Assignment 1

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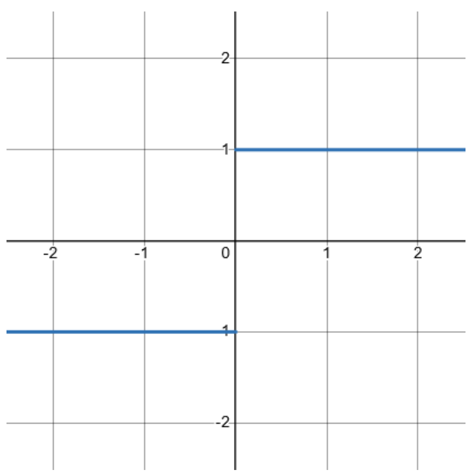
Problem 1

(a) What do you mean by domain and range of a function? Sketch and find the domain and range of the following functions:

Ans: The Domain of a function is the set of values for which the function is defined, any acceptable input for a function is a part of its domain. The Range of a function is the set of values that the function can return, any output received from a function after inputting a value from its domain is a part of its range.

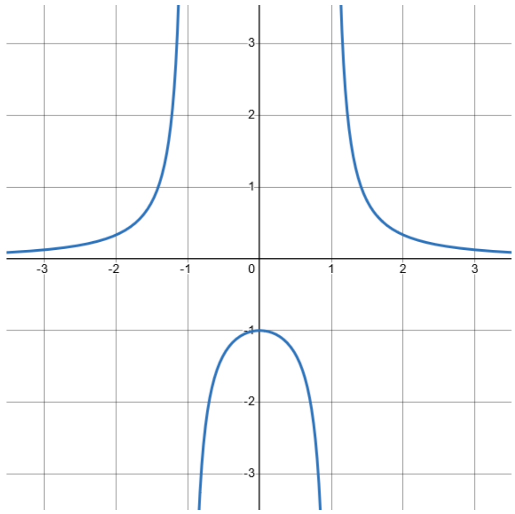
i. $f(x) = \frac{x}{|x|}$



Domain: $\mathbf{x \in R - \{0\}}$

Range: $\{-1, 1\}$

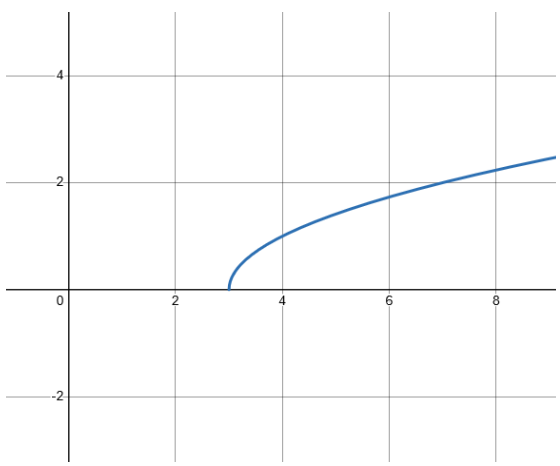
ii. $f(x) = \frac{1}{x^2-1}$



Domain: $\mathbf{x} \in \mathbf{R} - \{\mathbf{1}, \mathbf{-1}\}$

