin{bmatrix} -2i & -4-3i \\ 4-3i & -2i \end{bmatrix}$$

$$(iA)^{\theta} = -\begin{bmatrix} 2i & 4+3i \\ -4+3i & 2i \end{bmatrix}$$

$$(iA)^{\theta} = -(iA)$$

Hence iA is a skew Hermitian matrix

An important property

Every Square matrix can be uniquely expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix

Let A be any Square matrix

$$A = \frac{1}{2}A + \frac{1}{2}A$$

$$A = \frac{1}{2}A + \frac{1}{2}A^{\theta} + \frac{1}{2}A - \frac{1}{2}A^{\theta}$$

$$A = \underbrace{\frac{1}{2}(A+A^{\theta})}_{P} + \underbrace{\frac{1}{2}(A-A^{\theta})}_{Q}$$

$$A = \underbrace{P}_{\text{Hermitian Matrix}} + \underbrace{Q}_{\text{Skew-Hermitian Matrix}}$$

$$\boxed{P = \frac{1}{2}(A+A^{\theta}) \qquad Q = \frac{1}{2}(A-A^{\theta})}$$

Q.4 Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$, as a sum of Hermitian and Skew-Hermitian matrix

$$A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$A + A^\theta = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$A + A^\theta = \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

$$P=\begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 4 \end{bmatrix}$$

P is a Hermitian matrix

$$A-A^{\theta}=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}-\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$A-A^{\theta}=\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$Q=\frac{1}{2}(A-A^{\theta})$$

$$Q=\begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & 2i \end{bmatrix}$$

Q is a skew-Hermitian

$$A=P+Q$$

Hence,
A can be expressed as a sum of Hermitian and skew Hermitian

3. Unitary Matrix

A Square matrix A is said to be unitary if

$$\boxed{A^{\theta} A = AA^{\theta} = I}$$

$$AA^{\theta} = I$$

$$A^{-1}AA^{\theta} = A^{-1}I$$

$$IA^{\theta} = A^{-1}$$

$$A^{\theta} = A^{-1}$$

$$\boxed{A^{-1} = A^{\theta}}$$

Q.5 Show that the following matrix is unitary: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

AKTU 2021

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AA^\theta = \frac{1}{3}\begin{bmatrix} 1+1^2-i^2 & \cancel{1}+\cancel{i}-\cancel{1}-\cancel{i} \\ \cancel{1}-\cancel{i}-\cancel{1}+\cancel{i} & 1^2-i^2+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+1+1 & 0 \\ 0 & 1+1+1 \end{bmatrix}$$

$$AA^\theta = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{\cancel{3}} \times \cancel{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{AA^\theta = I}$$

Hence,
A is unitary matrix

Q.6 Show that the matrix $\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}$ is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$ AKTU 2018

$$A=\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}\alpha-i\gamma & -\beta-i\delta \\ \beta-i\delta & \alpha+i\gamma\end{bmatrix}$$

$$A^\theta=\begin{bmatrix}\alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma\end{bmatrix}$$

$$AA^\theta=\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}\begin{bmatrix}\alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma\end{bmatrix}$$

$$AA^\theta=\begin{bmatrix}\alpha^2+\gamma^2+\beta^2+\delta^2 & \cancel{(\alpha+i\gamma)(\beta-i\delta)}-\cancel{(\beta-i\delta)(\alpha+i\gamma)} \\ \cancel{(\beta+i\delta)(\alpha-i\gamma)}-\cancel{(\beta+i\delta)(\alpha-i\gamma)} & \beta^2+\delta^2+\alpha^2+\gamma^2\end{bmatrix}$$

$$AA^\theta=\begin{bmatrix}\alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2\end{bmatrix}$$

If $\alpha^2+\beta^2+\gamma^2+\delta^2=1$

$$AA^\theta=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}=I$$

Hence, A is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$

Complex Cube Root of Unity (ω)

$$x = (1)^{1/3}$$

$$x^3 = 1$$

$$x^3 - 1 = 0$$

$$x^3 - 1^3 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$$x = 1$$

$$x^2 + x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$x = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence

$$x = 1$$

$$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = \omega$$

$$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \omega^2$$

Note

$$\overline{\omega} = \omega^2$$

$$\overline{\omega^2} = \omega$$

$$\omega^3 = 1$$

$$1 + \omega + \omega^2 = 0$$

$\omega^4 = \omega\,\omega^3$

Q.7 Show that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where ω is complex cube root of unity **AKTU 2017**

