Revisiting the dimension truncation error of parametric elliptic PDEs

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Consider the elliptic PDE problem:

$$\begin{cases} -\nabla \cdot \big(a(\boldsymbol{x}) \nabla u(\boldsymbol{x})\big) = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in D, \\ +\text{boundary conditions.} \end{cases}$$

In practice, one or several of the material/system parameters may be uncertain or incompletely known and modeled as random fields:

- PDE coefficient a may be uncertain;
- Source term f may be uncertain;
- Boundary conditions may be uncertain;
- The domain D itself may be uncertain.

In forward uncertainty quantification, one is interested in assessing how uncertainties in the inputs of a mathematical model affect the output.

$\Rightarrow$ If the uncertain inputs are modeled as random fields, then the output of the PDE is also a random field. One may be interested in assessing the statistical response of the system, for example, the expectation or variance of the PDE solution (or some other quantity of interest thereof).

Background

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A popular model in the literature:

Uncertain diffusion coefficient

Find $u\colon D \times \Omega \to \mathbb{R}$ that satisfies

$$-\nabla \cdot \big(a(\boldsymbol{x}, \boldsymbol{\omega}) \nabla u(\boldsymbol{x}, \boldsymbol{\omega})\big) = f(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in D,$$
$$u(\cdot, \boldsymbol{\omega})|_{\partial D} = 0,$$

for almost all events $\boldsymbol{\omega} \in \Omega$. Here, the diffusion coefficient $a(\cdot, \boldsymbol{\omega}) \in L^\infty_+(D)$ is *uncertain*. Two popular models in the literature:

- **(Uniform and affine model)** For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a(\boldsymbol{x}, \boldsymbol{\omega}) = a_0(\boldsymbol{x}) + \sum_{j \geq 1} Y_j(\omega) \psi_j(\boldsymbol{x}), \quad Y_j \text{ i.i.d. uniform on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

- **(Lognormal model)** For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a(\boldsymbol{x}, \boldsymbol{\omega}) = a_0(\boldsymbol{x}) \exp\left(\sum_{j \geq 1} Y_j(\omega) \psi_j(\boldsymbol{x})\right), \quad Y_j \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

It is often desirable to truncate the input random field:

- **(Uniform and affine model')** For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a_s(\boldsymbol{x}, \boldsymbol{\omega}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{s} Y_j(\omega)\psi_j(\boldsymbol{x}), \quad Y_j \text{ i.i.d. uniform on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

- **(Lognormal model')** For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a_s(\boldsymbol{x}, \boldsymbol{\omega}) = a_0(\boldsymbol{x}) \exp\left(\sum_{j=1}^{s} Y_j(\omega)\psi_j(\boldsymbol{x})\right), \quad Y_j \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

Let u_s be the PDE response corresponding to dimensionally-truncated a_s.

Question: What is $\|u - u_s\| =?$

Remarks: Dimension truncation can be avoided in some cases: e.g., circulant embedding method in the lognormal case (Graham, Kuo, Sloan), best N-term approximation (Cohen, DeVore, Schwab 2010)...

Basic setting (Kuo, Schwab, Sloan 2012; Gantner 2018)

Consider

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = f(\boldsymbol{x}) & \boldsymbol{x} \in D, \\ u(\cdot, \boldsymbol{y})|_{\partial D} = 0 \end{cases}$$

with $a(\boldsymbol{x}, \boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j \geq 1} y_j \psi_j(\boldsymbol{x})$, $y_j \in [-1/2, 1/2]$. Let $U := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$, suppose $a(\boldsymbol{x}, \boldsymbol{y})$ uniformly elliptic for all $\boldsymbol{x} \in D, \boldsymbol{y} \in U$, and $(\psi_j) \in \ell^p$ is a nonincreasing sequence for some $p \in (0, 1)$.

We can write this as $Bu = f$, where $B = -\nabla \cdot a_0 \nabla - \sum_{j \geq 1} y_j \nabla \cdot \psi_j \nabla$ (and analogously for the dimensionally-truncated version B_s). Define $u_s(\boldsymbol{x}, \boldsymbol{y}) := u(\boldsymbol{x}, y_1, \ldots, y_s, 0, 0, \ldots)$.

Trick #1: For sufficiently large s, $\|B - B_s\|$ can be made arbitrarily small. As long as B_s and B are boundedly invertible, we can make $\|B_s^{-1}(B - B_s)\| < 1$ for $s \gg 1$. Neumann series:

$$B^{-1} = (I + B_s^{-1}(B - B_s))^{-1} = \sum_{k \geq 0} (-B_s^{-1}(B - B_s))^k B_s^{-1}.$$

For any bounded linear functional G,

$$\begin{aligned} \int_U G(u(\cdot, \boldsymbol{y}) - u_s(\cdot, \boldsymbol{y}))\, \mathrm{d}\boldsymbol{y} &= \int_U G((B^{-1} - B_s^{-1}) f)\, \mathrm{d}\boldsymbol{y} \\ &= \sum_{k \geq 1} G \int_U (-B_s^{-1}(B - B_s))^k B_s^{-1} f\, \mathrm{d}\boldsymbol{y}. \end{aligned}$$

Trick #2: The $k = 1$ term *also* vanishes!

Why does the $k = 1$ term vanish?

$$\begin{aligned}
&\int_U (-B_s^{-1}(B - B_s))B_s^{-1} f \,\mathrm{d}\boldsymbol{y} \\
&= \int_U B_s^{-1} \sum_{k>s} y_k \nabla \cdot \psi_k \nabla u_s \,\mathrm{d}\boldsymbol{y} \\
&= \sum_{k>s} \int_U \underbrace{y_k}_{y_{s+1}, y_{s+2}, \cdots} \cdot \underbrace{B_s^{-1}(\nabla \cdot \psi_k \nabla u_s)}_{y_1, \ldots, y_s} \,\mathrm{d}\boldsymbol{y} \qquad \text{(use Fubini)} \\
&= \sum_{k>s} \underbrace{\left(\int_{-1/2}^{1/2} y_k \,\mathrm{d}y_k \right)}_{=0} \left(\int_{U_{\le s}} B_s^{-1}(\nabla \cdot \psi_k \nabla u_s) \,\mathrm{d}\boldsymbol{y}_{\le s} \right) = 0.
\end{aligned}$$

Trick #3: Balancing the error terms:

$$\begin{aligned}
&\left| \int_U G(u(\cdot, \boldsymbol{y}) - u_s(\cdot, \boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} \right| \\
&= \left| \sum_{k=2}^{k'-1} G \int_U (-B_s^{-1}(B - B_s))^k B_s^{-1} f \,\mathrm{d}\boldsymbol{y} + \sum_{k \ge k'} G \int_U (-B_s^{-1}(B - B_s))^k B_s^{-1} f \,\mathrm{d}\boldsymbol{y} \right| \\
&\lesssim \underbrace{\sum_{k=2}^{k'-1} \sum_{\substack{|\boldsymbol{\nu}|=k \\ \nu_j = 0 \ \forall j \le s, \ \nu_j \ne 1 \ \forall j > s}} \boldsymbol{b}^{\boldsymbol{\nu}}}_{= \mathcal{O}(s^{-2/p+1}) \text{ Gantner 2018}} + \underbrace{\sum_{k=k'}^{\infty} \sum_{\substack{|\boldsymbol{\nu}|=k \\ \nu_j = 0 \ \forall j \le s}} \boldsymbol{b}^{\boldsymbol{\nu}}}_{= \mathcal{O}(s^{k'(-1/p+1)}) \text{ Kuo, Schwab, Sloan 2012}}.
\end{aligned}$$

Taking $k' = \lceil (2-p)/(1-p) \rceil$ yields the overall dimension truncation error $\mathcal{O}(s^{-2/p+1})$.

This is nice, but the Neumann series argument has difficulties when the dependence of B on $\boldsymbol{y}$ becomes (sufficiently) nonlinear:

- Lognormal setting.
- Spectral eigenvalue problems (Gilbert, Graham, Kuo, Scheichl, Sloan 2019)$^{(*)}$.
- Domain uncertainty quantification (Hakula, Harbrecht, K., Kuo, Sloan; in preparation).
- Input random field is modeled using a gPC expansion.
- The quantity of interest is nonlinear.

