Sneha Bhadauria
A004

Tutorial Unit III

1. Examine following vectors for Linear dependance / Independance.

a) $\{(2,1,0), (3,1,-1), (0,-1,1)\}$ in R^3

$$k_1(2,1,0) + k_2(3,1,-1) + k_3(0,-1,1) = 0$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_2 \to 2R_2 - R_1$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$R_3 \to R_2 - R_3$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad k_1 = k_2 = k_3 = 0$$

It is linearly independent

b) $\{(1,1,-1,1)\ (1,-1,2,-1)\ (3,1,0,1)\}$ in R^4

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 + R_1 \quad R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 \to 2R_3 + 3R_2$ $\quad R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

let $\boxed{k_3 = t}$

$2k_2 + 2k_3 = 0$

$\boxed{k_2 = -t}$

$\therefore \boxed{k_1 = -2t}$

They are linearly dependent

Q2. Check that set $S = \{(1,2,-1), (1,-1,2), (2,-1,1)\}$ is basis of v.s. $\mathbb{R}^3(\mathbb{R})$ or not.

$k_1(1,2,-1) + k_2(1,-1,2) + k_3(2,-1,1) = 0$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$k_1 = k_2 = k_3 = 0$

Solving we get, it is linearly independent

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -5 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 - 2k_1 \\ k_3 + k_1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -5 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 - 2k_1 \\ k_2 - k_1 + k_3 \end{bmatrix}$$

$-2a_3 = k_2 - k_1 + k_3 \Rightarrow a_3 = (k_1 - k_2 - k_3)/2$

$-3a_2 - 5a_3 = k_2 - 2k_1 \Rightarrow a_2 = -k_1 + 3k_2 - 5k_3/6$

$a_1 + a_2 + 2a_3 = k_1 \Rightarrow a_1 = (-2k_1 + 6k_2 + 4k_3)/6$

$\therefore$ Set S is a basis of vector space R^3

3. Show that $S = \{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2\}$ is basis of vector space P_2

$k_1(-4 + x + 3x^2) + k_2(6 + 5x + 2x^2) + k_3(8 + 4x + x^2) = 0$

$-4k_1 + 6k_2 + 8k_3 = 0$

$k_1 + 5k_2 + 4k_3 = 0$

$3k_1 + 2k_2 + k_3 = 0$

$$\begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_1 \to R_1 + 4R_2$ $\qquad$ $R_3 \to 3R_1 + 4R_3$

$$\begin{bmatrix} -4 & 6 & 8 \\ 0 & 26 & 24 \\ 0 & 26 & 28 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -4 & 6 & 8 \\ 0 & 26 & 24 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$k_3 = 0$ $\quad k_2 = 0$ $\quad k_1 = 0$

$\therefore$ It is linearly independent.

$$\begin{bmatrix} -4 & 6 & 8 \\ 0 & 26 & 24 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 + 4k_2 \\ 2k_1 + 4k_2 + 4k_3 \end{bmatrix}$$

$\because$ a, b, c cane be expressed in k_1, k_2, k_3 terms

$\therefore$ Set S is a basis of vector space P_2

4. $W = \{ (x, y) \in R^2 \mid ax + by = 0 \}$ Show that W is a subspace of R^2

let $(0, 0) \in W$

$a.0 + b.0 = 0$

$0 \in W$

let $u = (x_1, y_1) \mid ax_1 + by_1 = 0 \in W$ — (1)

$v = (x_2, y_2) \mid ax_2 + by_2 = 0 \in W$ — (2)

$\alpha u + \beta v = \alpha(x_1, y_1) + \beta(x_2, y_2)$

$= \{ (\alpha x_1 + \beta x_2), (\alpha y_1 + \beta y_2) \}$

$= a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2)$

$= \alpha(ax_1 + by_1) + \beta(ax_2 + by_2)$

$= \alpha \cdot 0 + \beta \cdot 0$ [by (1) & (2)]

