$\therefore S(a) \rightarrow P(a)$

$\therefore D(a) \wedge P(a)$

$\therefore (\exists x) [D(x) \wedge P(x)]$

Hence the result.

Example 19 : Draw all Hasse diagrams of posets with three elements. **MU - Dec. 09**

Solution :

Following diagram is hasse diagram of poset with three elements.

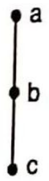

Fig. 2.1

Syllabus topic : Mathematical Induction / Relation.

2.10 Mathematical Induction :

2.10.1 Principle of Mathematical Induction :

Mathematical Induction is a powerful technique in Mathematics; especially in number theory, where many properties of natural numbers are established by this method.

In Mathematics, we are often required to generalise a particular solution. In order to do this, we look for a pattern in the particular solution. Mathematical induction generalises this pattern of solutions by proving that it is always possible to extend the solution to a group that is one larger than the previous. The generalisation is acheived by using a statement involving a variable natural number.

The logic underlying the principle of mathematical induction, makes it an extremely suitable method to solve problems related to real life. A few examples of this type will be discussed in this section.

To the software engineer, mathematical induction is an important tool in algorithm verification, to check whether a program statement is loop invariant, that is, whether it is true before and after every pass through a programming loop.

Statement of the principle of mathematical induction :

Let P(n) be a statement involving a natural number n.

1. If P(n) is true for $n = n_0$ and
2. Assuming P(k) is true, $(k \geq n_0)$ we prove P(k + 1) is also true,

   then P(n) is true for all natural numbers $n \geq n_0$

Step (1) is called as the **Basis of induction.**

Step (2) is called as the **Induction step.**

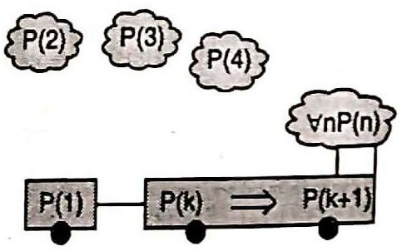

Fig. 2.1(a) : The principle of induction

The assumption that P(n) is true for n = k is called as the **Induction hypothesis.**

Refer Fig. 2.1 (a) for principle of induction.

2.10.2 Exercise Set - 8 (Solved) :

Example 1 : Prove by induction :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \text{ for all natural number values of n.}$$

Solution : Let P(n) be the statement : $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

(i) Basis of induction :

for n = 1,

$$P(1) : 1 = \frac{1(2)}{2} = 1$$

Hence P(1) is true.

(ii) Induction step :

Assume P(k) is true,

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \quad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P(k + 1) is also true.

$$P(k+1) : 1 + 2 + 3 + \ldots + k + (k+1)$$

$$= \frac{(k+1)[(k+1)+1]}{2} \quad \ldots(ii)$$

$$= \frac{(k+1)(k+2)}{2}$$

Using induction hypothesis (i), ... $\left[1 + 2 + 3 + \ldots k = \frac{k(k+1)}{2} \right]$

$$1 + 2 + 3 + \ldots k + (k+1) = \frac{k(k+1)}{2} + (k+1) \quad \ldots(iii)$$

Put Equation (iii) in Equation (ii)

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also ture. Therefore P(n) is true for all natural number values of n.

Example 2 : Prove by Mathematical induction method

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

MU - Dec. 09, May 13

Solution : Let P (n) be the statement : $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(i) **Basis of induction :**

For n = 1

$P(1) : 1 = \frac{6}{6} = 1$

Hence P (1) is true.

(ii) **Induction Step :**

Assume P (k) is true,

$$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = k(k+1)(2k+1) \qquad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P (k + 1) is also true.

$$P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$$

$$= \frac{k+1(k+2)(2k+3)}{6} \qquad \ldots(ii)$$

Using induction hypothesis (i) …. $\left[1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}\right]$

Put Equation (i) in Equation (ii) we get

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{2k2 + k + 6k + 6}{6}\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{2k2 + 7k + 6}{6}\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= \frac{k+1(k+2)(2k+3)}{6} = \frac{k+1(k+2)(2k+3)}{6}$$

Hence assuming P (k) is true, P (k + 1) is also true. Therefore P (n) is true by principle of mathematical induction for all n.

Example 3 : Prove that : $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

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Solution :

Let P (n) be the statement :

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

(i) **Basis of induction :**

For n = 1 $\quad P(1) : \frac{1}{1.4} = \frac{1}{4}$

Hence P(1) is true.

(ii) **Induction step :**

Assume P(k) is true, and prove P(k + 1) is also true.

$$P(k) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1} \quad \dots(ii)$$

$$P(k+1) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Using induction hypothesis (i),

$$\text{L.H.S. of Equation (ii)} = \frac{k}{3k+1} + \frac{1}{(3k+1)(3n+4)}$$

$$\dots\left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}\right]$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$$

$$= \frac{k+1}{3(k+1)+1}$$

$\therefore$ L.H.S = R.H.S. of Equation (ii)

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Example 4 : Use Induction to show that :

$1 + 3 + 5 + \dots + (2n - 1) = n^2$

MU - May 08

Solution :

Let P (n) be the statement : $1 + 3 + 5 + \dots\dots\dots + (2n - 1) = n^2$

(i) **Basis of induction :** For n = 1

$P(1) : 1 = 1^2$

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.,

$1 + 3 + 5 + \dots\dots\dots + (2n - 1) = k^2 \quad \dots(i)$

i.e. Induction hypothesis

Assuming P(k) is true we add 2k + 1 to both sides of P(n).

As the n^{th} add number is $2k - 1$ and the $n + 1^{st}$ add number is $2k + 1$

$\therefore\ 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + 2k + 1$

(ii) $\therefore\ 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$ which is $P(k + 1)$

That is, $P(k + 1)$ is true whenever $P(k)$ is true. By principle of mathematical induction is true for all n.

Example 5 : Use Mathematical Induction to show that. **MU - May 11**

$1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$

Solution :

Step 1 : Let P_n be the statement $1 + 5 + 9 + \text{----} + (4n - 3) = n(2n - 1)$.

Step 2 : P1 is true because $(4(1) - 3) = 1(2(1) - 1)$ [The Anchor.]

Step 3 : Assume that Pk is true, so that Pk: $1 + 5 + 9 + \text{----} + (4k - 3) = k(2k - 1)$

[The Inductive Hypothesis.]

Step 4 : The next term on the left-hand side would be $(4(k + 1) - 3)$.

Step 5 : $1 + 5 + 9 + \text{----} + (4k - 3) + (4(k + 1) - 3) = k(2k - 1) + (4(k + 1) - 3)$

Step 6 : $= 2k2 - k + 4k + 1$

Step 7 : $= 2k2 + 3k + 1$

Step 8 : $= (k + 1)2 + k(k + 1)$

Step 9 : $= (k + 1)(2k + 1)$ [Factor.]

Step 10 :$= (k + 1)(2(k + 1) - 1)$

Step 11 :$= Pk + 1$

Step 12 : So, the equation is true for $n = k + 1$.

Step 13 : Therefore, Pk is true for all positive integers, by mathematical induction.

Example 6 : Show that $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$

MU - May 01, May 04, Dec. 04, May 07

Solution :

Let P(n) be the statement :

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

(i) Basis of induction :

for $n = 1$, $P(1) : 1 + 2^1 = 2^{1+1} - 1$

$$3 = 3$$

$\therefore$ P (1) is true.

(ii) Induction step :

Assume P(k) is true i.e.,

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1 \quad \ldots(i)$$

i.e. Induction hypothesis.

Prove P(k + 1) is also true.

$P(k+1) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1+1} - 1$ ...(ii)

L.H.S. of Equation (ii) is,

$1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$ ...Using induction hypothesis (i)

$[1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1]$

$$= 2^{k+1} + 2^{k+1} - 1$$

$$= 2(2^{k+1}) - 1$$

$$= 2^{(k+1)+1} - 1$$

$\therefore$ L.H.S. = R.H.S. of Equation (ii)

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true.

Example 7 : Show that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

MU - Dec. 98, May 99, May 2000, Dec. 12

Solution : Let P(n) be the statement :

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = n(2n-1)(2n+1)/3$$

(i) Basis of Induction : For n = 1

$$P(1) : 1 = 1(2-1)(2+1)/3$$

$$1 = 1$$

$\therefore$ P(1) is true.

(ii) Induction step : Assume P(k) is true i.e.

$1^2 + 3^2 + 5^2 + \ldots (2k-1)^2 = k(2k-1)(2k+1)/3$ ...(i)

i.e. Induction hypothesis.

Assume P(k) is true, prove P(k + 1) is also true.

$P(k+1) : 1^2 + 3^2 + 5^2 + \ldots + (2(k+1)-1)^2 = (k+1)(2(k+1)-1)(2(k+1)+1)/3$ ...(ii)

L.H.S of (ii) is

$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2$

$= k(2k-1)(2k+1)/3 + (2(k+1)-1)^2$ ...Using induction hypothesis

$$= k(2k-1)(2k+1)/3 + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1) + 3(2k+1)}{3}\right] = (2k+1)\left[\frac{(2k^2 - k + 6k + 3)}{3}\right]$$

$$= (2k+1)\left[\frac{(2k^2 + 5k + 3)}{3}\right] = (2k+1)\left[\frac{(2k+3)(k+1)}{3}\right]$$

$$= \frac{(k+1)(2k+2-1)(2k+2+1)}{3}$$

$$= (k+1)[2(k+1)-1][2(k+1)+1]/3$$

$$= (k + 1) (2 (k + 1) - 1) (2 (k + 1) + 1)/3$$

$$= \text{R.H.S of Equation (ii)}$$

P (k + 1) is true whenever P(k) is true. Hence P(k) is true for all n by the principle of mathematical induction.

Example 8 : Prove that following by mathematical induction :

$2 + 5 + 8 + \dots + (3n - 1) = n(3n + 1)/2$ **MU - Dec. 97, May 17**

Solution :

Let P(n) : $2 + 5 + 8 + \dots (3n - 1) = n(3n + 1)/2$

(i) **Basis of induction :**

For $n = 1$;

$$(3(1) - 1) = 1 (3(1) + 1)/2$$

$$2 = \frac{4}{2}$$

$$2 = 2$$

$\therefore$ L.H.S. = R.H.S.

Hence, the given statement is true for n = 1

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.

$$2 + 5 + 8 + \dots + (3k - 1) = k (3k + 1)/2 \quad \dots(i)$$

To prove that the statement is true for P(k + 1)

i.e. $2 + 5 + 8 + \dots + [3(k + 1) - 1] = (k + 1) [3(k + 1) + 1] / 2$

$$\text{L.H.S} = 2 + 5 + 8 + \dots + (3k - 1) + [3(k + 1) - 1]$$

$$= \frac{k(3k + 1)}{2} + 3(k + 1) - 1 \quad \dots\text{from Equation (i)}$$

$$= \frac{(3k^2 + k + 6k + 6 - 2)}{2} = \frac{(3k^2 + 7k + 4)}{2}$$

$$= \frac{(3k^2 + 3k + 4k + 4)}{2} = \frac{(3k + 4) (k + 1)}{2}$$

$$= (k + 1) [\, 3(k + 1) + 1 \,] / 2$$

$\therefore$ L.H.S. = R.H.S.

Hence, given that the statement is true for n = k, it is true for n = k + 1.

Hence it has been proved by mathematical induction.

Example 9 : Show that $1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + ... + n)^2$ **MU - May 94, Dec. 95**

Solution :

Let $P(n) : 1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + ... + n)^2$

(i) **Basis of induction :**

For $n = 1$ $\quad (1)^3 = (1)^2$ which is true

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.,

$$1^3 + 2^3 + 3^3 + ... + k^3 = (1 + 2 + ... + k)^2 \quad ...(i)$$

To prove that P(k + 1) is true i.e.;

$$1^3 + 2^3 + 3^3 + ... + (k+1)^3 = [1 + 2 + ... + (k + 1)]^2$$

$$\begin{aligned}
\text{L.H.S.} &= 1^3 + 2^3 + 3^3 + ... k^3 + (k+1)^3 \\
&= [1^3 + 2^3 + 3^3 + ... k^3] + (k+1)^3 \\
&= (1 + 2 + ... + k)^2 + (k+1)^3 \quad \text{...by induction hypothesis, From Equation (i)} \\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \quad ...\left[\text{Recall that } 1 + 2 + ... + k = \frac{k(k+1)}{2}\right] \\
&= \frac{k^2 (k+1)^2}{4} + (k+1)^3 \\
&= (k+1)^2 \left[\frac{k^2}{4} + (k+1)\right] \\
&= (k+1)^2 \left[\frac{k^2 + 4k + 4}{4}\right] \\
&= \frac{(k+1)^2 (k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2 \\
&= [1 + 2 + ... + (k+1)]^2
\end{aligned}$$

Hence, L.H.S = R.H.S.

