Double Integral

Eg.

Evaluate following integrals.

1. $f(x,y) = x^2+y^2$

$R: [-1, 1] \times [0, 1]$

$$\int_0^1 \int_{-1}^1 (x^2+y^2)\, dx\, dy$$

$$\int_0^1 \left(\left[\frac{x^3}{3}\right]_{-1}^1 + y^2 [x]_{-1}^1 \right) dy$$

$$\int_0^1 \left[\frac{2}{3} + 2y^2 \right] dy$$

$$\frac{2}{3}[y]_0^1 + 2\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

2. $$\int_0^1 \int_0^y xy\, e^{-x^2}\, dx\, dy$$

$$\int_0^1$$ let $x^2 = t$

$2x \cdot dx = dt$

$x\, dx = \frac{dt}{2}$

$$\int_0^1 y \int_0^{y^2} e^{-t} \frac{dt}{2}\, dy$$

$$\frac{1}{2} \int_0^1 \left[\frac{e^{-t}}{-1}\right]_0^y dy$$

$$y^2 = t$$
$$2y\,dy = dt$$
$$y\,dy = \frac{dt}{2}$$

$$\frac{1}{\cancel{2}4}\int_0^1 (1-e^{-t})\,dt$$

$$\frac{1}{4}\left[[t]_0^1 - \left[\frac{e^{-t}}{-1}\right]_0^1 \right]$$

$$\frac{1}{4}\left[1 + \left(e^{-1} - 1 \right) \right] = \frac{1}{4e}$$

Eg
$$\int_0^1 \int_0^{\sqrt{1+x^2}} \left(\frac{1}{1+x^2+y^2} \right) dy \, dx$$

1st

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2} \, dy \, dx$$

$$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} \right] dx$$

$$\frac{\pi}{4} \int_0^1 (1+x^2)^{-1/2} \, dx$$

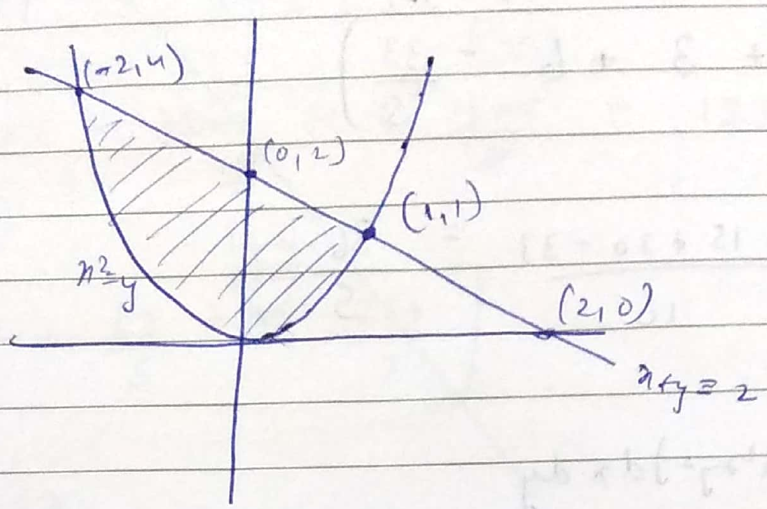
$1 + \sqrt{2}$

$$\frac{\pi}{4} \left[\log \left(x + \sqrt{1+x^2} \right) \right]_0^1$$

$$\frac{\pi}{4} \left[\log (1+\sqrt{2}) \right]$$

Evaluate,

$\iint y \, dx \, dy$, where R is region b/w parabola, $x^2 = y$ & line $x+y=2$



~~$y = 2 - x$~~ $\quad x+y=2$

$x^2 + x - 2 = 0$

$x = 1, -2$

$$\int_{-2}^{1} \int_{x^2}^{2-x} y \, dy \, dx$$

$$\int_{-2}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$\int_{-2}^{1} \frac{(2-x)^2 - x^4}{2} \, dx$$

$4 - 4x$

~~$\int_{-2}^{1} 2 - 2x \, dx = 2[x]_{-2}^{1} - 2\left(\frac{x^2}{2}\right)_{-2}^{1}$~~

$= 6 - 0 = 6$

$$\int_{-2}^{1}\left(\frac{4+x^2-4x-x^4}{2}\right)dx$$

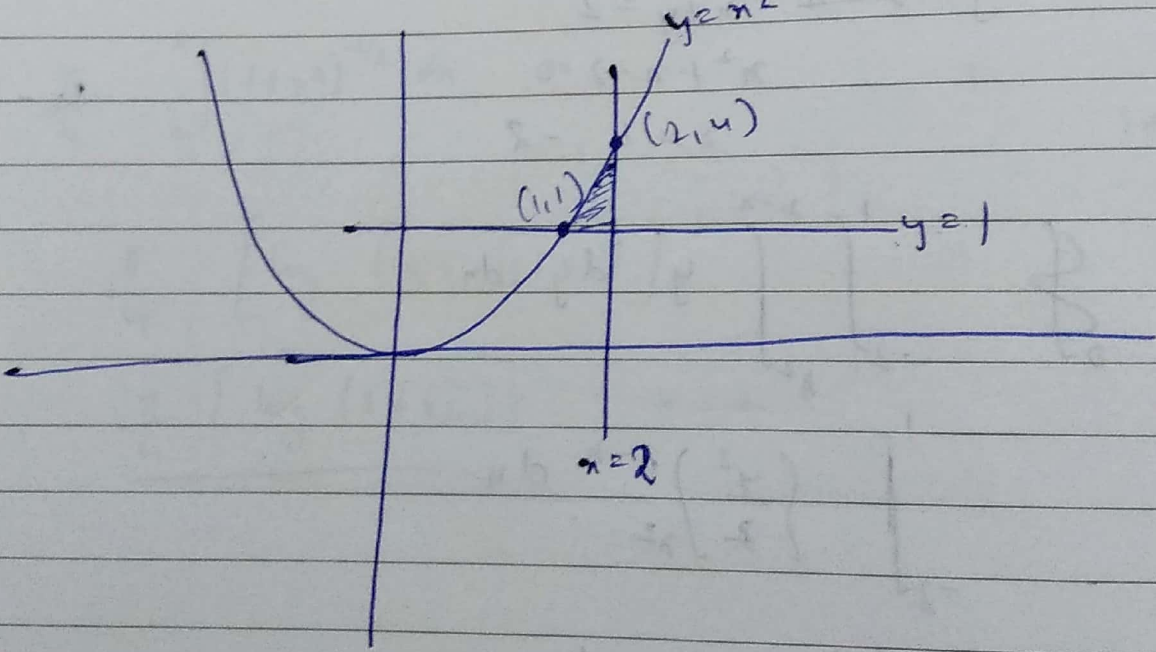
$$\frac{1}{2}\left(4[x]_{-2}^{1}+\left[\frac{x^3}{3}\right]_{-2}^{1}-2[x^2]_{-2}^{1}-\left[\frac{x^5}{5}\right]_{-2}^{1}\right)$$