Range: $(-\infty, -1] \cup [0, \infty)$

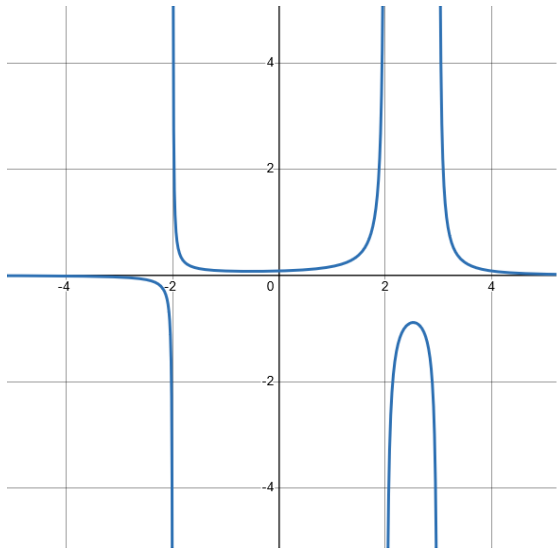
iii. $f(x) = \sqrt{x-3}$



Domain: $\mathbf{x} \in [\mathbf{3}.\infty)$

Range: $[\mathbf{0}, \infty)$

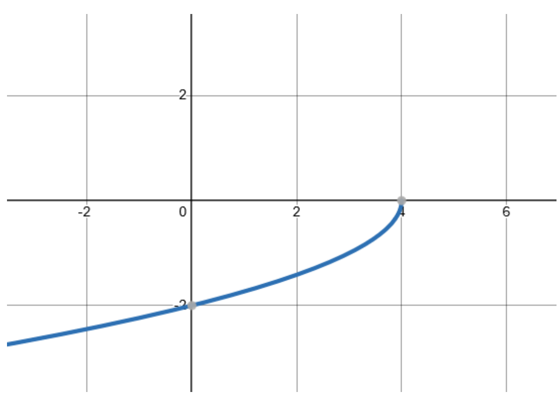
iv. $f(x) = \frac{1}{(4-x^2)(3-x)}$



Domain: $\mathbf{x} \in (-\infty, -\mathbf{2}) \cup (-\mathbf{2}, \mathbf{2}) \cup (\mathbf{2}, \mathbf{3}) \cup (\mathbf{3}, \infty)$

Range: $(-\infty, \mathbf{0}) \cup (\mathbf{0}, \infty)$

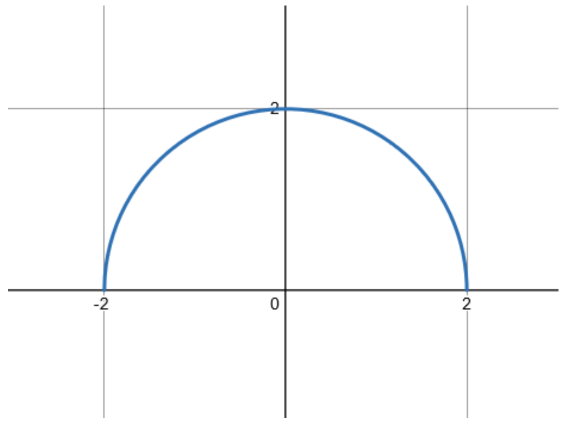
v. $f(x) = \frac{x-4}{\sqrt{4-x}}$



Domain: $\mathbf{x} \in (-\infty, \mathbf{4})$

Range: $(-\infty, \mathbf{0})$

vi. $f(x) = \sqrt{4 - x^2}$

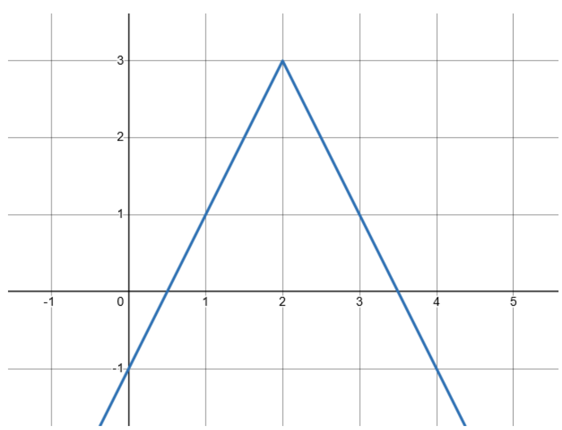


Domain: $\mathbf{x \in [-2, 2]}$

Range: $\mathbf{[0, 2]}$

(b) Sketch the graph of the following functions and hence write down the domain and range for them:

