Bulk Hilbert Space Factorization in AdS/CFT

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Abstract

In this paper, we will present a "strong" No Transmission Principle. Inspired from the split property of QFTs, we will discuss the emergent type III boundary algebras in the form of locally covariant QFTs, which are a kind of functorial QFTs. We will show that the dependence of bulk duals to CFTs is highlighted on the basis of the factorization of the bulk Hilbert space, which is only possible when the boundary algebras are not type I. We make use of the recent works by Liu and Leutheusser, and show that this can be seen from the duality between the bulk Fock space and the boundary GNS Hilbert spaces. We will show that this is inspired from a kind of toned-down split property, referred to as the "intermediate factoriality". We also provide a brief discussion on the relations to conformal completion in AdS/CFT, found by CPT-reflecting and gluing across extremal surfaces.

Note: In a recent Strings talk by Netta Engelhardt on an algebraic ER=EPR [1], a version of the results in this paper were presented in the sense of "connectedness". This is synonymous to saying that the bulk duals are connected, and the result of this paper, i.e., for bulk duals to two CFT copies to be dependent they must not be of type I, can be extended to covering the cases of type II_∞ and III algebras. We have added a section discussing the similarities and further prospects to this paper.

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Contents

1 Introduction

AdS/CFT has been a very successful holographic theory, with many advancements in the directions of holographic entanglement entropy [2, 3, 4, 5, 6, 7, 8], subregion-subregion duality [9, 10, 11, 12, 13, 14], and more recently, operator algebraic descriptions of the bulk-boundary duality and the algebra type [10, 11, 14]. The Ryu-Takayanagi formula states that the entanglement entropy of a boundary subregion A is given by the area of a minimal spacelike geodesic connecting the two boundary points [2, 3]:

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N} \,, \tag{1}$$

where γ_A is the Ryu-Takayanagi surface. This provides a very solid motivation for subregion duality, which is to say that the entanglement wedge $\mathcal{E}_W(A)$ of A is the dual bulk region associated to the boundary subregion A. To make this a more precise statement in the operator algebraic sense, one can say that in the finitely large N limit, the bulk Hilbert space $\mathcal{H}_{\text{bulk}}$ and the CFT Hilbert space $\mathcal{H}_{\text{CFT}}$ are equivalent, i.e.

$$\mathcal{H}_{\text{bulk}} = \mathcal{H}_{\text{CFT}} \,.$$

The work of Liu-Leutheusser [11, 10] showed that in the large N limit, the emergence of type III$_1$ subalgebra formulates a subregion-subalgebra duality, which states that the emergent III$_1$ von Neumann subalgebra corresponds to a causally complete bulk subregion.

In this fashion, the subregion-subalgebra duality states that in the $G_N \to 0$ expansion the Fock space $\mathcal{H}_{\text{bulk}}$ and the GNS Hilbert space $\mathcal{H}_{\text{GNS}}$ are equivalent, i.e. $\mathcal{H}_{\text{Fock}} = \mathcal{H}_{\text{GNS}}$, where we start by picking a CFT state Ψ for the construction. Then, the bulk algebra $\mathcal{B}_\Psi$ and the boundary single-trace operator algebra $\mathfrak{B}_\Psi$ corresponding to Ψ become equivalent:

$$\mathcal{B}_\Psi = \mathfrak{B}_\Psi \ . \tag{2}$$

In the parallel, there exists a *No Transmission Principle* (NTP) [15], which states that when two CFTs are independent, their bulk duals must also be independent. This is interesting in some nice contexts and has been discussed in a strong cosmic censorship aspect as well. However, the general statement of the NTP is somewhat mathematically vague, since we do not know anything about the specific cases in which the NTP holds, nor whether or not a violation can exist. In AdS/CFT, a mathematical statement would seem to arise from the operator algebraic formulation of subregion-subregion duality. In this paper, we provide a "strong" version of NTP by drawing inspiration from subregion-subalgebra duality and type III von Neumann algebras, and show that an algebraic formulation of NTP implies that type I spacetimes cannot be *complete*.

Before continuing, it is necessary to first indicate why we only discuss type I or type III algebras. While part of the answer is for emphasising the emergent type III_1 algebra into the story, we do not fully formalize the type II_∞ part of the discussion since there are a number of subtleties with the situation. This is not an issue of boundary or bulk algebras; for that matter, taking $1/N$ corrections into account takes us from a type III_1 to type II_∞ algebra [16]. However, for the sake of this paper we will stick to the large N limit. Future works on type II aspects as well may incorporate that part of the discussion.

This paper is structured as follows: in section 2, we will state the strong NTP, and discuss some of the subtleties surrounding such an algebraic statement. We first start by a discussion of the boundary operator algebras perspective in subsection 2.1, after which we state the proposal of the strong NTP in subsection 2.2. In subsection 2.3, we will discuss an intermediate implication of the strong NTP, which is that conformal completion is only possible in non-type I situations. In subsection 2.4, we comment on the relation between subregion-subalgebras and conformal completion in the case of entanglement wedge transitions. In section 3, we will discuss the relation between the split property in QFTs with the strong NTP proposal. We further take this to the curved spacetime setting by making use of intermediate factoriality condition from locally covariant QFTs[17, 18] in section 3.2.