$$\frac{1}{2}\left(12+3+6-\frac{33}{5}\right)$$

$$\frac{60+15+30-33}{10}=\frac{36}{5}$$

Q. $\iint_R (x^2+y^2)\,dx\,dy$

$R:\ y=x^2,\ x=2,\ y=1$



$$\int_{1}^{4}\int_{\sqrt{y}}^{2}(x^2+y^2)\,dx\,dy$$

$$\int_{1}^{4}\left(\left[\frac{x^3}{3}\right]_{\sqrt{y}}^{2}+y^2[x]_{\sqrt{y}}^{2}\right)dy$$

16

$$\int_1^4 \left(\frac{8 - y^{3/2}}{3} + 2y - y^{5/2}\right) dy$$

$$\frac{1}{3}\left[\frac{8}{8}[y]_1^4 - \left[\frac{y^{5/2}}{5/2}\right]_1^4 + 6\cdot\left[\frac{y^3}{3}\right]_1^4 - \left[\frac{y^{7/2}}{7/2}\right]_1^4\right] \quad 2?$$

$$\frac{1}{3}\left[8\times 3 - \frac{(32-1)2}{5} + \cancel{6}^{2}\cdot\frac{64-1}{\cancel{3}} - \frac{127\times 2}{7}\right]$$

$$\frac{1}{3}\left[24 - \frac{62}{5} + \overset{126}{\cancel{63\times 2}} - \frac{254}{7}\right]$$

$$\frac{237}{35}$$

$91\times\frac{2}{3}$

62

127

$\times 2$

237

$\times 3$

711

$2+\frac{1}{2}$

$$\int_1^4 \left(\frac{8 - y^{3/2}}{3} + y^2\left[2 - y^{1/2}\right]\right) dy$$

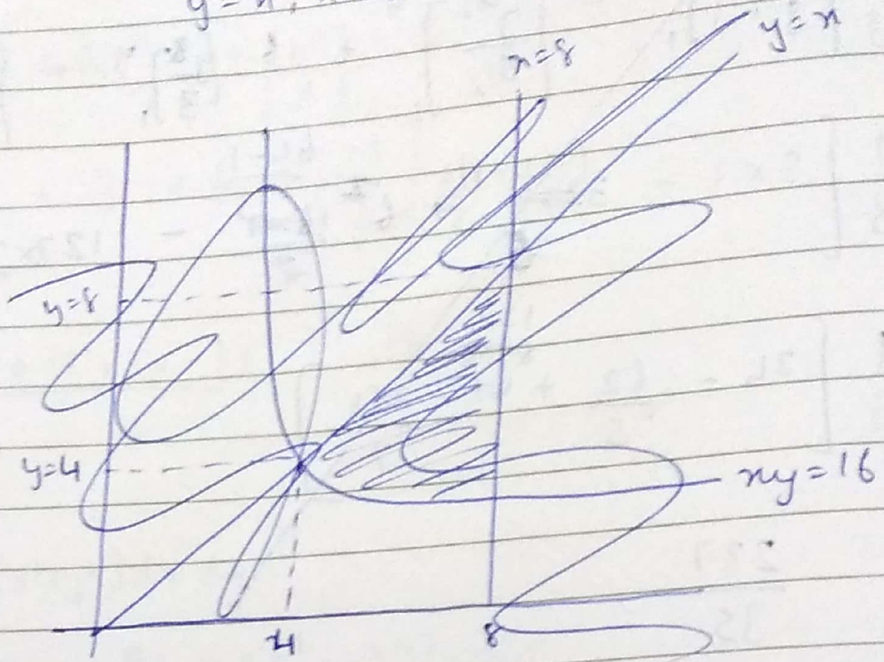
$$\frac{1}{3}\int_1^4 8 - y^{3/2} + 6y^2 - 3y^{5/2}\, dy$$

