Doubling the rate in high-dimensional function approximation with application to numerical integration

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- K–Kuo–Sloan (2024): Lattice-based kernel approximation and serendipitous weights for parametric PDEs in very high dimensions. To appear in *Monte Carlo and Quasi-Monte Carlo Methods 2022*. Preprint arXiv:2303.17755 [math.NA].

Part II: Doubling the rate for high-dimensional function approximation

- Sloan–K (2023+): Doubling the rate – improved error bounds for orthogonal projection in Hilbert spaces. Preprint 2023, arXiv:2308.06052 [math.NA].

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- K–Klebanov–Schillings (ongoing work)

Kernel interpolation over lattice point sets

Lattice rules

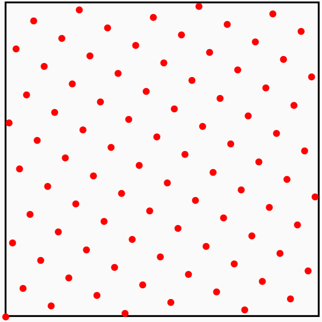
Rank-1 lattice rules

$$Q_{s,n}(f) = \frac{1}{n}\sum_{k=1}^{n} f(\boldsymbol{t}_k) \approx \int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = I_s(f)$$

have the nodes

$$\boldsymbol{t}_k = \mathrm{mod}\left(\frac{k\boldsymbol{z}}{n}, 1\right), \quad k \in \{1, \ldots, n\},$$

where the entire point set is determined by the *generating vector* $\boldsymbol{z} \in \mathbb{N}^s$, with all components *coprime* to n.



Lattice rule with $\boldsymbol{z} = (1, 55)$ and $n = 89$ nodes in $[0, 1]^2$

The quality of the lattice rule is determined by the choice of $\boldsymbol{z}$.

In [K–Kazashi–Kuo–Nobile–Sloan (2022)], we studied *kernel interpolation of smooth, periodic functions based on lattice point sets*. We considered the following setting:

Let $\alpha \geq 2$ be an even integer and let $H_{s,\alpha,\boldsymbol{\gamma}}$ be the Hilbert space containing absolutely continuous, somewhat smooth periodic functions $f : [0,1)^s \to \mathbb{R}$ endowed with the norm

$$\|f\|^2_{H_{s,\alpha,\boldsymbol{\gamma}}} = \sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \frac{1}{(2\pi)^{\alpha|\mathfrak{u}|}\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \bigg| \int_{[0,1]^{s-|\mathfrak{u}|}} \bigg(\prod_{j\in\mathfrak{u}} \frac{\partial^{\alpha/2}}{\partial y_j^{\alpha/2}} \bigg) f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} \bigg|^2 \, \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}$$

provided that f has mixed partial derivatives of order $\alpha/2$.

The space $H_{s,\alpha,\boldsymbol{\gamma}}$ is actually a *reproducing kernel Hilbert space* (RKHS), with an explicitly known and analytically simple reproducing kernel:

$$K(\boldsymbol{y},\boldsymbol{y}') = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \gamma_{\mathfrak{u}} \prod_{j\in\mathfrak{u}} \eta_\alpha(y_j,y_j'), \quad \boldsymbol{y},\boldsymbol{y}'\in[0,1]^s,$$

where

$$\eta_\alpha(y,y') = \frac{(2\pi)^\alpha}{(-1)^{\alpha/2+1}\alpha!} B_\alpha(|y-y'|), \quad y,y'\in[0,1],$$

where $B_2(y) = y^2 - y + \frac{1}{6}$, $B_4(y) = y^4 - 2y^3 + y^2 - \frac{1}{30}$, and so on, are the *Bernoulli polynomials* provided that $\alpha \geq 2$ is an even integer. In particular,

$$\langle f, K(\cdot,\boldsymbol{y})\rangle_{H_{s,\alpha,\boldsymbol{\gamma}}} = f(\boldsymbol{y}) \quad \text{for all } f\in H_{s,\alpha,\boldsymbol{\gamma}},\ \boldsymbol{y}\in[0,1]^s.$$

Example: If $(\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ are *product weights*, i.e.,

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \gamma_j, \quad \mathfrak{u}\subseteq\{1,\ldots,s\},$$

then the kernel has a computationally attractive form

$$K(\boldsymbol{y},\boldsymbol{y}') = \prod_{j=1}^s (1 + \gamma_j \eta_\alpha(y_j,y_j')).$$