While this draft was in progress, a Strings talk by Netta Engelhardt discussed an algebraic ER=EPR proposal, which finds that type I algebras must render bulk disconnectedness, in that the Cauchy slices must be disconnected. This is in agreement with the proposal in this paper, which can be understood in the Cauchy slices perspective by seeing that the initial data $\mathfrak{h}(\Sigma, h, \dots)$ for a suitable wedge (in the conformal completion sense) must not be such that one can CPT-reflect and glue across an extremal surface. We briefly allude to this connection in section 4.

Conventions: In this paper, we will use the following conventions. For a boundary subregion, we will use capital Latin letters A. For bulk subregions, we will use small gothic letters $\mathfrak{a}$, although we will mostly be interested in specific settings of these bulk subregions. For data about the spacelike slice, induced metric and other relevant quantities, we use $\mathfrak{h}$, and for CPT-reflected quantities, we will Tilde, i.e. $\tilde{\mu}$ for the CPT-reflection of some codimension-2 surface μ. Bulk and boundary subalgebras will be indicated by $\mathcal{X}$ and $\mathfrak{X}$ correspondingly.

2 Strong NTP

In this section, we will introduce the strong No Transmission Principle. This is the statement that the factorization of the boundary Hilbert space (the GNS Hilbert space) has a direct implication from the correspondence of the operator algebras on the bulk and boundary sides. In order to get to arrive at this statement, we will first start by making some motivations from the boundary operator algebras clear.

2.1 Boundary Operator Algebras

The starting point of the discussion is that of the boundary operator algebras. Here, the algebra of the boundary sides CFT_R and CFT_L have a tensor product overall algebra, $\mathcal{A} = \mathcal{A}_R \otimes \mathcal{A}_L$. The way to think of the boundary side in the Hilbert space context is to use the GNS construction, where we define a state $|\mathsf{a}\rangle$ for operators $\mathsf{a} \in \mathcal{A}_{\mathrm{TFD}}$, which is the thermal field double algebra $\mathcal{A}$ in the large N limit. In this limit, we can impose an inner product definition and endow a completion. Post-GNS construction, we obtain the GNS Hilbert space $\mathcal{H}_{\mathrm{GNS}}$, so that for some boundary algebra $\mathfrak{B}^{\mathrm{TFD}}_{R/L}$, each state in $\mathcal{H}_{\mathrm{GNS}}$, say $|\Pi_{R/L}\rangle$ belongs to the corresponding boundary algebra[1]. A relevant related aspect is that of the GNS representation of some state Ψ, which is useful in this paper when discussing the split property in subsection 3. For some Ψ in the GNS construction, the collection $(\mathcal{H}, \pi, \Omega)$ will be referred to as the GNS representation in the usual sense, see for example [19, 18].

The first hint towards the boundary operator algebraic picture of the strong NTP is as observed in Liu and Leutheusser's work on emergent type III_1 algebras. the boundary algebras in the surroundings of the Hawking-Page temperature T_{HP} have a characteristic alternation, in that for some temperature t below T_{HP}, the algebras $\mathfrak{B}_R$ and $\mathfrak{B}_L$ are type I, whereas they are type III_1 for some $T > T_{HP}$. Such a behaviour is not only limited

[1] See [10, 11] for details on the GNS Hilbert space, and section IIIA and Appendix A in [10] for a detailed discussion on the GNS Hilbert space construction.

to the case of the Hawking-Page temperature, but also the Page time, where one can pick times around the Page time to notice a subtle algebraic inclusion. This has been discussed later in section 2.3, where we make sense of this in terms of CPT-reflection and conformal completion. This was motivated from [20], where choosing times around the Page time with equal von Neumann entropy had different conformal completion. As we shall see, this can be explained in terms of the strong NTP, by noting that the type of algebra post-Page time is type III_1. Before moving on, we will recall why this boundary motivation is necessary; the baseline for the strong NTP is the (non-)factorization of the boundary Hilbert space, and the type of algebra that implies such a (non-)factorization. The inspiration here is from the type III nature of local QFTs, which are difficult in that the type III nature does not allow the definition of a trace, due to which entanglement entropy is ill-defined. In fact, AdS/CFT in itself indicates the holographic consequences of this type of von Neumann factors. When dealing with entanglement entropy, one can define a regulator (a lattice spacing) ϵ, which renders the type III algebra to a type I algebra, due to which the Ryu-Takayanagi formula makes sense. In this sense, the prescription contains the regulator, which in the $\epsilon \to 0$ limit diverges and gives back the original type III nature of the theory. Therefore, when stating that the two bulk duals to $\mathrm{CFT}_{R/L}$ are independent, it is obvious that the full problem ends with the question of what the boundary algebra is. If it were type I, the bulk Hilbert space would clearly factorize, implying the "independence" of the bulk duals from each other. If they were non-type I[2], this independence is lost, and the bulk Hilbert space $\mathcal{H}_{\text{bulk}}$ will no longer factorize. We adopt the boundary point of view since the general prescription in our situation requires a clarity of the boundary and not of the bulk, i.e. we require understanding the factorization of the CFT copies to imply the independence of the bulk. From the duality of the algebras in the bulk and boundary, it is clear that the GNS Hilbert space $\mathcal{H}_{\text{GNS}}$ being in equivalence with the bulk Fock space makes the converse true, since the boundary algebra ${}_{\Psi}\mathfrak{B}^{\mathfrak{X}}_{A}$ and bulk algebra ${}_{\Psi}\mathcal{B}^{\mathcal{X}}_{\mathfrak{a}}$ are also in correspondence. The tensor product structure then implies the strong NTP from the factorization $\mathcal{H}_{\text{bulk}} = \mathcal{H}_R \otimes \mathcal{H}_L$. The bulk tensor product is a global structure; on their own, $\mathcal{H}_{R/L}$ can be properly constructed by taking metric+matter perturbations as outlined in Liu-Leutheusser.