Therefore P(n) is true.

Example 10 : Prove that,

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)(2n+2)} = \frac{n}{2n+1} \quad \text{for } n \geq 1$$

Solution : Let P(n) :

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

(i) **Basis of induction :**

For $n = 1$ $\quad \frac{1}{1.3} = \frac{1}{3}$ which is true

$\therefore$ P(1) is true

(ii) Induction Step :

Assume P(k) is true i.e.;

$$\frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad \ldots(i)$$

To prove that P(k + 1) is true i.e.,

$$\frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}=\frac{(k+1)}{[2(k+1)+1]}$$

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{1.3}+\frac{1}{3.5}+\ldots\frac{1}{(2k-1)(2k+1)}+\frac{1}{[2(k+1)-1][2(k+1)+1]}\\
&= \frac{k}{2k+1}+\frac{1}{[2(k+1)-1][2(k+1)+1]} \quad \ldots\text{from Equation (i)}\\
&= \frac{k}{(2k+1)}+\frac{1}{(2k+1)(2k+3)}\\
&= \frac{1}{(2k+1)}\left[k+\frac{1}{(2k+3)}\right]\\
&= \frac{1}{(2k+1)}\left[\frac{2k^2+3k+1}{(2k+3)}\right]\\
&= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}\\
&= \frac{k+1}{2k+3}\\
\text{L.H.S.} &= \text{R.H.S.}
\end{aligned}$$

Hence, the result is true for all n.

Example 11 : Prove by induction that for $n \geq 1$

$$1.2+2.3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

Solution :

$$\text{Let } P(n) = 1.2+2.3+\ldots+n(n+1) = \frac{n(n+1)(n+2)}{3}$$

(i) Basis of induction :

$$\begin{aligned}
\text{Let } n &= 1\\
\text{L.H.S} &= 1(1+1)=1(2)=2\\
\text{R.H.S} &= \frac{1(1+1)(1+2)}{3}=\frac{1(2)(3)}{3}=\frac{6}{3}=2
\end{aligned}$$

(ii) Induction step :

Assuming that it is true for n = k,

$$P(k): 1.2+2.3+\ldots\ldots\ldots+k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \ldots(i)$$

We have to prove it for $n = k + 1$

$$P(k+1) : 1.2 + 2.3 + \ldots + k(k+1) + (k+1)(k+2) \qquad \ldots(ii)$$

$$= \frac{(k+1)[(k+1)+1]\ [(k+1)+2]}{2}$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \qquad \ldots\text{from Equation (i)}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)[k+3]}{3}$$

$$= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}$$

Hence P(n) is true.

Example 12 : Prove that

$1.1! + 2.2! + 3.3! + \ldots n.n! = (n+1)! - 1,$

where n is a positive integer. **MU - Dec. 05**

Solution :

Let $P(n) = 1.1! + 2.2! + 3.3! + \ldots n.n! = (n+1)! - 1$

(i) **Basis of induction :**

For $n = 1$, $1.1! = 1$ and $(1+1)! - 1 = 2! - 1 = 1$

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true, i.e,

$$1.1! + 2.2! + 3.3! + \ldots k.k! = (k+1)! - 1 \qquad \ldots(i)$$

To prove that P(k + 1) is true,

$$P(k+1) : 1.1! + 2.2! + 3.3! + \ldots (k+1)\cdot(k+1)! = (k+2)! - 1$$

$$\begin{aligned}
\text{L.H.S} &= 1.1! + 2.2! + 3.3! + \ldots k\,k! + (k+1)\cdot(k+1)! \\
&= (k+1)! - 1 + (k+1)\cdot(k+1)! \ \text{(by induction hypothesis).} \\
&= (k+1)! + (k+1)\cdot(k+1)! - 1 \\
&= (k+1)!\,(1 + k + 1) - 1 \\
&= (k+1)!\ (k+2) - 1 \\
&= (k+2)! - 1 \\
\text{L.H.S.} &= \text{R.H.S.}
\end{aligned}$$

Hence the result is proved.

Example 13 : By using mathematical Induction, prove that,

$$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}, \text{ where } n \geq 0$$

MU - Dec. 02. May 16

Solution :

$$\text{Let } P(n) : 1 + a + a^2 + \dots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

(i) Basis of induction :

For $n = 0$, $P(0) = \frac{1-a}{1-a} = 1$

$\therefore$ It is true for $n = 0$

For $n = 1$, $P(1) = \frac{1 - a^{1+1}}{1-a} = \frac{1-a^2}{1-a} = 1 + a$

$\therefore$ True for $n = 1$

(ii) Induction step :

Assume P(k) is true i.e.

$$P(k) : 1 + a + a^2 + \dots + a^k = \frac{1 - a^{k+1}}{1-a} \quad \dots(i)$$

To prove that P(k + 1) is true.

$$P(k+1) : 1 + a + a^2 + \dots + a^{k+1} = \frac{1 - a^{(k+1)+1}}{1-a}$$

$$\text{L.H.S.} = 1 + a + a^2 + \dots + a^k + a^{k+1}$$

$$= \frac{1 - a^{k+1}}{1-a} + a^{k+1} \quad \text{...from Equation (i)}$$

$$= \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1-a} = \frac{1 - a^{k+2}}{1-a}$$

$$= \frac{1 - a^{(k+1)+1}}{1-a}$$

$\therefore$ It is true for $n = k + 1$. Hence, P(n) is true for all n.

Example 14 : Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$. **MU - May 09**

Solution :

(i) Basis of Induction : Since it is given that, for all $n \geq 1$, Let $n = 1$.

$$(1)^3 + 2(1) = 1 + 2 = 3$$

3 is divisible by 3.

(ii) Induction step :

Assuming that for $n = k$, it is true i.e. $k^3 + 2k$ is divisible by 3.

We will prove it for $n = k + 1$. ...(i)

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2(1) + 3k(1)^2 + 1^3 + 2k + 2$$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= k^3 + 3k^2 + 5k + 3$$

$$= k^3 + 2k + 3k + 3k^2 + 3$$

$$= k^3 + 2k + 3k^2 + 3k + 3$$
$$= k^3 + 2k + 3(k^2 + k + 1)$$
$$= (k^3 + 2k) + 3(k^2 + k + 1)$$

Note that, since $(k^3 + 2k)$ is divisible by 3 by induction hypothesis (i), and $3(k^2 + k + 1)$ is also divisible by 3. Each term is divisible by 3, we can say $n^3 + 2n$ is divisible by 3.

Example 15 : Prove by mathematical induction that, $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

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Solution :

Let P(n) : $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

(i) **Basis of induction :**

For $n = 2$

P(2) : $2^4 - 4(2)^2$ is divisible by 3

$\therefore$ P(2) is true.

(ii) **Induction step :**

Assume P(k) is true.

i.e. $(k^4 - 4k^2)$ is divisible by 3.

Assume P(k) is true, and prove P(k + 1) is also true.

For $n = k + 1$ we have,

$(k + 1)^4 - 4(k + 1)^2$

$$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4k^2 - 8k - 4$$
$$= k^4 + 4k^3 + 2k^2 - 4k - 3$$
$$= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3$$
$$= k^4 - 4k^2 + 3(2k^2 - 1) + 4k^3 - 4k$$
$$= k^4 - 4k^2 + 3(2k^2 - 1) + 4k(k^2 - 1)$$
$$= (k^4 - 4k^2) + 3(2k^2 - 1) + 4k(k - 1)(k + 1)$$

Now $(k^4 - 4k^2)$ is divisible by 3 by induction hypothesis, as P(k) is true $3(2k^2 - 1)$ is divisible by 3 and $4k(k - 1)(k + 1)$ is divisible by 3, as it has a factor which is the product of three integers which are consecutive.

Hence P(n) is divisible by 3 for all $n \geq 2$.

Example 16 : Prove by induction that $n^2 + n$ is an even number, for every natural number n.

Solution :

Let P(n) : $n^2 + n$ is an even number

(i) **Basis of induction :**

For $n = 1$

$P(1) : (1)^2 + (1)$ i.e. 2 is an even number

$\therefore$ P(1) is true.

(ii) Induction step :

Assume P(k) is true i.e. $(k^2 + k)$ is an even number, and prove P(k + 1) is also true.

$$\begin{aligned} P(k+1) \ : \ & (k+1)^2 + (k+1) \\ = \ & k^2 + 2k + 1 + k + 1 \\ = \ & (k^2 + k) + 2(k+1) \end{aligned}$$

Now $(k^2 + k)$ is even, by the induction hypothesis, and $2(k + 1)$ is even.

$\therefore$ The sum $(k^2 + k) + 2(k + 1)$ is even

$\therefore$ By induction, $k^2 + k$ is an even number, for every natural number n.

Example 17 : Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction $n \geq 1$.
MU - Dec. 96. Dec. 13. May 14

Solution :

Let $P(n) : 8^n - 3^n$ is a multiple of 5

(i) Basis of induction : For $n = 1$

$$P(1) : 8^1 - 3^1 = 5 \text{ which is divisible by } 5$$

$\therefore$ P(1) is true.

(ii) Induction step : Assume P(k) is true i.e. $(8^k - 3^k)$ is a multiple of 5, and prove P(k + 1) is also true.

$$\begin{aligned} P(k+1) \ : \ & 8^{k+1} - 3^{k+1} \\ = \ & 8^k \cdot 8 - 3^k \cdot 3 \\ = \ & 8^k \cdot 8 - 3^k \cdot 8 + 3^k \cdot 5 \\ = \ & 8(8^k - 3^k) + 3^k \cdot 5 \end{aligned}$$

Now $8(8^k - 3^k)$ is a multiple of 5 by induction hypothesis and $3^k \cdot 5$ is already a multiple of 5.

Hence $8^n - 3^n$ is a multiple of 5, for $n \geq 1$.

Example 18 : Show that $n(n^2 - 1)$ is divisible by 24, where n is any odd positive integer. **MU - Dec. 16**

Solution :

Let $P(n) : n(n^2 - 1)$ is divisible by 24.

(i) Basis of induction : For $n = 1$

$$P(1) : 1((1)^2 - 1) = 0 \text{ which is divisible by } 24.$$

$\therefore$ P(1) is true.

(ii) Induction step : Assume P(k) is true i.e;

$P(k) : k(k^2 - 1)$ is divisible by 24,

where k is odd integer.

Now, the next odd integer is (k + 2); hence prove that P(k + 2) is also true.

$$\begin{aligned}
P(k+2) \quad : \quad & (k+2)\left[(k+2)^2-1\right] \\
= \; & (k+2)\left[k^2+4k+3\right] \\
= \; & (k+2)(k+1)(k+3) \\
= \; & k(k+1)(k+3)+2(k+1)(k+3) \\
= \; & k(k+1)(k-1+4)+2(k+1)(k+3) \\
= \; & k(k+1)(k-1)+4k(k+1)+2(k+1)(k+3) \\
= \; & k(k^2-1)+2(k+1)[2k+k+3] \\
= \; & k(k^2-1)+2(k+1)(3k+3) \\
= \; & k(k^2-1)+6(k+1)^2
\end{aligned}$$

Now, $6(k+1)^2$ is divisible by 24 for all odd n and $k(k^2-1)$ is divisible by 24 by induction hypothesis. Hence the whole expression is divisible by 24.

Therefore P(n) is also true.

Example 19 : Prove that $5^n - 1$ is divisible by 4, for $n \geq 1$ **MU - May 05, Dec. 10**

Solution :

Let P(n) : $5^n - 1$

(i) Basis of induction :

For n = 1, $\quad 5^1 - 1 = 4$, divisible by 4

(ii) Induction step :

Assume that $5^k - 1$ is divisible by 4.