Suppose that one is interested in finding an approximation for the function $f \in H_{s,\alpha,\gamma}$ based on point evaluations $f(\boldsymbol{t}_1), \ldots, f(\boldsymbol{t}_n)$, $\boldsymbol{t}_j \in [0,1]^s$. We introduce the *kernel interpolant*

$$f_n(\boldsymbol{y}) := \sum_{k=1}^{n} c_k K(\boldsymbol{t}_k, \boldsymbol{y}), \quad \boldsymbol{t}_k := \mathrm{mod}\Big(\frac{k\boldsymbol{z}}{n}, 1\Big),$$

where the coefficients are determined by requiring the interpolation condition $f_n(\boldsymbol{t}_k) = f(\boldsymbol{t}_k)$ to hold for all $k = 1, \ldots, n$. Equivalently, the coefficients can be solved from the linear system

$$\mathcal{K}\boldsymbol{c} = \boldsymbol{f},$$

where $\boldsymbol{c} := [c_1, \ldots, c_n]^{\mathrm{T}}$,

$$\mathcal{K}_{k,\ell} = K(\boldsymbol{t}_k, \boldsymbol{t}_\ell) \quad \text{and} \quad \boldsymbol{f} := [f(\boldsymbol{t}_1), \ldots, f(\boldsymbol{t}_n)]^{\mathrm{T}}.$$

Note that $\mathcal{K}_{k,\ell} = K\big(\frac{(k-\ell)\boldsymbol{z}}{n}, \boldsymbol{0}\big)$, i.e., $\mathcal{K}$ is a *circulant matrix* $\Rightarrow$

$$\boldsymbol{c} = \texttt{ifft}\big(\texttt{fft}(\boldsymbol{f})./\texttt{fft}(\mathcal{K}_{:,1})\big)$$

This can be computed in $\mathcal{O}(n \log n)$ time!

The kernel interpolant is cheap to construct!

Proposition (K–Kazashi–Kuo–Nobile–Sloan (2022))
A generating vector $\boldsymbol{z} \in \{1, \ldots, n-1\}^s$ *can be constructed by a CBC algorithm such that*

$$\|f - f_n\|_{L^2([0,1]^s)} \leq \frac{\tau}{n^{1/(4\lambda)}} \Bigg(\sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \max\{1, |\mathfrak{u}|\} \gamma_{\mathfrak{u}}^{\lambda} (2\zeta(\alpha\lambda))^{|\mathfrak{u}|} \Bigg)^{1/\lambda} \|f\|_{H_{s,\alpha,\boldsymbol{\gamma}}}$$

for $\lambda \in (1/\alpha, 1]$, $\alpha > 1$, *prime* n, *and* $\zeta(x) := \sum_{k=1}^{\infty} k^{-x}$, $x > 1$. *Here,* $\gamma_{\varnothing} := 1$ *and* $\tau := \sqrt{2}(2.5 + 2^{2\alpha\lambda+1})^{1/(2\lambda)}$.

Remark: Generalized for composite n by Kuo–Mo–Nuyens (2022+).

Numerical experiment

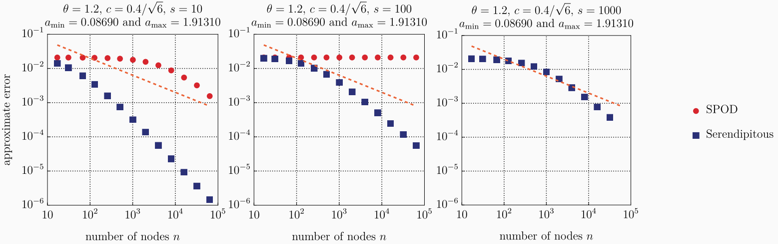
Let us consider the PDE problem

$$-\nabla \cdot (a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})) = x_2, \quad u(\cdot,\boldsymbol{y})|_{\partial D} = 0,$$

in the physical domain $D = (0,1)^2$ with the diffusion coefficient

$$a(\boldsymbol{x},\boldsymbol{y}) = 1 + \sum_{j=1}^{s} \sin(2\pi y_j)\psi_j(\boldsymbol{x}), \quad \boldsymbol{x} \in D,\ y_j \in [0,1],$$

where $\psi_j(\boldsymbol{x}) = cj^{-\theta}\sin(j\pi x_1)\sin(j\pi x_2)$. **Note that** $\|\psi_j\|_{L^\infty(D)} \propto j^{-\theta}$.