2.2 Proposal

The starting point for making sense of an algebraic statement of the NTP is to notice the bulk Hilbert space $\mathcal{H}_{\text{bulk}}$ factorizes into *left* and *right* Hilbert spaces when working in the

[2]We do not make any statement about whether the boundary algebra being type II or type III individually implies specific meanings, and instead speculate that in general, any non-type I algebra should not factorize. This is made a little more clear in section 2.3, when we discuss the pre- and post-Page time scenarios.

TFD state, as discussed above:

$$\mathcal{H}_{\text{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R \ . \tag{3}$$

The natural idea of the NTP here is that since the bulk duals are independent and they factorize, the corresponding CFTs must be independent as well. In general, two CFT copies that are independent must have independent bulk duals from the original proposition of the NTP.

One can pick some boundary subregion A on the left boundary and B on the right boundary and make the above observation clearer in the subregions sense. The end product would still be the same; the bulk Hilbert space $\mathcal{H}_{\text{bulk}}$ would be composed instead of the subregions $\mathfrak{a}$ and $\mathfrak{b}$ associated to A and B respectively, and the factorization of this Hilbert space would tell us the nature of the boundary operator algebra. This is evident, since one can take, for example, the equivalence for bulk subalgebra and boundary subalgebra:

$${}_{\Psi}\mathcal{B}_{\mathfrak{a}}^{\mathcal{X}} = {}_{\Psi}\, \mathfrak{B}_A^{\mathfrak{X}} \ , \tag{4}$$

where we take the full operator algebras on the bulk and boundary to be $\mathcal{B}_{\Psi}$ and $\mathfrak{B}_{\Psi}$ as before for each $\mathfrak{a}$ and $\mathfrak{b}$ and A and B respectively. We are now in a position to state the *strong No Transmission Principle*:

$$\text{If the boundary operator algebras are of type I, the bulk duals must be independent.} \tag{5}$$

This statement can be made more precise by noting that the key ingredient here is that the bulk independence is found when $\mathcal{H}_{\text{bulk}}$ factorizes as seen above. Since the type I nature of the boundary operator algebra can be seen in duality with this factorization, we can pick the left and right boundary algebras $\mathfrak{B}_L$ and $\mathfrak{B}_R$ and state (5) as follows:

$$\text{If } \mathfrak{B}_L \text{ and } \mathfrak{B}_R \text{ are type I, } \mathcal{H}_{\text{bulk}} \text{ factorizes.} \tag{6}$$

The bulk Fock spaces $\mathcal{H}_L^{\text{Fock}}$ and $\mathcal{H}_R^{\text{Fock}}$ are independently defined, and are found in the usual way by picking the state Ψ and the corresponding metric+matter perturbations, as seen from Liu-Leutheusser.

One can now try and see where the independence of the bulk and boundary sides is lost, by seeing that the independence is highlighted by the factorization of $\mathcal{H}_{\text{bulk}}$. Clearly, if one instead has a type III algebra, as can be seen in $G_N \to 0$ limit, the picture seems to change. In a type III situation, the factorization of $\mathcal{H}_{\text{bulk}}$ does not happen. Conversely, the non-type I nature of this limit would imply that the bulk independence is lost, and the original NTP is violated. The strong NTP restores this, and the bulk dependence is explained as the non-type I algebra effect on the bulk Hilbert space. Therefore, for type III algebra, this correctly implies the non-factorization of the bulk Hilbert space, and the strong NTP for type III boundary algebras is as follows:

$$\text{If } \mathfrak{B}_L \text{ and } \mathfrak{B}_R \text{ are type III, the bulk duals must not be independent.} \tag{7}$$

