Revisiting the spectrum of lattice-theoretic matrices

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From ICS to FDA and EVT – February 22, 2025

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Introduction

Let K_n be the set of all nonsingular $n \times n$ lower triangular $(0,1)$ matrices.

Example

K_3 consists of the matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}$$
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$\#K_n = 2^{n(n-1)/2}$.

$\rightarrow$ Curse of dimensionality!

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Let K_n be the set of all nonsingular $n \times n$ lower triangular $(0, 1)$ matrices.

- Hong and Loewy (2004)[1] introduced the numbers

$$c_n = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}},\ X \in K_n\}, \quad n \in \mathbb{Z}_+.$$

- Ilmonen, Haukkanen, and Merikoski (2008)[2] considered the numbers

$$C_n = \max\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}},\ X \in K_n\}, \quad n \in \mathbb{Z}_+.$$

[1] S. Hong and R. Loewy. Asymptotic behavior of eigenvalues of greatest common divisor matrices. *Glasgow Math. J.* **46** (2004), 551–569.

[2] P. Ilmonen, P. Haukkanen, and J. K. Merikoski. On eigenvalues of meet and join matrices associated with incidence functions. *Linear Algebra Appl.* **429** (2008), 859–874.

Some applications of c_n and C_n

$$c_n = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}},\ X \in K_n\}$$
$$C_n = \max\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}},\ X \in K_n\}$$

- Related to the extremal eigenvalues of lattice-theoretic matrices (say, GCD matrices; cf. Hong and Loewy 2004): if $A_{i,j} = f(x_i \wedge x_j)$, $x_1, \ldots, x_n \in P$, $f \colon P \to \mathbb{C}$, and $(P, \preceq)$ is a meet semilattice, then
$$c_n \min_{1 \le k \le n} |J_f(x_k)| \le |\lambda_{\min}(A)| \quad \text{and} \quad |\lambda_{\max}(A)| \le C_n \max_{1 \le k \le n} |J_f(x_k)|,$$
where J_f denotes the Jordan totient function on $\{x_i\}_{i=1}^n$ w.r.t. f.
- Spectrum of triangular Boolean matrices:
$$\min_{X \in K_n} \sigma_{\min}(X) = \sqrt{c_n} \quad \text{and} \quad \max_{X \in K_n} \sigma_{\max}(X) = \sqrt{C_n},$$
where $\sigma_{\min}$ and $\sigma_{\max}$ denote the smallest and largest singular value, respectively.

- Ilmonen, Haukkanen, and Merikoski (2008) demonstrated that

$$C_n = \lambda_{\max}(W_n W_n^{\mathrm{T}}), \quad \text{where } W_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

The value can be computed explicitly!

- Altınışık et al. (2016)[3] proved that

$$c_n = \lambda_{\min}(Y_n Y_n^{\mathrm{T}}), \quad \text{where } (Y_n)_{i,j} = \begin{cases} \frac{1-(-1)^{i+j}}{2} & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Behavior harder to characterize! (Main focus of this talk!)

[3] E. Altınışık, A. Keskin, M. Yıldız, and M. Demirbüken. On a conjecture of Ilmonen, Haukkanen and Merikoski concerning the smallest eigenvalues of certain GCD related matrices. *Linear Algebra Appl.* **493** (2016), 1–13.

The Ilmonen–Haukkanen–Merikoski numbers C_n

By Ilmonen, Haukkanen, and Merikoski (2008),

$$C_n = \lambda_{\max}(W_n W_n^{\mathrm{T}}), \quad \text{where } W_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

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$$W_n W_n^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}$$

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$$W_n W_n^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}$$

$$\Rightarrow \quad (W_n W_n^{\mathrm{T}})^{-1} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix} \quad \text{(FDM matrix!)}$$

Lemma

$$C_n = \frac{1}{4}\csc^2\left(\frac{\pi}{4n+2}\right) = \frac{4n^2}{\pi^2} + \frac{4n}{\pi^2} + \left(\frac{1}{12} + \frac{1}{\pi^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof. Let $A_n = (W_n W_n^{\mathrm{T}})^{-1}$. Then $C_n = \lambda_{\max}(W_n W_n^{\mathrm{T}}) = 1/\lambda_{\min}(A_n)$ for the $n \times n$ matrix $A_n = \mathrm{tridiag}(-1, 2, -1) - \boldsymbol{e}_n \boldsymbol{e}_n^{\mathrm{T}}$. The eigenvalues of this FDM matrix (equipped with mixed Dirichlet–Neumann boundary conditions) are well known, which yields the first equality.