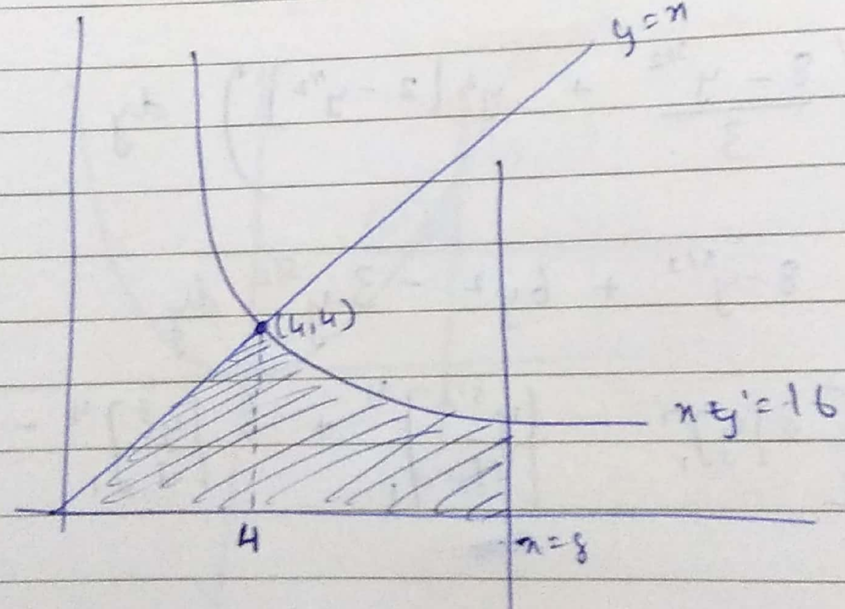
$$\frac{1}{3}\left[8[y]_1^4 - \left[\frac{y^{5/2}}{5/2}\right]_1^4 + 6\left[\frac{y^3}{3}\right]_1^4 - 3\left[\frac{y^{7/2}}{7/2}\right]_1^4\right]$$

Q. $\iint x^2 \, dx \, dy$

R: 1st quad bounded by $xy = 16$,

$y = x$, $x = 8$





$$\int_0^4 \int_0^x x^2 \, dy \, dx + \int_4^8 \int_0^{16/x} x^2 \, dy \, dx$$

$$\int_0^4 x^2 [y]_0^x \, dx + \int_4^8 x^2 [y]_0^{16/x} \, dx$$

PAGE No.
DATE / /

$$\int_0^4 x^3\,dx + \int_4^8 16x\,dx$$

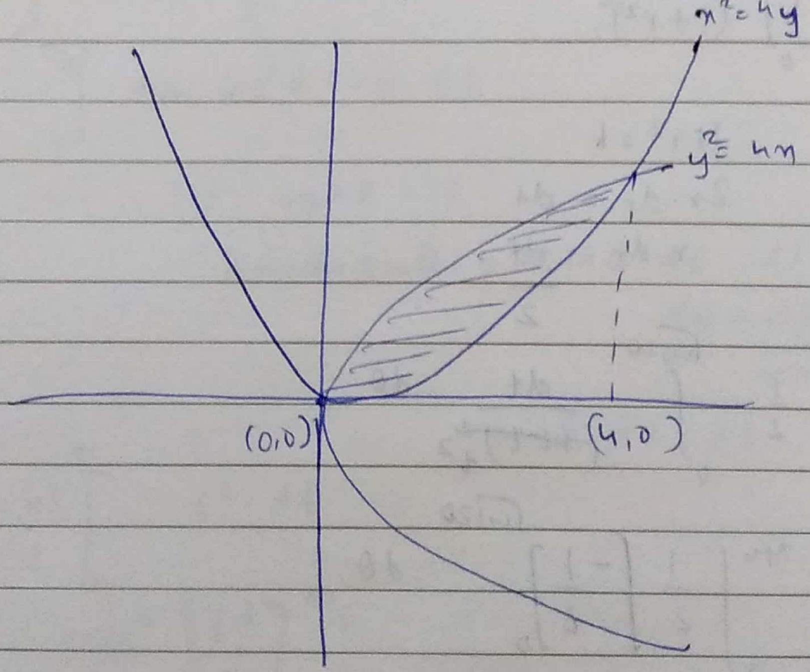
$$\left[\frac{x^4}{4}\right]_0^4 + \cancel{\int_4^8} \cancel{16}\left[\frac{x^2}{2}\right]_4^8$$

$$\frac{256}{4} + 8.(64-16)$$

$$64 + 384 = 448$$

Q. $\iint y\,dx\,dy$

$R: x^2 = 4y,\ y^2 = 4x$



$\cancel{y^2 = 4x}$, $\cancel{y = 2\sqrt{x}}$, $\cancel{\frac{x^2}{4} = 4y^2}$

$\cancel{x^2 = 4\sqrt{x}}$ x^2

$$\int_0^4\int_{x^2/4}^{2\sqrt{x}} y\,dy\,dx = \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^4 \left[4x - \frac{x^4}{16}\right] dx$$

Scanned by CamScanner

$$\frac{1}{2}\left[2\left[x^2\right]_0^4-\frac{1}{16}\left[\frac{x^5}{5}\right]_0^4\right]$$

$$\frac{1}{2}\left[2\times 16-\frac{1}{\cancel{16}}\times\frac{\cancel{16}\times 16\times 4}{5}\right]$$

$$\frac{1}{2}\left(\frac{160-\cancel{64}}{5}\right)=\frac{48}{5}$$

Evaluate double integral in polar coordinates

$$\iint f(r,\theta)\ dr\,d\theta$$

1:

$$\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\frac{r}{(1+r^2)^2}\,dr\,d\theta$$

$$1+r^2=t$$

$$2r\cdot dr=dt$$

$$r\cdot dr=\frac{dt}{2}$$

$$\int_0^{\pi/4}\frac{1}{2}\int_0^{\sqrt{\cos 2\theta}}\frac{dt}{\cancel{(1+t)^2}\ t^2}\cdot d\theta$$

$$\int_0^{\pi/4}\frac{1}{2}\left[-\frac{1}{t}\right]_0^{\sqrt{\cos 2\theta}}d\theta$$

$$\int_0^{\pi/4}-\frac{1}{2}\left[\frac{1}{1+\cancel{\sqrt{}}\cos 2\theta}-1\right]d\theta$$

$$-\frac{1}{2}\int_0^{\pi/4}\frac{\cancel{-\cos 2\theta}}{\cancel{1+\cos 2\theta}}\left(\frac{\sec^2\theta}{2}-1\right)d\theta$$