Here, the type II algebra arises as an intermediate situation, since clearly the strong NTP is not stringent in any way. One could naively guess that the type II situation does not indicate bulk independence, and so any non-type I scenario must indicate some sort of bulk dependence. However, we will defer a discussion on type II from this paper. The original NTP suggested that when the CFT copies are independent, the bulk Hilbert space factorizes. Naturally, this factorization indicates the type I operator algebraic nature that we made use of in defining the strong NTP. Consider the CFT copies on the left and right boundaries, denoted by CFT_L and CFT_R. Individually, the bulk duals to these two CFTs have a well-defined algebra $\mathcal{B}_L$ and $\mathcal{B}_R$, and from (2), the subregion-subalgebra duality indicates the factorization of $\mathcal{H}_{\text{bulk}}$ quite well in terms of the dual CFT Hilbert spaces as well. To see the factorization and the bulk independence more clearly, one can take some data $\mathfrak{h}_L$ and $\mathfrak{h}_R$ and evolve the data. If the bulk Hilbert space did not factorize, the geometric consequence would be that the evolution of the data $\mathfrak{h}_L + \mathfrak{h}_R$ to the future and past generates the proper combined geometry. However, this is not so; since the bulk duals are independent, the combined data $\mathfrak{h}_L$ and $\mathfrak{h}_R$ would not be continuous.

The factorization of the bulk Hilbert space indicated the type I nature. In the type III story, the bulk duals are not independent; in this case, one can take the initial data on the two sides and evolve continuously[3].

2.3 Conformal Completion

As mentioned previously, the bulk Hilbert space factorization leads to the strong NTP. Now, one could ask if the initial data problem discussed in the type I and type III discussions have any explicit meaning. This can be motivated by starting from a spacetime (M, g), with a type I boundary operator algebra. For various reasons that can be seen from coarse-graining [21, 22] and canonical purification [20, 23], it is necessary to *conformally complete* an operated spacetime by taking the CPT-reflection of (M, g), by CPT-reflecting CPT-odd quantities. This way, objects prescribed in the initial data $\mathfrak{h}$ such as the induced metric h on a Cauchy slice Σ, the trace of the extrinsic curvature K become CPT-reflected, which generates an auxiliary spacetime $(\tilde{M}, \tilde{g})$. By identifying a gluing codimension-2 surface, which is an extremal surface to identify the Barrabes-Israel junction conditions, we glue $(\tilde{M}, \tilde{g})$ and (M, g) to obtain a conformally completed spacetime (M', g'), which, for instance, is the coarse-grained spacetime corresponding to the outer wedge of a minimar surface along which the coarse-graining was done. However, conformal completion in itself also assumes that the reflected geometry gluing makes sense in the first place.

[3]In the next subsection, we will discuss the case of conformal completion, where we CPT reflect odd quantities after coarse-graining to find the initial data for the completion of the coarse-grained spacetime; this aspect of the full evolution of the combined initial data across these two sides becomes very prominent.

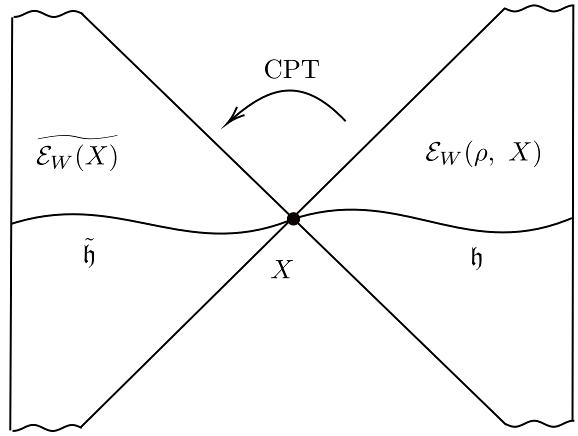

Figure 1: CPT reflection of the initial data from the original spacetime (M, g) to generate the conformally complete (M', g')

This geometry is interesting for a nice reason: the gluing across X works only when the initial data $\mathfrak{h} + \tilde{\mathfrak{h}}$ can be used to evolve to the future and past (generating the fiducial causal wedges). Clearly, such a thing only makes sense if the algebra of operators is type III. Here again, it would be nice to see the possibility of a type II situation here. However, in such cases, it is possible that the initial data evolution may not work. However, we do not provide a formal reason for this and will leave it as a possibility. There also may be relations to canonical purification, which also involves CPT-conjugation,, which we will allude to below. Conformal completion arises not necessarily due to coarse-graining, but in general in the purification aspect as well, where one can pick, for instance, an evaporating black hole and find times around the Page time to purify. In this way, conformal completion finds implications from the operator algebra, and as a result the type of the boundary algebra affects the full conformal completion. However, it is not the case that the algebra is type I for every disconnected geometry. Let us discuss this point a little more clearly. Consider an evaporating black hole. Then, pick two times t and T before and after the Page time t_P respectively, for which the von Neumann entropies are equal, as done in [20]:

$$S_\rho(t) = S_\rho(T)\ . \tag{8}$$

Applying the purification scheme at time T gives a conformally complete spacetime in that the purification is a full spacetime; clearly, the type III nature can be made sense of for $T > t_P$. For $t < t_P$, this is not so – $|\sqrt{\rho(t)}\rangle$ does not have a connected bulk dual [20]. However, this is *not* a type I situation, and so the possibility is that in general, any non-type I spacetime is dependent somehow. The most stringent criterion for this dependence to be decided is that of the factorization of the Hilbert space. This is somewhat close to the situation described in section 2.1 with temperatures below and above the Hawking-Page temperature. However, in that case, the below T_{HP} situation was a type I algebra, whereas seemingly the situation in pre-Page time black holes is somewhat unclear. We will not formalize any statement about the pre-Page time scenario, but we speculate that in some way, this is a non-type I case, and there exists a way of showing that the factorization of the Hilbert space is still the deciding factor.

2.4 Entanglement wedge transitions

Among many aspects we do not formally discuss in this paper is that of entanglement wedge transitions of two boundary regions. This is of interest if, continuing along the idea presented above in the form of conformal completion, one tries to make sense of the boundary algebra in the situation where a entanglement wedge transition occurs. The idea here is as follows: take two boundary subregions, A and B, with the boundary complement $\overline{AB}$ and bulk complement $\overline{\mathfrak{a} \cup \mathfrak{b}}$. If these two components are "small", then the bulk complement is just $\overline{\mathfrak{a} \cup \mathfrak{b}}$ as usual, and the entanglement wedges corresponding to each of A and B are $\mathcal{E}_W(A)$ and $\mathcal{E}_W(B)$. However, if these boundary subregions are "big", $\mathcal{E}_W(A)$ and $\mathcal{E}_W(B)$ become larger and become a large bulk wedge, which we will denote by $\mathcal{E}_W(AB)$. The bulk complement now is some $\overline{\mathfrak{ab}}$ instead. In the sense of conformal completion, it may be possible that based on the boundary subalgebra $\mathfrak{B}^{\mathfrak{X}}_{AB}$ in each of the cases, the CPT-reflected geometry cannot be glued to find the full conformal completion. In such scenario, one should expect type III_1 $\mathfrak{B}^{\mathfrak{X}}_{AB}$ to have a conformal completion, and could speculate that the type I situation is where such reflected geometry does not give a nice conformal completion. This does not invoke strong NTP; rather, this would be more or less from subregion-subalgebra duality, although it is not clear why one should work with conformal completions in the above situation.

We will finally discuss the implications of the "split property" of quantum field theories in AdS/CFT in the framework of the strong NTP. We see that the split property provides a very solid operator algebraic indication of the existence of a strong NTP by relating the emergence of type III_1 to a type I factor from a "collar" region separating the R and L regions. We will first give a quick recap of the split property, after which we will discuss the strong NTP result from the split property perspective, and also see the implications of a failure of the split property in the sense of holography of information.

3 Intermediate Factoriality

We will now discuss the split property in AdS/CFT, and discuss the links between the strong NTP and the split property. We will also remark on the failure of the split property, which closely has to do with holography of information in AdS/CFT, and present some musings on the strong NTP description of such a discrepancy. In the next section, we will discuss a more mathematical perspective on the *intermediate factorizability* condition, replacing the split property for gravitational settings.

3.1 Split Property

We will first start by giving a quick recap of the split property in general QFTs [19, 24]. For the sake of argument, we will start with a Minkowski spacetime[4], which contains a non-compact Cauchy slice Σ. Now, take a compact subregion $\mathcal{U}$ in Σ, and define the complement as $\overline{\mathcal{U}}$. Now, introduce a "collar" ε around $\mathcal{U}$, so that the complement is now defined for $\overline{\mathcal{U}\cup\varepsilon}$, denoted by $\overline{\mathcal{U}}^{\varepsilon}$. Assign an algebra of observables to each of $\mathcal{U}$ and $\overline{\mathcal{U}}^{\varepsilon}$, and denote them by $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\overline{\mathcal{U}}^{\varepsilon}}$ respectively. Then, the elements $\mathfrak{F}\in\mathcal{A}_{\mathcal{U}}$ and $\overline{\mathfrak{F}}\in\mathcal{A}_{\overline{\mathcal{U}}^{\varepsilon}}$ of these algebras must commute, i.e.

$$\left[\mathfrak{F}(\mathcal{A}_{\mathcal{U}}),\ \overline{\mathfrak{F}}(\mathcal{A}_{\overline{\mathcal{U}}^{\varepsilon}})\right]=0\ . \tag{9}$$

The split property then tells us that there is a type I factor $\mathcal{M}$ so that when describing the global Hilbert space $\mathcal{H}$, there is a factorization associated to $\mathcal{U}$ and $\overline{\mathcal{U}}^{\varepsilon}$:

$$\mathcal{H}=\mathcal{H}_{\mathcal{U}}\otimes\mathcal{H}_{\overline{\mathcal{U}}^{\varepsilon}}\ . \tag{10}$$