$= 0 \in W$

$\therefore \alpha u + \beta v \in W$

So W is a subset of R^2

5. $W = \{(x, y, z) \in R^3 \mid ax + by + cz = 0\}$ Show that W is a subspace of R^3.

let $(0, 0, 0) \in W$

$a.0 + b.0 + c.0$

$= 0 \in W$

let $u = (x_1, y_1, z_1) \mid ax_1 + by_1 + cz_1 = 0$ — ①

let $v = (x_2, y_2, z_2) \mid ax_2 + by_2 + cz_2 = 0$ — ②

$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2)$

$= \alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2)$

$= \alpha \cdot 0 + \beta \cdot 0$ — by ① & ②

$= 0$

$\therefore \alpha u + \beta v \in W$

So W is a subspace of R^3

6: Is $W = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \; x, y \in R \right\}$ a vector sub space of M_{22}

let $(0,0) \in W$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$$

$$= 0 \in W$$

$\therefore$ W is non empty

let $u = (x_1, y_1) \in W \quad \Big| \quad \begin{bmatrix} x_1 & y_1 \\ 0 & x_1 \end{bmatrix} \in W$

$v = (x_2, y_2) \in W \quad \Big| \quad \begin{bmatrix} x_2 & y_2 \\ 0 & y_2 \end{bmatrix} \in W$

$$\alpha u + \beta v = \alpha(x_1, y_1) + \beta(x_2, y_2)$$

$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$\begin{bmatrix} \alpha x_1 + \beta x_2 & \alpha y_1 + \beta y_2 \\ 0 & \alpha x_1 + \beta x_2 \end{bmatrix} \in W$$

$\therefore \alpha u + \beta v \in W$

So W is a vector subspace of M_{22}

7. $S = \{(x, y, z) \in R^3 \mid ax = by = cz\}$ Show that S is a subspace of R^3

let $(0,0,0) \in W$

$a \cdot 0 + b \cdot 0 = c \cdot 0$

$= 0 \in W$

let $u = (x_1, y_1, z_1) \mid ax_1 = by_1 = cz_1$ — ①

let $v = (x_2, y_2, z_2) \mid ax_2 = by_2 = cz_2$ — ②

$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= a(\alpha x_1 + \beta x_2) \cancel{+} b(\alpha y_1 + \beta y_2) \cancel{=} c(\alpha z_1 + \beta z_2)$

$= \alpha(ax_1) + \beta(ax_2) = \alpha(by_1) + \beta(by_2) = \alpha(cz_1) + \beta(cz_2)$

This holds true by eq ① & ②

$\therefore \alpha u + \beta v \in W$

So W is a subspace of R^3

8. find Orthogonal projection of V onto subspace W spanned by vectors u

$v = (3, 1, -2)$ $u_1 = (1, 1, 1)$ $u_2 = (1, -1, 0)$

$v = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

∴ The orthogonal projection of v onto W is

$$\text{proj}_W(v) = \left(\frac{u_1 v}{u_1 u_1}\right)u_1 + \left(\frac{u_2 v}{u_2 u_2}\right)u_2$$

$$u_1 u_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 1+1+1 = 3$$

$$u_2 u_2 = \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = 1+1+0 = 2$$

$$u_1 v = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix} = 3+1-2 = 2$$

$$u_2 v = \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix} = 3-1-0 = 2$$

$$\text{proj}_W(v) = \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{2}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}5/3\\-1/3\\2/3\end{bmatrix}$$

b) $v = (1,2,3)$ $\quad u_1 = (2,-2,1)$ $\quad u_2 = (-1,1,4)$

Orthogonal projection of v onto W is :

$$\text{proj}_W(v) = \left(\frac{u_1 v}{u_1 u_1}\right)u_1 + \left(\frac{u_2 v}{u_2 u_2}\right)u_2$$

$$u_1 u_1 = \begin{bmatrix}2 & -2 & 1\end{bmatrix}\begin{bmatrix}2\\-2\\1\end{bmatrix} = 4+4+1 = 9$$

$$u_2 u_2 = \begin{bmatrix} -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = 1 + 1 + 16 = 18$$

$$u_1 v = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 - 4 + 3 = 1$$

$$u_2 v = \begin{bmatrix} -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -1 + 2 + 12 = 13$$

$$\text{Proj}_W (v) = \left(\frac{u_1 v}{u_1 u_1}\right) u_1 + \left(\frac{u_2 v}{u_2 u_2}\right) u_2$$

$$= \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{13}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -9/18 \\ 9/18 \\ 54/18 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}$$

Q9. W be plane on R^3 with eqn $x - y + 2z = 0$, $v = (3, 1, 2)$ find orthogonal projection of v onto W and its components.

