Consider the measurement model

$$y_\gamma = a_\gamma x_\gamma, \text{ where } \log a_\gamma \sim \mathcal{N}(\log a_0, \sigma^2), \; a_i \perp a_j \text{ for } i \neq j, \; a_\gamma \perp x_\gamma \text{ and } x_\gamma > 0.$$

We can write this as

$$y = a \circ x, \text{ where } a \circ x := [a_1 x_1, \dots, a_n x_n]^T \text{ (elementwise product)}.$$

Then we obtain the likelihood by marginalizing over the nuisance parameter a:

$$\mathbb{P}(y|x) = \frac{\mathbb{P}(y,x)}{\mathbb{P}(x)} = \int_{\mathbb{R}^n} \frac{\mathbb{P}(y,x,a)}{\mathbb{P}(x)}\, da = \int_{\mathbb{R}^n} \frac{\mathbb{P}(y,x,a)}{\mathbb{P}(x)\mathbb{P}(a)} \mathbb{P}(a)\, da \quad \Bigg| \begin{array}{l} x \perp a \\ \Rightarrow \mathbb{P}(x,a) = \mathbb{P}(x)\mathbb{P}(a) \end{array}$$

$$= \int_{\mathbb{R}^n} \frac{\mathbb{P}(y,x,a)}{\mathbb{P}(x,a)} \mathbb{P}(a)\, da = \int_{\mathbb{R}^n} \mathbb{P}(y|x,a)\, \mathbb{P}(a)\, da.$$

What is $\mathbb{P}(y|x,a)$? Since we now condition on both x and a, we obtain from the measurement model that

$$y = \underbrace{a \circ x}_{\text{these are now fixed}} \quad \Rightarrow \quad \mathbb{P}(y|a,x) = \delta(y - a \circ x),$$

when δ is (formally) the Dirac delta distribution.

Therefore

$$\mathbb{P}(y|x) = \int_{\mathbb{R}^n} \delta(y - a \circ x)\, \mathbb{P}(a)\, da.$$

Change of variables: $v_i = a_i x_i \Rightarrow dv_i = x_i\, da_i$

$$\Rightarrow \mathbb{P}(y|x) = \frac{1}{x_1 \cdots x_n} \int_{\mathbb{R}^n} \delta(y - v)\, \mathbb{P}\left(\frac{v}{x}\right) dv \quad \Bigg| \; \frac{v}{x} := \left[\frac{v_1}{x_1}, \dots, \frac{v_n}{x_n}\right]^T$$

$$= \frac{1}{x_1 \cdots x_n} \mathbb{P}\left(\frac{y}{x}\right). \qquad (*)$$

By the independence of a_i and a_j for $i \neq j$, we obtain

$$\mathbb{P}\left(\frac{y}{x}\right) = \prod_{\gamma=1}^{n} p_{a_\gamma}\left(\frac{y_\gamma}{x_\gamma}\right).$$

Since $a_j \sim \log N(\log a_0, \delta^2)$, it follows by change of variables that

$$p_{a_j}(a) = p_{\log a_j}(\log a) \frac{d \log a}{da} = \frac{1}{\sqrt{2\pi\delta^2}} \frac{1}{a} \exp\left(-\frac{1}{2\delta^2}(\log a - \log a_0)^2\right).$$

Hence

$$\mathbb{P}\left(\frac{y}{x}\right) = \left(\frac{1}{\sqrt{2\pi\delta^2}}\right)^n \frac{x_1 \cdots x_n}{y_1 \cdots y_n} \exp\left(-\frac{1}{2\delta^2}\sum_{j=1}^{n}\left(\log\frac{y_j}{a_0 x_j}\right)^2\right).$$

Plugging this into (*) yields

$$\mathbb{P}(y|x) = \left(\frac{1}{\sqrt{2\pi\delta^2}}\right)^n \frac{1}{y_1 \cdots y_n} \exp\left(-\frac{1}{2\delta^2}\sum_{j=1}^{n}\left(\log\frac{y_j}{a_0 x_j}\right)^2\right).$$