We have
$$\begin{aligned}
5^{k+1}-1 &= (5^k \cdot 5 - 5) + 4 \\
&= 5(5^k - 1) + 4
\end{aligned}$$

By induction hypothesis $5^k - 1$ is divisible by 4.

$\therefore$ Each term on the RHS is divisible by 4.

$\therefore$ $5^{k+1} - 1$ is divisible by 4.

Hence $5^n - 1$ is divisible by 4 for $n \geq 1$.

Example 20 : Use induction to prove that : $7^n - 1$ is divisible by 6 for n = 1, 2, 3, ... **MU - May 06, May 12**

Solution : Let $\quad P(n) = 7^n - 1$

(i) Basic induction :

For n = 1, $7' - 1 = 6$ divisible by 6.

(ii) Induction step :

Assume that 7^{k-1} is dividible by 6.

Now put $n = k+1$

$$P(k+1) = 7^{k+1} - 1$$

We have, $$7^{k+1} = (7^k \cdot 7 - 7) + 6$$
$$= 7(7^k - 1) + 6$$

By induction hypothesis $7^k - 1$ is divisible by 6.

$\therefore$ Each term on the R.H.S. is divisible by 6.

$\therefore$ $7^{k-1} - 1$ divisible by 6.

Hence 7^{n-1} is divisible by 6.

Example 21 : Show that for any positive integer n, $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

Solution :

Let $P(n) : (11)^{n+2} + (12)^{2n+1}$

(i) Basis of induction :

For $n = 1, (11)^{n+2} + (12)^{2n+1} = 3059$ divisible by 133, since $133 \times 23 = 3059$.

(ii) Induction step :

Assume the result for n = k i.e. P(k) is true

i.e. $(11)^{k+2} + (12)^{2k+1}$ is divisible by 133. ...(i)

To prove that P(k + 1) is true,

$$(11)^{k+1+2} + (12)^{2(k+1)+1}$$
$$= 11 \cdot (11)^{k+2} + (12)^{2k+1} \cdot 144$$
$$= 11 \cdot (11)^{k+2} + (133 + 11)(12)^{2k+1}$$
$$= 11 \cdot (11)^{k+2} + (12)^{2k+1} + 133 \cdot (12)^{2k+1}$$

$11 \cdot (11)^{k+2} + (12)^{2k+1}$ is divisible by 133 by induction hypothesis (from (i)). $133 \cdot (12)^{2k+1}$ is also divisible by 133.

Since both these terms are divisible by 133, the result follows.

Example 22 : Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n. **MU - Dec. 11. Dec. 15**

Solution : Solve this Example same as Example 21.

Example 23 : Prove that $n^3 - n$ is divisible by 3, for a positive integer n.

Solution :

Let P(n) be $n^3 - n$.

(i) **Basis of induction :**

For $n = 1, 1^3 - 1 = 0$ is divisible by 3.

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.;

$(k^3 - k)$ is divisible by 3 ...(i)

To prove that P(k + 1) is true

For $n = k + 1$, we have

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= (k^3 - k) + 3(k^2 + k)$$

Since $k^3 - k$ is divisible by 3, by induction hypothesis (i), and $3(k^2 + k)$ is divisible by 3. The result follows.

Example 24 : Prove that $n! \geq 2^n$ for $n \geq 4$ **MU - Dec. 03**

Solution : Let $P(n) : n! \geq 2^n$

(i) **Basis of induction :**

For $n = 4$

$$P(4) : 4! \geq 2^4$$
$$: 24 \geq 16$$
$$: 4! \geq 2^4$$

$\therefore$ P(4) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.; $k! \geq 2^k$.

Prove that P(k + 1) is also true. ...(i)

$$\text{For } n = k+1$$
$$P(k+1) : (k+1)! \geq 2^{k+1} \quad \text{...(ii)}$$
$$\text{L.H.S.} = (k+1)\,k! \geq 2^k(k+1) \quad \text{...from Equation (i)}$$
$$\geq 2^k(2) \geq 2^{k+1}$$
$$(k+1)! \geq 2^{k+1}$$

i.e. $\geq$ R.H.S. of Equation (ii)

Hence $n! \geq 2^n$ for $n \geq 4$

Example 25 : Prove induction to show that $n! \geq 2^{n-1}$ for $n = 1, 2, \ldots\ldots$ **MU - May 03**

Solution :

Let P (n) be the given statement

(i) **Basis of induction :**

For $n = 1$, $1! \geq 2^{1-1}$

i.e. $1 \geq 1$ which is true

$\therefore$ P (n) is true for n = 1.

(ii) **Induction step :** We now assume that it is true for n = k.

i.e. $k! \geq 2^{k-1}$

To prove that statement is true for n = k + 1

Now $(k+1)! = (k+1) \cdot k!$

$\geq (k+1)2^{k-1} \geq 2 \cdot 2^{k-1}$ Since $k + 1 \geq 2$

$= 2^k$

Hence P (n) is true for n = m + 1

$\therefore$ P (n) is true for all $n \geq 1$.

Example 26 : Use mathematical induction to prove the following inequality : $n < 2^n$ for all positive integers n. **MU - Dec. 06. May 10**

Solution :

Basis step : P (1) is true since $1 < 2' = 2$ for n = 1

Inductive step : Assume P (k) is true for the positive integer k.

Assume P (k) i.e. $k < 2^k$ is true. We need to show that P (k + 1) is true i.e. $k + 1 < 2^{k+1}$ is true. Adding 1 to both sides of $k < 2^k$ we get,

$$k + 1 < 2^k + 1$$

We know that $1 \leq 2^k$, we get,

$$k + 1 < 2^k + 1 \leq 2^k + 2^k \leq 2^{k+1} = 2^{k+1}$$

So we have shown that P (k + 1) is true. The inductive step is complete.

Example 27 : Show that $n^4 < 10^n$, for $n \geq 1$

Solution : Let P(n) : $n^4 < 10^n$

(i) **Basis of induction :**

For n = 1

P (1) : $(1)^4 < 10^1$ i.e. 1 < 10 which is true.

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true i.e. $k^4 < 10^k$, ...(i)

Prove that P(k + 1) is also true.

For n = k + 1

$$\begin{aligned}
\text{L.H.S.} &= (k+1)^4 \\
&= k^4 + 4k^3 + 6k^2 + 4k + 1 \\
&< 10^4 + 4k^3 + 6k^2 + 4k + 1 && \text{...From Equation (i)} \\
&< 10^k + 10 && \text{...}(\because k \geq 1 \therefore 4k^3 + 6k^2 + 6k + 1 \geq 10) \\
&= 10^{k+1}
\end{aligned}$$

$\therefore$ The result is true for all n.

i.e. $n^4 < 10^n$, for $n \geq 1$

Example 28 : Prove by induction that the sum of the cubes of three consecutive integers is divisible by 9.

Solution :

We have to show that $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9.

Let $P(n) \ : \ (n-1)^3 + n^3 + (n+1)^3$

(i) Basis of induction :

For $n = 1$, we have,

$0 + 1 + (2)^3 = 9$, divisible by 9.

$\therefore$ P(1) is true

(ii) Induction step :

Assume P(k) is true i.e.,

$$(k-1)^3 + k^3 + (k+1)^3 = 3k^3 + 6k \text{ is divisible by 9} \qquad \text{...(i)}$$

To prove that P(k + 1) is true.

For $n = k + 1$

$$\begin{aligned}
[(k+1)-1]^3 + (k+1)^3 + [(k+1)+1]^3 &= k^3 + (k+1)^3 + (k+2)^3 \\
&= 3k^3 + 9k^2 + 15k + 9 \\
&= (3k^3 + 6k) + 9(k^2 + k + 1)
\end{aligned}$$

$(3k^3 + 6k)$ is divisible by 9 by Equation (i) and $9(k^2 + k + 1)$ is also divisible by 9.

Hence $(3k^3 + 6k) + 9(k^2 + k + 1)$ is divisible by 9. Hence the result is proved.

Example 29 : Show that $2^n > n^3$, for $n \geq 10$

Solution :

(i) Basis of induction :

For $n = 10$ $\quad 2^{10} = 1024 > 10^3$

(ii) Induction step : Assume that $2^k > k^3$

Then $\quad 2^{k+1} = 2^k \cdot 2 = 2^k(1+1) > 2^k\left(1 + \frac{1}{10}\right)^3$

$$\geq 2^k\left(1+\frac{1}{k}\right)^3 > k^3\left(1+\frac{1}{k}\right)^3$$

$$= k^3\frac{(k+1)^3}{k^3}$$

$$= (k+1)^3$$

Therefore $2^n > n^3$, for $n \geq 10$.

Example 30 : Show that any integer composed of 3^n identical digits is divisible by 3^n.

Solution :

(i) Basis of induction :

For n = 1, the result is true, since on integer is divisible by 3 if the sum of the digits is divisible by 3. For example 111, 444, 555 etc. are divisible by 3.

(ii) Induction step :

Let x be an integer composed of 3^{k+1} identical digits. Then we may express x as x = yz, where y is an integer composed of 3^k identical digits and

$$z = 10^{2\cdot 3^k} + 10^{3^k} + 1$$

For example,

$$777777777 = 777\,(10^6 + 10^3 + 1), \text{ putting } k = 1.$$

In this case y = 777 and z = 1001001

Now both y and z are divisible by 3 so that 777777777 is divisible by $3^2 = 9$.

In the general case y is divisible by 3^k, by induction hypothesis,

and $$z = \underbrace{10000...0}_{3^k - 1\ 0\text{'s}}\ \underbrace{1000...01}_{3^k - 1\ 0\text{'s}}$$

Clearly z is divisible by 3 Hence x = yz is divisible by $3^k \cdot 3 = 3^{k+1}$

Thus the result is proved for any value of n.

Example 31 : For a positive integer n > 1, prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Solution :

(i) Basis of induction :

For n = 2,

$$1 + \frac{1}{2^2} = 1 + \frac{1}{4} < 2 - \frac{1}{2} \text{ as } 1 + \frac{1}{4} = \frac{5}{4} < \frac{3}{2}$$

(ii) Assume for n = k, i.e.

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$$

i.e. induction hypothesis.

For $n = k + 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \text{ (by induction hypothesis)}$$

$$= 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k+1)^2 - k}{k(k+1)^2}$$

$$= 2 - \frac{k^2 + k + 1}{k(k+1)^2} = 2 - \left(\frac{k(k+1)+1}{k(k+1)^2}\right)$$

$$= 2 - \left(\frac{1}{k+1} + \frac{1}{k(k+1)^2}\right)$$

$$= \left(2 - \frac{1}{k+1}\right) - \frac{1}{k(k+1)^2}$$

$$< 2 - \frac{1}{k+1}$$ Hence the result

Example 32 : Consider the following function given in psudocode. **MU - Dec. 03**

FUNCTION SQ (A)

1. $C \leftarrow 0$
2. $D \leftarrow 0$
3. WHILE $(D \neq A)$
   (a) $C \leftarrow C + A$
   (b) $D \leftarrow D + 1$
4. RETURN (C)

END OF FUNCTION SQ

Above function computes the square of A. C_n and D_n be the values of variable C and D after passing through while loop n times. Let P(n) be the predicate $C_n = A \times D_n$. Prove by principle of Mathematical induction that $\forall\, n \geq 0$, P(n) is true.

Solution : Here $P(n) = A \times D_n$

(i) Basis of induction :

For $n = 1$, $P(1) = A \times D_1$

$= A \times 1$

$\therefore$ P(1) is true.

(ii) Induction Step :

Now assume P(n) is true for $n = k$

$\therefore P(k) = A \times D_k$

To prove that $P(k + 1)$ is true.

$\therefore$ $P(k+1) = [A \times D_k] \times D = A \times D_{k+1}$

$\therefore$ True for $n = k+1$

$\therefore$ True for all n.

Example 33 : Show that for any positive integer $n > 1$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

Solution :

Let $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$

(i) Basis of induction :

For $n = 2$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + 0.7071 = 1.7071$$

$$\sqrt{2} = 1.4142$$

$$\therefore \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$$

(ii) Induction step :

Assume that for $n = k$, $P(k)$ is true

i.e., $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$

Now for $n = k+1$

$$P(k+1) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

$$\text{L.H.S.} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

To show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

We have to show $\sqrt{k} > \sqrt{k+1} - \frac{1}{\sqrt{k+1}}$

i.e. $\sqrt{k}\sqrt{k+1} > k+1-1 = k$

i.e. $k(k+1) > k^2$ which is true for $k \geq 1$.