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}\frac{1}{\sqrt{3}}$$

$$AA^\theta = \frac{1}{3}\begin{bmatrix} 1+1+1 & 1+\omega^2+\omega & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega^3+\omega^3 & 1+\omega^2+\omega^4 \\ 1+\omega^2+\omega & 1+\omega^4+\omega^2 & 1+\omega^3+\omega^3 \end{bmatrix}$$

$$AA^\theta = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \frac{\cancel{3}}{\cancel{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{AA^\theta = I}$$

Hence, A is unitary matrix

Matrix : DPP-9

Topic : Complex Matrices

Q .1 Show that the matrix A is Hermitian and iA is skew- Hermitian where A is $\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ AKTU 2014

Q .2 If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, verify that A*A is a Hermitian matrix where A* is the conjugate transpose of A.

Q .3 Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as P + iQ , where P is a real symmetric and Q is real skew-symmetric .

Q.4 Show that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary. Also find A^{-1}

GATEWAY CLASSES

AKTU : B.Tech (Semester – 1)

GATEWAY CLASSES

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture -10

Today's target

- Characteristics Equations
- Eigen Values
- Properties of Eigen Values
- PYQs
- DPP

By Gulshan sir

Characteristic Equation

For 2 X 2 order matrix A

Method - 1

$$|A - \lambda I| = 0$$

Expand the determinant

Method - 2

$$|A - \lambda I| = 0$$

$$\boxed{\lambda^2 - S\lambda + |A| = 0}$$

Where

$S =$ Trace of the matrix

$|A| =$ Determinant of the matrix

For 3 X 3 order matrix A

Method - 1

$$|A - \lambda I| = 0$$

Expand the determinant

Method - 2

$$|A - \lambda I| = 0$$

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0}$$

Where

$S_1 =$ Trace of the matrix

$S_2 = M_{11} + M_{22} + M_{33}$

$|A| =$ Determinant of the matrix

Eigen Values OR Latent root OR Characteristic roots : Eigen values are the roots of Characteristic Equation

Q.1 Find the eigen values of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Method-1

$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

Expand

$(5-\lambda)(2-\lambda) - 4 = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\boxed{\lambda^2 - 7\lambda + 6 = 0}$

Eigen values

$\boxed{\lambda = 6, 1}$

Method-2

$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$\lambda^2 - S\lambda + |A| = 0$

S = Trace of matrix
$= 7$

$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix}$
$= 10 - 4$
$= 6$

$\boxed{\lambda^2 - 7\lambda + 6 = 0}$

Eigen values

$\boxed{\lambda = 6, 1}$

$ax^2+bx+c=0$

Q. 2 Find the eigen values of the matrix $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$A=\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$|A-\lambda I|=0$$

$$\lambda^2-S\lambda+|A|=0$$

S = Trace of Matrix

$$= -5-2$$

$$= -7$$

$$|A|=\begin{vmatrix} -5 & 2 \\ 2 & -2 \end{vmatrix}$$

$$|A|=(-5)(-2)-4$$

$$=10-4$$

$$=6$$

$$\lambda^2-(-7)\lambda+6=0$$

$$\boxed{\lambda^2+7\lambda+6=0}$$

Eigen values

$$\boxed{\lambda=-1,-6}$$

$ax^3+bx^2+cx+d=0$

Q.3 Find the eigen values of the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

$$|A-\lambda I| = 0$$

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0} \quad (1)$$

$S_1 =$ Trace of matrix

$$S_1 = 1+2-3$$

$$S_1 = 0$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 3 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

$$S_2 = -6-0+(-3-12)+(2-0)$$

$$S_2 = -6-15+2$$

$$S_2 = -19$$

$$|A| = \begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{vmatrix}$$

Expand along R_1

$$|A| = 1(-6-0)-0+4(0-6)$$

$$|A| = -6-24$$

$$|A| = -30$$

$$\lambda^3 - 19\lambda + 30 = 0$$

Eigen Values

$$\boxed{\lambda = 3, -5, 2}$$

Some important properties of eigen values

1. The sum of eigen values of a matrix is equal to trace of the matrix.

2. Determinant of matrix is equal to the product of eigen values.

3. If $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots \lambda_n$ are the eigen values of A , then the eigen values of

