A NOTE ON GNS CONSTRUCTION AND DUALITIES

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Abstract. The purpose of this note is to make a quick comment on the GNS construction from the sense of C^*-categories and how holography could be interpreted as a natural transformation.

Note: *This is a work in progress.*

The main proposition of this note is as follows:

Proposition for LCQFTs: Let $\mathcal{F}_M$ and $\mathcal{F}_N$ be two functors associated with LCQFTs of D and $D-1$ dimensions respectively. Then, holography is a natural transformation between $\mathcal{F}_M$ and $\mathcal{F}_N$ such that it preserves the Haag-Kastler conditions. Moreover, there exists a $*$-preserving functor (called a *representation*) through which the subsequent algebras in C^*-categories $A_M, A_N \in \text{Obj } \mathbf{C}_{M \text{ or } N}$ are *concrete categories*. This gives a duality of Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B \in \text{Obj } \mathbf{Hilb}$.

Recall that the representation of $\mathbf{C}$ is a $*$-functor $\mathbf{C} \to \mathbf{Hilb}$. For some ψ linear positive functional on (A, A) for $A \in \text{Obj } (\mathbf{C})$, there exists a representation $R_\psi(\mathbf{C})$ with some χ_ψ cyclic vector so that

$$\psi_a = \langle \chi_\psi,\ F_{\psi_a} \chi_\psi \rangle \quad \textbf{for } a \in (A, A)\ . \tag{1}$$

This extends up to a unitary equivalence, i.e. given some other representation $\tilde{R}_\psi(\mathbf{C})$, there exists some $U : R_\psi(\mathbf{C}) \to \tilde{R}_\psi(\mathbf{C})$. This is important because, when constructing concrete categories from such state-representation pairs, we would obtain Hilbert spaces that are unique up to isomorphism. Then, we take a (naturally faithful) F

$$\bigoplus_\psi F_\psi \ \text{ over all Obj } (\mathbf{C}) \tag{2}$$

that takes $\mathbf{C} \to \mathbf{Hilb}$. However, note that you don't naturally get states to select in general LCQFTs. We then identify Hadamard states instead of assuming specific vacuum states for the GNS construction. Define $\mathcal{H}$ad so that

$$\mathcal{H}ad(A) = \bigoplus_{\omega \in \text{Had}(A)} \mathcal{H}_\omega \tag{3}$$

are the objects and the morphisms are corresponding $*$-homomorphisms. We then consider $\mathcal{H}ad(\mathcal{F}_M)$ selection to get

$$\mathcal{H}ad(\mathcal{F}_M) = \bigoplus_{\omega \in \mathrm{Had}^*(\mathcal{F}_M)} \mathcal{H}_\omega \ . \tag{4}$$

Note that this in itself does not mean anything concrete, since one could have as many Hadamard states and therefore representations inclusive in this direct sum. The natural resolution is that we consider not an auxiliary selection, in refined algebraic quantization terminology, but a *physical* selection where these also satisfy constraints, particularly such as the isometry group constraint. By then taking the $\mathcal{H}ad$ selection over the $\mathcal{F}_M$ or $\mathcal{F}_N$ algebras, we would specifically be left with those states that are well-defined, and by GNS-completing these states, we obtain Hilbert spaces. By positing that holography is a natural transformation between these Hilbert spaces, we recover the fairly simpler version of Liu-Leutheusser type holography (conceived in AdS/CFT). Since we have isotony conditions, we also have corresponding "nesting" for algebras of observables associated to subregion wedges. As a simple case, if $A \subseteq B$, $\mathcal{F}_M(A) \subseteq \mathcal{F}_M(B)$ and correspondingly, there are boundary subregions with subalgebras $\mathcal{F}_N(\tilde{A}) \subseteq \mathcal{F}_N(\tilde{B})$. Beyond this regime, assuming stronger conditions like the categorial bicommutant theorem, one could make a stronger case for W^*-categories. However, the overall formalization rests with C^*-categories.

Note that there are two subtleties here: first, this only makes sense in AdS/CFT for the obvious reason that it is the only case in which a holographic dual is well-established. de Sitter, for instance, has issues both at EFT level and at a duality level where the dual QFT does not have a convenient boundary to live on. Second, this only has a weak form of AdS/CFT where we do not obtain a state-correlator correspondence. In fact, correlators are not even obtained formally in this case, at least to my knowledge. On the other hand, for the TFT-CFT duality at least at the 2d level, we use conformal blocks, and in that case it makes far more sense as to what holography means. However, this "functorial holography" is to an extent better than taking holography as just an equivalence between local nets of observables, in the sense that we now have a prescription for picking states for computing correlators and not just an abstract statement that algebras of observables are dual to each other. How exactly this can be computed is still yet to be done.

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