Holographic Principle and Quantum Gravity

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Part One and Two from
Bulk Physics, Algebras and All That

January, 2024

Abstract

This is a collection of Part One and Two from the Bulk Physics, Algebras and All That series on vkalvakotamath.github.io. Our emphasis will be on some information theoretic aspects of AdS/CFT and a few key things in de Sitter quantum gravity. We will discuss aspects of operator algebras and how such things correspond to the more physical picture of the duality.

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1 Introduction

Holography is a very fascinating subject. AdS/CFT is arguably one of the most powerful results of string theory, that there exists a form of holography for anti-de Sitter spacetimes arising from type IIB string theory. This was established by Juan Maldacena in 1997 in his celebrated paper, *Large N limit of Superconformal Field Theories*, where he showed AdS/CFT as the duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and an $\mathcal{N} = 4$ super Yang-Mills theory on the boundary. From this, there have spiralled many aspects of AdS/CFT, which have allowed us to make sense of things like entanglement entropy, AdS black holes, canonical purification, relative entropy, and modular theory. AdS/CFT also motivated Strominger's famous 2001 paper on the dS/CFT correspondence, which gives a form of holography for de Sitter spacetimes by putting the CFT on the conformal boundary $\mathcal{I}^{\pm}$. In these two settings, there have been an innumerable count of papers that have given descriptions using operator algebras, modular theory, deformations, and many other interesting tools. The purpose of this thesis[1] is to provide a very pedagogical account of some of these results.

This thesis is primarily based from my series of notes, titled *Bulk Physics, Algebras and All That*, which arise from a larger review, of which a part is now this thesis:

1. Part One: Bulk Reconstruction and Subregions. https://vkalvakotamath.github.io/files/Bulk_Physics__Algebras_and_All_That_Part_One.pdf
2. Part Two: Black Hole Information Paradox. https://vkalvakotamath.github.io/files/Bulk_Physics__Algebras_and_All_That_Part_Two.pdf.

The de Sitter discussion is based on a review that I had written in collaboration with Aayush Verma, titled *Revering Musings on de Sitter and Holography*, which is available at the following URLs:

1. https://vkalvakotamath.github.io/files//dSnote.pdf
2. https://aayushayh.github.io/dSnote.pdf.

The structure of this thesis is as follows. In section 2, we will discuss light-sheets and entropy bounds, where we will discuss how entropy bounds provide the description of singularity theorems, semiclassical quasilocal masses such as the ADM mass and the corresponding Penrose inequality, and a rough idea of holographic entanglement entropy. In section 3, we will discuss some of the foundations of AdS/CFT, and discuss holographic entanglement entropy proposals. We will review Ryu-Takayanagi and the Hubeny-Rangamani-Takayanagi formulas. We then discuss bulk reconstruction and subregion duality, where we will start by discussing the extrapolate dictionary in AdS/CFT and asymptotic bulk reconstruction by Hamilton, Kabat, Lowe and Lyfschitz. We will

[1] A larger account could also be made, although being for the sake of eligibility I did not deem a more thorough and mathematical review important.

then talk about subregion duality, which finds a mathematical construction from modular flows by Faulkner and Lewkowycz and the Jafferis-Lewkowycz-Maldacena-Suh formula for relative entropy. We do not discuss more modular theory things for the sake of brevity. We will finally talk about operator algebras and a recent work by Engelhardt and Liu, and an equivalent work that I had proposed.

In section 4, we will discuss the black hole information paradox, where we will start from the flat space construction with the Page curve, generalized second law, complementarity and the monogamy paradox and firewalls. We then discuss the AdS/CFT counterparts to the BHIP, starting by the Raju-Papadodimas construction of mirror operators and an infalling observer, entanglement wedge reconstruction aspects, ER=EPR paradox and firewalls. In section 5, we will discuss the dS/CFT correspondence, timelike entanglement entropy and pseudo entropy. We will then discuss entanglement entropy in global dS/CFT and in static patch holography, where in the latter we use bit threads as proposed by Susskind and Shaghoulian. Finally, in section 6, we briefly comment on holography in the form of canonical quantum gravity, in the sense of holography of information and Cauchy slice holography.

2 Light-Sheets and Entropy Bounds

Our starting point is that of light-sheets and entropy bounds. In general, we will denote by I a codimension-2 surface on a Cauchy slice $I \subset \Sigma$ in a globally hyperbolic time-orientable $D = 4$ manifold (M, g). Recall that the null expansion θ_C along a null congruence C is defined by making infinitesimal I "lifts" along the congruence, so that the rate of change of the area of I is measured by differentiating with the affine λ. One can attribute null hypersurfaces $K^{\pm}$ and $L^{\pm}$, which denote the null I-orthogonal future and past ingoing and outgoing congruences, with expansions $\theta^{\pm}_{K/L}$ respectively. By marginally trapped surface I, we imply that one of the congruences, say $\theta^{+}_{K} < 0$ and $\theta^{+}_{L} = 0$. By a future (or respectively past) trapped surface I, we imply that both congruences are negative, whereas by an extremal surface, we mean one for which the expansion along both the congruences is zero. When *at least* one of the expansions is negative in the future direction (which will be our general convention, although the same is implied by a negative past null expansion), I is *at least* future marginally trapped, assuming the null energy condition

$$T_{\mu\nu} k^{\mu} k^{\nu} \geq 0 \; , \tag{1}$$

the Raychaudari equation can be used to show that the converging null hypersurface(s) are closed. The null hypersurface bounded by I and the terminating point (which could be a caustic due to dense matter regions) is referred to as a **lightsheet**. These light-sheets are of interest since one can easily see that the domain of dependence of I $D(I)$ and the domain of dependence of the lightsheet $D(\mathcal{L})$ are equal, and timelike curves intersecting I also necessarily intersect $\mathcal{L}$. This also implies that the entropy of I is equal to that measured on the lightsheet due to the domains of dependence being equal. Bousso suggested that this describes a **covariant entropy bound**:

$$S(I) \leq \frac{\text{Area of } I}{4 G_N \hbar} \; , \tag{2}$$

where $\hbar$ will be suppressed more often than not. This bound is of a lot of interest, since this suggests that the holographic principle is very fundamental. Close relations can also be found to the Bekenstein bound, which is a weaker version of the covariant entropy bound (2) and will be discussed in the black hole information problem section.

The purpose of this section is only to give some hints of motivation to using light-sheets and how these are intrinsic to most of the results in AdS/CFT, de Sitter and beyond.

2.1 Singularity Theorems from Entropy*

Singularity theorems have been around from Penrose's 1965 theorem, which was the starting point of the question, *"how can we predict curvature singularities in a spacetime?"* The 1965 theorem tells us that the way of describing these singularities is by finding null geodesic incompleteness; the theorem can be stated as:

Penrose's Theorem: *A globally hyperbolic spacetime (M, g) is null geodesically incomplete if (1) the null energy condition is satisfied for all null vectors, (2) there exists a closed trapped surface, and (3) there exists a non-compact Cauchy surface.*

The proof of this theorem is left as a trivial exercise to the reader. This theorem can be further improvised into removing the global hyperbolicity condition by instead replacing it with a slightly weaker causal condition, the *chronology condition*, which prevents the existence of closed timelike curves. For the Hawking-Penrose theorem, we use the generic energy condition, so that there exists (*at least*) one of the following three:

1. A closed trapped surface,
2. An edgeless compact achronal set,
3. A point $p \in M$ for which null geodesics reconverge.

Again, as before, the proof is left to the reader as a nice exercise. Refer to Hawking and Ellis, 1973 for a detailed explanation of the proof. Now, one could ask if these singularity theorems could be made closer-to-physicality by explicitly dealing with the way that the matter curves affect the expansions. The works of Wall, Bousso and Moghaddam show that this can be made very precise.

The point of starting is the semiclassical version of the Bousso bound, which is obtained by replacing the area term in the expansion with the generalized entropy, which has semiclassical exterior corrections to the area term:

$$S_{gen}(I, \Sigma) = \frac{\text{Area of } I}{4G_N} + S_{ext}(I, \Sigma) \ . \tag{3}$$

In this way, the **quantum expansion** is defined by making infinitesimal lifts along C and measuring the rate of change of $S_{gen}(I, \Sigma)$:

$$\Theta = \lim_{A \to 0} = \frac{4G_N \hbar}{A} \frac{\mathrm{d}S_{gen}(I, \Sigma)}{d\lambda} \ . \tag{4}$$

One could now make use of the generalized second law (GSL), which requires the generalized entropy $S_{gen}(I, \Sigma)^\lambda$ defined at points along the null generator to be monotonically non-decreasing. One can easily see that the semiclassical condition of $\delta S_{gen}(I, \Sigma)^\lambda \leq 0$ (i.e. the quantum expansion is non-positive) is necessarily implied when the classical expansion is non-positive. This is because the exterior entropy can never dominate over the area term due to the order of $\hbar$ contributions. Therefore, the classical Penrose theorem can be extended into the semiclassical limit by requiring the GSL to be true. In this way, we arrive at Wall's singularity theorem:

Wall's theorem: *A globally hyperbolic spacetime (M, g) is null geodesically incomplete if the null energy condition is satisfied for all null vectors and there exists a quantum trapped surface in M.*

This is proved using the same approach as in the classical Penrose theorem, where we require the compactness of light-sheets to show that there exists an incomplete null generator.

The Bousso bound in itself implies incomplete null generators when the entropy $S(I)$ violates the Bousso bound when the light-sheets are compact. We will now discuss some definitions and propositions that will help us in reviewing this theorem and discuss the implications of the proof [1].