(i) kA are $k\lambda_1, k\lambda_2, k\lambda_3 \ldots k\lambda_n$

(ii) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m \ldots \lambda_n^m$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots \frac{1}{\lambda_n}$

4. Any square matrix A and its transpose A' have the same Eigen values.

5. The Eigen values of a Hermitian matrix are all real

6. The Eigen values of a Skew-Hermitian matrix is either zero or purely an imaginary number

7. The eigen values of a diagonal matrix are just the diagonal elements.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\lambda = 2, -4, 6$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1, 1$

8. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 5 \end{bmatrix}$$

$\boxed{\lambda = 1, 4, 5}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 3 & 3 & 5 \end{bmatrix}$$

$\lambda = 1, 2, 5$

Q.4 Find the product and sum of the Eigen values for $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ **AKTU 2024**

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Product of Eigen values $= |A|$

$$= \begin{vmatrix} 8 & -4 \\ 2 & 2 \end{vmatrix}$$

$$= 16 + 8$$

$$= 24$$

Sum of Eigen values = Trace of A

$$= 8 + 2$$

$$= 10$$

Q.5 If 2 and 3 are two eigen values of matrix A and determinant of A is 24. Find the third eigen value.

Let third Eigen value $= \lambda$

Product of Eigen values $= |A|$

$2 \times 3 \times \lambda = 24$

$\lambda = \frac{\cancel{24}^{4}}{\cancel{6}}$

$\boxed{\lambda = 4}$

Q.6 Find the value of x, the Eigen values of the Given matrix A are real

$$A = \begin{bmatrix} 10 & 5+i & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

AKTU 2017

We know that

Eigen values of a Hermitian matrix are real

Hence, A is a Hermitian matrix

$\Rightarrow x$ is the conjugate of $5+i$

$\boxed{x = 5 - i}$

Q.7 Find the eigen values of A^3 where $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\boxed{\lambda^2 - S\lambda + |A| = 0}$$

S = Trace of matrix

$$S = 5$$

$$|A| = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$$

$$|A| = 6 - 2$$
$$= 4$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = 4, 1$$

Eigenvalues of $A = 1, 4$

$$A^3 = 1^3, 4^3$$
$$= 1, 64$$

Q.8 Two eigen values of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the eigen values of A^{-1}

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$\lambda_1 = 1$

$\lambda_2 = 1$

Let $\lambda_3 = \lambda$

We know that

$\lambda_1 + \lambda_2 + \lambda_3 = $ Trace of matrix

$1 + 1 + \lambda = 2 + 3 + 2$

$2 + \lambda = 7$

$\boxed{\lambda = 5}$

Hence

$\lambda_1 = 1$

$\lambda_2 = 1$

$\lambda_3 = 5$

Eigen values of $A = 1, 1, 5$

" " $A^{-1} = \frac{1}{1}, \frac{1}{1}, \frac{1}{5}$

$= 1, 1, \frac{1}{5}$

Q.9 If the matrix $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ then find the eigen values of $A^3 + 5A + 8I$ AKTU 2022

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$$

Since, A is a triangular matrix

Eigen values of A

$\lambda = -1, 3, -2$

When $\lambda = -1$

$$A^3 + 5A + 8I = (-1)^3 + 5(-1) + 8(1)$$
$$= -1 - 5 + 8$$
$$= 2$$

When $\lambda = 3$

$$A^3 + 5A + 8I = (3)^3 + 5\times 3 + 8\times 1$$
$$= 27 + 15 + 8$$
$$= 50$$

When $\lambda = -2$

$$A^3 + 5A + 8I = (-2)^3 + 5(-2) + 8(1)$$

$$A^3 + 5A + 8I = -18 + 8$$
$$= -10$$

Hence

Eigen values are

$= 2, 50, -10$

Q.10 If the Eigen values of matrix A are 1, 1, 1 then find the Eigen values of $A^2 + 2A + 3I$ AKTU 2019