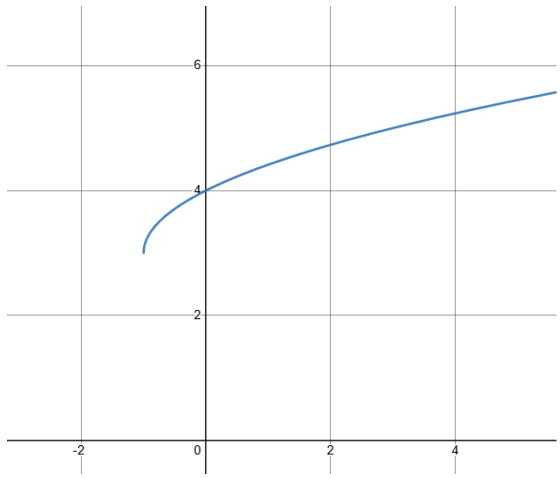
i. $f(x) = 3 - |2x - 4|$



Domain: $\mathbf{x \in R}$

Range: $\mathbf{(-\infty, 3]}$

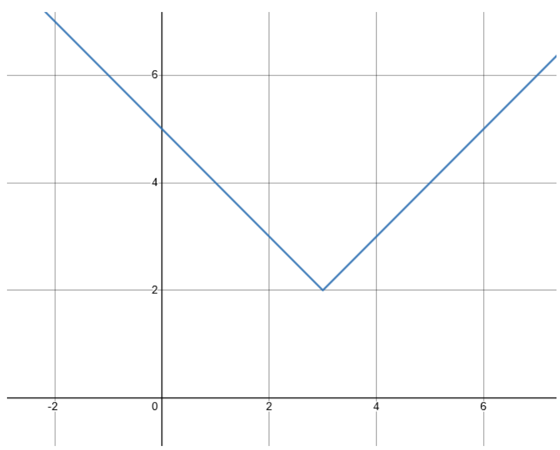
ii. $f(x) = 3 + \sqrt{x+1}$



Domain: $\mathbf{x} \in [\mathbf{-1}, \infty)$

Range: $[\mathbf{3}, \infty)$

iii. $k(x) = |x-3| + 2$



Domain: $\mathbf{x} \in \mathbf{R}$

Range: $[\mathbf{2}, \infty)$

(c) Verify that the following functions are inverses of each other

$f(x) = \sqrt[3]{2x+10}$

$g(x) = \frac{x^3+10}{2}$

We know that f(f$^{-1}(x)) = x$,

$g(f^-1(x)) = \frac{(\sqrt[3]{2x+10})^3+10}{2}$
$g(f^-1(x)) = \frac{2x+10+10}{2}$
$g(f^-1(x)) = \frac{2x}{2} = x$
[As g(f(x))=x, f and g must be inverses of each other]

Problem 2

(a) Find the following limits:

i.

$$\begin{aligned}
\lim_{t\to 1} \frac{t-1}{\sqrt{t}-1} &= \lim_{t\to 1} \frac{t-1}{\sqrt{t}-1} \cdot \frac{\sqrt{t}+1}{\sqrt{t}+1} \\
&= \lim_{t\to 1} \frac{(t-1)(\sqrt{t}+1)}{t-1} \\
&= \lim_{t\to 1} \sqrt{t}+1 \\
&= \sqrt{1}+1 \\
&= \boxed{2}
\end{aligned}$$

ii.

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1} &= \lim_{x\to 0}\frac{(1-2\sin^2 x)-1}{\cos x-1}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x}{\cos x-1}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x(\cos x+1)}{(\cos x-1)(\cos x+1)}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x(\cos x+1)}{\cos^2 x-1}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x(\cos x+1)}{\sin^2 x}\\
&= \lim_{x\to 0}\frac{-2\sin^2 x(\cos x+1)}{\sin^2 x}\\
&= \lim_{x\to 0} 2\cos x+2\\
&= 2+2=\boxed{4}.
\end{aligned}$$

iii.

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan 7x}{\sin 3x} &= \lim_{x\to 0}\frac{sin 7x}{\cos 7x.\sin 3x}\\
&= \lim_{x\to 0}\frac{7\sin 7x}{7x}.\frac{3}{3\sin 3x}.\frac{1}{\cos 7x}\\
&= \boxed{\frac{7}{3}}
\end{aligned}$$

[We know as x approaches 0, Sin(x)/x =0 and cos (x) =1]

iv. $\lim_{x\to 0}\frac{\sqrt{2-3x}-\sqrt{3x+2}}{x}$

$= \lim_{x\to 0}\frac{(\sqrt{2-3x}-\sqrt{3x+2})(\sqrt{2-3x}+\sqrt{3x+2})}{x(\sqrt{2-3x}+\sqrt{3x+2})}$

$= \lim_{x\to 0}\frac{(2-3x)-(3x+2)}{x(\sqrt{2-3x}+\sqrt{3x+2})(\sqrt{2-3x}+\sqrt{3x+2})}$

$= \lim_{x\to 0}\frac{-6x}{x(\sqrt{2-3x}+\sqrt{3x+2})}$

$= \lim_{x\to 0}\frac{-6}{\sqrt{2}+\sqrt{2}}$

$= \boxed{-\dfrac{3\sqrt{2}}{2}}$

v.