SPOD: Kernel interpolation using weights derived in [K–Kazashi–Kuo–Nobile–Sloan (2022)].
Serendip.: Kernel interpolation using product weights advocated in [K–Kuo–Sloan (2024)].

Doubling the rate for high-dimensional function approximation

Let $U \subseteq \mathbb{R}^s$ be a nonempty domain such that

- H is a Hilbert space continuously embedded in $L^2(U)$.
- B is a "smoother" normed space continuously embedded in H.

Critical assumption:

$$|\langle f, g\rangle_H| \leq \|f\|_{L^2(U)}\|g\|_B \quad \text{for all } f \in H,\ g \in B.$$

Theorem (Sloan–K 2023+)
Let P_n be the H-orthogonal projection operator onto a finite-dimensional space $V \subset H$, $n := \dim(V)$. Assume that for some $C, \kappa > 0$, there holds

$$\|f - P_n f\|_{L^2(U)} \leq Cn^{-\kappa}\|f\|_H \quad \text{for all } f \in H.$$

Then for all $g \in B$, we have

$$\|g - P_n g\|_{L^2(U)} \leq C^2 n^{-2\kappa}\|g\|_B$$

and

$$\|g - P_n g\|_H \leq Cn^{-\kappa}\|g\|_B.$$

Proof. Let $g \in B$.

Since $f = g - P_n g \in H$ and $P_n(g - P_n g) = P_n g - P_n g = 0$, we obtain

$$\|g - P_n g\|_{L^2(U)} = \|(g - P_n g) - P_n(g - P_n g)\|_{L^2(U)} \leq C n^{-\kappa} \|g - P_n g\|_H$$
$$\Rightarrow \quad \|g - P_n g\|_{L^2(U)}^2 \leq C^2 n^{-2\kappa} \|g - P_n g\|_H^2. \tag{1}$$

Now

$$\|g - P_n g\|_H^2 = \langle g - P_n g, g - P_n g\rangle_H \overset{g - P_n g \perp P_n g}{=} \langle g - P_n g, g\rangle_H$$
$$\leq \|g - P_n g\|_{L^2(U)} \|g\|_B. \tag{2}$$

Plugging this into (1) yields

$$\|g - P_n g\|_{L^2(U)}^2 \leq C^2 n^{-2\kappa} \|g - P_n g\|_{L^2(U)} \|g\|_B$$
$$\Rightarrow \quad \|g - P_n g\|_{L^2(U)} \leq C^2 n^{-2\kappa} \|g\|_B \tag{3}$$

as desired.

The inequality $\|g - P_n g\|_H \leq C n^{-\kappa} \|g\|_B$ follows (in squared form) by plugging (3) into (2). $\square$

Application to kernel interpolation

For $H = H_{s,\alpha,\boldsymbol{\gamma}}$ and $B = H_{s,2\alpha,\boldsymbol{\gamma}^2}$, there holds

$$|\langle f, g\rangle_{H_{s,\alpha,\boldsymbol{\gamma}}}| \leq \|f\|_{L^2(U)}\|g\|_{H_{s,2\alpha,\boldsymbol{\gamma}^2}}.$$

The kernel interpolant

$$f_n(\boldsymbol{y}) = \sum_{k=1}^{n} c_k K(\boldsymbol{t}_k, \boldsymbol{y})$$

is precisely the H-orthogonal projection of $f \in H$ onto $V = \mathrm{span}\{K(\boldsymbol{t}_1, \cdot), \ldots, K(\boldsymbol{t}_n, \cdot)\}$.

$\therefore$ If $f \in B$, then

$$\|f - f_n\|_{L^2(U)} \lesssim n^{-1/(2\lambda)},$$
$$\|f - f_n\|_{H_{s,\alpha,\boldsymbol{\gamma}}} \lesssim n^{-1/(4\lambda)},$$

for $\lambda \in (1/\alpha, 1],\ \alpha > 1$.

Application to numerical integration?

- Given a sufficiently smooth function, we are able to obtain twice the approximation rate using our kernel interpolant!
- “Doubling the rate” phenomena known also in the context of numerical integration.
  - E.g., (randomly shifted) rank-1 lattice rules have convergence rate $\mathcal{O}(n^{-1})$ for functions with first-order dominating mixed smoothness, but for a sufficiently smooth integrand we can obtain $\mathcal{O}(n^{-2})$ convergence using *baker’s transform*.
- Can our method be applied to numerical integration?