The second equality follows from the Laurent expansion of the first expression developed at infinity. $\square$

Hong and Loewy's numbers c_n

Notable lower bounds for c_n in the literature

Theorem (Mattila (2015))
For odd n,

$$c_n \geq \left(\frac{48}{n^4+50n^2+48n-51}\right)^{\frac{n-1}{2}}.$$

For even n,

$$c_n \geq \left(\frac{48}{n^4+2n^3+2n^2+n}\right)^{\frac{n-1}{2}}.$$

Theorem (Altınışık et al. (2016))
Let $(F_n)_{n=1}^{\infty}$ be the Fibonacci sequence. Then

$$c_n \geq \frac{2}{2F_nF_{n+1}+(-1)^n+1}.$$

Theorem (K (2021))
Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. For odd n,

$$c_n \geq \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n}+\frac{2}{25}\varphi^{-2n}-\frac{2}{5\sqrt{5}}n\varphi^{-2n}-\frac{23}{25}+n+\frac{2}{25}\varphi^{2n}+\frac{2}{5\sqrt{5}}n\varphi^{2n}+\frac{1}{25}\varphi^{4n}}}.$$

For even n,

$$c_n \geq \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n}+\frac{4}{25}\varphi^{-2n}-\frac{2}{5\sqrt{5}}n\varphi^{-2n}+\frac{2}{5}+n+\frac{4}{25}\varphi^{2n}+\frac{2}{5\sqrt{5}}n\varphi^{2n}+\frac{1}{25}\varphi^{4n}}}.$$

Lemma (K (2021))
Let $(F_k)_{k=1}^{\infty}$ be the Fibonacci sequence. For $n \geq 1$, there holds

$$c_n \geq \frac{1}{\sqrt{1+\sum_{i=2}^{n}(1+F_iF_{i-1})^2+2\sum_{i=2}^{n}\sum_{j=2}^{i}\left(F_{j-1}+\sum_{k=j+1}^{i}F_{k-1}F_{k-j}\right)^2}}.$$

Proof. Key inequality:

$$\boxed{c_n = \lambda_{\min}(Y_nY_n^{\mathrm{T}}) = \frac{1}{\lambda_{\max}(Z_n)} = \frac{1}{\|Z_n\|_2} \geq \frac{1}{\|Z_n\|_{\mathrm{F}}},}$$

where $Z_n := (Y_nY_n^{\mathrm{T}})^{-1}$ is the $n \times n$ matrix given by (cf. Altınışık et al. (2016))

$$(Z_n)_{i,j} = \begin{cases} 1+\sum_{k=i+1}^{n}F_{k-i}^2 & \text{if } i=j, \\ (-1)^{j-i}(F_{j-i}+\sum_{k=j+1}^{n}F_{k-i}F_{k-j}) & \text{if } i<j, \\ (-1)^{i-j}(F_{i-j}+\sum_{k=i+1}^{n}F_{k-i}F_{k-j}) & \text{if } i>j. \end{cases}$$

It is enough to show that

$$\|Z_n\|_{\mathrm{F}}^2 = 1+\sum_{i=2}^{n}(1+F_iF_{i-1})^2+2\sum_{i=2}^{n}\sum_{j=2}^{i}\left(F_{j-1}+\sum_{k=j+1}^{i}F_{k-i}F_{k-j}\right)^2.$$

What do the matrices Z_n look like?

$$Z_1 = (1)\,,$$

$$Z_2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix},$$

$$Z_4 = \begin{pmatrix} 7 & -4 & 3 & -2 \\ -4 & 3 & -2 & 1 \\ 3 & -2 & 2 & -1 \\ -2 & 1 & -1 & 1 \end{pmatrix}.$$