However, the Neumann argument works well

- in the standard affine-parametric operator equation setting.
- with certain coupled PDE systems appearing in optimal control problems with PDE constraints under uncertainty (Guth, K., Kuo, Schillings, Sloan 2021).
- if the affine random field is replaced by

$$\widetilde{a}(\boldsymbol{x}, \boldsymbol{y}) = \widetilde{a}_0(\boldsymbol{x}) + \sum_{j \geq 1} \theta(y_j)\psi_j(\boldsymbol{x}),$$

  where $\theta \in L^\infty([-1/2, 1/2])$ s.t. $\int_{-1/2}^{1/2} \theta(y)\,\mathrm{d}y = 0$ (K., Kazashi, Kuo, Nobile, Sloan 2022). For example, the so-called "periodic model" $\theta(y) := \frac{1}{\sqrt{6}} \sin(2\pi y)$ (K., Kuo, Sloan 2020).
  $\rightarrow$ the dependence of B on $\theta(\boldsymbol{y})$ is still affine.

Note: A Taylor series based approach is used in $(*)$ to derive dimension truncation rates in a non-affine parametric operator equation setting.

Do we expect the rate $\mathcal{O}(s^{-2/p+1})$ in very nonlinear problems?

Consider the Poisson problem

$$\begin{cases} -\Delta u(\boldsymbol{x}, \omega) = f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in D(\omega), \\ u(\boldsymbol{x}, \omega) = 0 & \text{for } \boldsymbol{x} \in \partial D(\omega), \end{cases}$$

where the bounded domain $D(\omega) \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is assumed to be *uncertain*.

Domain mapping method: Let $D_{\text{ref}} \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a fixed reference domain. Define perturbation field $\boldsymbol{V}(\omega) \colon \overline{D_{\text{ref}}} \to \mathbb{R}^d$, which we assume is given explicitly.

Uncertain domains studied by many authors in the literature: Harbrecht, Peters, Siebenmorgen, Schwab, Zech...

Parameterization of domain uncertainty

Let $U := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ and let $\boldsymbol{V} : \overline{D_{\text{ref}}} \times U \to \mathbb{R}^d$ be a vector field such that

$$\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} + \frac{1}{\sqrt{6}} \sum_{i=1}^{\infty} \sin(2\pi y_i) \psi_i(\boldsymbol{x}), \quad \boldsymbol{x} \in D_{\text{ref}}, \ \boldsymbol{y} \in U,$$

with *stochastic fluctuations* $\psi_i : D_{\text{ref}} \to \mathbb{R}^d$. Denoting the Jacobian matrix of ψ_i by ψ_i', the Jacobian matrix $J(\cdot, \boldsymbol{y}) : D_{\text{ref}} \to \mathbb{R}^{d \times d}$ of vector field $\boldsymbol{V}$ is

$$J(\boldsymbol{x}, \boldsymbol{y}) = I + \frac{1}{\sqrt{6}} \sum_{i=1}^{\infty} \sin(2\pi y_i) \psi_i'(\boldsymbol{x}), \quad \boldsymbol{x} \in D_{\text{ref}}, \ \boldsymbol{y} \in U.$$

The family of *admissible domains* $\{D(\boldsymbol{y})\}_{\boldsymbol{y} \in U}$ is parameterized by

$$D(\boldsymbol{y}) := \boldsymbol{V}(D_{\text{ref}}, \boldsymbol{y}), \quad \boldsymbol{y} \in U,$$

and we define the *hold-all domain* by setting

$$\mathcal{D} := \bigcup_{\boldsymbol{y} \in U} D(\boldsymbol{y}).$$

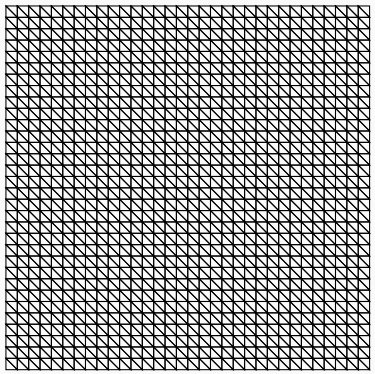


Figure 1: Reference domain

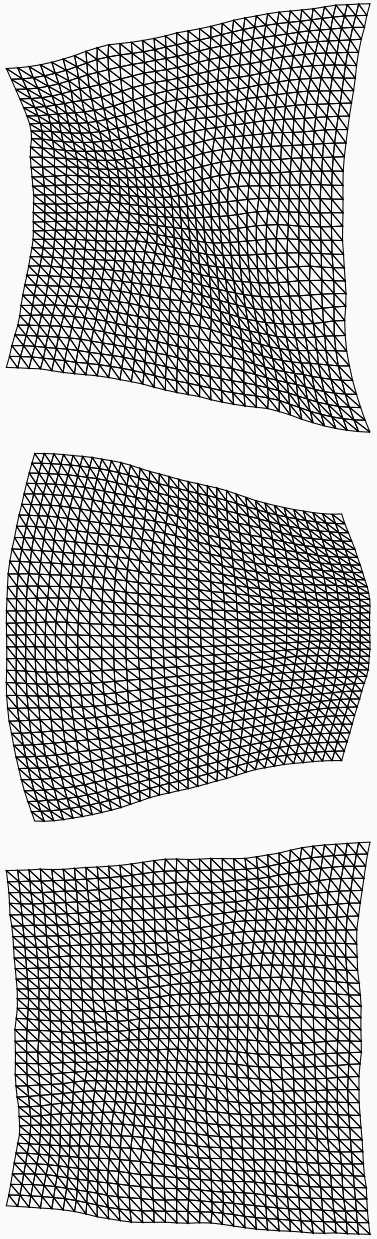


Figure 2: Three realizations of the random domain

Notations and assumptions

The reference domain $D_{\mathrm{ref}} \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is bounded with Lipschitz boundary.

(A1) For each $\boldsymbol{y} \in U$, $\boldsymbol{V}(\cdot, \boldsymbol{y})\colon \overline{D_{\mathrm{ref}}} \to \mathbb{R}^d$ is an invertible, twice continuously differentiable vector field.

(A2) For some $C > 0$, it holds that

$$\|\boldsymbol{V}(\cdot, \boldsymbol{y})\|_{C^2(\overline{D_{\mathrm{ref}}})} \leq C \quad \text{and} \quad \|\boldsymbol{V}^{-1}(\cdot, \boldsymbol{y})\|_{C^2(\overline{D(\boldsymbol{y})})} \leq C \quad \text{for all } \boldsymbol{y} \in U.$$

(A3) There exist constants $0 < \sigma_{\min} \leq 1 \leq \sigma_{\max} < \infty$ such that

$$\sigma_{\min} \leq \min \sigma(J(\boldsymbol{x}, \boldsymbol{y})) \leq \max \sigma(J(\boldsymbol{x}, \boldsymbol{y})) \leq \sigma_{\max} \quad \text{for all } \boldsymbol{x} \in D_{\mathrm{ref}},\ \boldsymbol{y} \in U,$$

where $\sigma(J(\boldsymbol{x}, \boldsymbol{y}))$ denotes the set of all singular values of matrix $J(\boldsymbol{x}, \boldsymbol{y})$,

(A4) It holds that $\|\psi_i\|_{W^{1,\infty}(D_{\mathrm{ref}};\mathbb{R}^d)} < \infty$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \|\psi_i\|_{W^{1,\infty}(D_{\mathrm{ref}};\mathbb{R}^d)} < \infty$.

(A5) For some $p \in (0, 1)$, it holds that

$$\sum_{i=1}^{\infty} \|\psi_i\|^p_{W^{1,\infty}(D_{\mathrm{ref}};\mathbb{R}^d)} < \infty.$$

The variational formulation of the model problem can be stated as follows: for $\boldsymbol{y} \in U$, find $u(\cdot, \boldsymbol{y}) \in H_0^1(D(\boldsymbol{y}))$ such that

$$\int_{D(\boldsymbol{y})} \nabla u(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla v(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} = \int_{D(\boldsymbol{y})} f(\boldsymbol{x}) \, v(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(D(\boldsymbol{y})), \tag{1}$$

where $f \in \mathcal{C}^\infty(\mathcal{D})$ is assumed to be an analytic function.