$$\begin{aligned}
\lim_{t\to\infty}\frac{2t^4-t^2+8t}{-5t^4+7} &= \lim_{t\to\infty}\frac{t^4(2-\frac{1}{t^2}+\frac{8}{t^3})}{t^4(-5+\frac{7}{t^4})} \\
&= \lim_{t\to\infty}\frac{2-\frac{1}{t^2}+\frac{8}{t^3}}{-5+\frac{7}{t^4}} \\
&= \frac{\lim_{t\to\infty}(2-\frac{1}{t^2}+\frac{8}{t^3})}{\lim_{t\to\infty}(-5+\frac{7}{t^4})} \\
&= \frac{2}{-5} \\
&= \boxed{-\frac{2}{5}}
\end{aligned}$$

vi.

$$\begin{aligned}
\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}}{x^2} &= \lim_{x\to 0}\frac{(a-\sqrt{a^2-x^2})(a+\sqrt{a^2-x^2})}{x^2(a+\sqrt{a^2-x^2})}\\
&= \lim_{x\to 0}\frac{a^2-(a^2-x^2)}{x^2(a+\sqrt{a^2-x^2})}\\
&= \lim_{x\to 0}\frac{x^2}{x^2(a+\sqrt{a^2-x^2})}\\
&= \lim_{x\to 0}\frac{1}{a+\sqrt{a^2-x^2}}\\
&= \boxed{\frac{1}{2a}}
\end{aligned}$$

vii.

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^x x}{x^2}\cdot\frac{1}{1+\cos x}\\
&= 1\cdot\frac{1}{1+1}\\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

viii.

$$\begin{aligned}
\lim_{x\to 0}\frac{\csc x-\cot x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x\sin x}\\
&= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x\sin x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x\sin x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x\sin x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}\\
&= 1\cdot\frac{1}{1+1}\\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(b) For the following function does $\lim_{x\to 0} f(x)$ **exist? If so, Find** $\lim_{x\to 0} f(x)$

$$f(x)=\begin{cases} x^2\cos(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

We know that

$$1 \geq cos(\frac{1}{x}) \geq -1$$
$$x^2 \geq x^2cos(\frac{1}{x}) \geq x^2$$
$$As \lim_{x\to 0} \quad 0 \geq x^2cos(\frac{1}{x}) \geq 0$$

Hence via the squeeze theorem we can prove that the limit does exist and is equal to 0.

Problem 3

(a) When is a function said to be continuous? Test the continuity of the following function at x=0

$$f(x) = \begin{cases} 1 + sinx & \text{if } 0 \leq x < \pi/2 \\ 0 & \text{if } x = 0 \end{cases}$$

A function $f(x)$ is said to be continuous at a point $x = a$ if $f(a)$ is defined and $\lim_{x\to a^-} f(x)$ along with $\lim_{x\to a^+} f(x)$ are defined and equal to $f(a)$.

$\lim_{x\to 0^-} f(x) = 0$[By definition]
$\lim_{x\to 0^+} f(x) = 1 + \sin 0 = 1$

As $1 \neq 0$ Right hand limit and left hand limit are not equal and hence the function is not continuous.

(b) Test the continuity of the following function at $x = 1$

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{x} & \text{if } x \geq 0 \end{cases}$$

$$\begin{aligned} f(1) &= 1[\text{Defined}] \\ \lim_{x \to 1^-} f(x) &= 1 \\ \lim_{x \to 1^+} f(x) &= 1/1 = 1 \end{aligned}$$

As $f(1)$ is defined and the right and left hand limits are equal the function is continuous.

(c) Determine whether the following functions are continuous at $x = 2$

i. $f(x) = \frac{x^2-4}{x-2}$
   $f(2) = \frac{4-4}{2-2} = \frac{0}{0} = \text{undefined}$
   As the function is not defined at $x = 2$ the function is not continuous.

ii. $g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

$$\begin{aligned} f(2) &= 3[\text{Defined}] \\ \lim_{x \to 1^-} \frac{x^{-}2^2}{x-2} &= \lim_{x \to 1^-} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x \to 1^-} x+2 \\ &= 2+2 = 4 \end{aligned}$$

As $4 \neq 3$ left hand limit is not equal to function value and hence the function is not continuous.

iii. $h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$

$$\begin{aligned}
f(2) &= 4[\text{Defined}] \\
\lim_{x \to 1^-} \frac{x^- 2^2}{x-2} &= \lim_{x \to 1^-} \frac{(x-2)(x+2)}{x-2} \\
&= \lim_{x \to 1^-} x + 2 \\
&= 2 + 2 = 4 \\
\lim_{x \to 1^+} \frac{x^- 2^2}{x-2} &= \lim_{x \to 1^-} x + 2 \\
&= 2 + 2 = 4
\end{aligned}$$