A spacetime (M, g) is said to be globally hyperbolic if $J^+(p) \cap J^-(q)$ is compact for $p, q \in M$, and if there exists a Cauchy surface Σ in (M, g). We will assume throughout this paper that (M, g) is globally hyperbolic. To indicate the boundary of a surface I in Σ, we use ∂I, and to indicate the boundary of a surface in (M, g), we will use $\dot{I}$. By a boundary $\partial\sigma$ we mean a compact Cauchy splitting codimension 2 submanifold for a surface σ that defines an interior and exterior for the Cauchy slice Σ and $\sigma - \partial\sigma \neq \varphi$. For $\sigma = I$, we would mean a surface for which at least the future ingoing congruence of the principle null $I-$orthogonal directions has a negative expansion. By the future (resp. past) domain of dependence of a set $A \subset M$, we mean the set of points $p \in M$ such that every past (resp. future) inextendible timelike curve γ passing through p also necessarily intersects A. The domain of dependence $D(A) = D^+(A) \cup D^-(A)$. We also have the following lemma, which will be useful in making sense of the exiting of null generators from a lightsheet:

Future boundary points: *If a point $b \in M$ is on the future boundary of a surface I, then the following conditions must hold:*

1. *there are no intersecting null geodesics before b,*
2. *b lies on a null orthogonal geodesic γ emanating from ∂I, and*
3. *there are no conjugate points on γ before b.*

Let X denote the complement of I in Σ. Then, $D^+(I) = D^+(\Sigma) - \mathcal{I}^+(X) - X$. Further, $D(I) = M - \mathcal{I}(X)$. This can be proved as follows. Let a point $p \in M$ be $p \in D^+(\Sigma) - \mathcal{I}^+(X) - X$. Then, a past inextendible timelike curve passing through p would necessarily be in $D^+(\Sigma)$, while the subtracted $\mathcal{I}^+(X) - X$ would mean that such a curve would not intersect X. Due to this, such curves would intersect I, and therefore $p \in D^+(I)$. Similarly, the same can be said for the past domain of dependence. Due to this, we can state that the domain of dependence of I is $M - \mathcal{I}(X)$, proving the result. Additionally, X has the property that $\dot{\mathcal{I}}^+(X) - X = \dot{\mathcal{I}}^+(\partial X) - \mathcal{I}^+(X) - X$. Using this, we have an interesting result, which can be stated as follows:

Termination of null congruences: *$\dot{\mathcal{I}}^+(X) - X$ is the future outgoing null congruence from I and terminates at either a caustic or at a point where neighbouring null geodesics intersect.*

This can be seen from the nature of the generators, where naturally the null geodesics that compose the future outgoing congruence do not enter $\mathcal{I}^+(X)$ (i.e. exist the future boundary of $K^+(I)$) unless we encounter a caustic or self-intersection. Due to this and the previous result, we can state that the set $\dot{\mathcal{I}}^+(X) - X$ is generated by the future null orthogonal geodesics originating from ∂I.

We are now in a position to state the singularity theorem on classical covariant entropy bound violation by Bousso and Shahbazi-Moghaddam [1]:

Bousso-Moghaddam: *A globally hyperbolic spacetime (M, g) is null geodesically incomplete if the null energy condition is satisfied and there exists a hyperentropic surface I, i.e. a surface for which the covariant entropy bound is violated.*

One can also extend this into semiclassical limit, where we can use the fact that $\theta \leq 0$ implies $\Theta \leq 0$ and work similarly to that of Wall's theorem [2] to show that there must be at least one incomplete null generator for some lightsheet $\mathcal{L}$. I had proposed this in [3]:

Semiclassical hyperentropic theorem: *Let a surface I have a boundary ∂I that is a compact codimension 2 submanifold with splitting, and let $\Theta_l^+ < 0$. If the surface is hyperentropic, there exists at least one null incomplete generator.*

We will now discuss another important aspect of light-sheets, which has to do with describing a semiclassical version of the Penrose inequality.

2.2 Semiclassical Quasilocal Mass Inequalities*

The Penrose inequality describes the relation between the ADM mass of a spacetime with the area of a minimal marginally trapped surface. Let Σ_0 be a closed spatial surface in a vacuum spacetime (M, g), and let $C^-_{\Sigma_0} \to \underline{C}$ be a hypersurface containing past-directed null geodesics through Σ_0. The ADM version of the inequality, which is the general reference when describing the Penrose inequality is that the ADM mass has a lower bound from $|\Sigma_0|$ similar to (**??**):

$$m_{ADM} \geq \sqrt{\frac{|\Sigma_0|}{16\pi}}, \tag{5}$$

which is measured at the spatial infinity i^0. The null Penrose inequality has an interesting relation to the Hawking mass, namely that the nature of the null hypersurface $\underline{C}$ allows one to make a statement about the Hawking mass acting as a lower bound, defining the *Bondi energy*. This would seem to follow immediately from a *null Penrose inequality*, which would be the result that given a regular $\underline{C}$, the Bondi energy has a bound from $|\Sigma_0|$ by fixing a reference frame γ^∞ at $\mathcal{I}^-$:

$$E_B^{\gamma^\infty}(\underline{C}) \geq \sqrt{\frac{|\Sigma_n|}{16\pi}}, \tag{6}$$

where Σ_n are spherical slices of $\underline{C}$. Here we define the null frame $(l, \underline{l})$, where l and $\underline{l}$ are null vectors normal to the surface Σ satisfying $g(l, \underline{l}) = 2$, with the respective past analog definitions. The implication from the Hawking mass is straightforward, which is that given a Σ_0, the following quasi-local mass definition applies a lower bound on the Bondi mass:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma_0|}{16\pi}} \left(\frac{\chi(\Sigma)}{2} - \frac{1}{16\pi} \int_\Sigma \operatorname{tr}\chi \operatorname{tr}\underline{\chi} \, \mathrm{dvol}(\mathbf{g}) \right), \tag{7}$$

where the integral is defined in terms of g on Σ and $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ are the null expansions defined w.r.t the second fundamental forms. $\chi(\Sigma) = 2$, and setting integral to zero (since Σ is marginally trapped and therefore either $\mathrm{tr}\chi = 0$ or $\mathrm{tr}\underline{\chi} = 0$), one finds that this becomes equal to $\sqrt{\frac{|\Sigma_0|}{16\pi}}$. Then, the bound on the Bondi mass is given by:

$$m_H(\Sigma_0) \leq m_B(\underline{C}), \tag{8}$$

where the Hawking mass is monotonically non-decreasing at $\mathcal{I}^-$.

However, taking into account of semiclassical corrections, it is easy to violate this bound. One could take an infalling null shell and allow the black hole to evaporate, due to which In the quantum regime we consider a quantum-corrected area term in place of the classical area, which is in terms of the generalized entropy defined for the surface. For a surface Σ, the following substitution takes place:

$$A(\Sigma) \rightarrow A(\Sigma)_Q = 4G\hbar S_{gen}(\Sigma). \tag{9}$$

In the paper [4], it was proposed that a *quantum Penrose inequality* holds:

$$m_{ADM}(\Sigma) \geq \sqrt{\frac{\hbar S_{gen}(L_\Sigma^+)}{4\pi}}. \tag{10}$$

One can then also find the relation to the Bondi mass, although this has some slightly non-trivial implications and is an open problem. This is essentially (5) with the correction given by (9), although the calculation is no longer in terms of some spatial hypersurface $C_\Sigma^{i^0}$, but instead on a null hypersurface L_Σ^+ that lies inside the horizon. This is to account for the fact that the lower bound may be violated when there is *soft matter*, for which the mass contribution may be arbitrarily low while the entropy contribution may not be necessarily at the same cost. Refer to figure 1 for the identification of the hypersurfaces in the conformal diagram of the spacetime.

It must be mentioned here that the notion of a "Penrose inequality" in general applies to any definition of a quasilocal mass. The general thought when referring to the Penrose inequality is that of the ADM mass, whereas one can also work with the Bondi mass, Bartnik mass, etc. There is an AdS/CFT version of the semiclassical Bartnik mass inequality, which was proposed to be in equivalence with the notion of outer entropy in a sense. We will not comment on this here.

2.3 Holographic Entanglement Entropy

To start with something that will be used in the AdS/CFT discussion, we will discuss the light-sheets aspect of holographic entanglement entropy. The firsthand motivation of approaching holographic entanglement entropy is from that of the Ryu-Takayanagi formula, which states that the entropy of a boundary subregion A is proportional to the area of a unique minimal surface γ_A:

$$S(A) \sim \text{Area of } \gamma_A \; . \tag{11}$$

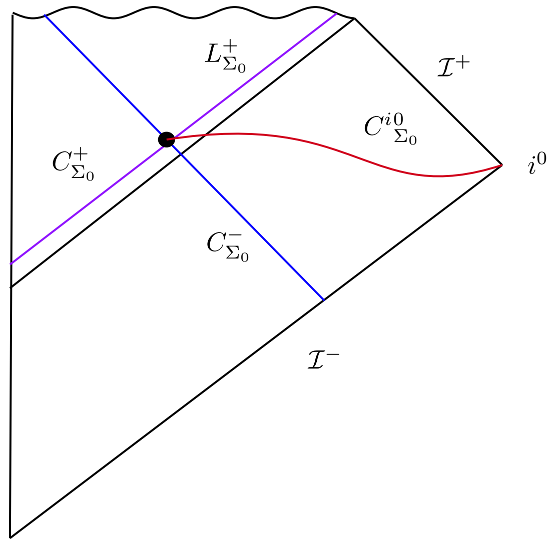

Figure 1: The Penrose diagram for the spacetime identifying the null hypersurfaces $C^{\pm}_{\Sigma_0}$ and the spatial hypersurface $C^{i^0}_{\Sigma_0}$. $L^+_{\Sigma_0}$ lies fully inside the horizon, and therefore faces no interference from matter away from the black hole.

The way to make sense of this is to see that in AdS/CFT, we work with dimensionality

$$\text{AdS}_D/\text{CFT}_{D-1} \ , \tag{12}$$

and the way of arriving at the holographic entanglement entropy is by using the Bekenstein-Hawking formula to the surface γ_A.

In the covariant holographic entanglement entropy proposal given by Hubeny, Rangamani and Takayanagi, adding up to the use of the minimal surface, we make use of the covariant entropy bound. This is done by taking the causal diamond formed by taking the union

$$\Xi(A) = \mathcal{J}^+(A) \cup \mathcal{J}^-(A) \ ,$$

and then by taking the boundaries $\partial^+\Xi(A) \cup \partial^-\Xi(A)$. These essentially allow one to enclose the light-sheets of the surface, so as to obtain the minimal surface that is obtained from the light-sheet construction. In other words, we want the surface corresponding to the boundary entanglement entropy to have the minimal area corresponding to light-sheets extended to the boundary, which give us ∂L^+ and ∂L^-, and the minimum area surface in this construction is the required surface. Through this, the entanglement entropy of a boundary subregion can be computed by the area of a minimal extremal surface (i.e. a surface that is extremal under perturbations):

$$S(A) = \frac{\text{Area of } \mathcal{X}_{HRT}}{4G_N} \ . \tag{13}$$

In this way, light-sheets play a key role in determining the entanglement entropy in AdS/CFT. These proposals paved way towards working with coarse-graining and canonical purification, where we construct the gravity dual via $\mathcal{CPT}$ reflecting around the extremal surface. Here, light-sheets become intrinsic to the construction, where we use the quantum null energy condition (QNEC) to find bounds and work with minimar surfaces. For instance, one can prove in the classical limit that the area of the extremal surface corresponding to some outer wedge always bounds the area of minimar surfaces from below:

$$\text{Area of } \mathcal{X}_{HRT} \leq \text{Area of } \sigma_i \ , \tag{14}$$

where one describes the family of minimar surfaces σ_i typically through a holographic screen. In the semiclassical limit, one has to add in bulk corrections, but a similar relation would hold. In general, the HRT proposal along with light-sheet constructions and some interesting tools, particularly those involving modular theory allow us to better understand how AdS/CFT describes information theory. In the next section, we will discuss some aspects along these lines, although we do not discuss Tomita-Takesaki theory and other operator algebraic results, such as the gravity dual of the Connes cocycle flow.