Scanned by CamScanner

PAGE No. DATE / /

$$-\frac{1}{2}\left(\frac{1}{2}[\tan\theta]_0^{\pi/4} - [\theta]_0^{\pi/4}\right)$$

$$-\frac{1}{2}\left(\frac{1}{2}\cdot 1 - \frac{\pi}{4}\right) = \frac{\pi}{8} - \frac{1}{4}$$

Q. $$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta$$

$$\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\cos\theta}\sin\theta \, d\theta$$

$$\int_0^{\pi/2}\frac{a^2\cos^2\theta\sin\theta}{2}\,d\theta$$

$$\frac{a^2}{2}\int_0^{\pi/2}\cancel{\cos}\ \cos^2\theta\sin\theta\,d\theta$$

let $\cos\theta = t$

$\cancel{2\cos\theta\sin\theta}$ $2 - \sin\theta \, d\theta = dt$

$\cancel{d\theta = \frac{dt}{\sin}}$

$-\sin\theta\,d\theta = dt$

$$-\frac{a^2}{2}\int_0^{\pi/2} t^2 . dt$$

$$-\frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^{\pi/2}$$

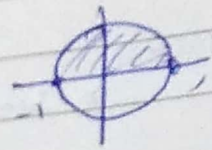
$$-\frac{a^2}{2}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$-\frac{a^2}{2}\left[0 - \frac{1}{3}\right] = \cancel{\frac{a^3}{6}} + \frac{a^2}{6}$$

Scanned by CamScanner

Changing order of integration

Eg. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$



$y = 0$

$y = \sqrt{1-x^2}$

$y^2+x^2=1$

$$\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Change order of integration & evaluate,

$$\int_{0}^{3}\int_{y^2/9}^{\sqrt{10-y^2}} dx\,dy$$

$x = y^2/9$, $x = \sqrt{10-y^2}$

$y^2 = 9x$, $x^2+y^2=10$

$(0,0)$, $r=\sqrt{10}$

$x=1$,

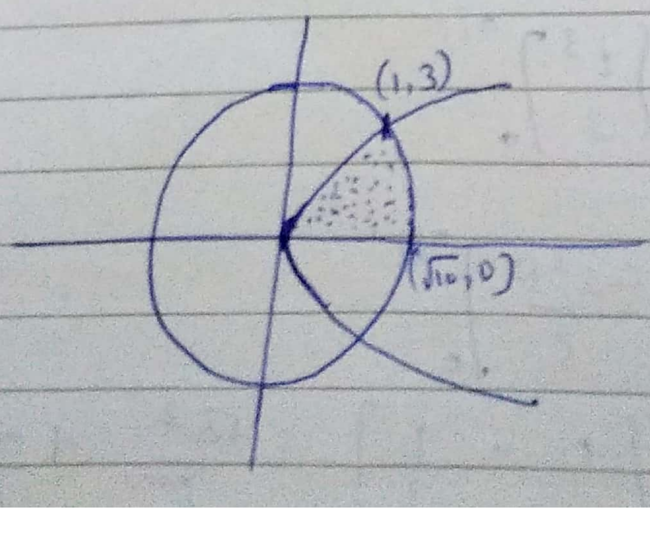
$y=\pm 3$

$x^2+9x-10=0$

$x^2+10x-x-10=0$

$x(x+10)-1(x+10)$

$x=-10$, $x=1$



4.75

$x^{1/2+1}$

After changing order of integration,

$$\iint dx\,dy = \int_0^1\int_0^{3\sqrt{x}} dy\,dx + \int_1^{\sqrt{10}}\int_0^{\sqrt{10-x^2}} dy\,dx$$

$$= \int_0^1 \{y\}_0^{3\sqrt{x}}\,dx + \int_1^{\sqrt{10}} (y)_0^{\sqrt{10-x^2}}\,dx$$

$$= 3\int_0^1 \sqrt{x}\,dx + \int_1^{\sqrt{10}} \sqrt{10-x^2}\,dx$$

$$= 2\left[x^{3/2}\right]_0^1 + \left(\frac{x}{2}\sqrt{10-x^2} + \cancel{10}\,5\sin^{-1}\frac{x}{\sqrt{10}}\right)_1^{\sqrt{10}}$$

$$= \cancel{x}2 + \left[\cancel{\frac{\sqrt{10}\sqrt{10-x^2}}{2}} + 5\cdot\frac{\pi}{2} - \frac{\overset{3}{\cancel{\sqrt{10-x^2}}}}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right.$$

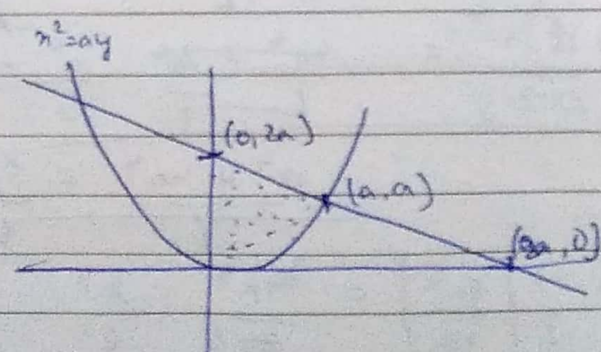
$$\cancel{= \frac{2}{2} + \frac{5\pi}{2}}$$

$$= \frac{1}{2} + \frac{5\pi}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