Idea: Let H be a RKHS over the computational domain $[0,1]^s$ with kernel K. Let the cubature point set $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n \in [0,1]^s$ be given (e.g., lattice points). We find the weights $w_1,\ldots,w_n$ of a cubature rule

$$\sum_{k=1}^n w_k f(\boldsymbol{t}_k) \approx \int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad f \in H,$$

by solving the linear system

$$\begin{cases} w_1 K(\boldsymbol{t}_1,\boldsymbol{t}_1) + \cdots + w_n K(\boldsymbol{t}_1,\boldsymbol{t}_n) = \int_{[0,1]^s} K(\boldsymbol{t}_1,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\ \vdots \\ w_1 K(\boldsymbol{t}_n,\boldsymbol{t}_1) + \cdots + w_n K(\boldsymbol{t}_n,\boldsymbol{t}_n) = \int_{[0,1]^s} K(\boldsymbol{t}_n,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \end{cases}$$

Let us focus on the weighted Sobolev space H of **non-periodic** functions $f\colon [0,1]^s \to \mathbb{R}$ with first-order dominating mixed smoothness, with norm

$$\|f\|_H^2 = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \bigg(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} \bigg)^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}$$

and kernel $K(\boldsymbol{y},\boldsymbol{y}') = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \gamma_{\mathfrak{u}} \prod_{j\in\mathfrak{u}} \omega(y_j,y_j')$, $\boldsymbol{y},\boldsymbol{y}' \in [0,1]^s$, where $\omega(y,y') = \frac{1}{2}B_2(|y-y'|) + (y-\frac{1}{2})(y'-\frac{1}{2})$, $y,y' \in [0,1]$.

Essentially, we are “reweighting” an existing cubature rule!

It is not difficult to check that

$$\int_{[0,1]^s} K(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = 1 \quad \text{for all } \boldsymbol{x} \in [0,1]^s.$$

Thus the weights $\boldsymbol{w} := [w_1, \ldots, w_n]^{\mathrm{T}}$ can be obtained from the linear system

$$\mathcal{K}\boldsymbol{w} = \mathbf{1},$$

where $\mathcal{K} \in \mathbb{R}^{n \times n}$ is defined elementwise by $\mathcal{K}_{i,j} = K(\boldsymbol{t}_i, \boldsymbol{t}_j)$.

Unfortunately, in the non-periodic case the system matrix is no longer circulant.

However, the construction of $\mathcal{K}$ is computationally feasible for large dimensions s if the weight structure is suitable.

- For product weights $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ (or equal weights), we can use the formula
$$K(\boldsymbol{y}, \boldsymbol{y}') = \prod_{j=1}^{s} (1 + \gamma_j \omega(y_j, y_j')).$$
- For POD or SPOD weights, we can assemble the matrix $\mathcal{K}$ using recursion formulae w.r.t. the dimension s (compare with the CBC construction in QMC analysis).

Special case: dimension $s = 1$

For simplicity, let us consider the *unweighted* 1-dimensional case. In this case,

$$\|f\|_H^2 = \left(\int_0^1 f(y)\,\mathrm{d}y \right)^2 + \int_0^1 |f'(y)|^2\,\mathrm{d}y$$

with kernel $K(y, y') = 1 + \frac{1}{2} B_2(|y - y'|) + (y - \frac{1}{2})(y' - \frac{1}{2})$.

Let $t_k = \frac{k}{n}$, $k = 0, \ldots, n-1$ be the 1d lattice points with $n \geq 2$. (This is precisely a left-Riemann rule.)

We can analytically solve the quadrature weights for our kernel quadrature $Q_n(f) = \sum_{k=0}^{n-1} w_k f(t_k)$ *in this case!* They are

$$\begin{aligned} w_0 &= \frac{1}{2n} \frac{12n^3}{12n^3 + n + 3}, \\ w_k &= \frac{1}{n} \frac{12n^3}{12n^3 + n + 3}, \quad k \in \{1, \ldots, n-2\}, \\ w_{n-1} &= \frac{3}{2n} \frac{12n^3}{12n^3 + n + 3}. \end{aligned}$$

Step 1

The quadrature error can be recast as

$$\begin{aligned}\int_0^1 f(y)\,\mathrm{d}y - \sum_{k=0}^{n-1} w_k f(t_k) &= \int_0^1 \langle f, K(\cdot, y)\rangle_H \,\mathrm{d}y - \sum_{k=0}^{n-1} w_k \langle f, K(\cdot, t_k)\rangle_H \\ &= \left\langle f, \int_0^1 K(\cdot, y)\,\mathrm{d}y - \sum_{k=0}^{n-1} w_k K(\cdot, t_k) \right\rangle_H \\ &= \langle f, h - h_n\rangle_H,\end{aligned}$$

where $h := \int_0^1 K(\cdot, y)\,\mathrm{d}y = 1$ and $h_n := \sum_{k=0}^{n-1} w_k K(\cdot, t_k)$ is the kernel interpolant of h.