We can transport the variational formulation (1) to the reference domain by a change of variable. Let us define

$$A(\boldsymbol{x}, \boldsymbol{y}) := (J(\boldsymbol{x}, \boldsymbol{y})^{\mathrm{T}} J(\boldsymbol{x}, \boldsymbol{y}))^{-1} \det J(\boldsymbol{x}, \boldsymbol{y})$$
$$f_{\mathrm{ref}}(\boldsymbol{x}, \boldsymbol{y}) := f(V(\boldsymbol{x}, \boldsymbol{y})) \det J(\boldsymbol{x}, \boldsymbol{y}),$$

for $\boldsymbol{x} \in D_{\mathrm{ref}}$, $\boldsymbol{y} \in U$. Then we can recast the problem (1) on the reference domain as follows: for $\boldsymbol{y} \in U$, find $\widehat{u}(\cdot, \boldsymbol{y}) \in H_0^1(D_{\mathrm{ref}})$ such that

$$\int_{D_{\mathrm{ref}}} \big(A(\boldsymbol{x}, \boldsymbol{y}) \nabla \widehat{u}(\boldsymbol{x}, \boldsymbol{y})\big) \cdot \nabla \widehat{v}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{D_{\mathrm{ref}}} f_{\mathrm{ref}}(\boldsymbol{x}, \boldsymbol{y}) \, \widehat{v}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad \forall \widehat{v} \in H_0^1(D_{\mathrm{ref}}). \tag{2}$$

The solutions to problems (1) and (2) are connected to one another by

$$u(\cdot, \boldsymbol{y}) = \widehat{u}\big(\boldsymbol{V}^{-1}(\cdot, \boldsymbol{y}), \boldsymbol{y}\big) \quad \Leftrightarrow \quad \widehat{u}(\cdot, \boldsymbol{y}) = u\big(\boldsymbol{V}(\cdot, \boldsymbol{y}), \boldsymbol{y}\big), \quad \boldsymbol{y} \in U.$$

Rank-1 lattice rules

Let f be an absolutely continuous periodic function w.r.t. each one of its variables. In our setting, we are interested in approximating high-dimensional integrals

$$I_s(f) = \int_{[0,1]^s} f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

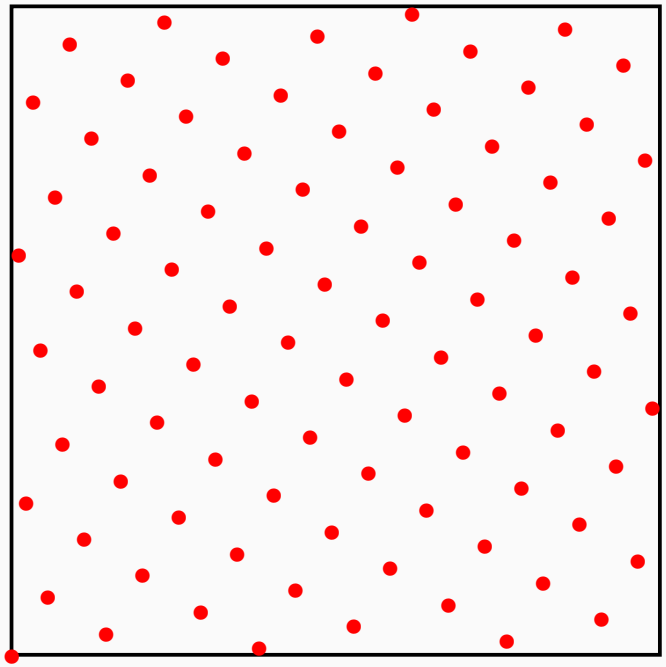
by using *rank-1 lattice rules*

$$Q_{s,n}(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(\boldsymbol{y}_i)$$

with

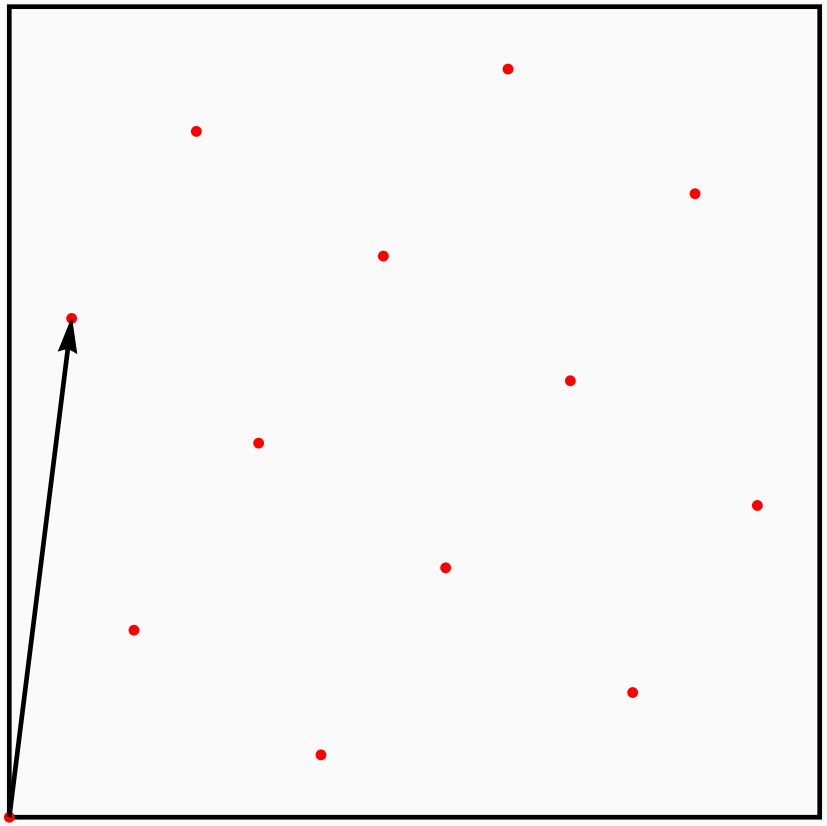
$$\boldsymbol{y}_i = \left\{ \frac{i\boldsymbol{z}}{n} \right\} := \mathrm{mod}\left(\frac{i\boldsymbol{z}}{n}, 1 \right),$$

where the entire point set is determined by the generating vector $\boldsymbol{z} \in \mathbb{N}^s$ and n.



Zaremba lattice with $\boldsymbol{z} = (1, 55)$ and $n = 89$ nodes in $[0, 1]^2$

Lattice rules and periodic functions are a match made in heaven!



A suitable generating vector for an integrand satisfying certain smoothness properties can be found using a *component-by-component* (CBC) algorithm (Nuyens and Cools 2006; Kuo, Nuyens, and Cools 2006).

- For integrands belonging to certain weighted Sobolev spaces of smooth functions, the CBC algorithm can be used to produce a generating vector satisfying a rigorous error bound. As input, the CBC algorithm takes the weights and smoothness parameter of the Sobolev space (and number of QMC nodes n).
- Fast CBC: FFT can be used to reduce the computational complexity of the CBC algorithm.

Theorem (Hakula–Harbrecht–K–Kuo–Sloan 2022)
It holds for all $\boldsymbol{y} \in U$ and all multi-indices $\boldsymbol{\nu} \neq 0$ that

$$\|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widehat{u}(\cdot, \boldsymbol{y})\|_{H_0^1(D_{\mathrm{ref}})} \lesssim (2\pi \widetilde{C})^{|\boldsymbol{\nu}|} \sum_{\boldsymbol{m} \leq \boldsymbol{\nu}} \frac{(|\boldsymbol{m}| + d - 1)!}{(d-1)!} \prod_{i \geq 1} \big(m_i! \beta_i^{m_i} S(\nu_i, m_i) \big),$$

where $\rho \geq 1$ satisfies $\|\partial_{\boldsymbol{x}}^{\boldsymbol{\nu}} f\|_{L^\infty(\mathcal{D})} \leq C_f \boldsymbol{\nu}! \rho^{|\boldsymbol{\nu}|}$, $\widetilde{C} := \frac{2\, d!\, (1+\sigma_{\max})^d\, \sigma_{\max}^3}{\sigma_{\min}^{d+4}}$,
$\beta_j := \frac{2+\sqrt{2}}{\sqrt{6}} \max(1+\sqrt{3}, \rho)\, \|\psi_j\|_{W^{1,\infty}(D_{\mathrm{ref}})}$, *and $S(n,k)$ denotes the Stirling number of the second kind.*

Plugging the above into the QMC error bound suggests choosing the (SPOD) weights

$$\gamma_{\mathfrak{u}} := \sum_{\boldsymbol{m}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} \frac{(|\boldsymbol{m}_{\mathfrak{u}}| + d - 1)!}{(d-1)!} \prod_{j \in \mathfrak{u}} \big(\widetilde{C}^{\alpha} m_j!\, \beta_j^{m_j} S(\alpha, m_j) \big), \quad \varnothing \neq \mathfrak{u} \subseteq \{1:s\},$$

and by setting $\lambda := p$ and $\alpha := \lfloor \frac{1}{p} \rfloor + 1$, we obtain QMC convergence rate $\|I_s(\widehat{u}_s) - Q_{s,n}(\widehat{u}_s)\|_{L^1(D_{\mathrm{ref}})} = \mathcal{O}(n^{-1/p})$ where the implied coefficient can be shown to be independent of the dimension s.