It follows from the definition of Z_n that this recurrence holds more generally:

$$Z_{n+1} = \begin{pmatrix} * & * \\ * & Z_n \end{pmatrix} \qquad \text{for all } n \in \mathbb{Z}_+.$$

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If we denote

$$Z_n = \begin{pmatrix} a_n & \boldsymbol{b}_n^{\mathrm{T}} \\ \boldsymbol{b}_n & Z_{n-1} \end{pmatrix},$$

then

$$\|Z_n\|_{\mathrm{F}}^2 = a_n^2 + 2\|\boldsymbol{b}_n\|^2 + \|Z_{n-1}\|_F^2.$$

In fact, from the elementwise definition of Z_n, we have

$$a_n^2 = (1 + F_n F_{n-1})^2, \quad \|\boldsymbol{b}_n\|^2 = \sum_{j=2}^{n} \left(F_{j-1} + \sum_{k=j+1}^{n} F_{k-1} F_{k-j} \right)^2.$$

We obtain the recurrence relation

$$\|Z_1\|_{\mathrm{F}}^2 = 1,$$

$$\|Z_n\|_{\mathrm{F}}^2 = \|Z_{n-1}\|_{\mathrm{F}}^2 + (1 + F_n F_{n-1})^2 + 2 \sum_{j=2}^{n} \left(F_{j-1} + \sum_{k=j+1}^{n} F_{k-1} F_{k-j} \right)^2$$

$$\Rightarrow \quad \|Z_n\|_{\mathrm{F}}^2 = 1 + \sum_{i=2}^{n} (1 + F_i F_{i-1})^2 + 2 \sum_{i=2}^{n} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-i} F_{k-j} \right)^2. \quad \square$$

To reiterate, we proved the following:

Lemma (K (2021))
Let $(F_k)_{k=1}^{\infty}$ be the Fibonacci sequence. Then

$$c_n \geq \frac{1}{\sqrt{1+\sum_{i=2}^{n}(1+F_iF_{i-1})^2+2\sum_{i=2}^{n}\sum_{j=2}^{i}\left(F_{j-1}+\sum_{k=j+1}^{i}F_{k-1}F_{k-j}\right)^2}}.$$

Is there a nice closed form for the expression inside the square root?

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Using symbolic computation or (tedious) hand-computations, one obtains

$$\begin{aligned}\|Z_n\|_{\mathrm{F}}^2 = \frac{1}{25}\varphi^{-4n} + \frac{3+(-1)^n}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} + 1 + \frac{13(-1)^n - 83}{50} \\ + n + \frac{3+(-1)^n}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}\end{aligned}$$

Theorem (K (2021))
For odd n,

$$c_n \geq \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{2}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} - \frac{23}{25} + n + \frac{2}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}}.$$

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Lower asymptotic bound: $c_n = \Omega(5\varphi^{-2n})$ as $n \to \infty$.

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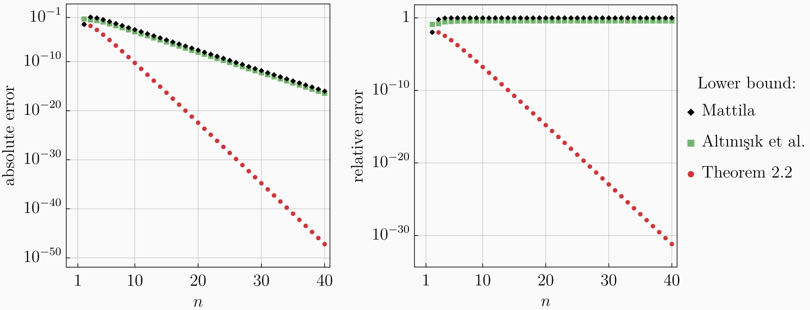
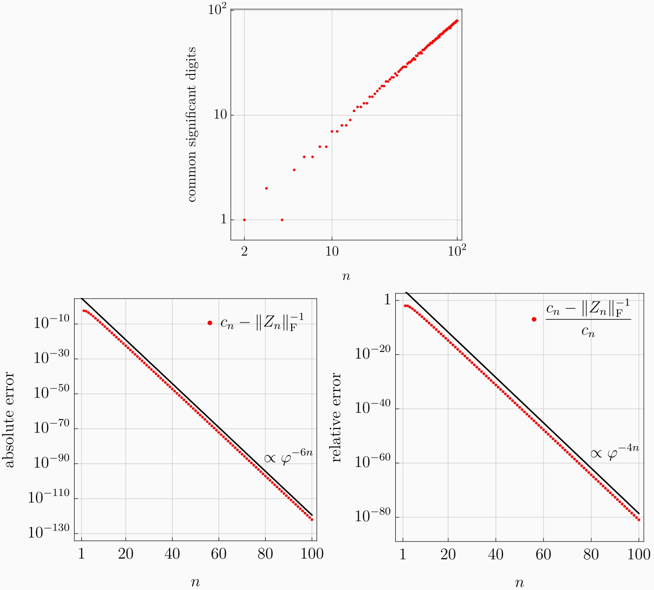
Numerical experiments