Q. $$\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$$

$y = \dfrac{x^2}{a}$ & $y = 2a - x$

$x^2 = ay$ $\qquad$ $x + y = 2a$



$x^2 = a(2a - x)$

$x^2 + ax - 2a^2 = 0$

$$x = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2}$$

$$= \frac{-a \pm 3a}{2}$$

$x = a \;,\; -2a$

$y = a \;,\; 4a$

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dy \, dx + \int_a^{2a} \int_0^{2a-y} xy \, dy \, dx$$

$$\int_0^a y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dx$$

$$\int_0^a \frac{ay^2}{2} dy + \int_a^{2a} \frac{y}{2}(4a^2 + y^2 - 4ay) \, dy$$

$$\frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\left(4a^2\left[\frac{y^2}{2}\right]_a^{2a} + \cancel{y^2}\left[\frac{y^4}{4}\right]_a^{2a} - 4a\left[\frac{y^3}{3}\right]_a^{2a}\right.$$

$$\frac{a^4}{6} + \frac{1}{2}\left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{32a^4}{3}\right\} - \frac{4a^4}{2} - \frac{a^4}{4} + \frac{4a^4}{3}\Big]$$

$$\frac{a^4}{6} + \frac{1}{2}\left[\frac{96a^4 + 48a^4 - 128a^4}{12}\right\} - 24a^4 - 3a^4 + 16a^4\Big]$$

96
48
144
128
16

$$\frac{a^4}{6} + \frac{\cancel{16}\,8\,a^4}{2 \times 12\,3} = \frac{\cancel{4a^4}}{6} = \frac{2a^4}{3} \quad \frac{a^4}{6} + \frac{2a^4}{3}$$

32
24
8

$$\frac{a^4 + 4a^4}{6} = \frac{5a^4}{6}$$

$$\frac{a^4}{6} + \frac{1}{2}\left[\frac{5a^4}{12}\right] = \frac{\cancel{3}9a^4}{\cancel{24}8} = \frac{3a^4}{6}$$

Calc. by converting to polar coordinates.

$$\iint f(x,y)\,dx\,dy$$

$$dx\,dy = |J|\,dr\,d\theta$$

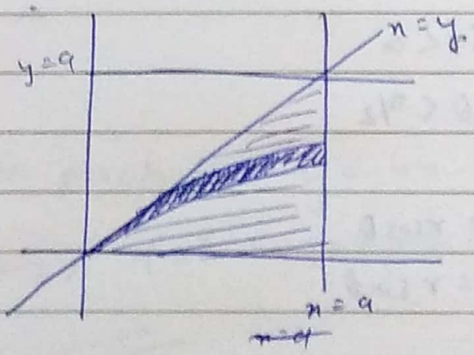
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$dx\,dy = r\,dr\,d\theta$$

Eg.

$$\int_0^a \int_y^a x\,dx\,dy$$



$$x = a$$
$$r\cos\theta = a$$
$$r = a/\cos\theta$$

$$\int_0^{\pi/4} \int_0^{a/\cos\theta} (r\cos\theta)\, r\,dr\,d\theta$$

$$\int_0^{\pi/4} \left(\frac{r^3}{3}\right)_0^{a/\cos\theta} \cos\theta \, d\theta$$

$$\frac{1}{3}\int_0^{\pi/4} \frac{a^3}{\cos^3\theta} \cdot \cos\theta \, d\theta = \frac{a^3}{3}\int_0^{\pi/4} \sec^2\theta \, . d\theta$$

$$= \frac{a^3}{3}\left[\tan\theta\right]_0^{\pi/4} = \frac{a^3}{3}$$

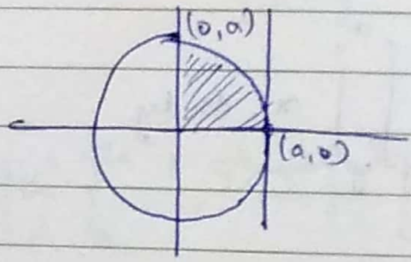
Q. Change to polar coordinates and evaluate.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} \, dy \, dx$$

$y = \sqrt{a^2-x^2}$, $y = 0$.

$x^2 + y^2 = a^2$

$x = a$, $x = 0$



$0 < r < a$

$0 < \theta < \pi/2$

$x = r\cos\theta$

$y = r\sin\theta$

$\cancel{dx = -r\sin\theta \, d\theta}$

$\cancel{dy = r\cos\theta \, d\theta}$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^{\pi/2} \int_0^a e^{-r^2} \, r \, dr \, d\theta$$

PAGE No.
DATE / /

$r^2 = t$

$2r \cdot dr = dt$

$r\,dr = \frac{dt}{2}$

$$\int_0^{\pi/2}\int_0^a e^{-t} \cdot \frac{dt}{2}\, d\theta$$

$$\frac{1}{2}\left[\int_0^{\pi/2}\left[\frac{e^{-r^2}}{-1}\right]_0^a d\theta\right]$$

$$-\frac{1}{2}\left[\int_0^{\pi/2}(e^{-a^2}-1)\,d\theta\right] = -\frac{1}{2}(e^{-a^2}-1)\cdot\frac{\pi}{2}$$