$W: x - y + 2z = 0$

W is a plane through orgin in R^3, So from eq of plane, $x = y - 2z$ So W consists of vectors of form:

$$\begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are basis of W

but they are not orthogonal. So
let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is vector in W, orthogonal to u_1
then $x - y + 2z = 0$.
$\therefore$ W is in plane W

$u_1 \cdot w = 0$ we have $x + y = 0$
Solving $x - y + 2z = 0$ & $x + y = 0$
$\Rightarrow x = -z$, $y = z$ $\Rightarrow w = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

So $w = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an orthogonal set in W & hence orthogonal basis

$\therefore$ Orthogonal basis for W are :-
$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ we have
$u_1 \cdot v = 2$, $u_2 \cdot v = -2$
$u_1 u_1 = 2$, $u_2 u_2 = 3$

$$\text{Proj}_W(v) = \left(\frac{u_1 v}{u_1 u_1}\right) u_1 + \left(\frac{u_2 v}{u_2 u_2}\right) u_2$$

$$= \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

Component of v orthogonal to w is vector
$$\text{Perp}_W(v) = v - \text{proj}_W(v)$$
$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$$

$\text{Proj}_W(v)$ is in W as it satisfies eqn of plane
$\text{Perp}_W(v)$ is ~~also~~ orthogonal to W, as it is a scalar multiple of normal vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ to W.

10 find range & kernel of $T: R^3 \to R^3$

$T(x,y,z) = (x-y+z,\ y-z,\ 2x-5y+5z)$

$x-y+z=0$

$y-z=0$

$2x-5y+5z=0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \Rightarrow R_3 \to R_3 + 3R_2 \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x-y+z=0$ $\qquad$ $x-y+y=0$

$\cancel{x}y-z=0$ $\qquad$ $x=0$

$y=z$

Kernel $= \{0, y, y \mid y \in R\}$ $\qquad$ Nullity $=1$

Range $= \{r, s, 2r-3s \mid r, s \in R\}$ $\qquad$ Rank $=2$

12. $T: R^3 \to R^3$ $\qquad$ $T(x,y,z) = (\underset{r}{x+z},\ \underset{s}{x+y+2z},\ \underset{r+s}{2x+y+3z})$

$x+z=0$

$x+y+2z=0$

$2x+y+3z=0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\qquad$ $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + z = 0$ $\quad$ $x = -z$

$y + z = 0$ $\quad$ $y = -z$

kernel = $\{(-z, -z, z) \mid z \in R\}$ $\quad$ Nullity = 1

Range = $\{r, s, r+s \mid r, s \in R\}$ $\quad$ Rank = 2

13. $T: R^2 \to R^3$ $\quad$ $T(x, y) = (x, x+y, y)$

for kernel

$x = 0$

$x + y = 0$ $\quad$ $\therefore x = 0, y = 0$

$y = 0$

kernel = $\{0, 0\}$

Range = $\{r, s, s-r \mid s, r \in R\}$

14. $T: R^2 \to R^2$ $\quad$ $T(x, y) = (2x-3y, 3x-4y)$

Show that T is invertible & find T^{-1}

$T(x, y) = (2x - 3y, 3x - 4y)$

$$T = \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \Rightarrow Det(T) = -8 + 9 = 1 \neq 0$$