As $f(4)$ is defined and right and left hand limits are equal the function is continuous at $x = 2$

Problem 4

(a) A function is defined by $h(x) = \begin{cases} 3 - 2x & \textbf{if } 0 \leq x < \frac{3}{2} \\ -3 - 2x & \textbf{if } x \geq \frac{3}{2} \end{cases}$

Test the continuity of the function at $x = \frac{3}{2}$

$$\begin{aligned} h(\frac{3}{2}) &= -3 - 2(\frac{3}{2}) \\ &= -3 - 3 \\ &= -6[\text{Defined}] \\ \lim_{x \to 3/2^-} 3 - 2x &= 3 - 2(\frac{3}{2}) \\ &= 3 - 3 = 0 \end{aligned}$$

As $-6 \neq 0$ left hand limit is not equal to function value and hence the function is not continuous.

(b) Discuss the continuity of f(x) at x = 0 where

$$f(x) = \begin{cases} xcos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f(0) = 0[\text{Defined}]$$
$$1 \geq cos(\frac{1}{x}) \geq -1$$
$$x \geq xcos(\frac{1}{x}) \geq x$$
$$\lim_{x \to 0} \quad 0 \geq xcos(\frac{1}{x}) \geq 0$$

$\lim_{x \to 0} xcos(\frac{1}{x}) = 0$

As the limit and function value are equal the function is continuous

c) Test the continuity and differentiability of $f(x)$ at $x = \frac{\pi}{2}$ where

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 + sinx & \text{if } 0 \leq x < \frac{\pi}{2} \\ 2 + (x - \frac{\pi}{2})^2 & \text{if } x \geq \frac{\pi}{2} \end{cases}$$

$$\begin{aligned} f(\frac{\pi}{2}) &= 2 + (\frac{\pi}{2} - \frac{\pi}{2})^2 \\ &= 2[\text{Defined}] \\ \lim_{x \to \frac{\pi}{2}^-} 1 + sinx &= 1 + 1 \\ &= 2 \\ \lim_{x \to \frac{\pi}{2}^+} 2 + (x - \frac{\pi}{2})^2 &= 2 + (\frac{\pi}{2} - \frac{\pi}{2})^2 \\ &= 2 \end{aligned}$$

As both limits and function value are equal the function is continuous.

We know from the limit definition of derivatives that for for a function f(x), its derivative is
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Hence the derivative can only exist when $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists when $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ and $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ are equal
At $x = \frac{\pi}{2}$,

$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{1+sin(\frac{\pi}{2}+h)-1-sin(\frac{\pi}{2}))}{h}$
$= \lim_{h\to 0^-} \frac{\sin\left(\frac{\pi}{2}-h\right)-1}{h}$
$= \lim_{h\to 0^-} \frac{\cos(h)-1}{h}$
$= \lim_{h\to 0^-} \frac{\cos(h)-1}{h} \cdot \frac{\cos(h)+1}{\cos(h)+1}$
$= \lim_{h\to 0^-} \frac{\cos^2(h)-1}{h(\cos(h)+1)}$
$= \lim_{h\to 0^-} \frac{\sin^2(h)}{h(1+\cos(h))}$
$= \lim_{h\to 0^-} \frac{\sinh(h)}{h} \cdot \frac{\sinh(h)}{1+\cosh(h)}$
$= \lim_{h\to 0^-} \frac{\sinh(h)}{h} \cdot sinh \cdot \frac{1}{(1+\cosh(h))}$
$= 1 \cdot sin(0) \cdot \frac{1}{(1+\cosh(0))} = 0$

$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{2+(x+h-\frac{\pi}{2})^2-(x-\frac{\pi}{2})^2}{h}$
$= \lim_{h\to 0^-} \frac{(x+h)^2-\pi(x+h)+\frac{\pi^2}{4}-x^2+\pi-\frac{\pi^2}{2}}{h}$
$= \lim_{h\to 0^-} \frac{x^2+2xh+h^2-\pi x-\pi h-x^2+\pi x}{h}$
$= \lim_{h\to 0^-} \frac{h^2+2xh-\pi h}{h}$
$= \lim_{h\to 0^-} h + 2x - \pi$
At $x = \pi/2$,
$= \lim_{h\to 0^-} h = 0$

As both left hand limit and right hand limits are equal we know that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and the function is differentiable.