3 Information Theoretics of AdS/CFT

There are two key reasons to believe that states Φ in the bulk have *some* relation to the boundary:

Localization of information: The bulk theory is a model (M, g) that satisfies the field equations with some additional settings (i.e. conditions on $T_{\mu\nu}$ and Λ), and is the *gravitating* part of the duality. If one associates to this bulk region some boundary ∂M (we still do not know *what* boundary this is!), one can ask, at least knowing about AdS/CFT, if there could be a time-band from $t = 0$ to $t = \epsilon$ of the bulk (i.e. a collection of Cauchy slices in a globally hyperbolic spacetime) such that the information inside this time-band is attributed to the boundary, or in some sense *localized* to the boundary. This is referred to as *holography of information*, where the states inside the bulk have some information associated to the boundary of the time-band. In principle, this can be made sense of by saying that states in the bulk correspond with operators in the boundary. This is not a precise statement, as we shall see below.

Boundaries exist: At least in a not-so-chronological way as we wished, we can see that infinities can "store" information. In flat space, this can be interpreted as follows: information about massless excitations are stored in an infinitesimal neighbourhood of $\partial^- \mathcal{I}^+$ as showed by [5]. Following these lines, the point of holography, like stated above, is that the information about the bulk states are stored on the boundary, and this boundary is meant to be a "suitable" one. One cannot say that these massless excitations are encoded on to the spacetial infinity i^0, for obvious causal reasons. Therefore, when talking about holography, we turn to causal physics to make sure that the information being encoded on a boundary *truly* describes a physical theory. Of course, this is not as "obvious" as I made it sound. In AdS, one can appeal to the nature of $\mathcal{N} = 4$ SYM and find AdS/CFT as Maldacena did, although this in itself is not a very straightforward idea. What was suggested by Maldacena is to take the Lagrangian in $D = 10$[2]

$$\mathcal{L} = \frac{1}{g_{\text{YM}}^2} \left(\text{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) + i \text{Tr} \left(\bar{\psi} \gamma^\mu D_\mu \psi \right) \right) \ , \tag{15}$$

and in the same dimensionality, we find the type IIB SUGRA "bulk" to be $\text{AdS}_5 \times S^5$. Both have similar symmetry groups, and this appeals to the initial statement of AdS/CFT: *that* $\text{AdS}_5 \times S^5$ *bulk is dual to* $\mathcal{N} = 4$ SYM *on the boundary.* Eventually, this statement was diluted from this really complicated string-theoretic statement to the more less-formidable statement, that AdS in D dimensions is dual to a CFT on the boundary, without making any explicit statements about SUGRA.

However, this clearly does not seem to be very "obvious" in regards to causal physics. Strikingly, this in itself is the distinction we made sense of in the introduction. AdS has a rather convenient boundary at i^0, but in the case of flat space, one has to understand where the excitations propagate to predict holography. In de Sitter, one uses the null

[2]Due to supersymmetric reasons on both boundary and bulk sides.

infinities $\mathcal{I}^{\pm}$, but this is somewhat complicated. As opposed to the naive expectation, there aren't two boundary duals to the bulk dS_D – instead, looking at the behaviour of null rays propagating to antipodal points on the planar slicing of de Sitter, one can see that the CFT lives on only one sphere. We will discuss this later.

In this way, one can define holography in the sense of state-operator correspondence between the bulk and boundary as the following dictionary for operator insertions in the n-point functions on the bulk and boundary side:

$$\langle \Phi(x_1) \dots \Phi(x_n) \rangle_{\text{bulk}} \sim \langle \mathcal{O}_\Phi(x_1) \dots \mathcal{O}_\Phi(x_n) \rangle_{\text{CFT}} \ . \tag{16}$$

Operators $\mathcal{O}_\Phi$ are dual to the bulk states Φ by a conformal weight Δ. In AdS/CFT, these are real-valued, whereas in something like global dS/CFT, these can take complex values, due to which one finds non-unitary CFTs. Due to this, things like entanglement entropy become rather complex (-valued), but this will be discussed in detail later.

3.1 Ryu-Takayanagi

The Ryu-Takayanagi formula tells us that the entanglement entropy of a boundary subregion R is given by the area of a spacelike surface (called the RT surface) connecting the boundary points of the boundary subregion. This can be safely assumed to be equal to 2, which would simplify the overall look of the formula. This way, the entanglement entropy of R would look something like

$$S(R) = \frac{c}{3} \log \frac{\gamma_R}{\epsilon} \ , \tag{17}$$

where ϵ is a UV cutoff. That is, we are working in the lattice limit, so that the entanglement entropy is defined properly. If we let $\epsilon \to 0$, we get a divergent entanglement entropy, which is consistent with the usual expectation of type III QFTs, where there is no "good" definition of entanglement entropy. Then, the work of Ryu and Takayanagi showed that this can be described by the area of the spacelike surface connecting the points (minimal when more than one):

$$S(R) = \frac{\text{Area of } \gamma_R}{4G\hbar} \ . \tag{18}$$

The RT formula can be used to obtain the Bekenstein-Hawking entropy of an AdS black hole. Fixing a finite temperature T and taking the $\mathcal{N} = 4$ SYM theory dual to an AdS black hole, so that the resulting entanglement entropy can be found to be the Bekenstein-Hawking entropy of the 5 black brane. The RT surface for this wraps the horizon, and gives the above result. In general, the RT formula pioneered most of the present landscape of AdS/CFT, extending to, arguably most famously, the use of RT surfaces in entanglement wedge reconstruction and the black hole information problem by Pennington [6].

One can also give a covariant description of holographic entanglement entropy using light-sheets, which was done by Hubeny, Rangamani and Takayanagi following Ryu-Takayanagi, which we will discuss next.

3.2 Hubeny-Rangamani-Takayanagi

As seen above, Ryu and Takayanagi's introduction of holographic entanglement entropy states that the entanglement entropy of a boundary subregion R is given by the area of a minimal spacelike geodesic joining the two distinct boundary points A and B defining the boundary R, which is beautiful and important enough to be re-iterating:

$$S(R) = \frac{\text{Area of } \gamma_R}{4G\hbar} , \tag{19}$$

where keep in mind that there is a UV cutoff ϵ so that the entanglement entropy is defined with lattice spacing. (As we shall discuss later, the continuum limit gives a divergent entanglement entropy since the von Neumann algebra is of type III.) This also paves way to introducing the notion of the *entanglement wedge*, which, as we shall see later, is the bulk subregion that can be fully reconstructed in terms of the corresponding bulk subregion. The entanglement wedge is defined as the domain of dependence of the homologous surface, so that

$$\mathcal{E}_W(R) = D(R \cup \gamma_R) + \Sigma(\gamma_R) , \tag{20}$$

where we explicitly denote $\Sigma(\gamma_R)$ to bear in mind that the homologous condition of the maximin prescription is encapsulated. That is, for the boundary subregion R, we find that the entanglement entropy is in general something like

$$S(R) = \max_{\Sigma} \min_{\gamma_R^i \sim R} \text{Area of } \gamma_R , \tag{21}$$

where we maximize over the Cauchy slices corresponding to the wedge and minimize over the surfaces γ_R^i that are homologous (denoted by $\sim$) to R. Using this maximin prescription, one can identify a covariant holographic entanglement entropy description, referred to as the Hubeny-Rangamani-Takayanagi (HRT) prescription:

$$S(R) = \frac{\text{Area of } \mathcal{X}_R}{4G\hbar} , \tag{22}$$

where $\mathcal{X}_R$ is an extremal surface in the bulk, so that the prescription is now something like

$$\text{Maximin} \leftrightarrow \text{Maxi-min-ext} , \tag{23}$$

where we maximise over the Cauchy slice, and find the minimal area extremal surface. In (22), $\mathcal{X}_R$ is the HRT surface corresponding to the boundary subregion R. The entanglement wedge is more appropriately defined in this case as

$$\mathcal{E}_W(R) = D^+(\gamma_R) \cup D^-(\gamma_R) \cup \Sigma(\gamma_R) . \tag{24}$$

It is obvious that there must be very striking applications to the issue of bulk reconstruction and subregion-subregion duality, but this is a discussion for the next section. For now, we will turn our attention to the nature of black holes, and the coarse-graining of such holographic black holes in the sense of Engelhardt-Wall geometry. This is a prescription to make sense of outer entropy and the relation of the von Neumann entropy of suitable marginally trapped surfaces (called *minimar* surfaces) to HRT.

3.3 Bulk Reconstruction and Subregions

The RT formula above dictates that the entanglement entropy of a boundary subregion A is given by the area of a minimal spacelike geodesic joining the boundary endpoints:

$$S(A) = \frac{c}{3}\log\frac{\gamma_A}{\epsilon} \equiv \frac{\text{Area of } \gamma_A}{4G_N\hbar} \ .$$

This was seen to extend into the HRT prescription, where the area of a bulk (minimal when more than one) extremal surface gives the entanglement entropy. The notion of $\mathcal{E}_W(A)$ in both the cases is still the same – although in the HRT prescription, it becomes more apparent as to what coarse-graining and other problems look like. From the FLM corrections, we can expect what bulk corrections would look like. For HRT, in the sense of quantum extremal surfaces, this becomes the generalized entropy

$$S_{gen} = \frac{\text{Area of } \mathcal{X}_{HRT}}{4G_N} + S_{bulk} + \dots \ . \tag{25}$$

This way, one could say that the entanglement entropy of A becomes something like the following:

$$S(A) = S_{gen}(\mathcal{X}_{\mathcal{HRT}}) \ . \tag{26}$$

This is at all orders in $\hbar$ – that is, sitting in perturbative quantum gravity, we have the following expansions in $g_{\mu\nu}$ for orders in $\hbar$[3]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1/2)}_{\mu\nu} + g^{(1)}_{\mu\nu} + \dots \ . \tag{27}$$

It is not hard to notice that, the $\mathcal{X}_{HRT}$ surface lies deeper in the bulk than the causal wedge $\mathcal{C}_W(A)$. Indeed, recall that the causal wedge is union of the future and past domains of dependence – that is, $\mathcal{C}_W(A) = I^+(D^+(A)) \cup I^-(D^-(A))$. In fact, it can be shown that $\mathcal{X}_{HRT}$ *always* lies deeper in the bulk than $\mathcal{C}_W(A)$. This is a very subtle yet important result – if one picks the entanglement wedge w.r.t this surface, clearly the picture of understanding bulk operators becomes affected. Since $\mathcal{E}_W(A)$ lies deeper in the bulk than $\mathcal{C}_W(A)$, this is a larger bulk wedge in which we can ask the duality question – "*given a bulk operator, what is the dual CFT operator?*"

Let me emphasise that the question of HRT in itself is a much larger set than you can expect. For instance, let us ask if the area of a suitable marginally trapped surface is related to the von Neumann entropy in the bulk-boundary prescription. The following definition will be useful.