In terms of density matrices, the property gives us two density matrices $\rho_{\mathcal{U}}$ and $\rho_{\overline{\mathcal{U}}^{\varepsilon}}$ for each of $\mathcal{U}$ and $\overline{\mathcal{U}}^{\varepsilon}$. For our discussion, the use of this is in finding a type I factor $\mathcal{M}$ that gives us this split property. Let $\mathscr{B}_{\mathcal{V}}$ be the set of bounded operators in the Hilbert space associated to region $\mathcal{V}$, which is either R or L. Then, there is, associated to the collar defined by ε, a type I factor $\mathcal{M}$ with a subset relation to $\mathscr{B}_R$ and the commutant $\mathscr{B}_L\,'$. There are many key correspondences to Tomita-Takesaki theory which have many important consequences which we will not point out here. A more elaborate discussion on the perspective adopted from locally covariant QFTs in the AdS/CFT sense has been provided in section 3.2. The way to see the correlation between the split property and the boundary algebra type is by noting that the regions R and L would have a divergent entanglement entropy, which would be evident from the type III_1 nature of the boundary algebra. Sitting with two CFT copies in the TFD state, the gravity dual is an eternal black hole. In this setting, lattice regularization is intrinsic, since defining a bulk short-distance cutoff ϵ allows us to compute the entanglement entropy in the form of the area plus bulk corrections prescription,

$$S_{RL}\sim\frac{\text{Area of Horizon}}{\epsilon}+S_{\text{bulk}}\ . \tag{11}$$

It is clear that the $\epsilon\to 0$ limit yields a "bad" entanglement entropy. Indeed, setting a non-zero cutoff as above gives us a nice type I algebra that factorizes the overall Hilbert space

[4] As it is pointed out later in this discussion, the usual arena of the split property is that of non-gravitational QFTs. Taking, for example, asymptotically Minkowskian space, the algebra of observables $\mathcal{A}(\mathcal{I}^{\pm})$ at the null infinities $\mathcal{I}^{\pm}$ are "squeezed" into a small band $\mathcal{I}^{\pm}_{\epsilon}$ near i^0, which in a way affects the natural statement of the split property [25, 26]. However, we do not comment on the relation of such a failure to strong NTP.

into something like (10). This was correlated with the Ryu-Takayanagi prescription, in the form of the type III_1 emergence from the type I factor $\mathcal{M}$. From the above discussion on the strong NTP, one can notice that the split property directly has implications on how we understand the bulk Hilbert space factorization. If the full Hilbert space factorized, i.e. the algebra was already type I, then we need not invoke the split property to find the factorization of the $\mathcal{N}$ and $\overline{N}$ Hilbert spaces. To see the relation between the bulk Hilbert space factorization and split property, we will stick to the discussion of the bulk Hilbert space, which from (2) can be translated to the boundary view that will be discussed next. In the type I situation, the factorization would give us the usual nature of $\mathcal{H}^{R/L}_{\text{Fock}}$ relating to the corresponding boundaries, with an overall tensor product factorization. To see this more clearly, start by seeing that the factorization of the R and L regions would imply that a bifurcation surface X does not join the two bulk duals, or they would be dependent. In order to define a continuous Cauchy slice Σ from CFT_R to CFT_L, there *must* exist some X. In order for this to be possible, $\mathcal{B}_{R/L}$ must be type III_1 as per the strong NTP (7). However, when the bulk algebras are type III_1, the split property plays a crucial role, from the fact that the $\epsilon \to 0$ limit recovers the divergent entanglement entropy S_{RL}. From the split property, the type III_1 nature of the bulk algebra $\mathcal{B}^{\mathcal{X}}_{R/L}$ is found as a result of the type I nature of the factor $\mathcal{M}$ in the sense of regularization, as discussed in [11, 23]. This also indicates that the dependence associated with type III_1 bulk algebras is intrinsically related to the split property. In the boundary perspective, if $\mathfrak{B}_{R/L}$ are type III_1, then this aspect of the split property can be similarly correlated to the strong NTP. Here, we must also point out the subtlety that the split property is built in a non-gravitational QFT. When in a gravitational QFT such as the discussion of AdS/CFT, it is necessary to notice that there exists *holography of information*, where the information of some bounded region $\mathcal{V} \subseteq \Sigma$ can be encoded on the boundary of Σ via complementarity. Due to this, one can see a failure of the split property in curved spacetime. In the full AdS/CFT picture without having to worry about R or L regions, one could pose the problem in terms of the Page curve when taking a collar region ε separating the bulk+boundary system with a non-gravitational bath, a prescription outlined in [25]. In the next section, we will point out the type III_1 setting with locally covariant QFTs.