Hence the result is true for all $n \geq 1$.

Example 34 : Use induction to prove that if F_n is the n^{th} fibonacci number, then

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

For all integers $n \geq 0$. **MU - Dec. 07**

Solution :

Let $$P(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

(i) Basis of induction :

For $n = 1$,

$$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

(ii) Induction step :

Assume that for $n = k$, $P(k)$ is true i.e.

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Now for $n = k+1$

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1}, \text{ probability} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}+1\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right] \\
F_{k+1} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{(k+1)+1}\right]
\end{aligned}$$

True for $n = k+1$

Hence the result is true for all $n \geq 0$.

Theorem :

Cardinality of a power set

MU - May 2000, Dec. 04

Example 35 : Let A be a finite set containing n elements and P(A) be a power set. Prove that cardinality of P(A) is 2^n.

Solution :

We prove the theorem by mathematical induction.

(i) **Basis of induction :**

For $n = 1$,

$A = \{a\}$, so that $P(A) = \{A, \phi\}$. Hence $|P(A)| = 2^1$ elements.

(ii) **Induction step :**

Assume that if $|A| = k$, $|P(A)| = 2^k$.

Let $|A| = k + 1$. For an element $a \in A$, consider the subset $B = A - \{a\}$. Since $|B| = k$, by induction hypothesis $|P(B)| = 2^k$, i.e. there are exactly 2^k subsets of B. Since every subset of B is also a subset of A, it follows that A contains atleast 2^k subsets. In addition, for each subset of B, say C, we have another subset $C \cup \{a\}$ of A. Hence the total number of subsets of A is

$2^k + 2^k = 2.2^k = 2^{k+1}$ subsets.

Hence by induction, it follows that

$$\text{if } |A| = n, \text{ then } |P(A)| = 2^n$$

De Morgan's Law

Example 36 : Verify De Morgan's law for n sets $A_1, A_2, \ldots A_n$.

i.e. $\overline{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \ldots \cap \bar{A}_n$

Solution :

Let P (n) be the statement : $\overline{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \ldots \cap \bar{A}_n$

(i) **Basis of induction :**

For $n = 1$, $\bar{A}_1 = \bar{A}_1$ which is true.

(ii) **Induction step :**

We assume that the result is true for $n = k$.

i.e. P (k) : $\overline{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \ldots \cap \bar{A}_k$ ...(i)

For $n = k + 1$

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}} = \overline{(A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}}$$

$$= \overline{(A_1 \cup A_2 \cup \ldots \cup A_k)} \cap \overline{A_{k+1}}$$

$$= \bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_{k+1} \cap \overline{A_{k+1}}$$

$$= \bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \overline{A_{k+1}} \quad \text{...By De Morgan's law for two sets.}$$

$$= \left(\bigcap_{i=1}^{k} \bar{A}_i\right) \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k+1} \bar{A}_i = \text{Right hand side of } P(k+1)$$

Mutual Inclusion and Exclusion

Example 37 : Establish the theorem an Mutual Inclusion and Exclusion.

Let $A_1, A_2, \ldots A_n$ be the collection of n sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_K| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \ldots \cap A_n|$$

Solution :

Proof is by induction on n. We have already proved the theorem for n = 2, 3. Hence, let us assume the theorem for (n – 1) numbers of sets and prove it for n sets.

Regarding, $A_1 \cup A_2 \cup \ldots \cup A_n$ as $(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n$,

We have, $|A_1 \cup A_2 \cup \ldots \cup A_n| = |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n|$

$$= |A_1 \cup A_2 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| \quad \text{...(i)}$$

By induction hypothesis

$$|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = \sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| \ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \quad \text{...(ii)}$$

Now $|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| = |(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)|$

$$= \sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| + \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k \cap A_n| - \ldots (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n| \quad \text{...(iii)}$$

Substituting Equations (ii) and (iii) in Equation (i) we obtain the equation.

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j < n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Example 38 : Prove by mathematical induction that if $A_1, A_2 \ldots A_n$ and B are any n sets then.

$$\left(\bigcap_{i=1}^{n} A_i\right) \cup B = \bigcap_{i=1}^{n} (A_i \cup B)$$

Solution :

Basis of induction :

For $n = 1$,

$P(1) : A_1 \cup B = A_1 \cup B$ is true

Induction step :

L.H.S of P (k + 1)

$$\left(\bigcap_{i=1}^{k+1} A_i\right) \cup B = \left(\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right) \cup B$$

$$= \left(\left(\bigcap_{i=1}^{k} A_i\right) \cup B\right) \cap (A_{k+1} \cup B) \quad \ldots \quad \text{Distributive law}$$

$$= \left(\bigcap_{i=1}^{k} (A_i \cup B)\right) \cap (A_{k+1} \cup B) = \left(\bigcap_{i=1}^{k+1} A_i \cup B\right)$$

Example 39 : Find how many palindromes of length n can be formed from an alphabet of k letters.

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Solution :

(i) n is odd i.e. palindrome is of form aca etc. here n = 3.

Now here as n is odd n – 1 is even.

Now we select (n –1)/2 letters before the midpoint i.e. 'c' in our notation from an alphabet of k letters (allowing repetitions) and arrange them in $k^{(n-1)/2}$ ways. The letter 'c' can be selected in k ways.

The remaining part is a reverse of the first part and is unique top every first part.

Therefore the number of palindromes

$$= k^{(n-1)/2} \cdot k = k^{(n-1+2)/2} = k^{(n+1)/2}$$

(ii) N is even i.e., palindrome is of form aa. Therefore the first part can be selected in $k^{n/2}$ ways and the 2^{nd} part is unique for every first half. Thus the total number of palindromes of length n from an alphabet of k letters is,

(i) $k^{(n+1)/2} \rightarrow$ n is odd (ii) $k^{n/2} \rightarrow$ n is even

Example 40 : Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be covered using L – shaped pieces, where each piece covers three squares at a time.

Solution :

Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L - shaped pieces.

(i) **Basis of induction :**

For n = 1, P(1) implies that any 2×2 chessboard with one square removed can be covered using L shaped pieces. P(1) is true, as seen in the following Fig. 2.2.

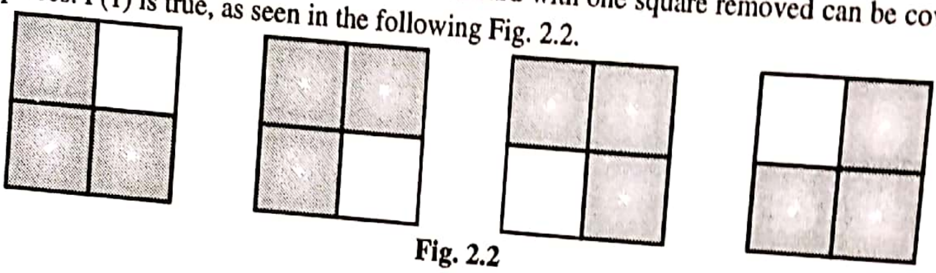


Fig. 2.2

(ii) **Induction step :**

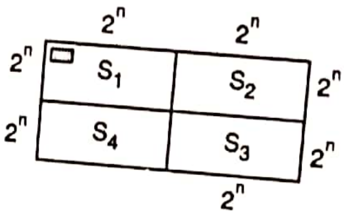


Fig. 2.3

Assume P(n) is true and prove that P(n + 1) is true. For this consider a $2^{n+1} \times 2^{n+1}$ chessboard with one square removed. Divide the chessborard into four equal halves of size $2^n \times 2^n$; as shown in the Fig. 2.3.

Then the square which has been removed, would have been removed from one of the four chessboards, say S_1. Then by induction hypothesis, S_1 can be covered using L-shaped pieces. Now from each of the 3 remaining chessboards, remove that particular square lying at the centre of the larger chessboard. This is illustrated in the Fig. 2.4.

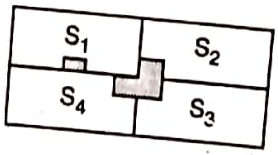


Fig. 2.4

Then by induction hypothesis, each of these $2^n \times 2^n$ chessborards with a square removed can be covered by the L-shaped pieces. Morever, the three squares that have been temporarily removed can be covered by one L-shaped piece. Hence, the entire $2^{n+1} \times 2^{n+1}$ chessboard can be covered by L-shaped pieces. This completes the proof.

Example 41 : Coin Problem : Suppose we have coins of two different denominations, 2 rupees and 5 rupees. It is possible to make up exactly any denomination of 7 rupees or more, using only these two denominations, assuming of course that we have an unlimited supply of these.

Solution :

For k = 7, we have one 5 rupee coin and one 2 rupee coin. For k = 8, we have four 2 rupee coins. Hence let us assume that we can make up a denomination of k rupees (k ≥ 7).We discuss two cases. Suppose there is a 5 rupee coin in the k-denomination, we have made up. Replacing the 5 rupee coin by three 2 rupee coins, we can make up a denomination of k + 1 rupees. On the other hand, suppose that the k - denomination coins, we have made up, consists of only 2 rupee coins, then replacing two 2 rupee coins by one 5 - rupee coin, we can still make up a denomination of k + 1 rupees. Thus the process can be continued till the supply runs out.

Example 42 : Solitaire game problem : For every integer i, there is an unlimited supply of balls marked with the number i. Initially a tray of balls is given and the balls are thrown away one at a time. If a ball marked i is thrown away, it is replaced by any finite number of balls marked 1, 2, ..., i – 1. There is no replacement for a ball marked 1. The game ends when the tray is empty. Show that the game always terminates for any tray of balls given initially.

Solution :

(i) **Basis of induction :**

For n = 1, there is a finite number of balls marked 1. Since, according to the rules of the game, there is no replacement if the balls are thrown away, the game terminated after a finite number of moves.

(ii) **Induction step :**

Let us assume that the game terminates if the largest number that appears on the balls is k, Suppose k + 1 is the largest number appearing an the balls. If all these balls are thrown away, they are replaced by balls marked 1, 2, …, k. Then the largest number appearing on the balls is k. Hence by induction hypothesis, the game has to terminate after a finite number of steps.

Example 43 : Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$. **MU - Dec. 14**

Solution : Let P (n) : $x^n - y^n$ is divisible by $x - y$

(i) Basis of induction : For n = 1

$$P(1) = x - y \text{ is divisible by } x - y$$

(ii) Induction step :

Assume p(k) is true. i.e. $x^k - y^k$ is divisible by $x - y$ and prove P(k + 1) is also true.

$$P(k+1) = x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$$

$x^k - y^k$ is divisible by $x - y$ induction by hypothesis and $x - y$ is divisible by $x - y$ by Basis of induction, hence $x^n - y^n$ is divisible by $x - y$.

2.10.3 Principle of Strong Mathematical Induction :

We shall now state a more powerful form of the principle of mathematical induction.

Statement :

Let P(n) be a statement involving a natural number n. If

(i) P(n) is true for $n = n_0$, and

(ii) P(n) is true for $n = k + 1$, **assuming that the statement is true for $n_0 \le n \le k$,** then the statement is true for all $n \ge n_0$.

Note that in the second principle of induction, we make a stronger assumption, that the statement is true for $n_0 \le n \le k$ (not merely for $n = k$)

2.10.4 Examples :

Example 1 : Show that any positive integer n greater than or equal to 2 is either a prime or a product of primes.

Solution :

(i) **Basis of induction :**

For $n = 2$, the statement is obviously true.

(ii) **Induction step :**

Assume the result for $2 \le n \le k$. Consider $k + 1$. If $k + 1$ is a prime, the statement is true. If $k + 1$ is not a prime then $k + 1 = pq$, where $p \le k$, $q \le k$. Hence by induction hypothesis, p is either a prime or product of primes. Similarly, q is either a prime or a product of primes. Hence $k + 1 = pq$ is a product of primes.

Example 2 : Jigsaw Puzzle Problem :

Show that for a jigsaw puzzle with n pieces, it will always take n – 1 moves to solve the problem.

Solution : (i) Basis of induction :

For $n = 1$, no moves are needed to solve the puzzle.