$\therefore$ It is invertible

$T(x, y) = (2x - 3y, 3x - 4y)$

$(x, y) = T^{-1}(2x - 3y, 3x - 4y)$

$= T^{-1}(u, v)$

$u = 2x - 3y$

$u - 2x = -3y$

$3y = \frac{2x-u}{3} = \frac{2(3v-4u)-u}{3}$

$= \frac{6v - 9u}{3} = 2v - 3u$

$v = 3x - 4y$

$4y = 3x - v$

$\cancel{4\left(\frac{2x-u}{3}\right) = 3x - v}$

$v = 3x - 4\left(\frac{2x-u}{3}\right)$

$T^{-1}(x,y) = \left(3v-4u, \frac{2v-3u}{3}\right)$

$3v = 9x - 8x + 4u$

$3v = x + 4u$

$x = 3v - 4u$

15. $T: R^2 \to R^2$ $\quad T(1,2) = (2,3) \quad T(0,1) = (1,4)$

find formulas for $T(x,y)$. Is T invertible

let $(x,y) = a(1,2) + b(0,1)$

$= (a, 2a+b)$

$\therefore x = a \qquad y = 2a + b$

$y = 2x + b$

$b = y - 2x$

$T(x,y) = x\,T(1,2) + (y-2x)\,T(0,1)$

$= x(2,3) + (y-2x)(1,4)$

$= (2x + y - 2x, 3x + 4y - 8x)$

$T(x,y) = (y, -5x + 4y)$

for invertible, T should be non singular

$y = 0$

$-5x + 4y = 0$

$T = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \Rightarrow Det(T) = 0 + 5 = 5 \neq 0$

$\therefore$ T is invertible

16. $T: R^2 \to R^2$ $\quad T(x,y) = (2x+y,\ 3x-5y)$.

$T(x,y) = (2x+y,\ 3x-5y)$

$T = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$ $\quad |T| = -10-3 = -13 \neq 0$

$\therefore$ T is invertible

$T(x,y) = (2x+y,\ 3x-5y)$

$(x,y) = T^{-1}(2x+y,\ 3x-5y)$

$= T^{-1}(u,v)$

$2x+y = u$ $\qquad\qquad 3x-5y = v$

$y = u-2x$ $\qquad\qquad v = 3x-5(u-2x)$

Using eq (1) $\qquad\qquad v = 3x-5u+10x$

$y = u - 2\left(\frac{v+5u}{13}\right)$ $\qquad\qquad \cancel{3x}\ v = 13x-5u$

$x = \frac{v+5u}{13}$ — (1)

$13y = 13u - 2v - 10u$

$y = \frac{3u-2v}{13}$

$T^{-1}(x,y) = \left(\frac{v+5u}{13}, \frac{3u-2v}{13}\right)$

17 Find QR factorization of

a) $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\therefore x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

$\because \{x_1, x_2, x_3\}$ is a Linearly Independent set so it forms basis for W

i) We set $v_1 = x_1$

ii) $v_2 = \text{perp}_{W_1}(x_2) = x_2 - \left(\frac{v_1 x_2}{v_1 v_1}\right) v_1$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$v_2' = 2v_2 = (3, 3, 1, 1)$

iii) $v_3 = \text{perp}_{W_2}(x_3) = x_3 - \left(\frac{v_1 x_3}{v_1 v_1}\right) v_1 - \left(\frac{v_2' x_3}{v_2' v_2'}\right) v_2'$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{15}{20} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

$v_3' = 2v_3 = (-1, 0, 1, 2)$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$q_2 = \frac{v_2'}{\|v_2'\|} = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}$$

$$q_3 = \frac{v_3'}{\|v_3'\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\0\\1\\2\end{bmatrix} = \begin{bmatrix}-1/\sqrt{6}\\0\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix} = \begin{bmatrix}-\sqrt{6}/6\\0\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix}$$

$A = QR$ for some upper triangular matrix R, To find R we use fact that

Q is orthogonal

$\therefore Q^T Q = I$

$\therefore Q^T A = Q^T Q R = IR = R$

$$\therefore R = Q^T A = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

b.) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\because \{x_1, x_2, x_3\}$ are L.I. so they forms basis for W

i.) we set $v_1 = x_1$

ii.) $v_2 = \text{perp}_{W_1}(x_2) = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$