Remark. The fast CBC construction cost to obtain the generating vector with SPOD weights is $\mathcal{O}(s\, n \log n + \alpha^2\, s^2\, n)$ operations.

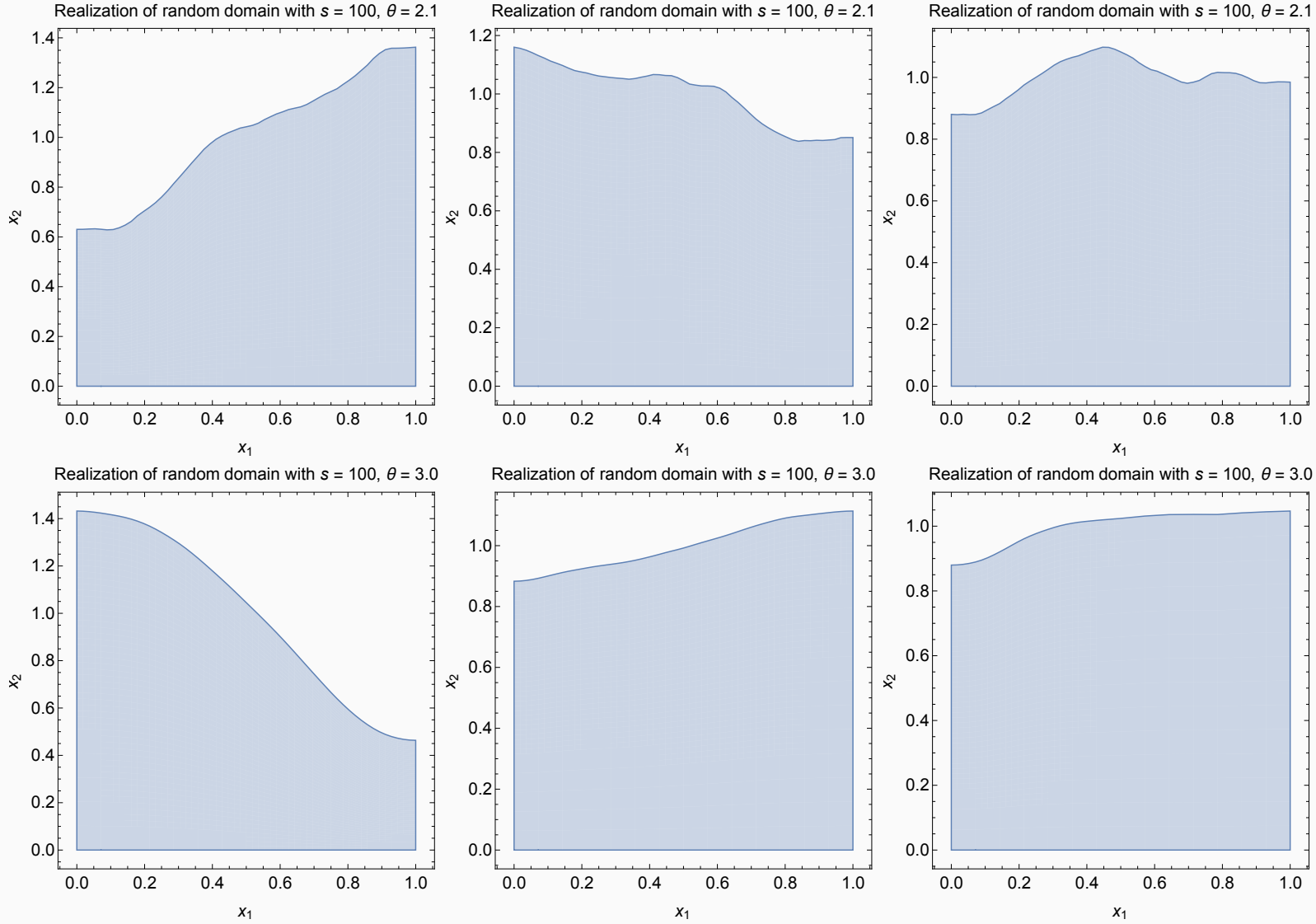


Figure 3: Realizations of the random domains in the numerical experiments. Here, we set $s = 100$ and $\theta \in \{2.1, 3.0\}$. Recall that the reference domain is $D_{\text{ref}} = (0, 1)^2$.

QMC error

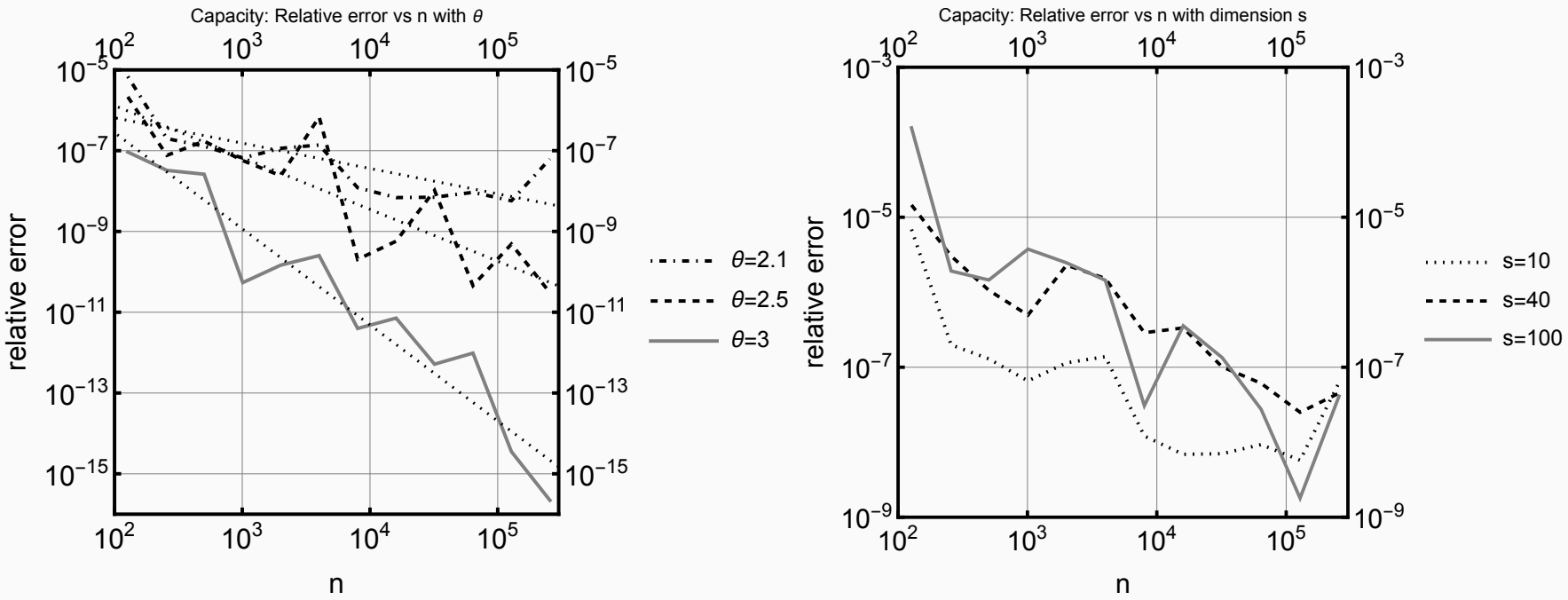
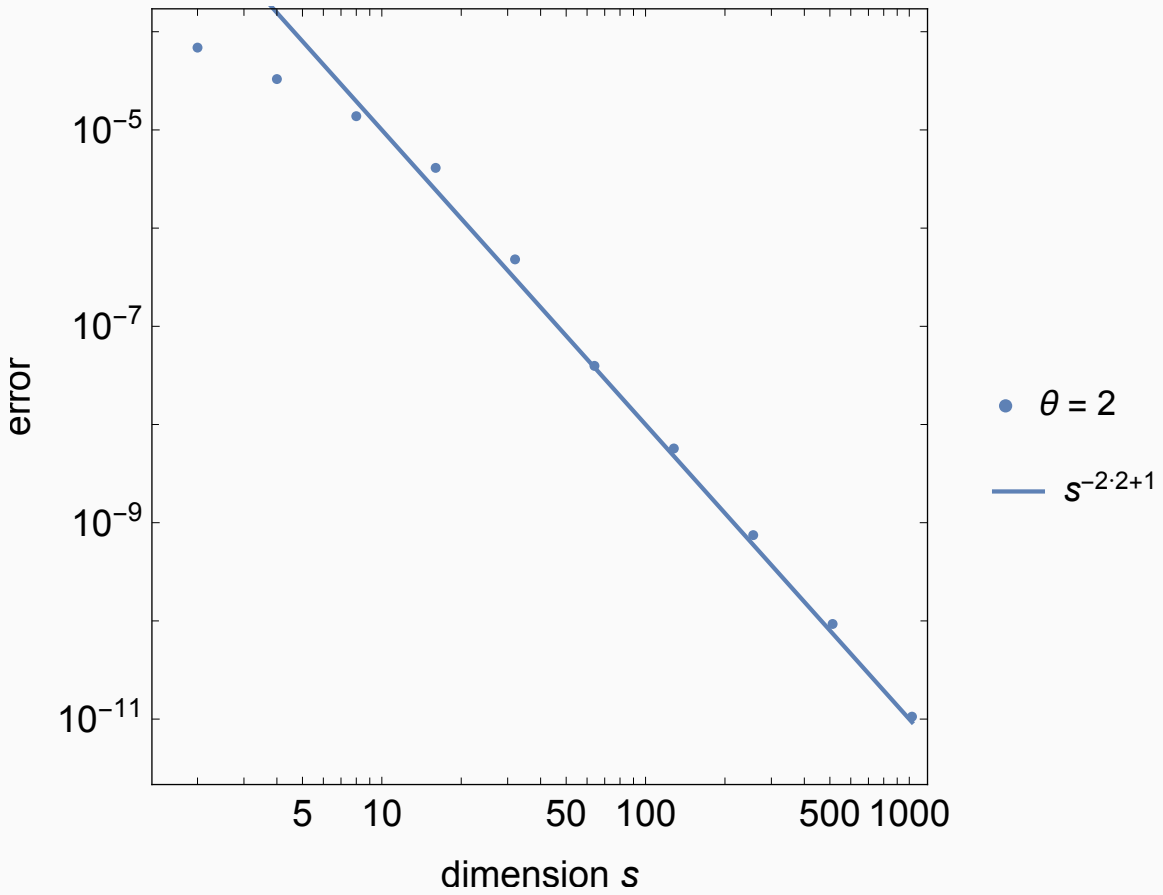