3.2 Locally Covariant QFT approach

A mathematically nicer restatement of the above discussion can be provided by adopting the view of locally covariant QFTs (LCQFTs), which are a kind of functorial QFTs (see Appendix A for a brief discussion on why such an aspect is necessary here). An example of functorial QFTs is that of topological QFTs, which are symmetrical monoidal functors from the n-bordisms category Bord_n to the category $\mathsf{Vect}_{\mathbb{K}}$, and a prominent example of such a theory is that of Chern-Simons theory. However, LCQFTs are functorial QFTs which incorporate the curved background g. A LCQFT is a functor defined as follows:

$$\mathcal{Z} \ : \ \mathfrak{Man}^D \ \longrightarrow \ \mathfrak{C}^* \ \mathfrak{Alg} \ . \tag{12}$$

Here, we will start by defining the two categories in consideration above. $\mathfrak{Man}^D$ is the category of $D \geq 2$ globally hyperbolic time-oriented manifolds (M, g). The morphisms in this category $\mathfrak{M}$ are in $\hom_{\mathfrak{Man}^D}((M_1, g_1),\ (M_2, g_2))$, and are isometric embeddings. The category $\mathfrak{C}^* \ \mathfrak{Alg}$ is made of objects that are C^*-algebras with unit elements. The morphisms $\mathfrak{N}$ are faithful unit preserving $\star$-homomorphisms. Then, as seen in (12), an LCQFT is a functor $\mathcal{Z}$ from $\mathfrak{Man}^D$ to $\mathfrak{C}^* \ \mathfrak{Alg}$. In the sense of a commutative diagram, we have:

$$\begin{array}{ccc} (M,g) & \xrightarrow{\quad \mathfrak{M} \quad} & (M',g') \\ {\scriptstyle \mathcal{Z}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{Z}} \\ \mathcal{Z}(M,g) & \xrightarrow{\mathcal{Z}(\mathfrak{M})} & \mathcal{Z}(M',(M',g')) \end{array}$$

The relevant identity property is that

$$\mathcal{Z}(\mathrm{id}_M) = \mathrm{id}_{\mathcal{Z}(M,g)} \ ,$$

and the composition satisfies $\mathcal{Z}(\mathfrak{M}') \circ \mathcal{Z}(\mathfrak{M}) = \mathcal{Z}(\mathfrak{M}' \circ \mathfrak{M})$.

While the present situation may seem more complicated than necessary, one can adopt the Haag-Kastler setting, where for an open bounded set $\mathcal{U}$ as before, we attribute a C^* algebra $\mathcal{A}(\mathcal{U})$. In taking isotony, we require that for some $\mathcal{V} \subset \mathcal{U}$, $\mathcal{A}(V) \subset \mathcal{A}(\mathcal{U})$. Then, in taking the causal nature of $\mathcal{Z}$, we require that for $\mathcal{U}_1, \mathcal{U}_2 \in \mathfrak{G}(M, g)$, where $\mathfrak{G}(M, g)$ is defined as discussed below, the commutator below vanishes, which follows from isotony:

$$[\mathcal{A}(\mathcal{U}_1),\ \mathcal{A}(\mathcal{U}_2)] = 0 \ . \tag{13}$$

$\mathfrak{G}(M, g)$ is defined as the collection of subsets $\in (M, g)$ such that they (1) are relatively compact and (2) contain for $p, q \in M$ the collection of causal curves $\gamma(p, q)$.

Then, for $M \in \mathfrak{Man}^D$, the split property states that for a state Ψ on $\mathcal{Z}(M, g)$ with the algebras $\mathfrak{R}_{\mathcal{P}}$ and $\mathfrak{R}_{\mathcal{Q}}$, where $\mathcal{P}$ and $\mathcal{Q}$ are regular Cauchy slices, there exists in the GNS representation triple $(\mathcal{H}, \pi, \Omega)$ a type I factor $\mathcal{M}$ satisfying:

$$\mathfrak{R}_{\mathcal{P}} \subset \mathcal{M} \subset \mathfrak{R}_{\mathcal{Q}}{}' \ . \tag{14}$$

This is essentially the split property discussed previously in section 3, but now this holds in a stronger formalism, in that we do not assume that the background spacetime (M, g) is necessarily the Minkowski space (M, η). However, the version we wish to attribute to the strong NTP is a slightly weaker argument than the split property, which is referred to as *intermediate factoriality*. This follows the same lines of the split property and can be stated as follows. Let Ψ be a state on $\mathcal{Z}(M, g)$. Then, for some $\mathcal{U} \in \mathfrak{G}(M, g)$, we define an algebra $\mathcal{M}_\Psi(\mathcal{U})$ attributed to $\mathcal{U}$ w.r.t Ψ. Then, if Ψ satisfies intermediate factoriality, then for each $\mathcal{U} \in \mathfrak{G}(M,g)$, there exists some $\mathcal{V} \in \mathfrak{G}(M,g)$ and a factor $\mathcal{N} = \mathbb{C}\mathbb{1}$ on $\mathcal{H}_{\Psi}^{\mathrm{GNS}}$ such that

$$\mathcal{M}_{\Psi}(\mathcal{U}) \subset \mathcal{N} \subset \mathcal{M}_{\Psi}(\mathcal{V}) . \tag{15}$$

The algebras $\mathcal{M}_{\Psi}(\mathcal{U})$ and $\mathcal{M}_{\Psi}(\mathcal{V})$ become $\mathscr{B}_R$ and $\mathscr{B}_L$, which play the role of the regular Cauchy pair. While there may be subtleties with how the argument proceeds, we expect this to be in parallel with that of the split property argument.

