Problem setting

Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space. We consider the Poisson problem

$$\begin{aligned} -\Delta u(\boldsymbol{x}, \omega) &= f(\boldsymbol{x}), \quad && \boldsymbol{x} \in D(\omega), \\ u(\boldsymbol{x}, \omega) &= 0, && \boldsymbol{x} \in \partial D(\omega), \end{aligned}$$

subject to an *uncertain domain* $D(\omega) \subset \mathbb{R}^d$, $d \in \{2, 3\}$, for almost every $\omega \in \Omega$.

Domain mapping method: Let $D_{\text{ref}} \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a fixed *reference domain*. Define perturbation field $\boldsymbol{V}(\cdot, \omega) \colon \overline{D_{\text{ref}}} \to \mathbb{R}^d$, which we assume is given explicitly.

Domain parameterization

Let $U := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ and let $\boldsymbol{V} \colon \overline{D_{\text{ref}}} \times U \to \mathbb{R}^d$ be a vector field such that, for $\boldsymbol{x} \in D_{\text{ref}}$ and $\boldsymbol{y} \in U$,

$$\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} + \frac{1}{\sqrt{6}} \sum_{i=1}^{\infty} \sin(2\pi y_i) \boldsymbol{\psi}_i(\boldsymbol{x}),$$

with *stochastic fluctuations* $\boldsymbol{\psi}_i \colon D_{\text{ref}} \to \mathbb{R}^d$.

The family of *admissible domains* $\{D(\boldsymbol{y})\}_{\boldsymbol{y} \in U}$ is parameterized for all $\boldsymbol{y} \in U$ by

$$D(\boldsymbol{y}) := \boldsymbol{V}(D_{\text{ref}}, \boldsymbol{y}),$$

and the *hold-all domain* is defined by setting

$$\mathcal{D} := \bigcup_{\boldsymbol{y} \in U} D(\boldsymbol{y}).$$

Main result

For all $\boldsymbol{y} \in U$, we find the transported solution $\widehat{u}(\cdot, \boldsymbol{y}) \in H_0^1(D_{\text{ref}})$ in the reference domain such that

$$\widehat{u}(\cdot, \boldsymbol{y}) = u(\boldsymbol{V}(\cdot, \boldsymbol{y}), \boldsymbol{y}) \quad \Leftrightarrow \quad u(\cdot, \boldsymbol{y}) = \widehat{u}(\boldsymbol{V}^{-1}(\cdot, \boldsymbol{y}), \boldsymbol{y}).$$

Let $\widehat{u}_{s,h}(\cdot, \boldsymbol{y}) := \widehat{u}_h(\cdot, (y_1, \ldots, y_s, 0, 0, \ldots))$ denote the *dimensionally-truncated, conforming first order finite element approximation* of $\widehat{u}(\cdot, \boldsymbol{y})$ subject to a regular uniform triangulation of D_{ref}. A *rank-1 lattice quasi-Monte Carlo (QMC) rule* is an equal weight cubature rule over the point set

$$\boldsymbol{y}^{(i)} = \operatorname{mod}(\tfrac{i\boldsymbol{z}}{n}, 1) - \tfrac{\mathbf{1}}{\mathbf{2}}, \quad i \in \{1, \ldots, n\},$$

completely determined by a *generating vector* $\boldsymbol{z} \in \mathbb{N}^s$ and the number of cubature nodes n.

Theorem [1]. *Let $f \in \mathcal{C}^\infty(\mathcal{D})$ be an analytic function. A rank-1 lattice QMC rule can be constructed by a fast component-by-component (CBC) algorithm such that*

$$\left\| \int_U \widehat{u}(\cdot, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} - \frac{1}{n} \sum_{i=1}^{n} \widehat{u}_{s,h}(\cdot, \boldsymbol{y}^{(i)}) \right\|_{L^1(D_{\text{ref}})} = \mathcal{O}(s^{-2/p+1} + n^{-1/p} + h^2),$$

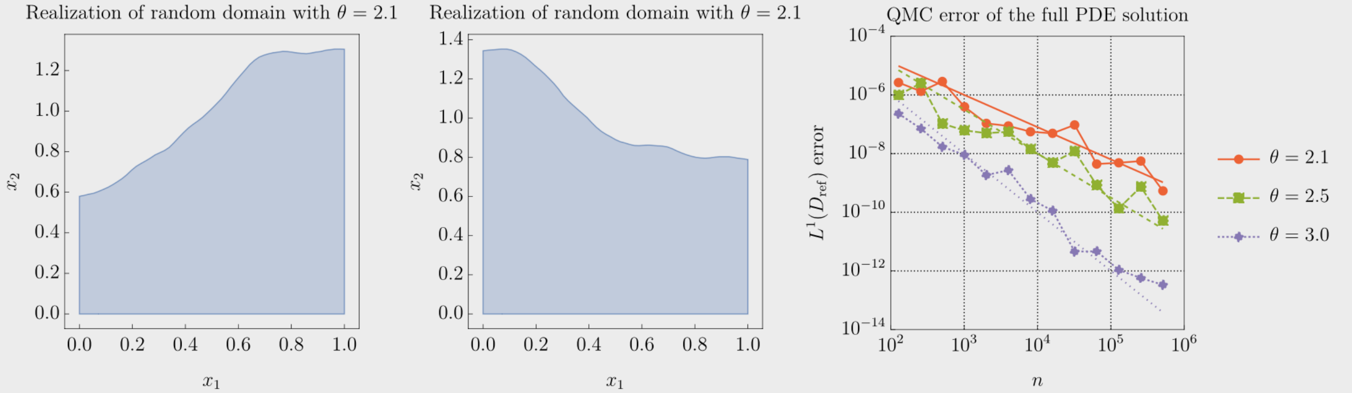
where the implied coefficient is independent of s, n, and the finite element mesh size h.

Numerical experiments

Let the reference domain be the unit square $D_{\text{ref}} = (0, 1)^2$. We consider the domain parameterization

$$D(\boldsymbol{y}) := \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1,\ 0 \le x_2 \le 1 + \tfrac{1}{\sqrt{6}} \textstyle\sum_{i=1}^{s} \sin(2\pi y_i) \psi_i(x)\right\}, \quad \boldsymbol{y} \in [-\tfrac{1}{2}, \tfrac{1}{2}]^s,$$

where only the top edge is uncertain, $\|\psi_i\|_{W^{1,\infty}(D_{\text{ref}})} \propto i^{-\theta+1}$, $s = 100$, and $\theta \in \{2.1, 2.5, 3.0\}$.



Left and middle: two realizations of the random domain corresponding to $\theta = 2.1$. Right: estimated QMC cubature errors corresponding to $\theta \in \{2.1, 2.5, 3.0\}$. Increasing θ results in a faster cubature convergence rate.