Minimar surfaces: *A surface is said to be a minimar surface if it is a marginally trapped surface and is homologous to the boundary subregion. Additionally, this homologous surface is also a weaker kind of a minimal HRT surface when extremal.*

[3] That is, contributions from graviton fluctuation expansions.

At this point we can make sense of where we are working in terms of the coarse-graining. It should be obvious that the conformal completion of the spacetime has to be taken after coarse-graining, so as to find the complete coarse-grained spacetime. This could be done by acting on the expansion and other relevant objects (such as the trace of the extrinsic curvature $\mathcal{K}$) with the $\mathcal{CPT}$ conjugation, which would generate the completion (M', g') to the coarse-grained (M, g) wedge. The main result in EW's paper is that the outer entropy for a minimar surface μ is given by a bound in terms of a corresponding HRT surface whose area coincides with μ, implying that

$$S^{outer}(\mu) = \frac{\text{Area of } \mu}{4G_N\hbar} \; . \tag{28}$$

In order to make sense of this, we have to keep in mind that to patch μ to the outer wedge $O_W(\mu)$, we have to consider the junction conditions for the initial data across $O_W(\mu)$. This is a somewhat tedious thing to do, but can be done nonetheless. The final construction leading to the core relation between μ and $\mathcal{X}_{HRT}$ is to make use of *representatives*, which are defined for minimar surfaces as $\bar{\mu}(\Sigma) = \mathcal{N}^{\pm}(\mu) \cup \Sigma$, where $\mathcal{N}^{\pm}$ are null congruences in the orthogonal outward null directions $k^{\mu}_{\pm}$. From the null energy condition (referred to as the null *convergence* condition in the paper), the area of the representative is bounded to μ as Area of $\bar{\mu} \leq$ Area of μ. By definition, outer entropy is the *maximized* entropy corresponding to some ρ attributed to μ, and we get

$$S^{outer}(\mu) \leq \frac{\text{Area of } \mu}{4G_N\hbar} \; . \tag{29}$$

Eventually, one can show that a unique extremal surface exists that defines the HRT surface, and by $\mathcal{CPT}$, one obtains the complete spacetime from the auxiliary spacetime (M', g') identified to match the extremal surfaces. This gies us the equality as a saturation of (29).

Now, to come back to the topic, keep in mind that the results in the above case is linked to that of reconstruction – of course, in the above discussion there was no need to worry about bulk-boundary operator dictionary. However, the general idea is that when one has a particular $\mathcal{E}_W(A)$, it must also be possible to reconstruct operators in the bulk given an understanding of the dual CFT operators. While the overall discussion did not assume an understanding of bulk reconstruction (or in fact any operator dictionary), the question as to "*can you identify the operators in* $\mathcal{E}_W(A)$?" still remains. In fact, one can motivate this from RT itself, without appealing to HRT! However, given that HRT in general is associated with the bulk more intrinsically than RT, we take it that the HRT prescription also requires us to understand bulk reconstruction and subregion duality in general.

3.3.1 Extrapolate Dictionary

Now we will take a parallel but slightly distinct route to bulk reconstruction. Recall that the general idea of AdS/CFT is that operators in the bulk are dual to operators in the CFT by a conformal weight Δ. The **extrapolate dictionary** is

$$\lim_{r\to\infty} r^{\Delta}\Phi(r,x) = \mathcal{O}_{\Phi}(x) \ , \tag{30}$$

which is to say that taking bulk operator insertions at the boundary gives us some dual CFT operator $\mathcal{O}_{\Phi}$. Then, one can also use the extrapolate dictionary in the sense of n-point functions by (30):

$$\lim_{r\to\infty} r^{n\Delta}\langle\Phi(r_1,x_1)\ldots\Phi(r_n,x_n)\rangle = \langle\mathcal{O}_{\Phi}(x_1)\ldots\mathcal{O}_{\Phi}(x_n)\rangle_{\text{CFT}} \ . \tag{31}$$

Keep in mind that the bulk operator insertions are *necessarily* at the near-boundary limit for the extrapolate dictionary. However, for the moment this will suffice. The Hamilton-Kabat-Lyfschytz-Lowe (HKLL) proposal from 2006 states that this dictionary can be used to reconstruct bulk operators by taking a bulk field Φ inserted at some $\mathbf{Y}$, and then taking the double-lightcone of this. Then, the spacelike patch that is bounded by the double-lightcone, say $\mathbf{S}$ contains a *smearing* function $K(\mathbf{Y}|x)$ – this Green's function has a support in $\mathbf{S}$, and $\Phi(\mathbf{Y})$ can be expressed as:

$$\Phi(\mathbf{Y}) \int d^D x\ f_{\Delta}(\mathbf{Y}|x)\mathcal{O}_{\Phi}(x) + \ldots \mathrm{O}(1/n) \ . \tag{32}$$

The interesting thing is that at leading order in $1/N$, the bulk locality can be expressed in terms of factorization in large N limit. The function $K(\mathbf{Y}|x)$ can be found by inverting (32). In this way, one comes back towards the question of reconstruction. Initially, like stated in the previous subsection, the problem seemed to be for $\mathcal{C}_W(A)$, which in itself is only a part of the bulk that can be reconstructed from CFT operators. However, after HRT, the picture was found to be incomplete, being a part of the entanglement wedge $\mathcal{E}_W(A)$. So, to make sense of these things, some notion of reconstruction must be first motivated – so that the notion of subregion-subregion duality can be found, either mathematically or physically. Of course, the latter is much more intuitive than the former, as we shall see soon. As of now, we wil shift our focus from purely talking about bulk reconstruction to a quick understanding of Quantum Error Correction (QEC).

3.3.2 HKLL-Reconstruction

In AdS/CFT, we can expect even naively that there is a duality between correlators in the bulk and correlators in the boundary corresponding to some n-point insertions, as detailed in (16). Suppose we start from the case of a free scalar field with the usual equations of motion,

$$\left(\Box - m^2\right)\Phi = 0 \ . \tag{33}$$

For this, we want to find the dual CFT operator in the boundary so that we can find the explicit dictionary for the correspondence. It is trivial to start from the general solutions to (33), which go as

$$f_{\omega lm}(r,t,\Omega) \sim e^{-i\omega t}\mathcal{S}_{lm}(\Omega) \ , \tag{34}$$

where $\mathcal{S}(\Omega)$ are spherical harmonics. Then, the soutions go like $r^{-\delta}$, where $\delta = \Delta,\ D = \Delta$. This becomes the scaling dimension,

$$\Delta = \frac{D}{2} + \frac{1}{2}\sqrt{D^2 + 4m^2}\ , \tag{35}$$

where note that $\Delta \in \mathbb{R}$. The correspondence between a bulk field Φ and its dual CFT operator $\mathcal{O}_\Phi$ is by Δ, and the real-valued nature of Δ indicates a unitary CFT. On the contrary, in de Sitter holography, this becomes complex-valued for certain values of m and the de Sitter length scale l_{dS}, due to which a non-unitary CFT is attributed. This gives entanglement entropy a complex-valued nature, which in turn complicates the geometric features associated to finding holographic entanglement entropy in global dS/CFT. However, more on this later.

Recall the extrapolate dictionary,

$$\lim_{r\to\infty} r^\Delta \Phi(x, r) = \mathcal{O}_\Phi(x)\ . \tag{36}$$

If one applies this dictionary to the case of the massive scalar field, the two-point function[4] takes the form of

$$\langle \mathcal{T}\Phi(x)\Phi(x')\rangle = \frac{\mathbf{y}^\Delta}{2^{2\Delta}\pi^{D/2}(2\Delta - D)}\frac{\Gamma(\Delta)}{\Gamma\left(\Delta - \frac{D}{2}\right)}\mathcal{F}\left(\Delta, \Delta + \frac{1-D}{2}, 2\Delta - D + 1, \mathbf{y}\right)\ , \tag{37}$$

where $\mathbf{y}$ is defined as $\cosh(l)^{-1}$. From the extrapolate dictionary, we will find that the two-point function in the boundary side becomes something like

$$\langle \mathcal{T}\mathcal{O}(t,\theta)\mathcal{O}(t',0)\rangle \propto \left(\frac{1}{\cos((t-t')(1-i\epsilon)) - \cos\theta}\right)^\Delta\ , \tag{38}$$

where we recognise the two-point correlator for the CFT. With the dictionary (36), we can rewrite the duality in terms of correlators (16) as:

$$\lim_{r\to\infty} r^\Delta \langle \Phi(r_1, x_1)\ldots\Phi(r_n, x_n)\rangle = \langle \mathcal{O}_\Phi(x_1)\ldots\mathcal{O}_\Phi(x_n)\rangle\ . \tag{39}$$

Let us try to understand the expansion of these correlators in detail.

3.3.3 Subregion Duality

As stated previously, at leading order in $1/N$, bulk locality can be expressed in terms of large N factorization. The entanglement wedge reconstruction aspect has a nice aspect in

[4] Usually, the two-point function suffices to work with, since higher-point functions boil down to a set of two-point functions.

the discussion, since we can cover it in AdS-Rindler coordinates and describe an HKLL-type reconstruction. We get something like the following, taking a vacuum state in CFT:

$$\Phi(\mathbf{Y}, \mathcal{E}_W(A)) = \int_{\mathcal{D}(A)} dx \; f_{\Delta}^{Rindler}(\mathbf{Y}|x)\mathcal{O}_{\Phi}(x) \;, \tag{40}$$

where $\mathcal{D}(A)$ is the double-lightcone system.

We can now define a **modular flow** based description of (40) by describing a modular flow on basis of ρ_A (dropping the Φ subscripts on all $\mathcal{O}$'s)

$$\mathcal{O}_s(x_A) = \rho_A^{-is/2\pi}\mathcal{O}(x_A)e^{is/2\pi} \;, \tag{41}$$

for which equation (40) becomes modified to include $\mathcal{O}_s$ in (41). To motivate what will be the case in the JLMS prescription, we will discuss aspects of the modular Hamiltonian as well. Define the **modular Hamiltonian** K_A as:

$$K_A = \frac{\mathbf{A}(\partial a)}{4G_N} + K_{bulk,\, a} + \mathrm{O}(G_N) \;, \tag{42}$$

where $K_\rho = -\log\rho$ (we will revisit this later as well). (42) can also be seen to motivate the equivalence between the commutators of K_{bdy} and K_{bulk} with bulk fields, which is essentially the motivation to the JLMS formula, which a little more explicitly takes into consideration of the *relative entropy*.