Figure 4: Left: Relative errors with increasing n, varying decay rate $\theta \in \{2.1, 2.5, 3.0\}$, and fixed dimension $s = 10$. Right: Relative errors with increasing n, varying dimension $s \in \{10, 40, 100\}$, and fixed decay rate $\theta = 2.1$. As the reference solution, we use a QMC approximation computed using $n = 1\,024\,207$ nodes.

Dimension truncation

Reference solution computed using $s = 2048$.
The error $\left\| \int_{[-\frac{1}{2},\frac{1}{2}]^{2048}} \widehat{u}_{2048}(\cdot, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \int_{[-\frac{1}{2},\frac{1}{2}]^{s}} \widehat{u}_{s}(\cdot, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \right\|_{L^2(D_{\mathrm{ref}})}$ was computed using an off-the-shelf QMC lattice rule with $n = 2^{19}$ nodes.



→Rate better than expected?!

How to resolve this?

The "affine and uniform" case has been studied by Harbrecht, Peters, Siebenmorgen 2016. Let $U := [-1/2, 1/2]^{\mathbb{N}}$ and let $\boldsymbol{V}_{\mathrm{aff}} : \overline{D_{\mathrm{ref}}} \times U \to \mathbb{R}^d$ be a vector field such that

$$\boldsymbol{V}_{\mathrm{aff}}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} + \sum_{i=1}^{\infty} y_i \boldsymbol{\psi}_i(\boldsymbol{x}), \quad \boldsymbol{x} \in D_{\mathrm{ref}}, \ \boldsymbol{y} \in U,$$

and denote the corresponding solution in the reference domain by $\widehat{u}_{\mathrm{aff}}$.

Compare with our periodic random field

$$\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} + \sum_{i=1}^{\infty} \theta(y_i) \boldsymbol{\psi}_i(\boldsymbol{x}), \quad \text{where } \theta(y) := \frac{1}{\sqrt{6}} \sin(2\pi y).$$

Key observation: $\widehat{u}_{\mathrm{per}}(\boldsymbol{x}, \boldsymbol{y}) := \widehat{u}(\boldsymbol{x}, \boldsymbol{y}) = \widehat{u}_{\mathrm{aff}}(\boldsymbol{x}, \theta(\boldsymbol{y}))$ (where θ is applied componentwise).

Key idea

Harbrecht, Peters, and Siebenmorgen show that the solution $\widehat{u}_{\mathrm{aff}}$ to the affine problem satisfies

$$\|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widehat{u}_{\mathrm{aff}}(\cdot, \boldsymbol{y})\|_V \leq C|\boldsymbol{\nu}|!\boldsymbol{b}^{\boldsymbol{\nu}},$$

where $V = H_0^1(D_{\mathrm{ref}})$, the constant $C > 0$ is independent of the dimension s, and $\boldsymbol{b} \propto (\|\psi_j\|_{W^{1,\infty}})_{j\geq 1} \in \ell^p$, $p \in (0,1)$. Suppose that $\boldsymbol{b}$ is a nonincreasing sequence.

Let $G \in V'$ be arbitrary, and define

$$\widetilde{F}(\boldsymbol{y}) = \langle G, \widehat{u}_{\mathrm{aff}}(\cdot, \boldsymbol{y})\rangle_{V',V}.$$

Evidently,

$$\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\boldsymbol{y}) = \langle G, \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widehat{u}_{\mathrm{aff}}(\cdot, \boldsymbol{y})\rangle_{V',V}$$

and it holds that

$$|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\boldsymbol{y})| \leq C|\boldsymbol{\nu}|!\boldsymbol{b}^{\boldsymbol{\nu}}.$$

Let us borrow a trick from the analysis of the spectral eigenvalue problem (Gilbert, Graham, Kuo, Scheichl, Sloan 2019): use Taylor series instead of Neumann series!

Develop the Taylor expansion first in the *affine setting:* let $\boldsymbol{y} \in U$ be arbitrary and denote $(\boldsymbol{y}_{\leq s}; 0) = (y_1, \ldots, y_s, 0, 0, \ldots) \in U$. By developing the Taylor expansion of $\widetilde{F}$ around this point, we obtain

$$\widetilde{F}(\boldsymbol{y}) - \widetilde{F}(\boldsymbol{y}_{\leq s}; 0) = \sum_{\ell=1}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \leq s}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\boldsymbol{y}_{\leq s}; 0)$$
$$+ \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \leq s}} \frac{k+1}{\boldsymbol{\nu}!} \boldsymbol{y}^{\boldsymbol{\nu}} \int_0^1 (1-t)^k \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\boldsymbol{y}_{\leq s}; t\boldsymbol{y}_{>s}) \, \mathrm{d}t.$$

Now that we have a representation for the pointwise dimension truncation error, we carry out a change of variable[1] $\boldsymbol{y} \leftarrow \theta(\boldsymbol{y})$:

$$F(\boldsymbol{y}) - F(\boldsymbol{y}_{\leq s}; 0) = \sum_{\ell=1}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \leq s}} \frac{\theta(\boldsymbol{y})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\theta(\boldsymbol{y}_{\leq s}); 0)$$
$$+ \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \leq s}} \frac{k+1}{\boldsymbol{\nu}!} \theta(\boldsymbol{y})^{\boldsymbol{\nu}} \int_0^1 (1-t)^k \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\theta(\boldsymbol{y}_{\leq s}); \theta(t\boldsymbol{y}_{>s})) \, \mathrm{d}t,$$

where $F(\boldsymbol{y}) = \langle G, \widehat{u}_{\mathrm{per}}(\cdot, \boldsymbol{y}) \rangle_{V',V}$ and we exploited $\widetilde{F}(\theta(\boldsymbol{y})) = F(\boldsymbol{y})$.