$$= (1-e^{-a^2})\frac{\pi}{4}$$

Area by double integral

Area $= \iint dy\,dx$

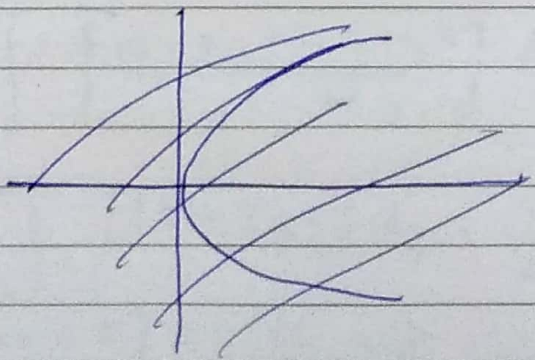
Area $= \iint r\,dr\,d\theta$

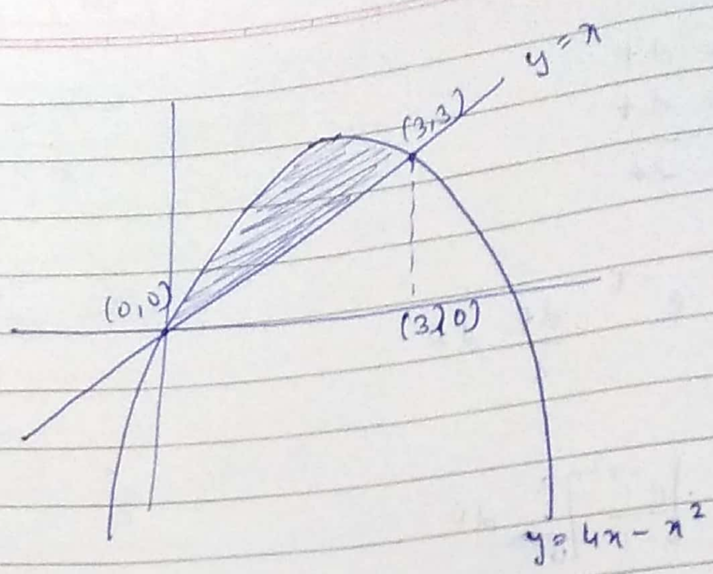
Find area lying b/w parabola $y = 4x - x^2$ and line $y = x$

$y = \cancel{(x-2)} - (x^2 - 4x + 4 - 4) = -(x-2)^2 + 4$

$y - 4 = -(x-2)^2$

vertex $(2, 4)$





$x = 4x - x^2$

$x^2 = 3x$

$x = 3, \; y = 3$

$$\int_0^3 \int_x^{4x-x^2} dy\, dx = \int_0^3 \left[\cancel{4x-x^2} \; 4\left[\frac{x^2}{2}\right] \right] dx$$

$$\int_0^3 (4x - x^2 - x)\, dx = \int_0^3 (3x - x^2)\, dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^3 - \left[\frac{x^3}{3}\right]_0^3$$

$$= 3 \times \frac{9}{2} - \frac{27}{3}$$

$$= \frac{27}{2} - 9 = \frac{9}{2}$$

PAGE No.
DATE / /

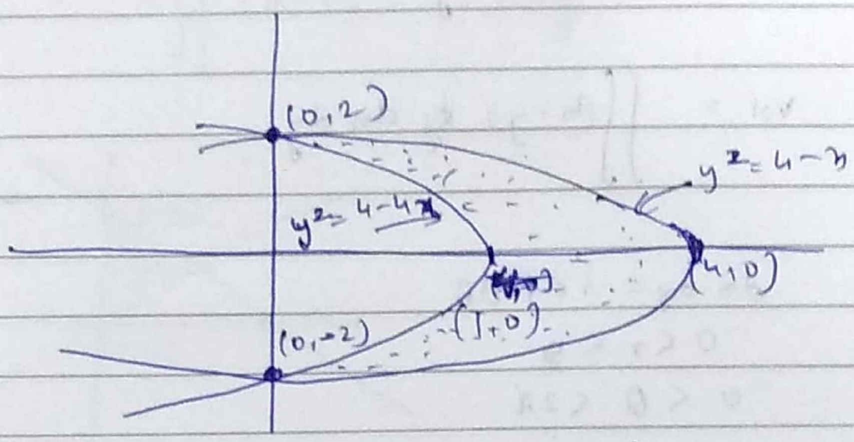
Q. Find area bounded by parabolas, $y^2 = 4-x$, $y^2 = 4-4x$

$y^2 = 4-x = 4-4x$

$x = 0$, $y = \pm 2$ $\Rightarrow (0,2), (0,-2)$

$y^2 = -(x-4)$ vertex $(4,0)$

$y^2 = -4(x-1)$ vertex $(1,0)$



Area $= 2\int_1^4 \int_{4-4x}^{4-x} ~~dy~~\, dx$ $\quad 2\int_0^2 \int_{\frac{4-y^2}{4}}^{4-y^2} dx\, dy$

$= 2\int_0^2 \left[4-y^2 - \left(\frac{4-y^2}{4}\right)\right] dy$

$= \cancel{2}\int_0^2 \left[\frac{16-4y^2-4+y^2}{\cancel{4}\ 2}\right] dy$

$= \frac{1}{2}\int_0^2 [12-3y^2]\, dy = \frac{1}{2}\left(12[y]_0^2 - \cancel{3}\left[\frac{y^3}{\cancel{3}}\right]_0^2\right)$

$= \frac{1}{2}[24-8] = \frac{16}{2} = 8.$

Q. Volume by double integral.

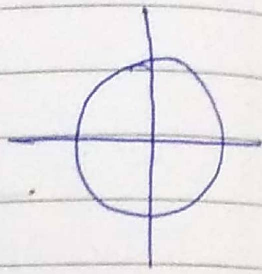
~~Q. Using~~

Vol. = $\iint z\,dx\,dy$

Q. Using double integration find vole bounded by cylinder, & plane, $y+z=4$, $z=0$.
$x^2+y^2=4$

$z = 4-y$

vol. = $\iint (4-y)\,dx\,dy$



$dx\,dy = r\,dr\,d\theta$

$0 < r < 2$

$0 < \theta < 2\pi$

$= \int_0^{2\pi}\int_0^{2} (4 - r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \left(4[r]_0^2 - \left[\frac{r^2}{2}\right]_0^2 \sin\theta\right) d\theta$