4 Engelhardt-Liu Proposal

While this paper was in preparation, a Strings talk by Netta Engelhardt presented the notion of an "Algebraic ER=EPR" in collaboration with Hong Liu, which resembles the strong NTP principle presented in this paper. The basis of algebraic ER=EPR is that the connectedness of the bulk geometry can be predicted by the operator algebra $\mathfrak{B}^{\mathfrak{X}}$. This connectedness is akin to the notion of the non-factorization of the bulk Hilbert space, which provides the synonymous phrasing of "dependence". Seemingly, type II factors have also been made sense of in the form of "quantum connectedness". The conformal completion aspects are also in agreement with each other; in fact, stronger in the algebraic ER=EPR proposal, where one explicitly works with canonical purification to identify connected and disconnected geometries. By focusing on the initial data $\mathfrak{h}$ in the wedges, one could find the equivalence of the two descriptions of connectedness and dependence. However, we must mention that the approach in this paper is more or less that of complete wedges, in that we do not pick bulk subregions to make sense of dependence, as such a thing is not the objective of the original NTP.

There are several interesting matches between Algebraic ER=EPR and the strong NTP described in this paper. While on the surface level both seem to be synonymous, strong NTP was first motivated from the split property and the emergence of type III_1 from the type I factor $\mathcal{M}$, whereas the way to motivate Algebraic ER=EPR is with the explicit use of purification. Due to this, Algebraic ER=EPR is much more precise in some cases. For instance, since it properly makes sense of purification, the entanglement wedge transitions situation described above is nicely incorporated, and agrees with the speculation in subsection 2.4. One can, in fact, also find the split property aspect discussed in subsection 3 by noting that the dependence of the two regions is found in the type III_1 situation, and this dependence requires that the prescription for the Cauchy slices provided by $\mathfrak{h}$ is continuous across X.

The Engelhardt-Liu paper was arXived recently before the submission of this paper [27].

5 Conclusion

In this paper, we presented and discussed a strong No Transmission Principle, which can be stated as the factorization of the bulk Hilbert space from type I boundary algebras. In the type III_1 setting, the Hilbert space no longer factorizes, and we have a dependence of the bulk duals to the boundary algebras. The starting point of this argument was that the emergence of type III_1 von Neumann algebras can be attributed to the split property, which gives this emergence from a type I factor $\mathcal{M}$. However, the split property is not very concrete in the gravitational regime, and so we instead adopt a slightly weaker condition, the intermediate factoriality argument, which follows in the strong NTP setting similarly to that of the split property. This is not a very strange approach; to this end, the intermediate factoriality condition arises as a result of the split property. We believe that alongside this, in the aforementioned Haag-Kastler scheme, functorial QFTs have a very important role in understanding AdS/CFT, and in a future work, we hope to make clear some aspects of such aspects.

A On Functorial and Algebraic QFT

In section 3.2, we discussed a *functorial* approach to QFT. Here, we will briefly mention why this approach is necessary.

QFTs in the mathematical sense are somewhat worrisome objects. In general we have two nice features in mathematical QFTs: firstly, in the operator algebraic sense, where naturally they are type III_1 and therefore pose an issue with defining a trace, and secondly, in the sense of category theory, where QFTs are defined in terms of functors between some category giving a physical interpretation, and an abstract category, giving a mathematical interpretation. The most famous example of functorial QFTs as mentioned in section 3.2 is that of topological QFTs, where they are functors

$$\mathscr{Z} \,:\, \mathsf{Bord}_n \longrightarrow \mathsf{Vect}_{\mathbb{K}} \,. \tag{16}$$

Another example of a prominent functorial QFT is that of homotopy QFT, where we work with, instead of the bordisms category, the n-cobordisms category, and make sense of $\mathbf{B}$-isomorphisms to define an $(n, \mathbf{B})$ homotopy QFT. However, the correlation we expect is to that of algebraic QFT, where we use the Haag-Kastler scheme used to make sense of regions $\mathcal{U} \in \mathfrak{G}(M, g)$. In this way, we can give an algebra of observables, and define the split property – which is somewhat natural for Minkowski space. However, in the curved spacetime setting, this becomes non-trivial for a very important reason: the notion of an "algebra of observables" becomes ill-defined, and usually requires "dressing" the operators. For this reason, we adopted the duality between the Haag-Kastler system and the locally covariant QFT description, which gives the expected link between C^* algebras and open regions back. The AdS/CFT aspect to all of this is very striking, as mentioned in the main text; since the emergence of type III_1 algebras is expected to be in correlation with the split property, the more appropriately defined split property in the curved background properly encapsulates the relation between the split property and this emergence of type III_1 boundary algebras.

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