(ii) **Induction step :**

Assume that for any jigsaw puzzle with m pieces, $1 \le m \le k$, it takes $m - 1$ moves to solve the puzzle.

Consider a jigsaw puzzle with $k + 1$ pieces. For the last move that products the solution of the puzzle two blocks with m_1 pieces and m_2 pieces respectively, where $m_1 + m_2 = k + 1$, are put together to form a single block. Since $1 \le m_1 \le k$, $1 \le m_2 \le k$, by induction hypothesis it takes $m_1 - 1$ moves to form one block and $m_2 - 1$ moves to form the other black. Hence the total number of moves to solve the puzzle.

$$= (m_1 - 1) + (m_2 - 1) + 1 = m_1 + m_2 - 1 = k + 1 - 1 = k \text{ moves}$$

2.11 University Questions and Answers :

Dec. 2008

Q. 1 Explain with an example different types of quantifiers.

Translate the Symbols into English

(i) Ex p(x) and (ii) $\exists$ (x) (~ p (x)) **(Sections 2.9.2 and 2.9.4, Example 3)** **(6 Marks)**

May 2009

Q. 2 Prove there is no rational number p/q whose square is 2. **(Section 2.5.2, Example 8)** **(4 Marks)**

Q. 3 Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$. **(Section 2.10.2, Example 14)** **(6 Marks)**

Q. 4 Negate the statement.

For all real numbers x, if $x > 3$ then $x^2 > 9$. **(Section 2.9.4, Example 6)** **(6 Marks)**

Dec. 2009

Q. 5 Prove by Mathematical induction method

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ **(Section 2.10.2, Example 2)** **(5 Marks)**

Q. 6 Explain Quantifiers. Negate the statement '$\sqrt{2}$ is not a rational number'.

(Sections 2.9.2 and 2.9.4, Example 19) **(4 Marks)**

May 2010

Q. 7 Use mathematical induction to prove the following inequality $n < 2^n$ for all positive integers n. **(Section 2.10.2, Example 26)** **(4 Marks)**

Q. 8 What is an Universal and existential quantifier? **(Sections 2.9.2.1 and 2.9.2.2)** **(4 Marks)**

Q. 9 Converse of statement is given.

Write inverse and contra positive of statement "If I come early then I can get a car".

(Section 2.3.1, Example 25) **(5 Marks)**

Q. 10 Prove that if x is a rational number and y is an irrational number, then x + y is an irrational number. **(Section 2.5.2, Example 5)** **(5 Marks)**

Dec. 2010

Q. 11 Using Mathematical Induction prove that $5^n - 1$ is divisible by 4 for n > = 1 **(Section 2.10.2, Example 19)** **(5 Marks)**

Syllabus

Relations, Paths and Diagraphs, Properties and types of binary relations, Operations on relations, closures, Warshall's algorithm, Equivalence and Partial ordered relations, Poset Hasse diagram and Lattice.

Introduction :

This chapter is concerned with relations. In the preceding chapter we dealt with sets, elements and general properties of sets. Now we progress further and study the various relationships that may exist between elements of a set. We study various properties of a relation, including its matrix and graphical representations.

The concept of relation is of primary importance in computer science, especially in the study of data structure such as linked list, array, relational models etc. Relations are also important in the analysis of algorithms, information system etc.

Syllabus topic : Relations

3.1 Relations :

The notion of a relation between two sets of objects is quite common and intuitively clear (a formal definition will be given later). If A is the set of all living humans males and B is the set of all living human females, then the relation F (father) can be defined between A and B. Thus, if $x \in A$ and $y \in B$, then x is related to y by the relation F if x is the father of y, and we write x F y. Because order matters here, we refer to F as a relation from A to B. We could also consider the relations S and H from A to B by letting x S y means that x is a son of y, and x H y means that x is the husband of y.

If A is the set of all real numbers, there are many commonly used relations from A to A . An example is the relation "less than", which is usually denoted by $<$, so that x is related to y if $x < y$, and the other order relations $>$, $\geq$, and $\leq$. We see that a relation is often described verbally and may be denoted by a familiar name or symbol. The problem with this approach is that we will need to discuss any possible relation from one abstract set to another. Most of these relations have no simple verbal description and no familiar name or symbol to remind us of their nature or properties. Furthermore, it usually awkward, and sometimes nearly impossible, to give any precise proofs of the properties that relation satisfies if we must deal with a verbal description of it.

To solve this problem, observe that the only thing that really matters about a relation is that we know precisely which elements in A are related to which elements in B. Thus suppose that A = { 1 , 2, 3, 4 } and R is a relation from A to A. If we know that 1 R 2, 1 R 3, 1 R 4, 2 R 3, 2 R 4, and 3 R 4, then we know everything we need to know about R. Actually, R is the familiar relation <, "less than", but we need not know this. It would be enough to be given the foregoing list of related pairs. Thus we may say that R is completely known if we know all R - related pairs. We could then write R = { (1, 2) , (1,3), (1, 4), (2, 3), (2, 4), (3, 4) }, since R is essentially equal to or completely specified by this set of ordered pairs. Each ordered pair specifies that its first element is related to its second element, and all possible related pairs are assumed to be given, at least in principle. This method of specifying a relation does not require any special symbol or description and so is suitable for any relation between any two sets.

3.1.1 Cartesian Product :

MU - Dec. 11

Let A and B be non-empty sets. The **Cartesian Product** of A and B, denoted as $A \times B$ is the set of all ordered pair of the form (a, b) where $a \in A$ and $b \in B$, i.e.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

$$\text{If } A = \phi \text{ or } B = \phi, \text{ then } A \times B = \phi$$

Example 1 :

Let $A = \{ a, b, c \}$ and

$B = \{ 1, 2 \}$

Then $A \times B = \{ (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \}$

Example 2 :

Let A = Set of students = { Shilpa, Ramesh, Aparna }

and B = Set of marks in Discrete Structures = { 65, 56, 72 },

then $A \times B$ = { (Shilpa, 65), (Shilpa, 56), (Shilpa, 72), (Ramesh, 65), (Ramesh, 56), (Ramesh, 72), (Aparna, 65), (Aparna, 56), (Aparna, 72) }

Note : From this point of view a relation from A to B is simply a subset of $A \times B$ (giving the related pairs), and, conversely, any subset of $A \times B$ can be considered a relation, even if it is an unfamiliar relation for which we have no name or alternative description. We choose this approach for defining relations.

3.1.2 Definition :

Let A and B be nonempty sets. A relation R from A 'to' B is a subset of $A \times B$. If $R \subseteq A \times B$ and $(a, b) \in R$, we say that a 'is related to' b by R, and we also write a R b. If a is not related to b by R, we write a $\not R$ b. Frequently, A and B are equal. In this case, we often say that $R \subseteq A \times A$ 'is a relation on' A, instead of a relation from A to A.

Example 1 :

Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$

Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B.

Example 2 :

Let A and B be sets of real numbers. We define the following relation R (equals) from A to B.

a R b if and only if $a = b$

Example 3 :

Let $A = \{1, 2, 3, 4, 5\}$. Define the following relation R (less than) on A :

a R b if and only if $a < b$.

Then $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Example 4 :

Let $A = \mathbb{R}$, the set of real numbers. We define the following relation R on A :

x R y if and only if x and y satisfy the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

The set R consists of all points on the ellipse shown in Fig. 3.1.

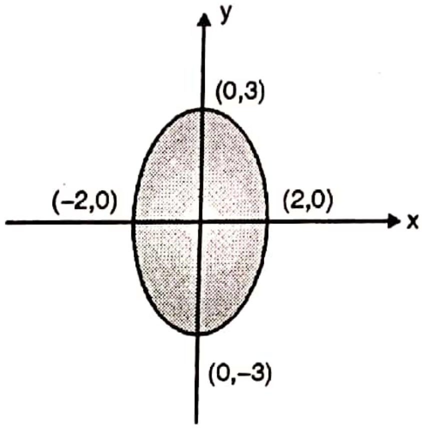


Fig. 3.1

Example 5 :

Let A = the set of all lines in the plane. Define the following relation R on A :

l_1 R l_2 if and only if l_1 is parallel to l_2.

Example 6 : An airline services the five cities c_1, c_2, c_3, c_4 and c_5. Table 3.1 gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is \$ 100, while the cost of going from c_4 to c_2 \$ 200.

Table 3.1

From \ To	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

We now define the following relation R on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\}$: c_i R c_j if and only if the cost of going from c_i to c_j is defined and less than or equal to \$ 180. Find R.

Solution :

The relation R is the subset of $A \times A$ consisting of all cities (c_i, c_j), where the cost of going from c_i to c_j is less than or equal to \$ 180. Hence

$$R = \{ (c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_4), (c_3, c_1), (c_3, c_2), (c_4, c_3), (c_4, c_5), (c_5, c_2), (c_5, c_4) \}$$

Definition :

Let $\{ A_1, A_2, \ldots, A_n \}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times \ldots \times A_n$ is called an **n-ary relation** on $A_1, A_2, \ldots, A_n$.

If $R = \phi$, then R is called **void** or **empty relation.**

If $R = A_1 \times A_2 \times \ldots \times A_n$, then R is called the **universal** relation.

If $A_i = A$ for all i, then R is called an 'n - ary relation on A'.

If $n = 1, 2$ or 3, then R is called a **unary, binary** or **ternary** relation respectively.

Among the relations, binary relations are the most important being widely used in various applications.

Example :

$$\text{Let } A = \{a, b\}, B = \{\alpha, \beta\}, \text{ and } C = \{1, 2\}$$

Ternary relation among 3 sets A, B and C is defined as subset of the cartesion product of the 2 sets $A \times B$ and C, denoted $(A \times B) \times C$.

$$A \times B = \{(a,\alpha), (a, \beta), (b, \alpha), (b, \beta)\}$$

$$(A \times B) \times C = \{((a, \alpha), 1), ((a, \beta), 1), ((b, \alpha), 1), ((b, \beta), 1), ((a, \alpha), 2), ((a, \beta), 2), ((b, \alpha), 2), ((b, \beta), 2)\}$$

3.1.3 Set Arising from Relations :

3.1.3.1 Definition :

Domain of relation R :

Let $R \subseteq A \times B$ be a relation from A to B. The **domain** of R, denoted by **Dom (R)**, is the set of elements in A that are related to some element in B. In other words, Dom (R), a subset of A, is the set of all first elements in the pairs that make up R.

Range of relation R :

Similarly, we define the **range** of R, denoted by **Ran (R)**, to be the set of elements in B that are second elements of pairs in R, that is, all elements in B that are related to some element in A.

Elements of A that are not in Dom (R) are not involved in the relation R in any way. This is also true for elements of B that are not in Ran (R).

Example 1 :

Let $A = \{1, 2, 3\}$, $B = \{r, s\}$

and $R = \{(1, r), (2, s), (3, r)\}$

$\text{Dom}(R) = \{1, 2, 3\} = A$

$\text{Ran}(R) = \{r, s\} = B$

Example 2 :

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$

$a\,R\,b$, if and only if $a < b$

$\therefore$ $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

$\text{Dom}(R) = \{1, 2, 3, 4\}$

$\text{Ran}(R) = \{2, 3, 4, 5\}$

If R is a relation from A to B and $x \in A$, we define R(x), the **R-relative set of x**, to be the set of all y in B with the property that x is R - related to y. Thus, in symbols.

$$R(x) = \{y \in B \mid x\,R\,y\}$$

Similarly, if $A_1 \subseteq A$, then $R(A_1)$, the **R-relative set of A_1**, is the set of all y in B with the property that x is R-related to y for some x in A_1. That is,

$$R(A_1) = \{y \in B \mid x\,R\,y \text{ for some } x \text{ in } A_1\}$$

From the preceding definitions, we see that $R(A_1)$ is the union of the sets $R(x)$, where $x \in A_1$. The sets R (x) play an important role in the study of many types of relations.

Example :

Let $A = \{a, b, c, d\}$, R is Relation from A to A.

$R = \{(a, a), (a, b), (b, c), (c, a), (d, c), (c, b)\}$

Then $R(a) = \{a, b\}$, $R(b) = \{c\}$, and if $A_1 = \{c, d\}$, then

$R(A_1) = \{a, b, c\}$

The following theorem shows the behaviour of the R-relative sets with regard to basic set operations.