$v_2' = 2v_2 = (1, 0, -1)$

iii.) $v_3 = \text{perp}_{W_2}(x_3) = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right) v_2'$

$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$q_2 = \frac{v_2'}{\|v_2'\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$R = Q^T A \qquad Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}$$

$$R = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$$

$$R = \begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & \sqrt{2}\\ 0 & 1/\sqrt{2} & \sqrt{2}\\ 0 & 0 & 1\end{bmatrix}$$

c) $A = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$ $\quad x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\quad x_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ $\quad x_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

$\because \{x_1\ x_2\ x_3\}$ are L.I. so they form basis for W

i) We set $v_1 = x_1$

ii) $v_2 = perp_W(x_2) = x_2 - \left(\frac{x_2\, v_1 x_2}{v_1 v_1}\right) v_1$

$$\begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix}$$

$v_2' = 4v_2 = (-3, 1, 1, 1)$

iii.) $V_3 = \text{perp}_{w_1}(x_3) = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right)v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right)v_2'$

$$\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \cancel{\begin{bmatrix}-5/4\\-11/12\\1/12\\1/12\end{bmatrix}} \begin{bmatrix}0\\-4/6\\2/6\\2/6\end{bmatrix}$$

$v_3' = 3v_3 = (0, -2, 1, 1)$ ~~$v_3' = 12 v_3 = (-15, -11, 1, 1)$~~

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$q_2 = \frac{v_2'}{\|v_2'\|} = \frac{\sqrt{3}}{6}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2\\\sqrt{3}/6\\\sqrt{3}/6\\\sqrt{3}/6\end{bmatrix}$$

$$q_3 = \frac{v_3'}{\|v_3'\|} = \cancel{\begin{bmatrix}-15\\-11\\1\\1\end{bmatrix}} \frac{1}{\sqrt{6}}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-2\sqrt{6}/6\\\sqrt{6}/6\\\sqrt{6}/6\end{bmatrix}$$

$$Q = \begin{bmatrix}1/2 & -3\sqrt{3}/6 & 0\\ 1/2 & \sqrt{3}/6 & -2\sqrt{6}/6\\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6\\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6\end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\\ -3\sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/6\\ 0 & -2\sqrt{6}/6 & \sqrt{6}/6 & \sqrt{6}/6\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

$$R = \begin{bmatrix}2 & 3/2 & 1\\ 0 & \sqrt{3}/2 & \sqrt{3}/3\\ 0 & 0 & \sqrt{6}/3\end{bmatrix}$$

18. Express following transformations in matrix form and find composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

$(x_1, x_2) = T(y_1, y_2) = (2y_1 - 3y_2, 4y_1 + y_2)$

$(y_1, y_2) = T(z_1, z_2) = (z_1 - 2z_2, 2z_1 + 3z_2)$

$(x_1, x_2) = T(y_1, y_2)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$X = AY$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$Y = BZ$

$X = ABZ$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & -13 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$(x_1, x_2) = (-4z_1 - 13z_2, 8z_1 - 5z_2)$

19. $Y = AX$ $\quad X_1 = (2, 2)'$ $\quad X_2 = (4, -1)'$ to
$\quad Y_1 = (3, 2)'$ $\quad Y_2 = (2, 3)'$

$Y_1 = AX_1$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$2a + 2b = 3$ — (1)

$2c + 2d = 2$ — (2)

$Y_2 = A X_2$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$4a - b = 2$ —— ③

$4c - d = 3$ —— ④

Solving eq ① ② ③ & ④

$a = \frac{7}{10}$ $b = \frac{4}{5}$

$c = \frac{4}{5}$ $d = \frac{1}{5}$

$$\therefore A = \begin{bmatrix} 7/10 & 4/5 \\ 4/5 & 1/5 \end{bmatrix}$$

Linear trans. $Y = AX$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7/10 & 4/5 \\ 4/5 & 1/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ — Ans