A comparison between different lower bounds for c_n

n	c_n	$\lVert Z_n \rVert_{\mathrm{F}}^{-1}$
1	1.000000000	1.000000000
2	0.381966011	0.377964473
3	0.198062264	0.196116135
4	0.087003112	0.086710997
5	0.037068335	0.037037037
6	0.014827585	0.014824986
7	0.005816999	0.005816805
8	0.002245345	0.002245332
9	0.000862203	0.000862202
10	0.000330004	0.000330004



The common significant digits (in decimal), absolute errors, and relative errors between the lower bound $\|Z_n\|_{\mathrm{F}}^{-1}$ and exact value of c_n for $n \in \{2, 3, \ldots, 100\}$. All computations were carried out by using 150 digit precision in Mathematica 10.

An upper bound for c_n

Some upper bounds for c_n have been developed in the literature (Loewy 2021; Altınışık 2021). However, I'll develop some sharper estimates by exploiting the observation that the second largest eigenvalue of Z_n is bounded from above by $\frac{4}{5}$ as $n \to \infty$.

The matrix Z_n is positive definite, so its eigenvalues are positive. Let $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_n^{(n)} > 0$ denote the eigenvalues of Z_n.

Before we are able to derive an upper bound on c_n, we need some information about the spectrum of Z_n. Specifically, we begin by proving that $\lambda_1^{(n)} > 1$ and $\lambda_2^{(n)} < \frac{4}{5}$.

The proof uses two ingredients:

- $\lambda_k^{(n)} \leq \lambda_{k+1}^{(n-1)} \leq \lambda_{k+1}^{(n)}$ (eigenvalue interlacing)
- $\det(\frac{4}{5}I - Z_n) < 0$ for all n (difficult part)

The first ingredient follows immediately from the Cauchy interlacing theorem.

$$\begin{bmatrix} -\frac{1}{5} - \sum_{k=2}^{n} F_{k-1}F_{k-1} & F_1 + \sum_{k=3}^{n} F_{k-2}F_{k-1} & -F_2 - \sum_{k=4}^{n} F_{k-3}F_{k-1} & \cdots & (-1)^n F_{n-1} \\ F_1 + \sum_{k=3}^{n} F_{k-1}F_{k-2} & -\frac{1}{5} - \sum_{k=3}^{n} F_{k-2}F_{k-2} & F_1 + \sum_{k=3}^{n} F_{k-3}F_{k-2} & \cdots & (-1)^{n-1} F_{n-2} \\ -F_2 - \sum_{k=4}^{n} F_{k-1}F_{k-3} & F_1 + \sum_{k=3}^{n} F_{k-2}F_{k-3} & -\frac{1}{5} - \sum_{k=4} F_{k-3}F_{k-3} & \cdots & (-1)^{n-2} F_{n-3} \\ F_3 + \sum_{k=5}^{n} F_{k-1}F_{k-4} & -F_2 - \sum_{k=4}^{n} F_{k-2}F_{k-4} & F_1 + \sum_{k=3}^{n} F_{k-3}F_{k-4} & \cdots & (-1)^{n-3} F_{n-4} \\ & & & \vdots & \end{bmatrix}$$