Eigen values of A

$\lambda = 1, 1, 1$

When $\lambda = 1$

$A^2 + 2A + 3I = (1)^2 + 2(1) + 3(1)$

$= 1 + 2 + 3$

$= 6$

Hence

Eigen values of $A^2 + 2A + 3I$

$= 6, 6, 6$

Matrix : DPP-10

Topic : Eigen Values

Q .1 Find the eigen values of the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Ans 0, 1, -2

Q .2 Find the eigen values of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Ans 1, 2, 3

Q.3 The matrix A is defined as $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$

Ans 4, 110, 10

Q .4 The matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ is given. Find the eigen values of $4A^{-1} + 3A + 2I$

Ans 6, 15

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 11

Today's target

- Eigen Values and Eigen Vectors
- PYQs
- DPP

By Gulshan sir

Eigen vectors corresponding to an Eigen value is not unique(i.e. Eigen vectors are infinite)

Q.1 Find the eigen values and eigen vectors the matrix $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^2 - S\lambda + |A| = 0$$

S = Trace of matrix

$$S = -5 - 2$$

$$S = -7$$

$$|A| = 10 - 4$$

$$|A| = 6$$

$$\boxed{\lambda^2 + 7\lambda + 6 = 0}$$

$$\lambda = -1, -6$$

Non-zero eigen vectors corresponding to eigen values

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

When $\lambda = -1$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \longrightarrow 2R_2 + R_1$$

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 = 0$$

Let $\boxed{x_1 = K_1}$

$$-4K_1 + 2x_2 = 0$$

$$2x_2 = 4K_1$$

$$\boxed{x_2 = 2K_1}$$

Eigen vector

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} K_1 \\ 2K_1 \end{bmatrix}$$

$$\boxed{X_1 = K_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$K_1 \in R$

When $\lambda = -6$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + (-2R_1)$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$

Let $x_2 = K_2$

$$x_1 = -2K_2$$

Eigen vector

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2K_2 \\ K_2 \end{bmatrix}$$

$$\boxed{X_2 = K_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Q.2 Find the Eigen values and corresponding Eigen Vectors of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$

AKTU 2023

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = Trace of matrix

$$= \cancel{2} + 3 - \cancel{2}$$

$$\boxed{S_1 = 3}$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = \begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$$S_2 = -6 + 4 + (-4 + 1) + (6 - 2)$$

$$S_2 = -2 - 3 + 4$$

$$\boxed{S_2 = -1}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{vmatrix}$$

$$|A| = 2(-6+4) - 1(-4+4) + 1(-2+3)$$

$$|A| = -4 + 1$$

$$\boxed{|A| = -3}$$

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\boxed{\lambda = 3, -1, 1}$$

Eigen vectors corresponding to eigen values

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

When $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 4 \\ -1 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1 \quad R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \quad \text{— (2)}$$

$$2x_2 + 6x_3 = 0 \quad \text{— (3)}$$

Let $\boxed{x_3 = K_1}$

$$2x_2 + 6K_1 = 0$$

$$2x_2 = -6K_1$$

$$\boxed{x_2 = -3K_1}$$

From (2)

$$-x_1 - 3K_1 + K_1 = 0$$

$$-x_1 - 2K_1 = 0$$

$$\boxed{x_1 = -2K_1}$$

Eigen vectors

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -2K_1 \\ -3K_1 \\ K_1 \end{bmatrix}$$

$$\boxed{X_1 = K_1 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}}$$

$K_1 \in R$

When $\lambda = -1$

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_3$

$$\begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1 \quad R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_2 - x_3 = 0 \quad \text{--- (3)}$$

$$2x_2 + 2x_3 = 0 \quad \text{--- (4)}$$

Let $\boxed{x_3 = K_2}$

$$\boxed{x_2 = -K_2}$$

$$-x_1 + \cancel{K_2} - \cancel{K_2} = 0 \qquad \boxed{x_1 = 0}$$

Eigen vector

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 \\ -K_2 \\ K_2 \end{bmatrix}$$

$$\boxed{X_2 = K_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

$K_2 \in R$

When $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + (-2R_1)$

$R_3 \longrightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$ — (5)

$2x_3 = 0$ — (6)