3.1.3.2 Theorem :

Theorem 1 :

Let R be a relation from A to B, and let A_1 and A_2 be subsets of A. Then

(i) If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$ (ii) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(iii) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Proof :

(i) If $y \in R(A_1)$,

then $x\,R\,y$ for some $x \in A_1$.

Since $A_1 \subseteq A_2$, $x \in A_2$

Thus, $y \in R(A_2)$, which proves part (i)

(ii) If $y \in R(A_1 \cup A_2)$, then by definition $x\,R\,y$ for some x in $A_1 \cup A_2$.

If x is in A_1,

then, since x R y, we must have $y \in R(A_1)$.

By the same argument, if x is in A_2,

then $y \in R(A_2)$.

In either case, $y \in R(A_1) \cup R(A_2)$. Thus we have shown that,

$R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2)$.

Conversely, since $A_1 \subseteq (A_1 \cup A_2)$,

Part (i) tells us that $R(A_1) \subseteq R(A_1 \cup A_2)$.

Similarly, $R(A_2) \subseteq R(A_1 \cup A_2)$.

Thus $R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$, and therefore part (ii) is true.

(iii) If $y \in R(A_1 \cap A_2)$,

then for some x in $A_1 \cap A_2$, x R y.

Since x is in both A_1 and A_2, it follows that y is in both $R(A_1)$ and $R(A_2)$; that is

$y \in R(A_1) \cap R(A_2)$. Thus part (iii) holds.

It is a useful and easily seen fact that the sets R (a), for a in A, completely determine a relation R. We state this fact precisely in the following theorem.

Theorem 2 : Let R and S be relations from A to B, If R (a) = S (a) for all a in A, then R = S.

Proof : If a R b, then $b \in R(a)$. Therefore $b \in S(a)$ and a S b. A completely similar argument shows that, if a S b, then a R b. Thus R = S.

3.1.4 Representation of Relation :

Graphical and tabular form :

A relation between two finite sets can be presented in tabular form and Graphical form.

For example :

Let $A = \{a, b, c, d\}$

$B = \{\alpha, \beta, \zeta\}$

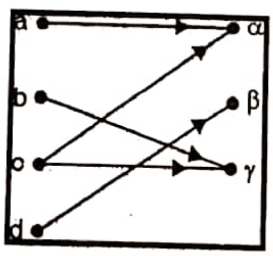
and R is a relation from A to B.

$R = \{(a, \alpha), (b, \zeta), (c, \alpha), (c, \zeta), (d, \beta)\}$

Above relation we can represent in tabular form and graphical form

Tabular form :

	α	β	ζ
a	√		
b			√
c	√		√
d		√	



Matrix Form :

We can represent a relation between two finite sets with matrix as follows. If $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets containing m and n elements, respectively, and R is a relation from A to B, we represent R by the $m \times n$ matrix $M_R = [m_{ij}]$, which is defined by

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the **Matrix of R** often M_R provides an easy way to check whether R has a given property.

3.1.5 Examples :

Example 1 :

Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$

Let R is a relation from set A to set B.

$R = \{(1, r), (2, s), (3, r)\}$

Then the matrix of R is,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Conversely, given sets A and B with $|A| = m$ and $|B| = n$, an $m \times n$ matrix whose entries are zeros and ones determines a relation, as is illustrated in the following example.

Example 2 :

Consider the matrix.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Since M is 3×4, we let

$A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

Then $(a_i, b_j) \in R$ if and only if $m_{ij} = 1$

Thus $R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$

Example 3 : Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$

Let $R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$

Find the relation matrix.

Example 13 : If π_1 : 1, 2, 4, 3 and π_2 : 3, 5, 6, 4, find the composition $\pi_2 \circ \pi_1$.

Solution :

The composition of π_1 and π_2 is the path $\pi_2 \circ \pi_1$: 1, 2, 4, 3, 5, 6, 4 from vertex 1 to vertex 4 of length 6.

Example 14 : If π_1 : 1, 7, 5 and π_2 : 5, 6, 7, 4, 3 find the composition $\pi_2 \circ \pi_1$.

Solution :

The composition of π_1 and π_2 is the path $\pi_2 \circ \pi_1$: 1, 7, 5, 6, 7, 4, 3 from vertex 1 to vertex 3 of length 6.

Example 15 : Let R be the relation on set A. Prove that there is a path of length n from a to b if and only if $(a, b) \in R^n$. **MU - May 03**

Solution :

Given R is relation on set A.

$\therefore$ The path of length n in R from A to B is a finite sequence.

$\pi_1 : a_1\ x_1, x_2, x_3, \ldots\ x_{n-1}\ b$

beginning with a and ending with b such that $a\ R\ x_1, x_1, Rx_2 \ldots x_{n-1}\ R\ b$

The path of length n involves n + 1 elements of A which may or may not be distinct.

For 'n' a positive integer we define relation R^n on A as $x\ R^n\ y$ which means that there is a path of length n from x to y in R.

It follows that path of length n from a to b.

if $a\ R^n\ b$ i.e. $(a, b) \in R^n$.

Syllabus Topic : Properties and types of binary Relations

3.4 Properties of Relations :

In many applications to computer science, we deal with relations on a set A, rather than relations from A to B. These relations have certain properties which are useful in storing data, more efficiently, on the computer.

3.4.1 Reflexive Relations :

A relation R on a set A is **reflexive** if for 'every' element $a \in A$, a R a, i.e. $(a, a) \in R$. R is not a reflexive relation if for 'some' element $a \in A$, $a \not R a$, i.e. $(a, a) \notin R$.

Examples 1 :

Let A = {a, b} and let R = {(a, a), (a, b), (b, b)}.

Then R is reflexive.

Example 2 :

Let A = {1,2} and let R = {(1,1), (1, 2)}.

R is not reflexive since (2,2) $\notin$ R.

We can identify a reflexive relation by its matrix as follows. The matrix of a reflexive relation must have all 1's on its main diagonal.

Similarly, we can characterize the digraph of reflexive relation as follows. A reflexive relation has a cycle of length 1 at every vertex.

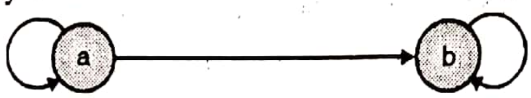


Fig. 3.28

Finally, we may note that if R is reflexive on a set A, then Dom (R) = Ran (R) = A.

Diagraph of reflexive relation.

Let A = {a, b} and let R = {(a, a), (a, b), (b, b)}.

Following Fig. 3.28 shows the digraph of R.

3.4.2 Irreflexive Relations :

A relation R on a set A is **irreflexive** if $a \not R a$, i.e. (a,a) $\notin$ R for every a $\in$ A.

Thus R is irreflexive if no element is related to itself.

Example 1 :

Let A = {1, 2} and let R = {(1, 2), (2, 1)}.

Then R is irreflexive since (1, 1) (2, 2) $\notin$ R.

Example 2 :

Let A = {1, 2} and let R = {(1, 2), (2, 2)}.

Then R is not irreflexive since (2, 2) $\in$ R.

Note that R is not reflexive either; since (1, 1) $\notin$ R.

Example 3 :

Let A be a nonempty set. Let R = $\phi \subseteq A \times A$, the 'empty relation'. Then R is not reflexive, since (a, a) $\notin$ R for all a $\in$ A (the empty set has no elements.) However, R is irreflexive.

We can identify irreflexive relation by its matrix as follows. The matrix of irreflexive relation must have all o's on its main diagonal.

Similarly we can characterize the digraph of a irreflexive relation as follows. A irreflexive relation has no cycle of length 1.

3.4.3 Symmetric Relations :

A relation R on a set A is **symmetric** if whenever a R b, then b R a. It then follows that R is not symmetric if we have some a and b $\in$ A with a R b, but $b \not R a$.

Examples :

1. Let A be set of p le. Let a R b if a is a friend of b. Then obviously b is related to a. Hence the relation of being :nd" is a symmetric relation.
2. Let A be set of s in a plane. For lines $l_1, l_2 \in A$, Let l_1 R l_2 if l_1 is parallel to l_2. Then l_2 R l_1, since the ion of being "parallel to" is a symmetric relation.
3. Let A be set of pe and let a R b if a is brother of b. Then this is not a symmetric relation since b can be the siste a. This relation will be symmetric only if A is the set of males.
4. Let A = {1, 2} a t R = {(1, 1), (2, 2)}. This is an example of a symmetric relation which is also reflexive.
5. Let A = {1, 2, } and let R = {(1, 2), (2, 2), (3, 4), (4, 1)}. R is not symmetric, since (1, 2) ∈ R, but (2, ∉ R.

The relation mat corresponding to a symmetric relation is a symmetric matrix. The matrix $M_R = [m_{ij}]$ of a symmetr lation satisfies the property that.

if = 1, then $m_{ji} = 1$.

Moreover, if m_{ji} = then $m_{ij} = 0$.

Thus M_R is a mat uch that each pair of entries, symmetrically placed about the main diagonal, are either both 0 or both

3.4.3.1 Digraph of S metric Relation :

We consider the d phs of symmetric relations in more detail.

The digraph of a metric relation R has the property that if there is an edge from vertex i to vertex j, then there is an e from vertex j to vertex i. Thus, if two vertices are connected by an edge, they must always be con ed in both directions. Because of this, it is possible and quite useful to give a different representation a symmetric relation. We keep the vertices as they appear in the digraph, but if two vertices a and b connected by edges in each direction, we replace these two edges with one undirected edge or a "tw vay street". This undirected edge is just a single line without arrows and connects a and b. The res ng diagram will be called the 'graph' of the symmetric relation. (Graph will be given a more general n ning in Chapter 5).

Example :

Let A = {a, b, , e} and let R be the symmetric relation given by,

R = {(a, b), , a), (a, c), (c, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c)}

The usual digraph R is shown in Fig. 3.29 while Fig. 3.30 shows the graph of R. Note that each undirected edge corr onds to two ordered pairs in the relation R.

An undirected edg etween a and b, in the graph of a symmetric relation R, corresponds to a set {a, b} such that (a, b) ∈ and (b, a) ∈ R. Sometimes we will also refer to such a set {a, b} as an 'undirected edge' of the rel on R and call a and b 'adjacent vertices'.

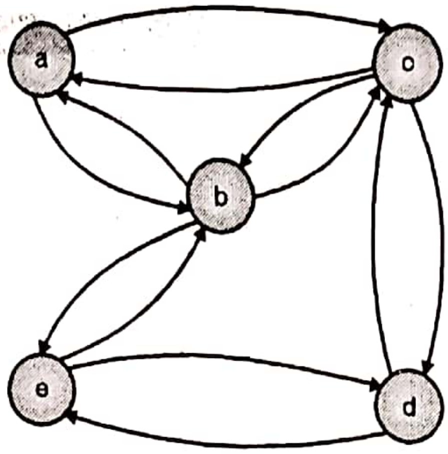


Fig. 3.29 : Diagraph of R

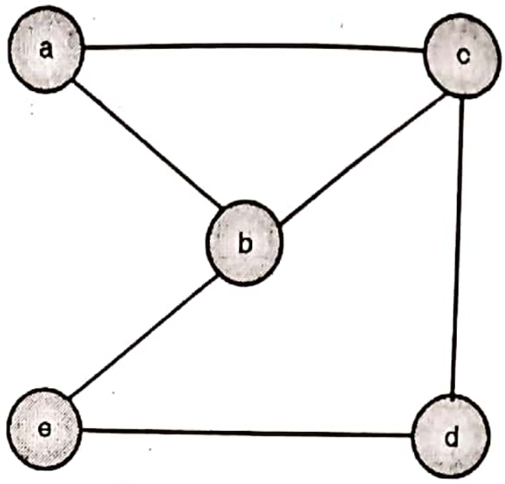


Fig. 3.30 : Graph of R

3.4.3.2 Connected Relation :

A symmetric relation R on a set A is called **connected** if there is a path from any element of A to any other element of A. This simply means that the graph of R is all in one piece. In Fig. 3.31. We show the graphs of two symmetric relations. The graph in Fig. 3.31 (a) is connected, whereas that in Fig. 3.31 (b) is not connected.