As mentioned in the earlier discussion on HKLL reconstruction, one can take the extrapolate dictionary (30) and the invert construction (32) for a bulk-boundary correlator, to find the function $K(\mathbf{Y}|x)$. In this way, this distribution can be formally understood. To see how this all adds up, one can take the above mentioned equivalence of the modular Hamiltonian on both sides with the $1/N$ corrections:

$$[\Phi(\mathbf{Y}_A), K_{A,\, bdy}] = [\Phi(\mathbf{Y}_A), K_{A,\, bulk}] + \mathrm{O}(1/N\;,) \tag{43}$$

which gives the bulk modular flow that Faulkner and Lewkowycz found:

$$e^{iK_A s}\Phi(\mathbf{Y})e^{iK_A s} = e^{iK_{A,\, bulk}s}\Phi(\mathbf{Y}_A)e^{iK_{A,\, bulk}s} \;. \tag{44}$$

This has some subtleties with the exponential of the modular flow and modular Hamiltonian, which can be put aside in the present discussion in the sense of low energy theory. One can then explicitly make sense of this modular flow and the exponentiated JLMS formula, although a very formal discussion on this has not been provided in this paper (see section 2.6 for an informal set of remarks on this).

Finally, we will discuss the notion of relative entropy and why it is natural to find relative entropy in AdS/CFT. As we shall discuss, relative entropy in a sense is more fundamental than the usual entanglement entropy computed by the analytic continuation $n \to 1$ of Renyi entropy S^n due to the type III nature of general local quantum field theories. We will discuss relative entropy in Araki's approach in terms of Tomita-Takesaki theory.

One of the most direct implications of AdS/CFT is that of "subregion-subregion duality", where a boundary CFT subregion corresponds to some bulk subregion. From bulk reconstruction, one can suspect that such a description holds. Bulk locality is a very interesting thing. One usually attributes this to the notion of *emergence* of the bulk in AdS/CFT, although the mathematical aspects of these usually has to do with a very nice aspect of operator algebras associated to the boundary CFT. In general, the idea of the type of von Neumann algebra on the bulk and boundary sides is a very fascinating thing; the boundary CFT is of type III_1, but one can simply introduce a lattice spacing ϵ as discussed previously to turn it into a type I algebra, thereby giving us a nice regulated holographic entanglement entropy. Taking into account of $1/N$ corrections, the type III_1 nature becomes type II_∞ – this has also been the center of interest in Chandrasekaran, Longo, Pennington and Witten's work on the algebra of observables in static patch de Sitter holography.

However, a much more mathematically fundamental notion has to do with understanding the subregion-subregion duality of the bulk and boundary. At finitely large N, the bulk Hilbert space $\mathcal{H}_{bulk}$ and the CFT Hilbert space $\mathcal{H}_{CFT}$ are equivalent in the sense of identifying bulk and boundary states equivalently:

$$\mathcal{H}_{bulk} = \mathcal{H}_{CFT} \ . \tag{45}$$

In fact, this should not be a surprise – the entire program of bulk reconstruction and operator dictionaries comes from this! From RT itself, one can see that bulk subregions have an interesting property: *they emerge from boundary operator subalgebras*. This is the starting point for our discussion on subregion-subregion and subregion-subalgebra duality. The example of RT is in fact quite fundamental, since the idea of entanglement wedge duality to boundary subregion subalgebra is derived from the fact that the entanglement wedge contains operators that contain information about the boundary subregion. This way, bulk properties like locality arise from purely boundary subalgebra properties – that is, the *bulk is emergent from boundary subalgebra*.

For the sake of discussion, consider also the case of an eternal black hole in AdS. The "right" external region in AdS is dual to operators[5] in the algebra of the "right" boundary in the thermofield double (TFD). In the same way, for some bulk subregion inside the right wedge, we have a dual subalgebra of CFT operators in the corresponding boundary subregion. This extends to all sorts of subregions in AdS/CFT, although keep in mind that the implication so far is that we are working in the large N limit. Subregion-subalgebra duality also is in this limit, although we will not exactly emphasise on this aspect in the discussion.

We will discuss two aspects of subregion duality in this subsection; (1) on entanglement wedge reconstruction in the sense of modular flows, and (2) on subregion-subalgebra duality. The former is a motivation towards using more algebraic aspects in the sense of reconstruction and bulk subregion emergence, while the latter is an explicit description of Liu and Leutheusser's work on subregion-subalgebra duality.

[5]By this in general we will refer to single-trace operators.

3.3.4 Modular Flows: Faulkner-Lewkowycz

We will start from the modular Hamiltonian. More precisely, let me rewrite this to include the fact that the modular Hamiltonians corresponding to the bulk and boundary sides are dual is the case at leading order in $1/N$:

$$[K_{bdy}, \Phi] = [K_{bulk}, \Phi] + \mathrm{O}(1/N) \ . \tag{46}$$

Now, by again ignoring the aspect of exponentiation in JLMS, we will rewrite this in the sense of a modular flow, something like

$$e^{-iK_A s}\Phi(\mathbf{Y})e^{iK_A s} \ . \tag{47}$$

Now why would we want to do something like this? Because the modular Hamilotonian has a natural interpretation as the generator of automorphism. This plays a very important role in understanding type III algebras from Tomita-Takesaki as pointed out before. What we could do now is to express an operator $X \in \mathcal{A}_A$ in the form of a modular flow, like

$$X_s = e^{iK_A s} X e^{-iK_A s} \in \mathcal{A}_A \ . \tag{48}$$

One can naturally see that the commutator (46) looks somewhat fishy – it looks like something you would get also from the boundary side. Then, the duality reduces to something of the form

$$e^{-iK_A s}\Phi(\mathbf{Y})e^{iK_A s} = e^{-iK_{A,\, bulk} s}\Phi(\mathbf{Y})e^{iK_{A,\, bulk} s} \ . \tag{49}$$

Now this seems nice. Clearly, this also has some nice subregion aspects, and this is most explicit when considering the case of entanglement wedge reconstruction. However, notice something interesting about the entire argument.

Clearly, the boundary CFT algebra is of type III_1. And this entire argument seems to be intrinsically based on this fact; one could now conjecture that there is a type III operator algebra for such subregion-duality. In fact, one could go ahead and see for the case of RT or eternal black hole case mentioned previously in discussion. If this really were the case, could one enforce this statement by explicitly computing the subregion-subalgebra duality?

3.3.5 Jafferis-Lewkowycz-Maldacena-Suh

We are now in a position to talk about the full nature of the JLMS argument. For the derivation of the JLMS result,we will adopt the perspective provided by Dong, Harlow and Wall's use of the first law of entanglement to derive the JLMS result.

Start by noting that the relative entropy $S(\rho|\sigma)$ can be written as

$$-S(\rho) + \mathrm{Tr}(\rho K_\sigma) \ . \tag{50}$$

In order to proceed, we will make use of the so-called *first law of entanglement*, where for some $\rho \to \rho + \delta\rho$, we have

$$S(\rho + \delta\rho) - S(\rho) = \mathrm{Tr}(\delta\rho K_\sigma) + \mathrm{O}(\delta\rho^2) \ . \tag{51}$$

Faulkner's paper showed that one can define a bulk operator $\mathcal{A}_{loc}$ given by

$$\mathcal{A}_{loc} = \frac{\text{Area of } \mathcal{X}_{HRT}}{4G_N} \tag{52}$$

at leading order in G_N. Using this, one finds that

$$S(\rho_A) = S(\rho_a) + \mathrm{Tr}(\rho_a \mathcal{A}_{loc}) \ . \tag{53}$$

Here the n-th Renyi entropy is computed by a bulk path integral to find the analytic continuation $n \to 1$ for finding von Neumann entropy. (53) in the sense of quantum extremal surfaces also holds at $1/N$ orders, by extremizing over the generalized entropy contribution. Then, we have

$$S(\rho + \delta\rho) - S(\rho) = \mathrm{Tr}\left((\sigma_a + \delta\rho_a)\mathcal{A}_{loc}\right) + S\left(\sigma_a \delta\rho_a\right) \ . \tag{54}$$

Now, we can find the terms like $\mathrm{Tr}(\rho_A K_{\sigma_A})$. This term looks like

$$\mathrm{Tr}\left(\rho_A K_{\sigma_A}\right) = \mathrm{Tr}\left(\rho\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ . \tag{55}$$

From the first law of entanglement (51), for some perturbation of σ_A in $\mathcal{H}_{code}$, we have

$$\mathrm{Tr}(\delta\sigma_A K_{\sigma_A}) = \mathrm{Tr}\left(\delta\sigma_a\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ . \tag{56}$$

All this boils down to

$$\mathrm{Tr}(\rho_A K_{\sigma_A}) = \mathrm{Tr}\left(\rho_a\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ , \tag{57}$$

which using (50) finally gives us the required **JLMS formula**,[6]

$$S(\rho_A|\sigma_A) = S(\rho_a|\sigma_a) \ . \tag{58}$$

For the time being, I will discuss some of the aspects of bulk reconstruction in the sense of subregion-subregion duality. I will not put this discussion as a subsection since this is a naive formulation of what subregion duality is, but in the next section we will delve deeper into some of the mathematical aspects of this, specifically, in the direction of Liu and Leutheusser's work on subregion-subalgebra duality and emergence.

[6] I know that I have skipped over a lot of details, but the point of this derivation is to capture some essentials and not have to reproduce the entire discussion for our purposes.

3.4 Operator Algebras

As said above, the notion of a type III subalgebra on the boundary side seems to correspond to bulk subregion emergence. This is what was referred to as the **subregion-subalgebra duality** in a paper by Hong Liu and Samuel Leutheusser. Simply stated, the following discussion is the key result of their work, which we will discuss in this bit of the subsection.

The starting point now is (45). If we start from some CFT state $|\Psi\rangle$, we can express a relation between the bulk Fock space and the GNS-constructed Hilbert space boundary HIlbert space,

$$\mathcal{H}_{\Psi}^{Fock} = \mathcal{H}_{\Psi}^{GNS} \ . \tag{59}$$

Let us denote by χ (same convention as Liu-Leutheusser) the collection of bulk fields + metric. By χ_Ψ we mean χ corresponding to Ψ, and choose the vacuum state to be $|0\rangle_{\chi_\Psi}$ defined by bulk fields + metric perturbations around $|0\rangle_{\chi_\Psi}$. The example cited in Liu-Leutheusser is that of a bulk field, which looks like [7]

$$\Phi(\mathbf{Y}) = \Phi_0(\mathbf{Y}) + \sum_n \left(u_n(\mathbf{Y})a_n + u_n^*(\mathbf{Y})a_n^\dagger\right) \ . \tag{60}$$

Here u_n denote mode functions. The Fock space $\mathcal{H}_{\Psi}^{Fock}$ then is found by operating successively on $|0\rangle_{\chi_\Psi}$ with $a_n^\dagger$. The GNS Hilbert space is obtained using the GNS construction from single-trace operators $\mathcal{S}$'s on $|\Psi\rangle$.