[1] Why no chain rule? Consider: $\mathrm{e}^x = \sum_{n \geq 0} \frac{x^n}{n!} \Rightarrow \mathrm{e}^{\sin x} = \sum_{n \geq 0} \frac{(\sin x)^n}{n!}$.

Integrate both sides over $\boldsymbol{y} \in U$ and use triangle inequality to obtain

$$\begin{aligned}
&\left| \int_U (F(\boldsymbol{y}) - F(\boldsymbol{y}_{\le s}; 0))\,\mathrm{d}\boldsymbol{y} \right| \\
&\le \sum_{\ell=1}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \le s}} \frac{1}{\boldsymbol{\nu}!} \left| \int_U \theta(\boldsymbol{y})^{\boldsymbol{\nu}} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\theta(\boldsymbol{y}_{\le s}); 0)\,\mathrm{d}\boldsymbol{y} \right| \\
&\quad + \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \le s}} \frac{k+1}{\boldsymbol{\nu}!} \left| \int_U \int_0^1 \theta(\boldsymbol{y})^{\boldsymbol{\nu}} (1-t)^k \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \widetilde{F}(\theta(\boldsymbol{y}_{\le s}); \theta(t\boldsymbol{y}_{>s}))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y} \right| \\
&\lesssim \sum_{\ell=2}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \le s \\ \nu_j \ne 1 \ \forall j > s}} \boldsymbol{b}^{\boldsymbol{\nu}} + \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \le s}} \boldsymbol{b}^{\boldsymbol{\nu}},
\end{aligned}$$

where the implied coefficient is independent of s. Since $F(\boldsymbol{y}) = \langle G, u_{\mathrm{per}}(\cdot, \boldsymbol{y}) \rangle_{V',V}$ with arbitrary $G \in V'$, by taking the supremum over $\{G \in V' : \|G\|_{V'} = 1\}$ we obtain

$$\left\| \int_U (\widehat{u}_{\mathrm{per}}(\cdot, \boldsymbol{y}) - \widehat{u}_{\mathrm{per},s}(\cdot, \boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} \right\|_{H_0^1(D_{\mathrm{ref}})} \lesssim \underbrace{\sum_{\ell=2}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \le s \\ \nu_j \ne 1 \ \forall j > s}} \boldsymbol{b}^{\boldsymbol{\nu}}}_{= \mathcal{O}(s^{-2/p+1}) \text{ Gantner 2018}} + \underbrace{\sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \le s}} \boldsymbol{b}^{\boldsymbol{\nu}}}_{= \mathcal{O}(s^{(k+1)(-1/p+1)}) \text{ Kuo, Schwab, Sloan 2012}}.$$

By choosing $k = \lceil \frac{1}{1-p} \rceil$ we obtain the overall rate $\mathcal{O}(s^{-2/p+1})$.

Dimension truncation in the lognormal setting (K., Guth 2022)

Let V be a separable Banach space and suppose that $u(\cdot, \boldsymbol{y}) \in V$ is such that

$$\|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y})\|_V \leq C \frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}} \boldsymbol{b}^{\boldsymbol{\nu}} \prod_{j \geq 1} \mathrm{e}^{b_j |y_j|}$$

for some sequence $\boldsymbol{b} = (b_j)_{j \geq 1} \in \ell^p$, $p \in (0, 1)$. Especially, this is the regularity bound in the lognormal case.

We are interested in

$$\left\| \int_{\mathbb{R}^{\mathbb{N}}} (u(\cdot, \boldsymbol{y}) - u_s(\cdot, \boldsymbol{y}))\, \mu_G(\mathrm{d}\boldsymbol{y}) \right\|_V.$$

We again use Taylor series instead of Neumann series!

Let $G \in V'$ and define

$$F(\boldsymbol{y}) = \langle G, u(\cdot, \boldsymbol{y}) \rangle_{V',V}.$$

Evidently

$$\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}) = \langle G, \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y}) \rangle_{V',V}$$

and it holds that

$$|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y})| \leq C \frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}} \boldsymbol{b}^{\boldsymbol{\nu}} \prod_{j \geq 1} \mathrm{e}^{b_j |y_j|},$$

where we merged $\|G\|_{V'}$ into the constant C and $\boldsymbol{b} = (b_j)_{j \geq 1} \in \ell^p$ is a nonincreasing sequence for some $p \in (0, 1)$.

Denote the space of admissible parameters by

$$U_{\boldsymbol{b}} := \left\{ \boldsymbol{y} \in \mathbb{R}^{\mathbb{N}} : \sum_{j \geq 1} b_j |y_j| < \infty \right\}.$$

Let $\boldsymbol{y} \in U_{\boldsymbol{b}}$ be arbitrary and denote $(\boldsymbol{y}_{\le s}; 0) = (y_1, \dots, y_s, 0, 0, \dots) \in U_{\boldsymbol{b}}$. By developing the Taylor expansion of F around this point, we obtain

$$F(\boldsymbol{y}) = F(\boldsymbol{y}_{\le s}; 0) + \sum_{\ell=1}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \le s}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0)$$
$$+ \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \le s}} \frac{k+1}{\boldsymbol{\nu}!} \boldsymbol{y}^{\boldsymbol{\nu}} \int_0^1 (1-t)^k \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; t\boldsymbol{y}_{>s}) \, \mathrm{d}t.$$

Rearranging this equation and integrating it against a product Gaussian measure,

$$\int_{U_{\boldsymbol{b}}} (F(\boldsymbol{y}) - F(\boldsymbol{y}_{\le s}; 0)) \prod_{j \ge 1} \phi(y_j) \, \mathrm{d}\boldsymbol{y}$$
$$= \sum_{\ell=1}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j \le s}} \frac{1}{\boldsymbol{\nu}!} \int_{U_{\boldsymbol{b}}} \boldsymbol{y}^{\boldsymbol{\nu}} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0) \prod_{j \ge 1} \phi(y_j) \, \mathrm{d}\boldsymbol{y}$$
$$+ \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j \le s}} \frac{k+1}{\boldsymbol{\nu}!} \int_{U_{\boldsymbol{b}}} \int_0^1 (1-t)^k \boldsymbol{y}^{\boldsymbol{\nu}} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; t\boldsymbol{y}_{>s}) \prod_{j \ge 1} \phi(y_j) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{y}.$$

Again, the summand in the first term vanishes whenever $\nu_k = 1$ with $k > s$ since

$$\int_{U_{\boldsymbol{b}}} \boldsymbol{y}^{\boldsymbol{\nu}} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0) \prod_{j \ge 1} \phi(y_j) \, \mathrm{d}\boldsymbol{y}$$
$$= \left(\int_{\mathbb{R}^s} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0) \prod_{j=1}^{s} \phi(y_j) \, \mathrm{d}\boldsymbol{y} \right) \left(\int_{\mathbb{R}} y_k \phi(x_k) \, \mathrm{d}y_k \right) \left(\int_{U_{\boldsymbol{b}}} \boldsymbol{y}^{\boldsymbol{\nu}} \prod_{\substack{j > s \\ j \ne k}} \phi(y_j) \, \mathrm{d}\boldsymbol{y}_{\{s+1:\infty\} \setminus \{k\}} \right) = 0.$$

We arrive at

$$\left| \int_{U_{\boldsymbol{b}}} (F(\boldsymbol{y}) - F(\boldsymbol{y}_{\le s}; 0)) \prod_{j\ge 1} \phi(y_j)\,\mathrm{d}\boldsymbol{y} \right|$$

$$\le \sum_{\ell=2}^{k} \sum_{\substack{|\boldsymbol{\nu}|=\ell \\ \nu_j=0 \ \forall j\le s \\ \nu_j\ne 1 \ \forall j>s}} \frac{1}{\boldsymbol{\nu}!} \int_{U_{\boldsymbol{b}}} |\boldsymbol{y}^{\boldsymbol{\nu}}| \cdot \ |\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0)| \prod_{j\ge 1} \phi(y_j)\,\mathrm{d}\boldsymbol{y}$$

$$+ \sum_{\substack{|\boldsymbol{\nu}|=k+1 \\ \nu_j=0 \ \forall j\le s}} \frac{k+1}{\boldsymbol{\nu}!} \int_{U_{\boldsymbol{b}}} \int_0^1 (1-t)^k |\boldsymbol{y}^{\boldsymbol{\nu}}| \cdot |\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; t\boldsymbol{y}_{>s})| \prod_{j\ge 1} \phi(y_j)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}.$$