$= \int_0^{2\pi} (8 - 2\sin\theta)\,d\theta = 8\times 2\pi + 0$

$= \cancel{16\pi + 2} = 16\pi$

$= \int_0^{2\pi} \left(4\left[\frac{r^2}{2}\right]_0^2 - \left[\frac{r^3}{3}\right]_0^2 \sin\theta\right) d\theta$

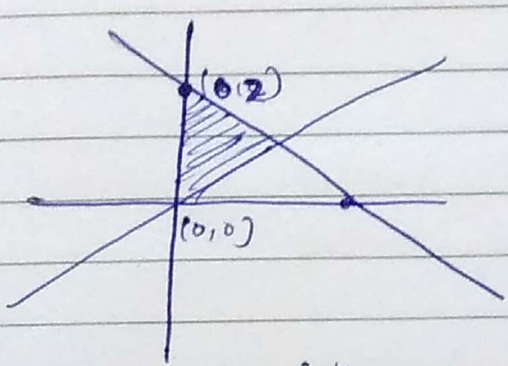
PAGE No.
DATE / /

$$= \int_0^{2\pi} \left(8 - \frac{8}{3}\sin\theta\right) d\theta$$

$$= 8 \times 2\pi = 16\pi$$

Q. Find vol. of region that lies under parabola, $z = x^2 + y^2$ and above triangle enclosed by lines, $y = x$, $x = 0$, $x + y = 2$ in xy plane.

$$\text{vol.} = \iint (x^2 + y^2)\, dx\, dy$$



$-6y(2-y)$

$-12y + 6y^2$

$$= \int_0^2 \int_y^{2-y} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^2 \left[\left(\frac{x^3}{3}\right)_y^{2-y} + y^2 (x)_y^{2-y} \right) dy$$

$$= \int_0^2 \left(\frac{(2-y)^3 - y^3}{3} + y^2 (2 - y - y) \right) dy$$

$$= \int_0^2 \left(\frac{8 - y^3 - 12y + 6y^2 - y^3 + 6y^2 - \cancel{6}3y^3}{3} \right) dy$$

$$= \frac{1}{3} \int_0^2 (-5y^3 + 12y^2 - 12y + 8)\, dy$$

$$= \frac{1}{3} \left(-5\left(\frac{y^4}{4}\right)_0^2 + \cancel{12}4\left(\frac{y^3}{\cancel{3}}\right)_0^2 - \cancel{12}6\left(\frac{y^2}{\cancel{2}}\right)_0^2 + 8(y)_0^2 \right.$$

Triple Integration

$$\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^a\int_0^x e^{x+y}\left[e^z\right]_0^{x+y} dy\,dx$$

$$\cancel{= \int_0^a\int_0^x e^{2x}\cdot e^{2y}\cdot dy\,dx = \int_0^a e^{2x}\cdot\left[\frac{e^{2y}}{2}\right]_0^x}$$

$$\cancel{= \frac{1}{2}\int_0^a e^{4x}\cdot dx = \frac{1}{2}\left[\frac{e^{4x}}{4}\right]_0^a = \frac{e^{4a}}{8}}$$

$$\cancel{= \int_0^a \frac{e^{2x}}{2}\cdot\left[e^{2x}-1\right] dy}\ dx$$

$$= \int_0^a\int_0^x e^{x+y}\left[e^{x+y}-1\right] dy\,dx$$

$$= \int_0^a\int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy\,dx$$

$$= \int_0^a \left(e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x - e^x\cdot\left[e^y\right]_0^x\right) dx$$

$$= \int_0^a \left(\frac{e^{2x}}{2}\left[e^{2x}-1\right] - e^x\left[e^x-1\right]\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\right) dx$$

$$= \frac{e^{4a}}{2} - \frac{3e^{2a}}{2} + e^a - \left[\frac{1}{2} - \frac{3}{2} + 1\right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$$

Eg.

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} 1.\, dz\, dy\, dx$$

$$\int_{x=0}^{2}\int_{y=0}^{6} [z]_0^{4-x^2} dy\, dx = \int_{x=0}^{2}\int_{y=0}^{6} (4-x^2)\, dy\, dx$$

$$\int_{x=0}^{2} \left(4[y]_0^6 - x^2[y]_0^6\right) dx$$

$$\int_0^2 (24-6x^2)\, dx = 24[x]_0^2 - 6\left[\frac{x^3}{3}\right]_0^2$$

$$= 24\times 2 - 2\times 8$$

$$= 48 - 16 = 32$$

Eg.

$$\int_0^{\pi/2}\int_0^{2a\cos\theta}\int_0^{r^2/2a} r.\, d\theta .\, dr\, dz$$

$\theta \quad r \quad z$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r.\ [z]_0^{r^2/2a}\ \cancel{d\theta\, dz}\, dr\, d\theta$$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r.\frac{r^2}{2a} .dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} \frac{1}{2a} . d\theta$$

$$= \int_0^{\pi/2} \frac{8a^4\cos^4\theta}{4} . \frac{1}{2a} . d\theta$$

$\cos^2\theta \cdot \cos^2\theta$ $\cos^2\theta \cdot \cos\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$\cos^2\theta = \cos$

$$\cancel{\frac{8}{3}} \int_0^{\pi/2} \cos^4\theta \cdot d\theta$$

$$\cancel{\frac{8}{3}} \int_0^{\pi/2}$$

Eq $$\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \int_{z=0}^{r^2/2a} r \cdot dz\, d\theta\, dr$$

$$\int_0^{\pi/2} \int_0^{2a\cos\theta} r \cdot \frac{r^2}{2a} \cdot dr\, d\theta$$

$$\frac{1}{2a} \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$\frac{1}{8a} \int_0^{\pi/2} 16 a^4 \cos^4\theta \cdot d\theta$$