Now, let the algebra of single-trace operators acting on $\mathcal{H}_{\Psi}^{GNS}$ be denoted by $\mathcal{A}_{\Psi}^{GNS}$, and the algebra of bulk fields on $\mathcal{H}_{\Psi}^{Fock}$ be denoted by $\tilde{\mathcal{A}}_{\chi_\Psi}^{Fock}$. Then, looking at (59), we must have a correspondence of the form

$$\mathcal{A}_{\Psi}^{GNS} = \tilde{\mathcal{A}}_{\chi_\Psi}^{Fock} \ . \tag{61}$$

This motivates subregion-subalgebra duality. If one picks a bulk subregion a and denotes by $\mathcal{Y}_a$ the bulk algebra, one can notice that in the usual large-N limit it must be of type III_1. From this, one can see from the previous discussions that there must exist a boundary subalgebra $\bar{\mathcal{Y}} \in \mathcal{A}_{\Psi}^{GNS}$ of type III_1, so that we have

$$\mathcal{Y}_a = \bar{\mathcal{Y}} \ . \tag{62}$$

This has some interesting implications. Firstly, it tells us that the boundary operator algebras and the the bulk operator algebras are intricately related to each other. Secondly, it has close relations with the large-N and $1/N$ corrections, as worked by Witten in Gravity and the Crossed Product and the work by Chandrasekharan, Pennington and Witten.

The $1/N$ corrections result in the type II_∞ nature of the algebra, which Witten showed to be the natural expectation. However, one could look at this in the sense of algebrafication of ER=EPR in the AdS/CFT sense, where a Cauchy slice Σ connecting the two boundaries may be considered in the sense of the type of (say) boundary operator algebra. Clearly, in order for there to exist a kind of type I operator algebra, the two corresponding Hilbert spaces must factorize. Therefore, the GNS Hilbert space factorizes like

$$\mathcal{H}^{GNS} = \mathcal{H}_L \otimes \mathcal{H}_R \ . \tag{63}$$

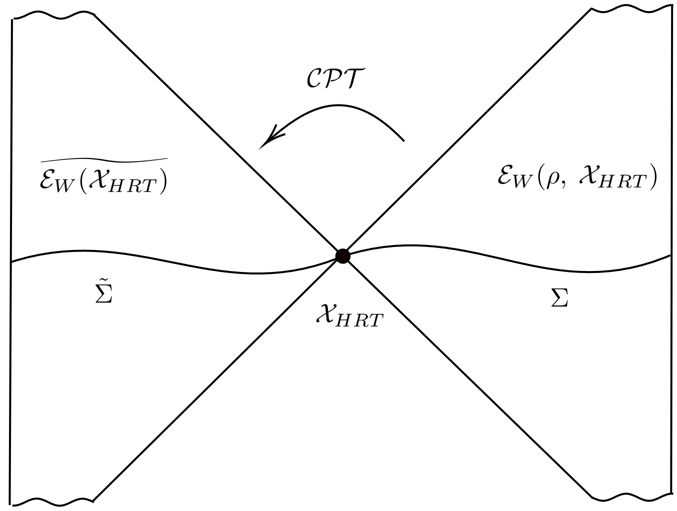

Figure 2: $\mathcal{CPT}$ reflection and continuity around the QES.

When this fails, we expect the algebra to be "at least" type III, which can be upgraded to type II in the sense mentioned above. The proposal by Engelhardt and Liu in algebraic ER=EPR, and the proposal I had put forward around the same time look at the operator algebras in the sense of this connectedness (or independence in my work, where I look at the geometries being independent in the sense of a strong No Transmission principle[7]). The story starts from the QES landscape.

The discussion on QES physics has an interesting aspect of canonical purification, which involves purifying and constructing a gravity dual by CPT reflection and gluing across the extremal surface. What one does is to take a $\mathcal{CPT}$ reflection around a "constructed" (in the sense of EW construction this is the coarse-grained) spacetime with initial data specified. That is, for some initial data on the constructed $\mathfrak{H}(\Sigma, h, \dots)$. By $\mathcal{CPT}$ reflecting $\mathfrak{H}$, one obtains a "mirror" of the initial data on the wedge, giving us a complete geometry from $\Sigma \cup \tilde{\Sigma}$, where Tilde-d quantities are the $\mathcal{CPT}$ reflected ones. With this, the entire coarse-grained spacetime can be constructed by taking these completed wedges $O_W(\sigma) \cup \tilde{O}_W(\tilde{\sigma})$ (glued across the HRT surface by identifying the codimension 2 junction conditions) and evolving them. In the sense of canonical purification, one now first finds the gravity dual for the CP (canonically purified) boundary, and then one does the $\mathcal{CPT}$ reflection across $\mathcal{X}_{HRT}$ to obtain the "full" spacetime. For instance, taking the CFT in a mixed state ρ, the CP-completed spacetime gives the appropriate full Schwarzschild-AdS spacetime. This happens around the QES, which plays a very important role: the connectedness of Σ and $\tilde{\Sigma}$ is determined by this nontrivial QES (see fig:2). However, there are some very interesting algebraic aspects to this.

[7]The NTP is the statement that when CFTs are independent, their bulk duals must also be independent. The strong NTP proposal I had put forward has to do with identifying type III algebras and the split property from the original NTP.

In general, when one says that the bulk emerges because of entanglement between the two boundaries in TFD, this aspect of connectedness arising from a common QES is the determining factor. If the two boundaries were to be disconnected, reconstruction from $\mathcal{CPT}$ reflection around the QES would not "connect" the two boundaries. If two boundary subregions, A_1 on CFT_L and A_2 on CFT_R (where CFT_L and CFT_R depict left and right boundaries respectively), then the entanglement wedges would have a common edge as seen previously. Recollect that in the large N limit, the algebra of bulk operators is type III. The usual way of purifying is to double the Hilbert space, which gives us the TFD state,

$$|TFD\rangle = \frac{1}{Z}\sum e^{-\beta E_n/2}|n_i\rangle|n_j\rangle \ . \tag{64}$$

We recover this from the Gibbs state, which gives us $|TFD\rangle$. As said before, the connectedness of this purified state is based on whether or not the full geometry can be constructed from $\mathcal{CPT}$ reflection and evolution from the constraints imposed on $\mathfrak{H}$. If one considers the cases of a pre-Page time and a post-Page time, one can see that the full spacetime geometry in the former case is different from that of the latter. On the basis of connectedness, we see that one is not connected, whereas the other one is connected. In order to better make sense of this, we will use the type III nature and (59).

First, start by noticing two interesting details: if the two geometries associated with A_1 and A_2 (and corresponding entanglement wedges $\mathcal{E}_W(A_1)$ and $\mathcal{E}_W(A_2)$) are connected, then they *must* share an edge, as evident from the constructed geometry. Similarly, it must also be that there must not exist states in the bulk Fock space $\mathcal{H}^{Fock}$ that factorize into product states in $\mathcal{E}_W(A_1)$ or $\mathcal{E}_W(A_2)$. From (59), it must also be that the same goes for the GNS Hilbert space $\mathcal{H}^{GNS}$. From subregion-subalgebra and emergent type III factors, it is also clear that the operator algebra of the bulk is type III. If the geometry were to be connected, $\mathcal{H}^{GNS}$ must *also* be type III from subregion-subalgebra. Therefore, one can conjecture that the connectedness of these two boundary sides is based on the algebraic emergence in the large N limit. It should also be clear from these observations that if the algebra of bulk operators is type I, then they must be disconnected.

Before remarking further, I wish to briefly take a detour to point out the **No Transmission principle** (NTP), which states that if two CFTs are independent, then the corresponding bulk duals must *also* be independent [8]. This has some nice implications, for instance in the strong cosmic censorship situation in AdS/CFT [8, 9], but this also has quite a bit of relevance to the present discussion on the relation between the type of operator algebras and connectedness. However, I will not provide a description of the proposal here, and will instead direct the interested reader to [10].

Finally, the statement of the Engelhardt-Liu speculation of **algebraic ER=EPR** is that if the algebras $\mathcal{A}(A_1)$ and $\mathcal{A}(A_2)$ associated to A_1 and A_2 are of type III, they are connected. If they are type I, they are disconnected. However, there is an intermediate case here – what about type II algebras? Does there exist a case that satisfies *some* of the properties outlined above, but maybe not all? Clearly, if there were to exist a type II situation, it could be possible that there are divergent entropies but ones for which a trace can be defined.

The pre-Page time case is *exactly* an example we can use here. This cannot be type III or type I, since in the $G_N \to 0$ limit there is clearly a divergence of entropy, but trace can be defined nonetheless. This becomes the type II example we were looking for, and therefore it must also be that this is "somewhat" connected. The speculation by Engelhardt and Liu is that there is a classical ER=EPR description in type III, a quantum connectedness description for type II and no connectedness for type I cases [11]. There is also a further description of phase transitions between bulk subregions being type III, but we will not discuss that here.

4 Black Hole Information Problem

We will now talk about the black hole information problem (BHIP) and the features that AdS/CFT has that closely provide resolutions to the BHIP. The reader surely must be familiar with the paradox regarding information, black holes and the famous Hawking-Preskill-Thorne wager. Of course, black holes are not easy things to make sense of, and have been the center of attention in hep-th for quite some time now. AdS/CFT has brought an entirely new perspective on this problem, and the bulk-boundary physics in this theory has shed light on this important problem.

While the traditional phrase to this entire scheme is the "black hole information *paradox*", we are obliged to call it a problem to (slightly) exaggerate the look of this landscape as a problem. Black holes are, as it must be clear to anyone having picked up some hep-th papers on the arXiv, definitely not an easy thing to make sense of. A good deal of information theory is involved when dealing with the BHIP in general, and most importantly in the AdS/CFT counterpart to the BHIP, there are many reconstruction aspects that are involved. For instance, one could ask if the interior of the black hole is something that can be made sense of by instead making measurements to the CFT side. The purpose of these notes is to address these things in a nice pedagogical way. The structure of these notes is as follows.