By performing the elementary row and column operations

$$\mathrm{row}_1 \leftarrow \mathrm{row}_1 + \mathrm{row}_2 - \mathrm{row}_3$$

$$\mathrm{col}_1 \leftarrow \mathrm{col}_1 + \mathrm{col}_2 - \mathrm{col}_3,$$

we obtain an "arrowhead" type matrix

$$\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 \\ -\frac{1}{5} & -\frac{1}{5} - \sum_{k=3}^{n} F_{k-2}F_{k-2} & F_1 + \sum_{k=3}^{n} F_{k-3}F_{k-2} & \cdots & \cdots & \cdots & (-1)^{n-1} F_{n-2} \\ \frac{1}{5} & F_1 + \sum_{k=3}^{n} F_{k-2}F_{k-3} & -\frac{1}{5} - \sum_{k=4} F_{k-3}F_{k-3} & \cdots & \cdots & \cdots & (-1)^{n-2} F_{n-3} \\ 0 & -F_2 - \sum_{k=4}^{n} F_{k-2}F_{k-4} & F_1 + \sum_{k=3}^{n} F_{k-3}F_{k-4} & \cdots & \cdots & \cdots & (-1)^{n-3} F_{n-4} \\ 0 & & \vdots & & & & \end{bmatrix}$$

Importantly, these elementary row and column operations preserve the determinant.

$$\det H_n = \det \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 \\ -\frac{1}{5} & & & & & & \\ \frac{1}{5} & & & & & & \\ 0 & & & & & & \\ 0 & & & H_{n-1} & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}$$

Schur's determinant lemma:

$$\det H_n = \det H_{n-1}\left(\tfrac{2}{5} - [-\tfrac{1}{5}, \tfrac{1}{5}, 0, \cdots, 0] H_{n-1}^{-1} \begin{bmatrix} -\frac{1}{5}, \\ \frac{1}{5} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

$$= \det H_{n-1}\big(\tfrac{2}{5} - \tfrac{1}{25}\big((H_{n-1}^{-1})_{1,1} - (H_{n-1}^{-1})_{1,2} - (H_{n-1}^{-1})_{2,1} + (H_{n-1}^{-1})_{2,2}\big)\big).$$

With a bit more work, by iterating the row/column operations, we find that

$$
\begin{aligned}
\det H_n &= \det H_{n-1}\left(\frac{2}{5} - \frac{1}{25}((H_{n-1}^{-1})_{1,1} - (H_{n-1}^{-1})_{1,2} - (H_{n-1}^{-1})_{2,1} + (H_{n-1}^{-1})_{2,2})\right) \\
&= \det H_{n-1}\left(\frac{2}{5} - \frac{1}{25}\left(\frac{\det H_{n-3}}{\det H_{n-1}}\left(\frac{2}{5} - \frac{1}{25}\frac{\det H_{n-4}}{\det H_{n-3}}\right)\right)\right) \\
&= \frac{2}{5}\det H_{n-1} - \frac{2}{125}\det H_{n-3} + \frac{1}{625}\det H_{n-4}. \qquad (1)
\end{aligned}
$$

For $n = 1, \dots, 4$, the determinants can be easily computed as

$$
\det H_1 = -\frac{1}{5}, \quad \det H_2 = -\frac{19}{25}, \quad \det H_3 = -\frac{41}{125}, \quad \det H_4 = -\frac{79}{625},
$$

so we can use induction on (1) to prove that

$$
\boxed{\det H_n = \det(\tfrac{4}{5}I - Z_n) = -\frac{1}{2}\cdot\frac{1}{5^n}(3(-1)^n + 10n^2 - 5) \quad \text{for all } n.}
$$

Note especially that $\det H_n < 0$ for all n.

Lemma

The eigenvalues of Z_n satisfy $\lambda_2^{(n)} < \frac{4}{5}$ and $\lambda_1^{(n)} > 1$.

Proof. By induction w.r.t. n. In the base case $n = 2$, we can simply compute the eigenvalues of $Z_2 \in \mathbb{R}^{2\times 2}$ and verify that the claim holds.