$-2x_3 = 0$ — (7)

$$\boxed{x_3 = 0}$$

From (5)

$x_1 + x_2 + 0 = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$$\frac{x_1}{1} = \frac{x_2}{-1} = K_3$$

$$\boxed{\begin{aligned} x_1 &= K_3 \\ x_2 &= -K_3 \end{aligned}}$$

Eigen vector

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} K_3 \\ -K_3 \\ 0 \end{bmatrix}$$

$$\boxed{X_3 = K_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

Q.3 Find the Eigen values and corresponding Eigen Vectors of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **AKTU 2024**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = Trace of matrix

$$= 1 + 2 + 2$$

$$\boxed{S_1 = 5}$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = 2 + 4 + 2$$

$$\boxed{S_2 = 8}$$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$$|A| = 1(4-2) - 2(0+1) + 2(0+2)$$

$$|A| = \cancel{2} - \cancel{2} + 4$$

$$\boxed{|A| = 4}$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

Non-zero eigen vectors corresponding eigen values

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (1)}$$

When $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_3$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + (-2R_2)$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 + x_3 = 0 \quad \text{— (2)}$$

$$x_2 + x_3 = 0 \quad \text{— (3)}$$

Let $x_2 = K_1$

$$\boxed{x_3 = -K_1}$$

From (2)

$$-x_1 + 2K_1 - K_1 = 0$$

$$-x_1 + K_1 = 0$$

$$\boxed{x_1 = K_1}$$

Eigen vector

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} k_1 \\ k_1 \\ -k_1 \end{bmatrix}$$

$$\boxed{X_1 = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

When $\lambda = 2, 2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 + 2x_3 = 0$ — (4)

$x_3 = 0$ — (5)

$-2x_3 = 0$ — (6)

From (5) and (6)

$$\boxed{x_3 = 0}$$

From (4)

$-x_1 + 2x_2 + 0 = 0$

$-x_1 + 2x_2 = 0$

$x_1 = 2x_2$

$\frac{x_1}{2} = \frac{x_2}{1} = k_2$ (say)

$$\boxed{x_1 = 2k_2}$$

$$\boxed{x_2 = k_2}$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \\ 0 \end{bmatrix}$$

$$\boxed{X_2 = k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

Number of independent vector is less than 3

Q .4 Find all the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ AKTU 2021

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = Trace of matrix

$$= 2+3+2$$

$$= 7$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = 6 + 3 + 6$$

$$S_2 = 15$$

$$|A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$|A| = 2(6-0) + 1(0-3)$$

$$|A| = 12 - 3$$

$$|A| = 9$$

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

$$\boxed{\lambda = 1, 3, 3}$$

Non-zero eigen vectors corresponding to eigen values

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—(1)}$$

When $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 \quad \text{---} \quad (2)$$

$$2x_2 = 0 \quad \text{---} \quad (3)$$

From (2)

$$\boxed{x_2 = 0}$$

Let $\boxed{x_1 = K_1}$

$$\boxed{x_3 = -K_1}$$

Eigen vector

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ 0 \\ -K_1 \end{bmatrix}$$

$$\boxed{X_1 = K_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

When $\lambda = 3, 3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_3 = 0 \quad \text{---} \quad (4)$$

Let $\boxed{x_2 = K_2}$

Let $\boxed{x_1 = k_3}$

From (4)

$\boxed{x_3 = k_3}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X = \begin{bmatrix} k_3 \\ k_2 \\ k_3 \end{bmatrix}$$

$$X = \begin{bmatrix} 0k_2 + k_3 \\ k_2 + 0k_3 \\ 0k_2 + k_3 \end{bmatrix}$$

$$X = \begin{bmatrix} 0k_2 \\ k_2 \\ 0k_2 \end{bmatrix} + \begin{bmatrix} k_3 \\ 0k_3 \\ k_3 \end{bmatrix}$$

Where

$$X_2 = \begin{bmatrix} 0k_2 \\ k_2 \\ 0k_2 \end{bmatrix}$$

$$\boxed{X_2 = k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$X_3 = \begin{bmatrix} k_3 \\ 0k_3 \\ k_3 \end{bmatrix}$$

$$\boxed{X_3 = k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

Q.5 Find the eigen value of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 51 \\ 51 \end{bmatrix}$ **AKTU 2023**

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} 51 \\ 51 \end{bmatrix}$$

We know that

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} 51 \\ 51 \end{bmatrix} = 0$$

$$51(4-\lambda) + 51 \times 2 = 0$$

$$204 - 51\lambda + 102 = 0$$

$$+51\lambda = +306$$

$$\boxed{\lambda = 6}$$

?