Bounding the first sum:

$$\int_{U_{\boldsymbol{b}}} |\boldsymbol{y}^{\boldsymbol{\nu}}| \cdot |\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; 0)| \prod_{j\ge 1} \phi(y_j)\,\mathrm{d}\boldsymbol{y} \le C \frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}} \boldsymbol{b}^{\boldsymbol{\nu}} \int_{U_{\boldsymbol{b}}} \prod_{j\ge 1} |y_j|^{\nu_j} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y_j^2 + b_j|y_j|}\,\mathrm{d}\boldsymbol{y}$$

$$\le C \frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}} \boldsymbol{b}^{\boldsymbol{\nu}} \left(\prod_{j\in \mathrm{supp}(\boldsymbol{\nu})} \int_{\mathbb{R}} |y_j|^{\nu_j} \mathrm{e}^{-\frac{1}{2}y_j^2 + b_j|y_j|}\,\mathrm{d}y_j \right) \underbrace{\int_{U_{\boldsymbol{b}}} \prod_{j\ge 1} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y_j^2 + b_j|y_j|}\,\mathrm{d}\boldsymbol{y}}_{\text{bounded independently of } s \text{ and } \boldsymbol{\nu} \text{ (Gittelson 2010)}}.$$

To estimate the middle factor, we can use the following Lemma:

Lemma (Gradshteyn and Ryzhik formulae 3.462.1, 9.240, 9.210.1)

Let $\nu \in \mathbb{N}_0$ *and* $A, B > 0$. *Then*

$$\begin{aligned}
&\int_{\mathbb{R}} |y|^{\nu} \mathrm{e}^{-Ay^2+B|y|} \,\mathrm{d}y \\
&= 2\sqrt{\pi}(4A)^{-\frac{\nu+1}{2}} \nu! \left[\frac{1}{\Gamma(\frac{\nu}{2}+1)} {}_1F_1\left(\frac{\nu+1}{2}, \frac{1}{2}; \frac{B^2}{4A}\right) \right. \\
&\qquad \left. + \sqrt{\frac{1}{A}} \frac{B}{\Gamma(\frac{\nu}{2}+\frac{1}{2})} {}_1F_1\left(\frac{\nu}{2}+1, \frac{3}{2}; \frac{B^2}{4A}\right) \right],
\end{aligned}$$

where ${}_1F_1$ *denotes the* confluent hypergeometric function of the first kind.

Thus

$$\begin{aligned}
&\prod_{j \in \mathrm{supp}(\boldsymbol{\nu})} \int_{\mathbb{R}} |y_j|^{\nu_j} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y_j^2 + b_j|y_j|} \,\mathrm{d}y_j \\
&= \prod_{j \in \mathrm{supp}(\boldsymbol{\nu})} \left(\sqrt{2} 2^{-\frac{\nu_j+1}{2}} \nu_j! \left[\frac{1}{\Gamma(\frac{\nu_j}{2}+1)} {}_1F_1\left(\frac{\nu_j+1}{2}, \frac{1}{2}; \frac{b_j^2}{2}\right) \right. \right. \\
&\qquad \left. \left. + \sqrt{2} \frac{b_j}{\Gamma(\frac{\nu_j}{2}+\frac{1}{2})} {}_1F_1\left(\frac{\nu_j}{2}+1, \frac{3}{2}; \frac{b_j^2}{2}\right) \right] \right).
\end{aligned}$$

Note that here

$$\max\left\{ \frac{1}{\Gamma(\frac{\nu}{2}+1)}, \frac{1}{\Gamma(\frac{\nu}{2}+\frac{1}{2})} \right\} \leq 2 \quad \text{for all } \nu \in \mathbb{N}_0.$$

Since ${}_1F_1$ has the monotonicity properties

$$
\begin{aligned}
{}_1F_1(a,b;x) &< {}_1F_1(a',b;x) \quad \text{for } a<a',\\
{}_1F_1(a,b;x) &> {}_1F_1(a,b',x) \quad \text{for } b<b',
\end{aligned}
$$

and it is not too difficult to see that

$$
\begin{aligned}
\max\left\{{}_1F_1\left(\frac{\nu_j+1}{2},\frac{1}{2};\frac{b_j^2}{2}\right),{}_1F_1\left(\frac{\nu_j}{2}+1,\frac{3}{2};\frac{b_j^2}{2}\right)\right\} &\le {}_1F_1\left(\frac{\nu_j}{2}+1,\frac{1}{2};\frac{b_j^2}{2}\right)\\
&\le (\nu_j+1)\cdot\nu_j!\,\mathrm{e}^{\frac{5}{2}b_j^2},
\end{aligned}
$$

so we obtain (using $\boldsymbol{\nu}!\le|\boldsymbol{\nu}|!$ and $\prod(1+a_k)\le \mathrm{e}^{\sum a_k}$ repeatedly)

$$
\begin{aligned}
&\prod_{j\in\operatorname{supp}(\boldsymbol{\nu})}\int_{\mathbb{R}}|y_j|^{\nu_j}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}y_j^2+b_j|y_j|}\,\mathrm{d}y_j\\
&\le 2^{|\boldsymbol{\nu}|}(|\boldsymbol{\nu}|!)^2\pi^{|\boldsymbol{\nu}|/2}\left(\prod_{j\in\operatorname{supp}(\boldsymbol{\nu})}(\nu_j+1)\right)\mathrm{e}^{\frac{5}{2}\sum_{j\ge1}b_j^2}\prod_{j\in\operatorname{supp}(\boldsymbol{\nu})}(1+\sqrt{2}b_j)\\
&\le 2^{|\boldsymbol{\nu}|}(|\boldsymbol{\nu}|!)^2\pi^{|\boldsymbol{\nu}|/2}\mathrm{e}^{|\boldsymbol{\nu}|}\mathrm{e}^{\frac{5}{2}\sum_{j\ge1}b_j^2}\mathrm{e}^{\sqrt{2}\sum_{j\ge1}b_j}.
\end{aligned}
$$

Finally, note that by Jensen's inequality $\sum_{j\ge1}b_j^2\le(\sum_{j\ge1}b_j)^2$ and $|\boldsymbol{\nu}|\le k$, so this term can be bounded independently of s and $\boldsymbol{\nu}$.

$$
\therefore\sum_{\ell=2}^{k}\sum_{\substack{|\boldsymbol{\nu}|=\ell\\ \nu_j=0\ \forall j\le s\\ \nu_j\ne1\ \forall j>s}}\frac{1}{\boldsymbol{\nu}!}\int_{U_{\boldsymbol{b}}}|\boldsymbol{y}^{\boldsymbol{\nu}}|\cdot\ |\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}}F(\boldsymbol{y}_{\le s};0)|\prod_{j\ge1}\phi(y_j)\,\mathrm{d}\boldsymbol{y}\lesssim\sum_{\ell=2}^{k}\sum_{\substack{|\boldsymbol{\nu}|=\ell\\ \nu_j=0\ \forall j\le s\\ \nu_j\ne1\ \forall j>s}}\boldsymbol{b}^{\boldsymbol{\nu}},
$$

where implied coefficient is independent of s.

Bounding the second sum:

$$(k+1)\int_{U_{\boldsymbol{b}}}\int_0^1 (1-t)^k |\boldsymbol{y}^{\boldsymbol{\nu}}| \cdot |\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y}_{\le s}; t\boldsymbol{y}_{>s})| \prod_{j\ge 1}\phi(y_j)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}$$

$$\le (k+1)\frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}}\frac{\|G\|_{V'}\|f\|_{V'}}{a_{\min}}\boldsymbol{b}^{\boldsymbol{\nu}}\int_0^1 (1-t)^k \int_{U_{\boldsymbol{b}}} |\boldsymbol{y}^{\boldsymbol{\nu}}| \prod_{j=1}^{s}\frac{\mathrm{e}^{-\frac{1}{2}y_j^2+b_j|y_j|}}{\sqrt{2\pi}} \prod_{j>s}\underbrace{\frac{\mathrm{e}^{-\frac{1}{2}y_j^2+tb_j|y_j|}}{\sqrt{2\pi}}}_{\le \frac{\mathrm{e}^{-\frac{1}{2}y_j^2+b_j|y_j|}}{\sqrt{2\pi}}}\mathrm{d}\boldsymbol{y}\mathrm{d}t$$

$$\le \frac{|\boldsymbol{\nu}|!}{(\ln 2)^{|\boldsymbol{\nu}|}}\frac{\|G\|_{V'}\|f\|_{V'}}{a_{\min}}\boldsymbol{b}^{\boldsymbol{\nu}}\int_{U_{\boldsymbol{b}}} |\boldsymbol{y}^{\boldsymbol{\nu}}| \prod_{j\ge 1}\frac{\mathrm{e}^{-\frac{1}{2}y_j^2+b_j|y_j|}}{\sqrt{2\pi}}\,\mathrm{d}\boldsymbol{y}$$

and the integral can be bounded independently of $\boldsymbol{\nu}$ and s as in the first sum.