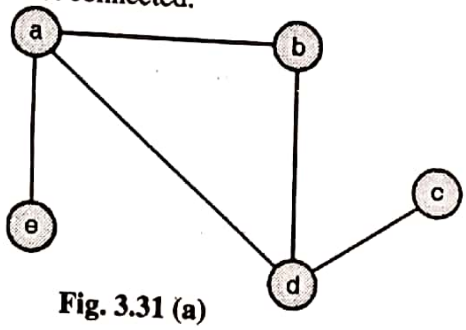


Fig. 3.31 (a)

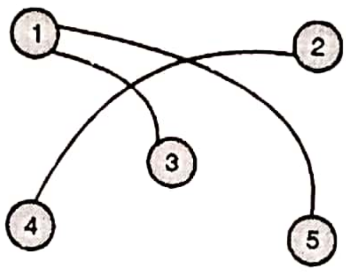


Fig. 3.31 (b)

3.4.4 Asymmetric Relations :

A relation R on a set A is **asymmetric** if whenever a R b, then b **R** a. It then follows that R is not asymmetric if we have some a and b $\in$ A with both a R b and b R a.

Examples :

1. Let $A = \mathbb{R}$, the set of real numbers and let R be the relation '<'.

   If $a < b$, then $b \not< a$ (b is not less than a), so '<' is asymmetric.

2. Let $A = \{1, 2, 3, 4\}$ and let

   $$R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$$

   Then, R is not asymmetric, since $(2, 2) \in R$.

3. Let $A = Z^+$, the set of positive integers, and let

   $$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}.$$

   If $a = b = 3$, say then a R b and b R a, so R is not asymmetric.

3 . Ar ymme : Rela ns :

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A rel n R on set A ntisym ric if never b and a, th = b. c positiv this d tion is R is ymmet f wher r a ≠ b n a R b R f s that not an metric e have d b in ≠ b, a oth a R d b R

E ple 1 : Let A = he set al nun and le be the ion '≤'. antisy tric ?

S on :

If a ≤ d b ≤ a = b ce '≤' antisy tric re n.

E ple 2 : Let A = e set egers, let

R = {(a, A × A < b}

So that the rel 'less t . Is R s etric, metric antisym tric ?

S on :

Symm : If a < en it is true th < a, so not sy etric.

Asym y : If a then b (b is n ss thar so R is mmetr

Antisy etry : I b, then her a ≮ r b ≮ a that R tisymr ic.

E ple 3 : Let A = 2, 3} let R = 1, 2), ( ), (2, s R sy etric, metric, ntisym ic?

Sc on :

Symm : R is ymmet ither si (2, 3) ∈ but (3, R

Asymi y : R is not as netric s both ( ) and (2 ∈ R.

Antisy etry : R ot anti metric (1, 2) (2, 1)

Ex ple 4 : Let A = , 3, 4} let R = 2), (2, 3, 4), (4 . Is R s etric, metric ntisym ic ?

Sc on :

Symm : R is n ymmet since (1 ∈ R, b 2, 1) ∉

Asymr y : R is asymm c, since 2) ∈ R.

Antisy etry : R ntisym ic, sinc a ≠ b, e r (a, b) R or (b, R.

Ex ple 5 : Let A = Z he set c ositive in ers, and R = {(a) ∈ A × a divide .

s R anti metric ?

So on :

If a | b b | a, th a = b, so R is antis metric.

3.4.6 Transitive Relations :

A relation R on a set A is **transitive** if whenever a R b and b R c, then a R c. It follows that a relation R is not transitive if there exist a, b and c in A so that a R b and b R c, but a $\not R$ c. If such a, b, and c do not exist, then R is transitive.

Example 1 :

Let A be set of people and let R be the relation of being "brother of". Then a is brother of b and b is brother of c implies a is brother of c.

Hence R is transitive.

Example 2 :

Let A = Z, the set of integers, and let R be the relation less than.

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To see whether R is transitive, we assume that a R b and b R c. Thus a < b and b < c. It then follows that a < c, so a R c. Hence R is transitive.

Example 3 : Let $A = Z^+$, the set of positive integers, and let

$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$

Is R transitive?

Solution :

Suppose that a R b and b R c, so that $a \mid b$ and $b \mid c$. It then does follow that $a \mid c$. Thus R is transitive.

Example 4 : Let A = {1, 2, 3, 4} and let

R = {(1, 2), (1, 3), (4, 2)}

Is R transitive ?

Solution :

Since there are no elements a, b, and c in A such that a R b and b R c, but a $\not R$ c, we conclude that R is transitive.

Example 5 : Let A = N, the set of natural numbers and let.

$R = \{(a, b) \mid a + b \text{ is an odd number}\}$

Is R transitive ?

Solution :

R is not transitive since (1, 2) and (2, 1) $\in$ R , but (1, 1) $\notin$ R, (since 1 + 1 is not an odd number).

A relation R is transitive if and only if its matrix $M_R = [m_{ij}]$ has the property.

if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$.

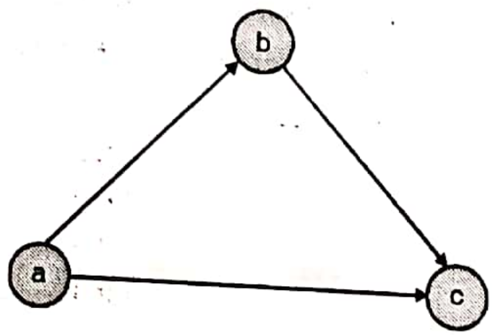

Fig. 3.32

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Digraph of transitive relation.

Let $A = \{a, b, c\}$ and let

$R = \{(a, b), (b, c), (a, c)\}$. Fig. 3.32 shows the digraph of R.

3.4.6.1 Theorem :

To see what transitivity means for the digraph of a relation, we translate the definition of transitivity into geometric terms.

If we consider particular vertices a and c, the conditions a R b and b R c mean that there is a path of length 2 in R from a to c. In other words, a R^2 c. Therefore, we may rephrase the definition of transitivity as follows : If a R^2 c, then a R c; that is , $R^2 \subseteq R$ (as subsets of $A \times A$). In other words, if a and c are connected by a path of length 2 in R, then they must be connected by a path of length 1.

We can slightly generalize the foregoing geometric characterization of transitivity as follows :

Theorem 1 : A relation R is transitive if and only if it satisfies the following property : If there is a path of length greater than 1 from vertex a to vertex b, there is a path of length l from a to b (that is, a is related to b). Algebraically stated, R is transitive if and only if $R^n \subseteq R$ for all $n \geq 1$.

Proof : The proof is left to the reader (For proof refer Example 19 of this section i.e. section 3.6.4). It will be convenient to have a restatement of some of these relational properties in terms of R-relative sets. We list these statements without proof.

Theorem 2 : Let R be a relation on a set A. Then

(i) Reflexivity of R means that $a \in R(a)$ for all a in A.

(ii) Symmetry of R means that $a \in R(b)$ if and only if $b \in R(a)$.

(iii) Transitivity of R means that if $b \in R(a)$ and $c \in R(b)$, then $c \in R(a)$.

3.4.7 Examples :

In Examples 1 through 4, let $A = \{1, 2, 3, 4\}$. Determine whether the relation is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive :

Example 1 : $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 3), (3, 4), (4, 4)\}$ **May 10**

Solution :

(i) Reflexive : Given relation R is reflexive,
Since $(1, 1), (2, 2), (3, 3), (4, 4) \in R$, i.e. all elements belonging to A are related to itself.

(ii) Symmetric : Given relation R is symmetric, since if $(1, 2) \in R$ then $(2, 1) \in R$ and if $(3, 4) \in R$ then $(4, 3)$ is also $\in R$.

(iii) Transitive : Given relation R is transitive

Since,

1 R 2	and	2 R 1	implies	1 R 1
2 R 2	and	2 R 1	implies	2 R 1
3 R 3	and	3 R 4	implies	3 R 4
4 R 4	and	4 R 3	implies	4 R 3

Hence R is reflexive, symmetric, transitive.

3.5 Equivalence Relations :

3.5.1 Definition :

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A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive. The following are some of the common but important examples of equivalence relations.

Examples :

(i) Let A = I R and R be 'equality' of numbers.

(ii) Consider all subsets of a universal set and R be the relation, 'equality' of sets.

(iii) A is the set of triangles and R is 'similarity' of triangles.

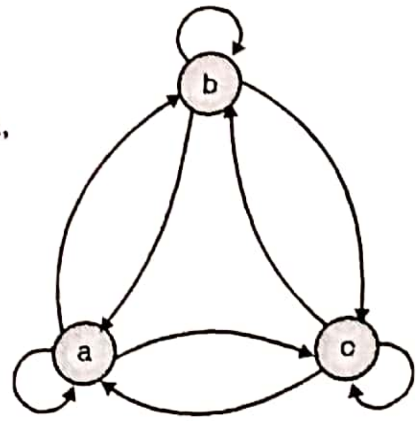

Fig. 3.40

(iv) A is a set of students and R is the relation of being in the same class or division.

(v) Let A be set of statement forms and R be the relation of 'Logical equivalence.'

(vi) A is set of lines in a plane and R is the relation of lines being 'parallel'.

The digraph of an equivalence relation will have the following characteristics.

(i) Every vertex will have a loop.

(ii) If there is an arc from a to b, there should be an arc from b to a.

(iii) If there is an arc from a to b and arc from b to c, there should be an arc from a to c

In short, the following is a typical digraph of an equivalence relation.

3.5.2 Equivalence Relations and Partitions :

If **P** is a partition of a set A, then **P** can be used to construct an equivalence relation on A.

Theorem 1 : Let **P** be a partition of a set A. (Sets in **P** are called the blocks of **P**)

Define the relation R on A as follows.

a R b if and only if a and b are members of the same block.

Then R is an equivalence relation on A.

Proof :

(i) If $a \in A$, then clearly a is in the same block as itself; so a R a.

(ii) If a R b, then a and b are in the same block; so b R a.

(iii) If a R b and b R c, then a, b and c must all lie in the same block of **P**. Thus a R c.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. R will be called the **equivalence relation determined by P**.

3.5.2.1 Refinement of Partition :

Definition : Let **P** and **P'** be partitions of a non-empty set A. The P' is called a refinement of **P** if every block (element) of P' is contained in a block of P.

Example :

Let $A = \{a, b, c\}$

Let $\mathbf{P} = \{\{a\}, \{b, c\}\}$ and

$\mathbf{P'} = \{\{a\}, \{b\}, \{c\}\}$

Then **P'** is a refinement of **P**.

Let R and R' be the equivalence relation induced by **P** and **P'** respectively. Then the following theorem relates R' and R.

Theorem 1 : Let **P** and **P'** be partitions of non empty set A and let R, R' be the equivalence relations induced by **P** and **P'** respectively. Then **P'** refines **P** if and only if $R' \subseteq R$.

Proof : Let **P '** be a refinement of **P**. We to prove that $R' \subseteq R$. Let a R' b. Then the some block $A_i' \in$ **P ' P'** such that a, b $\in$ A. Since **P '** refines **P**, $A_i' \subseteq A_i$ for some block $A_i \in$ **P**. Hence a, b $\in A_i$ which implies a R b. Hence $R' \subseteq R$.

Next, let $R' \subseteq R$. We have to prove **P'** is a refinement of **P**. Let $A_i' \in P'$ and $a \in A_i'$. Then $A_i' = R'(a)$. Let $x \in A_i'$. This implies xR′a and hence x Ra since $R' \subseteq R$. This means that $R'(a) \subseteq R(a)$. Denote by A_i the block R(a). Then $A_i' \subseteq A_i$ which means that **P'** is a refinement **P**.

3.5.3 Equivalence Classes :

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Let R be an equivalence relation on a set A. For every $a \in A$, let $[a]_R$ (or R (a)) denote the set $\{x \in A \mid x R a\}$. Then $[a]_R$ is called as the **equivalence class of a with respect to R.**

$[a]_R \neq \phi$ since $a \in [a]_R$

The 'rank' of R is the number of distinct equivalence classes of R if the number of classes is finite; otherwise the rank is said to be infinite.

In what follows, we will drop the suffix R, and denote the equivalence class of a simply as [a]. $[a]_R$ is also denoted as R(a).

The following theorem gives an important characterisation of the equivalence classes.