Step 2

Let us assume that $f \in B$, where B is a Sobolev space of functions with dominating mixed smoothness of order 2.

Since

$$\langle f, h - h_n \rangle_H = \left(\int_0^1 f(y)\,\mathrm{d}y \right) \left(\int_0^1 (h(y) - h_n(y))\,\mathrm{d}y \right) + \int_0^1 f'(y)(h'(y) - h_n'(y))\,\mathrm{d}y,$$

we can use integration by parts to obtain

$$\begin{aligned} &\int_0^1 f'(y)(h'(y) - h_n'(y))\,\mathrm{d}y \\ &= f'(1)\underbrace{(h(1) - h_n(1))}_{=\frac{6n}{12n^3+n+3}} - f'(0)\underbrace{(h(0) - h_n(0))}_{=0} - \int_0^1 f''(y)(h(y) - h_n(y))\,\mathrm{d}y \\ &\leq \frac{6n}{12n^3 + n + 3} f'(1) + \|f''\|_{L^2(0,1)} \|h - h_n\|_{L^2(0,1)}. \end{aligned}$$

Therefore

$$|\langle f, h - h_n \rangle_H| \leq \underbrace{\frac{6n}{12n^3 + n + 3}}_{=\mathcal{O}(n^{-2})} f'(1) + \|h - h_n\|_{L^2(0,1)} \left(\|f\|_{L^2(0,1)} + \|f''\|_{L^2(0,1)} \right).$$

It remains to assess the convergence rate of $\|h - h_n\|_{L^2(0,1)}$.

Step 3

Making use of the identities

$$\sum_{k=0}^{n-1} w_k = \frac{24n^2}{12n^3+n+3} + \frac{12n^2(n-2)}{12n^3+n+3},$$

$$\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} w_k w_\ell \bigg(\frac{46}{45} - \frac{1}{6}t_k^2 + \frac{1}{12}t_k^3 - \frac{1}{24}t_k^4 + \frac{1}{4}t_k^2 t_\ell - \frac{1}{6}t_\ell^2$$

$$+ \frac{1}{4}t_k t_\ell^2 - \frac{1}{4}t_k^2 t_\ell^2 + \frac{1}{12}t_\ell^3 - \frac{1}{24}t_\ell^4 + \frac{1}{12}|t_k - t_\ell|^3 \bigg) = \frac{720n^6 + n^2 + 60n - 45}{5(12n^3+n+3)^2},$$

where the latter identity is valid for $n \geq 2$, we obtain

$$\|h - h_n\|_{L^2(0,1)}^2 = \int_0^1 \bigg(1 - \sum_{k=0}^{n-1} w_k K(x, t_k)\bigg)^2 \mathrm{d}x$$

$$= \int_0^1 \bigg(1 - 2\sum_{k=0}^{n-1} w_k K(x,t_k) + \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} w_k w_\ell K(x,t_k)K(x,t_\ell)\bigg)\,\mathrm{d}x$$

$$= 1 - 2\sum_{k=0}^{n-1} w_k + \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} w_k w_\ell \bigg(\frac{46}{45} - \frac{1}{6}t_k^2 + \frac{1}{12}t_k^3 - \frac{1}{24}t_k^4 + \frac{1}{4}t_k^2 t_\ell - \frac{1}{6}t_\ell^2$$

$$+ \frac{1}{4}t_k t_\ell^2 - \frac{1}{4}t_k^2 t_\ell^2 + \frac{1}{12}t_\ell^3 - \frac{1}{24}t_\ell^4 + \frac{1}{12}|t_k - t_\ell|^3 \bigg) = \cdots = \frac{6n(n+15)}{5(12n^3+n+3)^2}$$

for $n \geq 2$.

$\therefore \|h - h_n\|_{L^2(0,1)} = \mathcal{O}(n^{-2})$ as desired.

Generalization to higher dimensions?