Matrix : DPP-11

Topic : Eigen Values and Eigen Vectors

Q .1 Find the eigen value of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ AKTU 2017

Q.2 Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Q .3 Find all the eigen values and eigen vectors of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Q .4 Find the eigen values and corresponding eigen vectors of the following matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ AKTU 2022

Q.5 Find all the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

GATEWAY CLASSES

AKTU : B.Tech (Semester – 1)

ENGINEERING MATHEMATICS - I

UNIT -1 : MATRICES

Lecture - 12

Today's target

- Cayley-Hamilton Theorem
- PYQs
- DPP

By Gulshan sir

GATEWAY CLASSES

AKTU : Engineering Mathematics-I

UNIT-1 : Matrices

- Inverse of a matrix by Elementary transformations (L-1)
- Rank of matrix
  - (i) By determinant method (L-2)
  - (ii) By echelon form method (L-3)
  - (iii) By normal form method (L-4)
  - (iv) By normal form method (PAQ Form) (L-5)
- Solution of system of linear equations
  - (i) Non Homogenous System of linear equations (L-6)
  - (ii) Homogenous System of linear equations (L-7)
- Linear Dependence and Independence of vectors (L-8)
- Complex Matrices (L-9)
  - (i) Hermitian
  - (ii) Skew-Hermitian
  - (iii) Unitary Matrices
- Eigen values and Eigen vectors (L-10 + L-11)
- Cayley Hamilton Theorem (L-12)
- Applications to Engineering problems (L-12)

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation

A

Application of Cayley-Hamilton Theorem

- To find the inverse of the matrix
- To evaluate polynomial and value of polynomial
- To find adjoint of the matrix

$$\lambda^3 - 5\lambda^2 + 4\lambda - 10 = 0$$

$$\underbrace{A^3 - 5A^2 + 4A - 10I} = 0$$

$$0 = 0$$

$$A^{-1} = \frac{1}{|A|} \text{adj}.A$$

$$\boxed{\text{adj}.A = |A| A^{-1}}$$

Q.1 Verify Cayley – Hamilton theorem for a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}.

Also evaluate $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

OR

Q.1 Verify Cayley – Hamilton theorem for a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}.

Also express the polynomial $B = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ as quadratic polynomial in A and hence find B

OR

Q.1 Verify Cayley – Hamilton theorem for a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}.

Also find matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ **AKTU 2022**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$S_1 =$ Trace of matrix

$$= 2 + 1 + 2$$

$$\boxed{S_1 = 5}$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$S_2 = 2 + 3 + 2$$

$$\boxed{S_2 = 7}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\boxed{|A| = 3}$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{(1)}$$

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A \times A^2$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10+0+4 & 8+1+4 & 8+0+5 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 5+0+8 & 4+1+8 & 4+0+10 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} + \begin{bmatrix} -25 & -20 & -20 \\ 0 & -5 & 0 \\ -20 & -20 & -25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LH = RHS

Hence

Cayley-Hamiton theorem is

Verified

From ①

$$A^3 - 5A^2 + 7A - 3I = 0$$

Pre-multiply by A^{-1}

$$A^{-1}A^3 - 5A^{-1}A^2 + 7A^{-1}A - 3A^{-1}I = 0$$

$$A^{-1}AA^2 - 5A^{-1}AA + 7A^{-1}A - 3A^{-1} = 0$$

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 5A + 7I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -10 & -5 & -5 \\ 0 & -5 & 0 \\ -5 & -5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}}$$

$$\underbrace{A^8 - 5A^7 + 7A^6 - 3A^5} + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + \underbrace{A^4 - 5A^3} + 8A^2 - 2A + I$$

$$= A^5 \times 0 + A\left(A^3 - 5A^2\right) + 8A^2 - 2A + I$$

$$= 0 + A\left(-7A + 3I\right) + 8A^2 - 2A + I$$

$$= -7A^2 + 3A + 8A^2 - 2A + I$$

$$= A^2 + A + I$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q.2 Determine $A^{-1},\ A^{-2},\ A^{-3}$ if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ by using Caley – Hamilton theorem