Next, suppose that $\lambda_2^{(n-1)} < \frac{4}{5}$ and $\lambda_1^{(n-1)} > 1$ for some n. By interlacing, we know that $\lambda_1^{(n)} \geq \lambda_1^{(n-1)} > 1$ and $0 < \lambda_n^{(n)} \leq \cdots \leq \lambda_3^{(n)} \leq \lambda_2^{(n-1)} < \frac{4}{5}$. Consider the characteristic polynomial

$$p_n(\lambda) = \det(\lambda I - Z_n) = (\lambda - \lambda_1^{(n)})(\lambda - \lambda_2^{(n)})(\lambda - \lambda_2^{(n)}) \cdots (\lambda - \lambda_n^{(n)}).$$

Plugging in the value $\lambda = \frac{4}{5}$ reveals that

$$p_n(\tfrac{4}{5}) = \underbrace{(\tfrac{4}{5} - \lambda_1^{(n)})}_{<0}(\tfrac{4}{5} - \lambda_2^{(n)})\underbrace{(\tfrac{4}{5} - \lambda_2^{(n)}) \cdots (\tfrac{4}{5} - \lambda_n^{(n)})}_{>0}.$$

However, we already know that

$$p_n(\tfrac{4}{5}) = \det(\tfrac{4}{5} I - Z_n) < 0 \quad \text{for all } n$$

which implies that $\frac{4}{5} - \lambda_2^{(n)} > 0 \Leftrightarrow \lambda_2^{(n)} < \frac{4}{5}$. $\square$

The previous lemma implies that

$$
\begin{aligned}
& 0 < \lambda_n^{(n)} \leq \cdots \leq \lambda_2^{(n)} < \frac{4}{5} \\
\Rightarrow \quad & \|Z_n\|_{\mathrm{F}}^2 = \lambda_1^2 + \cdots + \lambda_n^2 < \|Z_n\|_2^2 + \frac{16}{25}(n-1).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\|Z_n\|_{\mathrm{F}}^2 < \|Z_n\|_2^2 + \frac{16}{25}(n-1) \quad & \Leftrightarrow \quad \frac{1}{\|Z_n\|_2^2} < \frac{1}{\|Z_n\|_{\mathrm{F}}^2} + \frac{\frac{16}{25}(n-1)}{\|Z_n\|_{\mathrm{F}}^2 \|Z_n\|_2^2} \\
& \Leftrightarrow \quad \frac{1}{\|Z_n\|_2^2}\left(1 + \frac{\frac{16}{25}(1-n)}{\|Z_n\|_{\mathrm{F}}^2}\right) < \frac{1}{\|Z_n\|_{\mathrm{F}}^2} \\
& \Leftrightarrow \quad \frac{1}{\|Z_n\|_2} < \frac{1}{\sqrt{\|Z_n\|_{\mathrm{F}}^2 + \frac{16}{25}(1-n)}}.
\end{aligned}
$$

Since $c_n = \frac{1}{\|Z_n\|_2}$ and we know $\|Z_n\|_{\mathrm{F}}$ with equality by K (2021), we obtain the following...

Theorem
There holds for all $n \in \mathbb{Z}_+$ *that*

$$c_n \geq \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n}+\frac{3+(-1)^n}{25}\varphi^{-2n}-\frac{2}{5\sqrt{5}}n\varphi^{-2n}+\frac{13(-1)^n-33}{50}+n+\frac{3+(-1)^n}{25}\varphi^{2n}+\frac{2}{5\sqrt{5}}n\varphi^{2n}+\frac{1}{25}\varphi^{4n}}}$$

and

$$c_n < \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n}+\frac{3+(-1)^n}{25}\varphi^{-2n}-\frac{2}{5\sqrt{5}}n\varphi^{-2n}+\frac{13(-1)^n-1}{50}+\frac{9}{25}n+\frac{3+(-1)^n}{25}\varphi^{2n}+\frac{2}{5\sqrt{5}}n\varphi^{2n}+\frac{1}{25}\varphi^{4n}}}.$$

In consequence,

$$c_n \sim 5\varphi^{-2n}$$

as $n \to \infty$.

Conclusions

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Thank you for your attention!