- It seems difficult to generalize the 1d argument to higher dimensions.
- Not clear if the result even holds for general $s > 1$ (does the "compatibility condition" $|\langle f, g\rangle_H| \leq \|f\|_{L^2}\|g\|_B$ hold, how to control $\sum_{k=1}^{n} w_k$, etc.?)
- Nevertheless, it is easy to implement the method numerically even in high dimensions.

$$\int_{[0,1]^s} \frac{1}{1+\sum_{j=1}^{s} j^{-2} y_j} \, \mathrm{d}\boldsymbol{y}$$

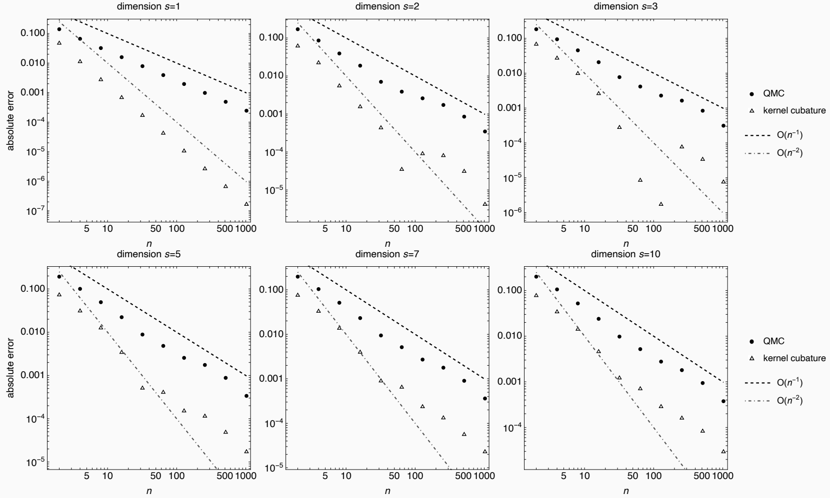


Figure 1: QMC vs. kernel cubature constructed using the weights $\gamma_{\mathfrak{u}} = \left(|\mathfrak{u}|! \prod_{j\in\mathfrak{u}} \frac{b_j}{\sqrt{2\zeta(2\lambda)/(2\pi^2)^{\lambda}}}\right)^{\frac{2}{1+\lambda}}$, $\lambda = \frac{1}{2-2\delta}$, $\delta = 0.05$, for all $\mathfrak{u} \subseteq \{1,\ldots,s\}$.

Let us consider the PDE problem

$$-\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = x_1, \quad u(\cdot, \boldsymbol{y})|_{\partial D} = 0,$$

in the physical domain $D = (0,1)^2$ with the diffusion coefficient

$$a(\boldsymbol{x}, \boldsymbol{y}) = 1 + \sum_{j=1}^{s} y_j \psi_j(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \ y_j \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

where $\psi_j(\boldsymbol{x}) = 0.1 j^{-2} \sin(j\pi x_1) \sin(j\pi x_2)$. We compute $\mathbb{E}[G(u)]$ using both QMC and "reweighted" QMC, where $G(v) = \int_D v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$.

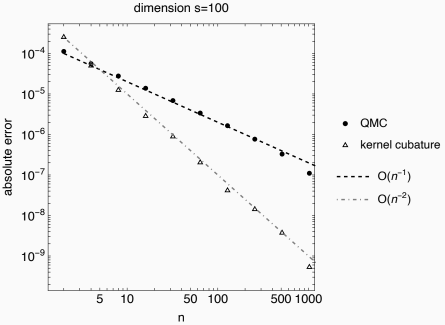


Figure 2: QMC vs. kernel cubature with $s = 100$ constructed using the weights $\gamma_{\mathfrak{u}} = \left(|\mathfrak{u}|! \prod_{j \in \mathfrak{u}} \frac{b_j}{\sqrt{2\zeta(2\lambda)/(2\pi^2)^\lambda}}\right)^{\frac{2}{1+\lambda}}$, $\lambda = \frac{1}{2-2\delta}$, $\delta = 0.05$, for all $\mathfrak{u} \subseteq \{1, \ldots, s\}$.

Conclusions

- Doubling the rate for kernel approximation.
- Preliminary work commenced to explore a similar phenomenon in numerical integration (1d case OK, but higher dimensions unclear).

K–Kazashi–Kuo–Nobile–Sloan (2022): Fast approximation by periodic kernel-based lattice-point interpolation with application in uncertainty quantification. *Numer. Math.* **150**:33–77.

K–Kuo–Sloan (2024): Lattice-based kernel approximation and serendipitous weights for parametric PDEs in very high dimensions. To appear in *Monte Carlo and Quasi-Monte Carlo Methods 2022*. Preprint arXiv:2303.17755 [math.NA].

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Thank you for your attention!