AKTU 2024

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = \text{Trace of matrix}$$
$$= 4 + 3 - 3$$
$$\boxed{S_1 = 4}$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$S_2 = \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$S_2 = (-9+8) + (-12+6) + (12-6)$$

$$S_2 = -1 - 6 + 6$$

$$\boxed{S_2 = -1}$$

$$|A| = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix}$$

$$|A| = 4(-9+8) - 6(-3+2) + 6(-4+3)$$

$$|A| = -4 + 6 - 6$$

$$\boxed{|A| = -4}$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

By Cayley-Hamilton theorem

$$A^3 - 4A^2 - A + 4 = 0 \quad \text{—— (1)}$$

Pre-multiply by A^{-1}

$$A^2 - 4A - I + 4A^{-1} = 0$$

$$4A^{-1} = -A^2 + 4A + I \quad \text{—— (2)}$$

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6-6 & 24+18-24 & 24+12-18 \\ 4+3-2 & 6+9-8 & 6+6-6 \\ -4-4+3 & -6-12+12 & -6-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix}$$

Put A^2, A and I in (2)

$$4A^{-1} = -\begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix} + 4\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} -16 & -18 & -18 \\ -5 & -7 & -6 \\ 5 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 16 & 24 & 24 \\ 4 & 12 & 8 \\ -4 & -16 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}}$$

Note

$A^{-2} = A^{-1} \times A^{-1}$

$A^{-3} = A^{-1} \times A^{-2}$

$$A^3 - 4A^2 - A + 4 = 0 \quad \text{—} (1)$$

Premultiply (1) by A^{-2}

$$A - 4I - A^{-1} + 4A^{-2} = 0$$

$$4A^{-2} = -A + 4I + A^{-1}$$

$$16A^{-2} = -4A + 16I + 4A^{-1}$$

$$16A^{-2} = -4\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + 16\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -24 & -24 \\ -4 & -12 & -8 \\ 4 & 16 & 12 \end{bmatrix} + \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

$$16A^{-2} = \begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$$

$$\boxed{A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}}$$

$$A^3 - 4A^2 - A + 4 = 0 \quad \text{--- (1)}$$

Premultiply (1) by A^{-3}

$$I - 4A^{-1} - A^{-2} + 4A^{-3} = 0$$

$$4A^{-3} = A^{-2} + 4A^{-1} - I$$

$$64A^{-3} = 16A^{-2} + 16 \times 4A^{-1} - 16I$$

$$64A^{-3} = \begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix} + 16\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix} - 16\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$64A^{-3} = \begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix} + \begin{bmatrix} 16 & 96 & 96 \\ -16 & 96 & 32 \\ 16 & -160 & -96 \end{bmatrix} + \begin{bmatrix} -16 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & -16 \end{bmatrix}$$

$$64A^{-3} = \begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$$

$$\boxed{A^{-3} = \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}}$$

Application of matrices to engineering problems

- Graph theory
- Computer Graphics
- Electric circuits with resistors and applied voltages
- A compound mass spring system

Matrix : DPP-12

Topic : Cayley-Hamilton Theorem

Q .1 Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and find A^{-1} **AKTU 2020.**

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -8 \\ -1 & 0 & 4 \end{bmatrix}$$

Q.2 Verify Cayley-Hamilton theorem, for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence compute A^{-1}.

Also evaluate $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$ **AKTU 2016, 2018, 2023**

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \qquad 5A - I = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$$

Q .3 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ find A^{-1} and A^4 using Cayley-Hamilton theorem. Also show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Q.4 Using Cayley-Hamilton theorem, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Also express the polynomial B = $A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$ as quadratic polynomial in A and hence find B

AKTU 2015, 2019

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$B = A^2 + A + I = \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{bmatrix}$$

Q .5 Express the polynomial $2A^5 - 3A^4 + A^2 - 4I$ as linear polynomial in A where $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ **AKTU 2017**

$138A - 403I$

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Thank You