In section 4.1, we will quickly recap some basics of the BHIP, and the setup used will be the usual Schwarzschild geometry. We will recap Hawking's arguments on the evaporation of black holes in semiclassical gravity, and the Page curve. We will talk about the generalized second law as formulated by Bekenstein [12], and some arguments therein. We will review complementarity [13, 14], and motivate AMPS [15] and firewalls via the monogamy paradox [16]. We do not, however, review in detail firewalls and fuzzballs and only allude to some arguments around them.

In section 4.2, we will discuss AdS/CFT aspects of the BHIP, particularly focusing on the Raju-Papadodimas proposal on black hole interior reconstruction [16, 17, 18]. We will discuss coupling an AdS black hole theory to a bath in 2D dilaton theory and motivate quantum islands [19]. We will then discuss an ER=EPR paradox with the TFD as argued by Marolf and Wall [20]. Finally, we will discuss briefly some aspects of ER=EPR and (non-)traversable wormholes [21, 22]. For a lighter introduction to BHIP, see Aayush's excellent notes [23].

4.1 In a Nutshell

Classical black holes are great. One can do nice things with classical black holes, like work with gravitational waves in mergers, or do lensing, and basically most of the astrophysical works. The No Hair theorem shows that the after evaporation, the only numbers required to describe black holes are M, Q and L, which in itself causes some suspicion. Hawking showed that assuming the null energy condition,

$$T_{\mu\nu}X^{\mu}X^{\nu} \geq 0 \ , \tag{65}$$

classical black holes always have an increasing horizon area. Following Bekenstein and Hawking's argument that the entropy of a black hole is proportional to the area of the horizon $S_{BH} \sim A$, it must, therefore, also be that the entropy of a black hole is always increasing. This seems to be kind of nice. However, classical black holes do not evaporate, which was shown to be a semiclassical issue.

But then, in semiclassical physics, would the evaporation and therefore decreasing horizon area imply a violation of the second law of thermodynamics? This seems so. However, Bekenstein argued that the correct quantity to work with in this situation is the generalized entropy S_{gen},

$$S_{gen} = \frac{\text{Area of Horizon}}{4G_N} + S_{ext} \ . \tag{66}$$

The generalized second law of thermodynamics would then concern this quantity's monotonicity properties. One should, therefore, expect that as the horizon area decreases, the exterior entanglement entropy compensates for this. However, there are subtleties with how this von Neumann entropy functions, as we shall discuss. We will first start by doing some quick analysis in the free scalar field situation across the horizon, and the origin of Hawking radiation.

Let us sit in the M^{D+1} spacetime with the metric

$$ds^2 = -\mathrm{d}U\mathrm{d}V + \delta_{\mu\nu}dx^{\mu}dx^{\nu} \ . \tag{67}$$

One also has to expect deviations to this form of the metric, but this is not very essential here. One could pick a semiclassical state $|\Psi\rangle$ and compute the two correlator $\langle\phi(x_1)\phi(x_2)\rangle_{\Psi}$. The general form looks like

$$\frac{\Gamma(D-1)}{2^D\pi^{D/2}\Gamma\left(\frac{D}{2}\right)}\frac{1}{\gamma^{\frac{D-1}{2}}}\left(1+\mathrm{O}(\gamma)\right) \ , \tag{68}$$

where γ is the geodesic distance between x_1 and x_2. The coordinates are nice to work with to emphasise on the importance of correlations across null surfaces. The Schwarzschild setup can be written in terms of the tortoise coordinate r^*, which blows up to negative infinity at the horizon $r = H$, by requiring

$$\frac{1}{f(r)} \equiv 1 - \frac{\mathcal{M}}{r^2} = \frac{dr^*}{dr} \ ,$$

where

$$\mathcal{M} = \frac{8GM\pi^{1-\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}{D-1} \ .$$

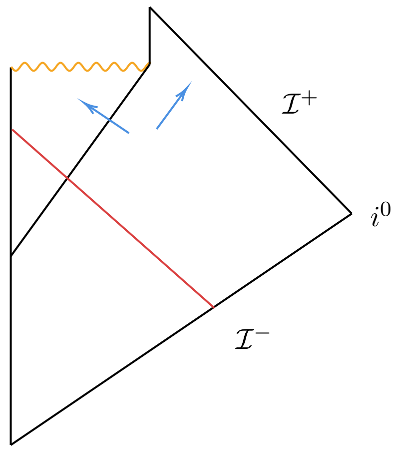

Figure 3: An evaporating black hole. The red line denotes an infalling shell of matter, whereas the blue arrows indicate negative and positive fluxes of particles falling into and escaping the horizon and escaping to $\mathcal{I}^+$ respectively.

Then, the metric becomes

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-1}^2 \, . \tag{69}$$

We now want to work with small scale correlations around the horizon. Taking a scalar field description, one has the Klein-Gordon solutions, denoted by $\xi^{\text{in}}(\omega, l, r^*)$ and $\xi^{\text{out}}(\omega, l, r^*)$. Here, l and ω are the angular quantum numbers and the frequency respectively. $\xi^{\text{in}}(\omega, l, r^*)$ in the r approaching H from outside limit, denoted by $r \to H^+$ looks like

$$\xi^{\text{in}}(\omega, l, r^*) = \chi_{\omega,l} e^{-i\omega r^*} \, , \tag{70}$$

whereas $\xi^{\text{out}}(\omega, l, r^*)$ in the $r \to H^-$ limit looks like

$$\xi^{\text{out}}(\omega, l, r^*) = e^{i\omega r^*} + \chi'_{\omega,l} e^{-i\omega r^*} \, . \tag{71}$$

The factors $\chi_{\omega,l}$ and $\chi'_{\omega,l}$ are not needed in this analysis, but can be computed nonetheless. Then, one can write a field in terms of these something like

$$\phi = \sum \int d\omega \, \left[\mathcal{A}\xi^{\text{in}}(\omega, l, r^*) + \mathcal{B}\xi^{\text{out}(\omega,l,r^*)} \right] e^{-i\omega t} Y(\Omega) + \text{ hermitian conjugates} \, . \tag{72}$$

In the Kruskal coordinates, we have

$$U = \frac{-1}{\kappa} e^{\kappa(r^* - t)} \, , \quad \text{and} \tag{73}$$

$$V = \frac{1}{\kappa} e^{\kappa(r^* + t)} \, . \tag{74}$$

With precise calculations of $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$, and the smearing functions associated to $a_{\omega,l}$ and $a^{\dagger}_{\omega,l}$, the reader is directed to 2012.05770 and 1910.02992. For now, it is only relevant that $\mathfrak{a} = a_{\omega,l}$ and $\tilde{\mathfrak{a}} = \tilde{a}_{\omega,l}$ with a normalized commutator. The two-point function for this then is

$$\langle a_{\omega,l} a^{\dagger}_{\omega,l} \rangle_{\Psi} = \frac{1}{1 - e^{-\beta\omega}} , \tag{75}$$

where β is the inverse temperature. **This implies that there is a flux at $\mathcal{I}^+$.**

4.1.1 Page Curve

Now, here is our situation: we know from Bekenstein and Hawking's famous formula,

$$S = \frac{A}{4} , \tag{76}$$

that the area of the horizon is proportional to the entropy of the black hole. Since Hawking radiation exists, the positive flux to $\mathcal{I}^+$ tells us that the black hole evaporates, leading to a decreasing area of the horizon. So, at the least, we expect that the generalized second law holds [12], and (66) is monotonically increasing. Here is where we encounter the *Page curve* dilemma.

The natural information theoretic interpretation of the Page curve [16, 24] is that one could take a full system $\mathcal{H}$, and pick a small subsystem S. If one takes the ratio of the von Neumann entropy in terms of the ratio of S to $\mathcal{H}$, one would find that it obeys a very characteristic nature: it increases upto a certain point, at which $S = \frac{1}{2}\mathcal{H}$. Then, since the subsystem S becomes the larger system and $S' = \mathcal{H} - S$ becomes smaller, the curve decreases steadily. This is the **Page curve** for $\mathcal{H}$.

What we expect of black holes can be seen from usual aspects of von Neumann entropy and states. The von Neumann entropy

$$S_{vN} = -\mathrm{Tr}(\rho \log \rho) \tag{77}$$

has the property that for pure states $|\psi\rangle$, it vanishes. So we would expect that the density matrix ρ looks like $|\psi\rangle\langle\psi|$, for which the eigenvalues look like

$$\rho = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} .$$

An additional property is that the maximum value of S_{vN} is given by the logarithm of the dimensionality of the Hilbert space $\mathcal{H}$.

Entropy bounds, however, also play a role here; one expects that the **Bekenstein bound** (which is a certain limit of the Bousso bound) holds:

$$S \leq 2\pi E \mathcal{R} , \tag{78}$$

where E is the energy and $\mathcal{R}$ is the radius of the smallest area sphere packing the system. Then, denoting by S_{rad} the entropy of Hawking radiation, we expect $S_{\text{rad}} = 0$ for the pure state. We would expect that this, eventually goes back to zero, i.e. a pure state. However, Hawking's famous calculation showed that this is not the case; instead, it increases, and saturates the Bekenstein bound without ever reaching zero again. See fig. 4.

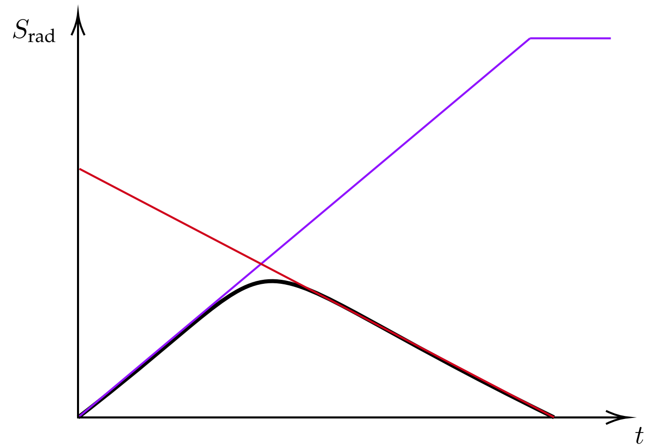

Figure 4: The Page curve. The red line is the thermodynamic entropy of the black hole, which is monotonically decreasing as the black hole evaporates. The thick black line is the Page curve, whereas Hawking's calculation is the violet line.

Now this is bad. This tells us that in some sense, unitarity is being lost. One can again see the usual expectation by noting that the dimension of the Hilbert spaces of Hawking radiation and the evaporating black hole are supposed to compensate for one another.