Altogether, we obtain

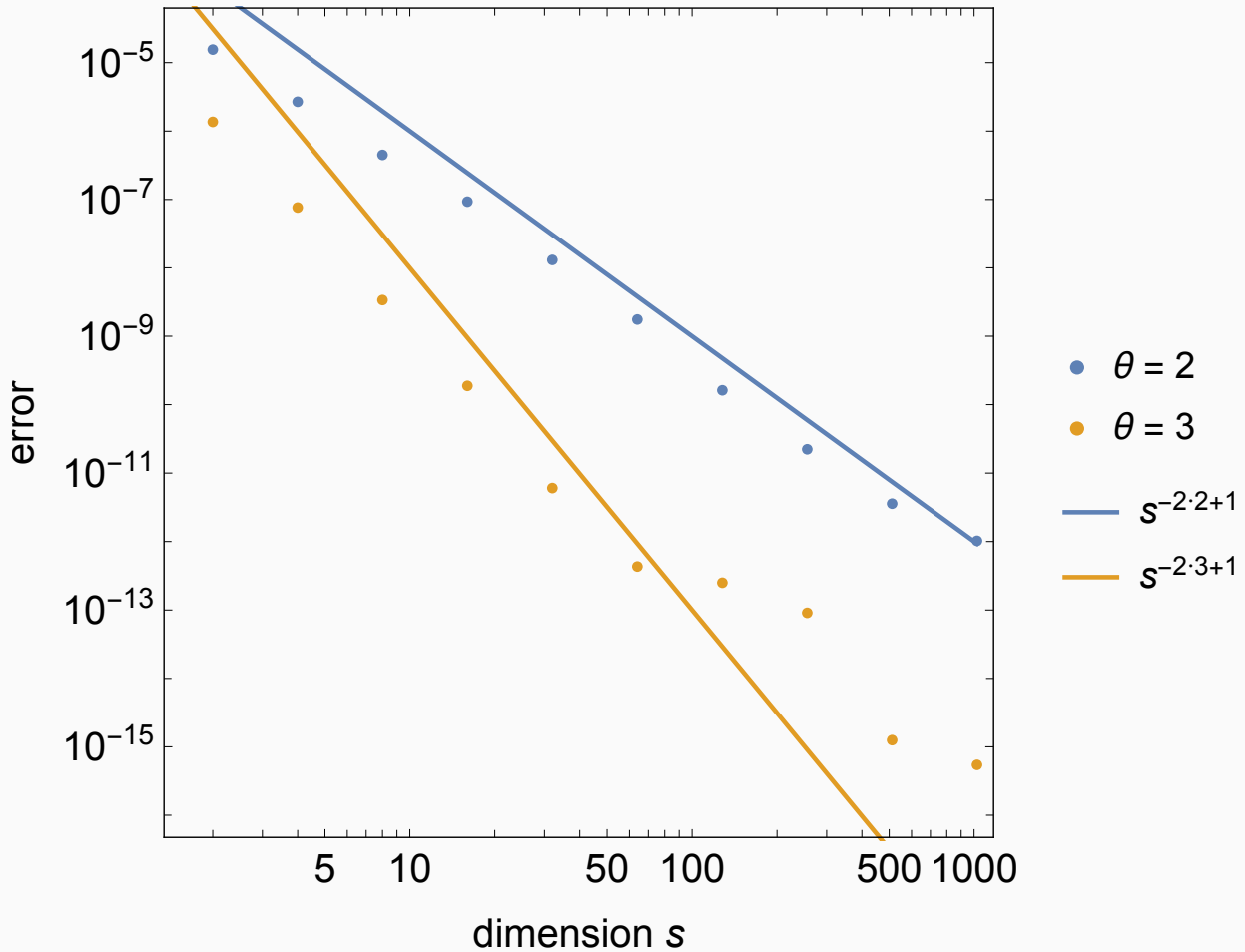
$$\left|\int_{\mathbb{R}^{\mathbb{N}}}(F(\boldsymbol{y})-F(\boldsymbol{y}_{\le s};0))\overline{\boldsymbol{\mu}}_G(\mathrm{d}\boldsymbol{y})\right| \lesssim \sum_{\ell=2}^{k}\sum_{\substack{|\boldsymbol{\nu}|=\ell\\ \nu_j=0\ \forall j\le s}}\boldsymbol{b}^{\boldsymbol{\nu}} + \sum_{\substack{|\boldsymbol{\nu}|=k+1\\ \nu_j=0\ \forall j\le s}}\boldsymbol{b}^{\boldsymbol{\nu}},$$

where the coefficient is independent of the dimension s. Finally, since $F(\boldsymbol{y}) = \langle G, u(\cdot,\boldsymbol{y})\rangle_{V',V}$ and $G \in V'$ was arbitrary, taking the supremum over $\{G \in V' : \|G\|_{V'} = 1\}$ yields that

$$\left\|\int_{\mathbb{R}^{\mathbb{N}}}(u(\cdot,\boldsymbol{y})-u_s(\cdot,\boldsymbol{y}))\overline{\boldsymbol{\mu}}_G(\mathrm{d}\boldsymbol{y})\right\|_V \lesssim \sum_{\ell=2}^{k}\sum_{\substack{|\boldsymbol{\nu}|=\ell\\ \nu_j=0\ \forall j\le s\\ \nu_j\neq 1\ \forall j>s}}\boldsymbol{b}^{\boldsymbol{\nu}} + \sum_{\substack{|\boldsymbol{\nu}|=k+1\\ \nu_j=0\ \forall j\le s}}\boldsymbol{b}^{\boldsymbol{\nu}} \overset{\substack{\text{Gantner 2018}\\ \text{Kuo, Schwab,}\\ \text{Sloan 2012}}}{=} \mathcal{O}(s^{-2/p+1}).$$

Numerical experiment

Input: $a(\boldsymbol{x}, \boldsymbol{y}) = \exp\left(\sum_{j\geq 1} j^{-\theta} y_j \sin(\pi j x_1)\sin(\pi j x_2)\right)$, $\theta \in \{2, 3\}$.
Quantity of interest: $G(u_s(\cdot, \boldsymbol{y})) = \int_{\mathbb{R}^s_*} u_s(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$; reference $s = 2048$.
The error $\left|\int_{\mathbb{R}^{2048}_*} G(u_{2048}(\cdot, \boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} - \int_{\mathbb{R}^s_*} G(u_s(\cdot, \boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}\right|$ computed using an off-the-shelf QMC lattice rule with $n = 2^{19}$ nodes.



Conclusions

- In many cases, the quantity of interest may be nonlinear (e.g., entropic risk measure for PDE constrained optimization under uncertainty) or the PDE itself may not fit into the affine parametric operator equation framework.
- Taylor series exploit the smoothness of the integrand and are thus suitable in cases where the application of Neumann series is difficult (or even impossible).
- The Taylor argument works, e.g., when the integrand $F(\cdot, \boldsymbol{y}) \in V$ belongs to a separable Banach space V such that

$$\|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\cdot, \boldsymbol{y})\|_V \lesssim [(|\boldsymbol{\nu}| + a_1)!]^{d_1} \prod_{j \geq 1} (a_2 b_j)^{\nu_j} \exp(a_3 b_j |y_j|)$$

  for integer $a_1 \geq 0$, real numbers $a_2 > 0$, $a_3, d_1 > 0$ and a nonincreasing sequence $\boldsymbol{b} \in \ell^p$, $p \in (0, 1)$. Integration against either uniform or Gaussian weights OK.
- Taylor series are robust with respect to smooth, nonlinear transformations $\boldsymbol{y} \leftarrow \theta(\boldsymbol{y})$ of the random variable $\boldsymbol{y}$ (as long as ellipticity is preserved).