Theorem 1 : Let R be an equivalence relation on a set A. The following hold :

(i) For all a, b, $\in$ A, either [a] = [b] or

$$[a] \cap [b] = \phi$$

(ii)
$$A = \bigcup_{a \in A} [a]$$

Proof :

(i) If $A = \phi$, there is nothing to prove. Hence assume $A \neq \phi$. If $A = \{a\}$, singleton set, the result is trivially true. Therefore consider elements a, b, $\in$ A. Suppose $[a] \neq [b]$. Then we have to show that $[a] \cap [b] = \phi$. Suppose this is not true. Let $c \in [a] \cap [b]$ then c R a and c R b. Since R is symmetric it follows that a R c and c R b. But R is transitive as well. Hence, we have a R b, i.e. $b \in [a]$ and $a \in [b]$, which mean that $[a] = [b]$, a contradiction. Hence, $[a] \cap [b] = \phi$.

(ii) Clearly for each $a \in A$, $[a] \subseteq A$.

Hence, $\bigcup\limits_{a \in A} [a] \subseteq A$. Conversely, let $x \in A$. Then $x \in [a]$ for some $a \in A$. This implies that x R a, i.e. a R x. Hence $a \in [x]$ which means that $[a] = [x]$. Therefore $A \subseteq \bigcup\limits_{a \in A} [a]$.

Hence it follows that $A = \bigcup\limits_{a \in A} [a]$

3.5.4 Exercise Set

Example 1 : Let A = {a, b, c} and let

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether R is an equivalence relation.

Solution :

$$R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$$

R is reflexive since (a, a), (b, b), (c, c) $\in$ R

R is symmetric since (b, c) $\in$ R $\rightarrow$ (c, b) $\in$ R

R is transitive since,

(b,b)	and	(b, c) $\in$ R	implies	(b, c) $\in$ R,
(b, c)	and	(c, b) $\in$ R	implies	(b, b) $\in$ R,
(c, c)	and	(c, b) $\in$ R	implies	(c, b) $\in$ R,
(c, b)	and	(b, b) $\in$ R	implies	(c, b) $\in$ R,
(c, b)	and	(b, c) $\in$ R	implies	(c, c) $\in$ R,
(b, c)	and	(c, c) $\in$ R	implies	(b, c) $\in$ R,

Hence R is an equivalence relation.

Example 2 : Let A = Z, the set of integers, and let R be defined by a R b if and only if $a \leq b$. Is R an equivalence relation? **MU - Dec. 08**

Solution :

Since $a \leq a$, R is reflexive.

If $a \leq b$, it need not follow that $b \leq a$, so R is not symmetric.

$$\therefore \; af = be$$

$\therefore$ (a, b) R (e, f)

Hence R is transitive.

Thus R is an equivalence relation.

Example 56 : Suppose that A is non empty set and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f(y). Show that R is an equivalence relation on A. **MU - Dec.16**

Solution :

- (x, x) $\in$ R since f(x) = f(x). Hence R is reflexive + (x, y) $\in$ R if and only if f(x) = f(y) which holds if and only if f(y) = f(x). Hence (x, y) $\in$ R if and only if (y, x) $\in$ R. Hence R is symmetric.
- If (x, y) $\in$ R and (y, z) $\in$ v R then f(x) = f(y) and f(y) = f(z)

  Hence f(x) = f(z). Thus (x, z) $\in$ R

Hence R is transitive.

Syllabus Topic : Operations on relations

3.6 Manipulation of Relations :

Now that we have investigated the classification of relations by properties they do or do not have, we next define some operations on relations.

3.6.1 Operations on Relations :

1 **Complement of a relations :** Let R and S be relations from a set A to a set B. (R and S are simply subsets of A $\times$ B). The complement of $\bar{R}$, R is referred to as the 'complementary relation', which is defined below,

a $\bar{R}$ b if and only if a Rb, i.e.

$$\bar{R} = \{(a, b) \mid (a, b) \in R\}$$

2. **Intersection of relations :** Let R and S be relation from a set A to set B. In Relational terms, we see that a $(R \cap S)$ b means that a R b and a S b.

3. **Union of relations :** Let R and S be relation from a set A to set B. We see that a $(R \cup S)$ b means that a R b or a S b.

4. **Inverse of a relation :** Let R is a relation from A to B. Inverse of a Relation, usually written R^{-1}. The relation R^{-1} is a relation from B to A (reverse order from R) defined by

   b R^{-1} a if and only if a R b.

It is clear from this that $(R^{-1})^{-1}$ = R. It is not hard to see that Dom (R^{-1}) = Ran (R) and Ran (R^{-1}) = Dom (R). We leave these simple facts for the reader to check.

3.6.2 Theorems :

Theorem 1 : Suppose that R and S are relations from A to B.

(a) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$ **MU - Dec. 02, Dec. 05**

(b) If $R \subseteq S$, then $\bar{S} \subseteq \bar{R}$

(c) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(d) $\overline{(R \cap S)} = \bar{R} \cup \bar{S}$ and $\overline{R \cup S} = \bar{R} \cap \bar{S}$

MU - May 06, May 12

Proof :

Parts (b) and (d) are special cases of general set properties (Refer chapter 1–set theory). The proof is straight forward and is left as an exercise.

We now prove part (a). Suppose that $R \subseteq S$ and let $(a, b) \in R^{-1}$. Then $(b, a) \in R$, So $(b, a) \in S$. This, in turn, implies that $(a, b) \in S^{-1}$. Since each element of R^{-1} is in S^{-1}, we are done.

We next prove part (c). For the first part, suppose that $(a, b) \in (R \cap S)^{-1}$. Then $(b, a) \in R \cap S$, so $(b, a) \in R$ and $(b, a) \in S$. This means that $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$, so $(a, b) \in R^{-1} \cap S^{-1}$. The converse containment can be proved by reversing the steps. A similar argument works to show that $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

The relations R and R^{-1} can be used to check if R has the properties of relations that we presented in section 3.4. For instance, we saw earlier that R is symmetric if and only if $R = R^{-1}$. Here are some other connections between operations on relations and properties of relations.

Theorem 2 : Let R and S be relations on a set A.

(a) If R is reflexive, so is R^{-1}.

(b) If R and S are reflexive, then so are $R \cap S$ and $R \cup S$.

(c) R is reflexive if and only if $\bar{R}$ is irreflexive. **MU - Dec. 03, Dec. 05, May 07**

Proof :

(a) Let Δ be the equality relation on A. We know that R is reflexive if and only if $\Delta \subseteq R$. Clearly, $\Delta = \Delta^{-1}$, so if $\Delta \subseteq R$, then $\Delta = \Delta^{-1} \subseteq R^{-1}$ by theorem 1, so R^{-1} is also reflexive.

(b) To prove part (b), we note that if $\Delta \subseteq R$ and $\Delta \subseteq S$, then $\Delta \subseteq R \cap S$ and $\Delta \subseteq R \cup S$. Thus if R and S are reflexive, then so are $R \cap S$ and $R \cup S$.

(c) To show part (c), we note that a relation S is irreflexive if and only if $S \cap \Delta = \phi$. Then R is reflexive if and only if $\Delta \subseteq R$ if and only if $\Delta \cap \bar{R} = \phi$ if and only if R is irreflexive.

Example :

$$\text{Let } A = \{1, 2, 3\} \text{ and consider the two reflexive relations.}$$

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\} \text{ and}$$

$$S = \{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\} \text{ Then}$$

(a) $R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 2), (3, 3)\}$;

R and R^{-1} are both reflexive.

(b) $\bar{R} = \{(2, 1), (2, 3), (3, 1), (3, 2)\}$ is irreflexive while R is reflexive.

(c) $R \cap S = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$ and

$R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 2), (3, 3)\}$ are both reflexive.

Theorem 3 : Let R be a relation on a set A. Then

(a) R is symmetric if and only if $R = R^{-1}$

(b) R is anti-symmetric if and only if $R \cap R^{-1} \subseteq \Delta$.

(c) R is asymmetric if and only if $R \cap R^{-1} = \phi$

Proof :

The proof is straight forward and is left as an exercise.

Theorem 4 : Let R and S be relations on A.

(a) If R is symmetric, so are R^{-1} and $\bar{R}$

(b) If R and S are symmetric, so are $R \cap S$ and $R \cup S$. **MU - May 05, Dec. 06**

Proof :

(a) If R is symmetric, $R - R^{-1}$ and thus $(R^{-1})^{-1} = R = R^{-1}$, which means that R^{-1} is also symmetric. Also $(a, b) \in (\bar{R})^{-1}$ if and only if $(b, a) \in \bar{R}$ if and only if $(b, a) \notin R$ if and only if $(a, b) \notin R^{-1} = R$ if and only if $(a, b) \in \bar{R}$ so $\bar{R}$ is symmetric and part (a) is proved.

(b) The proof of part (b) follows immediately from Theorem 1(c).

Example :

Let $A = \{1, 2, 3\}$ and consider the symmetric relations.

$R = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$

and $S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

Then

(a) $R^{-1} = \{(1, 1), (2, 1), (1, 2), (3, 1), (1, 3)\}$ and

$\bar{R} = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$;

R^{-1} and $\bar{R}$ are symmetric.

(b) $R \cap S = \{(1, 1), (1, 2), (2, 1)\}$ and

$R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$ which are both symmetric.

Theorem 5 : Let R and S be relations on A.

(a) $(R \cap S)^2 \subseteq R^2 \cap S^2$

(b) If R and S are transitive, so is $R \cap S$.

(c) If R and S are equivalence relations, so is $R \cap S$.

Proof :

(a) We prove part (a) geometrically. We have a $(R \cap S)^2$ b if and only if there is a path of length 2 from a to b in $R \cap S$. Both edges of this path lie in R and in S, so a R^2 b and a S^2 b, which implies that a $(R^2 \cap S^2)$ b.

(b) To show part (b), recall from section 3.4 that a relation T is transitive if and only if $T^2 \subseteq T$. If R and S are transitive, then $R^2 \subseteq R$, $S^2 \subseteq S$, so $(R \cap S)^2 \subseteq R^2 \cap S^2$ [by part (a)] $\subseteq R \cap S$. so $R \cap S$ is transitive.

(c) We next prove part (c). Relations R and S are each reflexive, symmetric, and transitive. The same properties hold for $R \cap S$ from Theorems 2(b), 4(b), and 5(b), respectively. Hence $R \cap S$ is an equivalence relation.

3.6.3 Composition :

Now suppose that A, B and C are sets, R is a relation from A to B, and S is a relation from B to C. We can then define a new relation, the 'composition' of R and S, written 'S o R'. The relation S o R is a relation from A to C and is defined as follows.

If a is in A and c is in C, then a (SoR) C if and only if for some b in B, we have a R b and b S c. In other words, a is related to C by S o R if we can get from a to c in two stages : first to an intermediate vertex b by relation R and then from b to c by relation S. The relation So R might be thought of as "S following R" since it represents the combined effect of two relations, first R, then S.

3.6.3.1 Theorems :

Theorem 1 : Let R be a relation from A to B and let S be a relation from B to C. Then, if A_1 is any subset of A, we have $(S \circ R)(A_1) = S(R(A_1))$

Proof : If an element $z \in C$ is in $(S \circ R)(A_1)$ then $x (S \circ R) z$ for some x in A_1. By the definition of composition, this means that x R y and ySz for some y in B. Thus $y \in R(x)$, so $z \in S(R(x))$. Since $\{x\} \subseteq A_1$, $S(R(x)) \subseteq S(R(A_1))$, since if $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$.

Hence $z \in S(R(A_1))$, so $(S \circ R)(A_1) \subseteq S(R(A_1))$.

Conversely, suppose that $z \in S(R(A_1))$. Then $z \in S(y)$ for some y in $R(A_1)$ and, similarly, $y \in R(x)$ for some x in A_1. This means that x R y and y S z, so $x (S \circ R) z$. Thus $z \in (S \circ R)(A_1)$, so $S(R(A_1)) \subseteq (S \circ R)(A_1)$. This proves the theorem.

Theorem 2 : Let A, B, and C be sets, R a relation from A to B, and S a relation from B to C. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof :

Let $c \in C$ and $a \in A$. Then $(c, a) \in (S \circ R)^{-1}$ if and only if $(a, c) \in S \circ R$, that is, if and only if there is a $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$. Finally this is equivalent to the statement that $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$ that is, $(c, a) \in R^{-1} \circ S^{-1}$.

Note : In general $R \circ S \neq S \circ R$.