$$2a^3 \int_0^{\pi/2} \cos^4\theta \cdot d\theta$$

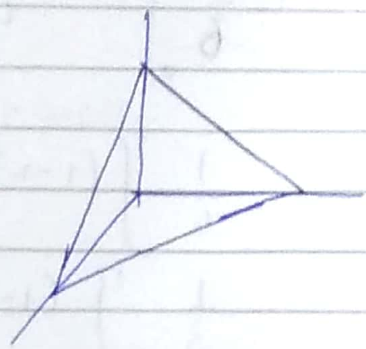
$$2a^3 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{\cancel{2}a^3}{\cancel{4}\,2} \int_0^{\pi/2} (1 + \cos^2 2\theta + 2\cos 2\theta)\, d$$

$$\frac{a^3}{2}$$

Eg. $\iiint (x+y+z)\,dz\,dy\,dx$

over the tetrahedron bounded by the planes

$x=0,\ y=0,\ z=0,\ x+y+z=1$



$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dz\,dy\,dx$$

$$\int_0^1\int_0^{1-x} \cancel{x+y+}\ x[z]_0^{1-x-y} + y[z]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y}$$

$$\int_0^1\int_0^{1-x}\left(x(1-x-y)+y(1-x-y)+\frac{(1-x-y)^2}{2}\right)dy\,dx$$

$$\int_0^1\int_0^{1-x}\left(x-x^2-xy+y-xy-y^2+\frac{1+(x+y)^2-2x-2y}{2}\right)dy\,dx$$

$$\frac{1}{2}\int_0^1\int_0^{1-x}\left(\cancel{2x}-2x^2+\cancel{2y}-2y^2-4xy+1+x^2+y^2+2xy-\cancel{2x}-\cancel{2y}\right)dy\,dx$$

$$\frac{1}{2}\int_0^1\int_0^{1-x}\left(-x^2-y^2-2xy+1\right)dy\,dx$$

$$\frac{1}{2}\int_0^1 -x^2[y]_0^{1-x}-\left[\frac{y^3}{3}\right]_0^{1-x}-\cancel{2}x\left[\frac{y^2}{\cancel{2}}\right]_0^{1-x}+[y]_0^{1-x}$$

$$\frac{1}{2}\int_0^1\left(\overset{-x^2(1-x)}{\cancel{-x^2+x^3}}-\frac{(1-x)^3}{3}-2x(1-x)^2+(1-x)\right)dx$$

$x^2 - 2x + 1$ $\quad (x-1)^2$

DATE

$$\frac{1}{2}\int_0^1 (1-x)\left[-x^2 - \frac{(1+x^2-2x)}{3} - 2x + 2x^2 + 1\right] dx$$

$$\frac{1}{6}\int_0^1 (1-x)\left[-3x^2 - 1 - x^2 + 2x - 6x + 6x^2 + 3\right] dx$$

$$\frac{1}{6}\int_0^1 (1-x)\left[2x^2 - 4x + 2\right] dx$$

$$\frac{1}{3}\int_0^1 (1-x)(x-1)^2\, dx = \frac{-1}{3}\int_0^1 \left(1 - x^3 - \overset{3x^2+3x}{\cancel{3x(x-1)}}\right) dx$$

$$= \frac{-1}{3}\left\{x - \frac{x^4}{4} - 8x^3 + \frac{3}{2}x^2\right\}_0^1$$

$$= \frac{-1}{3}\left[1 - \frac{1}{4} - 1 + \frac{3}{2}\right]$$

$$= \frac{1}{8}$$

Q. When region of integration is not bounded by planes,

1. when region of integration sphere,
use spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\cos\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

a. when region, whole sphere,

$$x^2+y^2+z^2 = a^2$$

$0<r<a$, $0<\theta<\pi$, $0<\phi<2\pi$

b. when hemisphere,

$0<r<a$, $0<\theta<\pi/2$, $0<\phi<2\pi$

c. First octet of sphere.

$0<r<a$, $0<\theta<\pi/2$, $0<\phi<\pi/2$

2. Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$x = ar\sin\theta\cos\phi$

$y = br\sin\theta\sin\phi$

$z = cr\,\cancel{\cos\theta}\,\cos\theta$

$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$

3. When region of integration is cylindrical with base radius a.

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,d\theta$

$0<r<a$, $0<\theta<2\pi$; $-\infty<z<\infty$

Eg.

$$\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z)^{3/2}}$$

where, $V \rightarrow$ vol. bounded by sphere

$x^2+y^2+z^2=a^2$

&

$x^2+y^2+z^2=\cancel{bc}\, b^2$

$b>a$

$a<r<b$

$0<\theta<\pi$

$0<\theta<2\pi$

$x = \cancel{r\cos\theta\sin\phi}\ r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = \cancel{r\cos\phi}\ \ r\cos\theta$

$\cancel{x^2+y^2+z^2 = r^2(\cos\sin^2\phi + \cos^2\phi)}$

$\cancel{= r^2(\sin^2\phi + \cos^2\phi)} = r^2$

$x^2+y^2+z^2 = r^2$

$dx\,dy\,dz = \cancel{r^2 d}\ r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\int_0^{2\pi}\int_0^{\pi}\int_a^{b} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(r^2)^{3/2}}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_a^{b} \frac{1}{r}\,dr$$

PAGE No.
DATE / /

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, (\log r)_a^b$$

$$2\pi \cdot (-\cos\theta)_0^{\pi} (\log b/a) = 2\pi \cdot \log(b/a)$$

Eg. $\iiint (x^2+y^2+z^2) \, dx \, dy \, dz$

over first octat of sphere

$$x^2+y^2+z^2=a^2$$

$0<r<a, \quad 0<\theta<\pi/2, \quad 0<\phi<\pi/2$