The more explicit calculation of the number of these particles generating this flux is due to Hawking. Note that from the first law of black hole thermodynamics,

$$dM = \frac{\kappa}{\pi} dA + \Phi dQ + \Omega dL \ . \tag{79}$$

Then, the number of these particles, $N_{j\omega lmp}$ is [25]

$$\Gamma_{j\omega lmp} \left[\exp\left(2\pi\kappa^{-1}\left(\omega - e\Phi - m\Omega\right) \mp 1\right)\right]^{-1} \ , \tag{80}$$

where the subscripts j, l, m and p denote species, spherical harmonic, angular quantum number and helicity. The minus and plus signs are for bosons or fermions respectively.

4.1.2 Generalized Second Law

Let us now concern ourselves with a little more on what the role of the GSL really is. Firstly, it is clear that this is a highly non-trivial law, since one has an evaporating black hole without a proper construct of the variation of S_{gen}. A nice modern approach to this is in terms of "holographic screens", where one starts by picking suitable marginally trapped surfaces inside the black hole and construct a holographic screen of such surfaces. In semiclassical gravity, one takes quantum marginally trapped surfaces, i.e. surfaces for which the quantum expansion is negative along one null congruence and zero along the null congruence orthogonal to it. One can define $\mathcal{F}_\lambda(a)$, a one-parameter family of functions that can be used along a null congruence parametrized by the affine λ and a along the spacelike codimension-2 surface I. This satisfies

$$\partial_\lambda \mathcal{F}_\lambda(y) \geq 0 \ .$$

Then, the quantum expansion is defined as

$$\Theta_k(\mathcal{F};\ a) = \frac{4G_N}{\sqrt{\eta}} \frac{\delta S_{gen}(I_\mathcal{F})}{\delta \mathcal{F}_a} \ , \tag{81}$$

with the interpretation that similar to the classical expansion, one is making infinitesimal deformations to the surface I along the null congruence to measure the variation of the generalized entropy. Here, η is the induced metric, and we assume that the surface is a compact surface (non-compactness is typically troublesome to work with and plays a key role in singularity theorems). k is the outgoing null congruence, whereas l will denote the ingoing null congruence. Associated to this would be a pair of future and past congruences, so on an all we have $k^\pm$ and $l^\pm$. Recall that the Bousso bound states that the most entropy that can pass through I is bounded by the area of it, assuming the null energy condition (so that the domains of dependence of the lightsheet and the surface are closed and equal):

$$S(I) \leq n \frac{A(I)}{4G_N} \ , \tag{82}$$

where n is the number of light-sheets. In this way, one expects that for (at least) marginally trapped surfaces, this entropy bound is *at most* saturated. The Bekenstein bound can be seen to be a slightly weaker form of this bound. A lot of work has been done for evaporating black holes and such entropy bounds, and in particular the holographic screen approach by Bousso and Engelhardt shows that the GSL is preserved by taking semiclassical corrections to the holographic screen (dubbed the Q-screen).

4.1.3 Complementarity

In regards to the full nature of physics in the black hole information paradox, Susskind, Thorlacius and Uglum [13, 14] postulated three points dealing with (1) what to expect of quantum field theory for an evaporating black hole, (2) the semiclassical approximation of the field theory and (3) the dimensionality of the subspace giving the black hole a description. These are fully expanded as follows:

1. **Postulate 1.** *An evaporating black hole can be described by usual QFT.* One could make this more precise by saying that there exists an S-matrix describing infalling matter and Hawking radiation:

$$S\,(\text{infalling matter} \mid \text{Hawking radiation}) \ .$$

In the face of this, there is a nice way of making sense of $\mathcal{I}^+$ in a holographic sense in the asymptotically flat setting. Among many things, in a noncompact situation, like Minkowski spacetime, one has some very fascinating observations, thanks to algebraic QFT. One such result is the split property, which states that one can define a collar region ε around a bounded region $\mathcal{U} \subset M$ on a noncompact Cauchy slice Σ, and for this there exists a type I factor $\mathfrak{R}$ so that the full Hilbert space factorizes like

$$\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_{\bar{N}} \ . \tag{83}$$

However, there exists a very interesting property of gravitational QFTs, called *holography of information*, due to which information about $\mathcal{U} \cup \varepsilon$ is *also* available on the boundary of the Cauchy slice $\partial\Sigma$. This is clearly in opposition to the assertion of the split property, which is that the state in $\mathcal{U} \cup \varepsilon$ can be prepared individually to that of the complement. These aspects also appear in the AdS/CFT discussion of the BHIP.

2. **Postulate 2.** *Physics to a good deal is semiclassical understood outside the stretched horizon.*

3. **Postulate 3.** *The dimensionality of the subspace of states describing this black hole is related to the Bekenstein bound:*

$$\text{Dim}\left(\mathcal{H}_{BH(M)}\right) = e^{S_{\textbf{Bek}}(M)} \ . \tag{84}$$

4.1.4 Monogamy and Firewalls

One could argue that there is a fourth postulate which largely encompasses the other three postulates of complementarity:

4. **Postulate 4.** *For an infalling observer, the horizon of the evaporating black hole should be natural.*

The argument of Almheiri, Marolf, Polchinski and Sully (AMPS) [15] is that this postulate, along with a bit of the others is somewhat dangerously naive. One expects that there is a bad counter-example of this in the form of "firewalls". Before going there, we will quickly revisit a monogamy paradox. (Monogamy is, after all, a good and ethical thing.)

One could follow this discussion intently, but at a point realise the meaning of this entire problem for old black holes. Take a black hole geometry so that one can find a "good" Cauchy slice Σ to i^0, so that it is cut into three sections: a region A that lies just inside the horizon, B which lies just outside, and C, which lies at $\partial\Sigma$; the idea being that the slice cuts the interior and the exterior of the black hole and the Hawking radiation. Now, we already know from the discussion above that across the horizon, modes are entangled, so

$$A \longleftrightarrow B \ .$$

Ok. Now, for an old black hole, the near horizon modes are entangled with Hawking radiation extending to $\mathcal{I}^+$. However, this now implies that

$$B \longleftrightarrow C \ ,$$

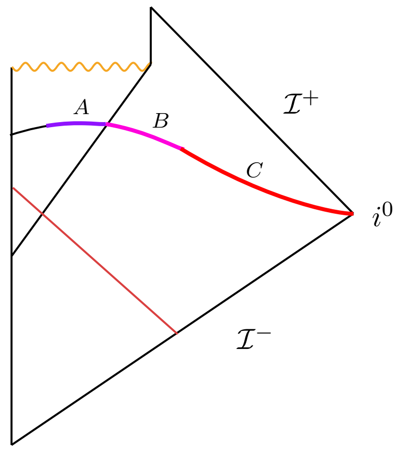

Figure 5: A suitable Cauchy slice, for which there are three sections of interest: A, which lies just inside the horizon, B, which lies just outside the horizon, and C, which extends to the boundary of the Cauchy slice $\partial\Sigma$ at i^0.

which goes badly with the **monogamy of entanglement**. Another way of arriving at this discrepancy is to assume that the Hilbert space of the full theory factorizes like [16]

$$\mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \ . \tag{85}$$

Clearly, this violates strong subadditivity entanglement entropy. One way of resolving this anti-monogamy problem is to go against one of the complementarity postulates. What AMPS presented in a paper titled "*Complementarity or Firewalls?*" is to replace the horizon with a firewall (see fig. 6). Of course, this is against the equivalence principle and complementarity in entirety. So, one could argue that perhaps in a loophole-ish way that after all, one has to *measure* the Hawking radiation first. In Susskind's argument, the two famous protagonists Alice and Bob are made to do these measurements in the *cloning* setup, where Alice jumps in with a qubit, with Bob just outside the horizon, and Bob waits for the information to come out as Hawking radiation, and jumps in. The non-firewall situation in the entanglement picture looks something like this: initially, we only have maximal entanglement between $A \longleftrightarrow B$, and Alice sits in B. In later times, we would have $A \longleftrightarrow C$, and one could get a conservation of entanglement, as Susskind coined it. What the firewall situation does is simplify the paradox by quite a bit – by simply saying that in the cloning setup, by the time Bob can jump in following Alice in early pre-Page time, Bob waits till about at or after the Page time, but he is destroyed at the firewall. Or, if Alice herself waits till the Page time, she herself is destroyed at the firewall. Essentially, one could motivate firewalls simply computationally, by saying that the correlator $\langle T_{\mu\nu}\rangle$ diverges at the firewall.

Susskind then goes on to argue that the firewall situation is not as "neat" as it seems. For instance, when Alice jumps in, what she observes is an *apparent horizon*, which changes the location of the actual horizon. Eventually, Bob jumping in would *also* change the location

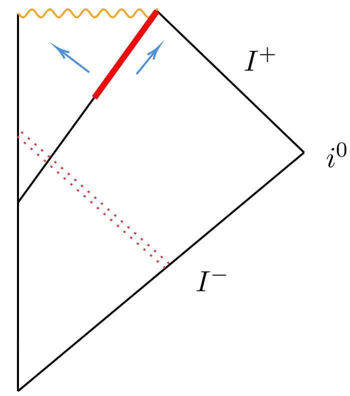

Figure 6: A firewall (thick red line) in a young black hole, with the Hawking radiation identified.

of the horizon, and seemingly it complicates the firewall solution. With the scrambling time vs Page time debate, there are many arguments to each side. However, even just the post-Page time situation looks rather complicated. If one views the firewall as an extension of the singularity, and taking into account of the changing horizon, the cloning setup would clearly give a different result without enough satisfaction from the firewalls picture (see fig. 7).

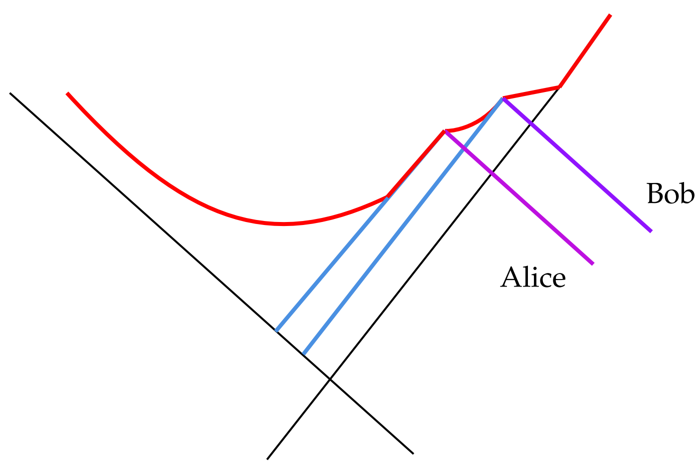

Figure 7: Adapted from Susskind's paper. The firewall, extended from the singularity, and the apparent horizons (thick blue lines) are identified with changes from infalling Alice and Bob.

So, the fuzzballs and firewalls proposals contain some pros and cons:

1. **Firewalls:** Looks good with the monogamy paradox – strong subadditivity of entropy is restored! Goes well with the AMPS, a little worrisome with complementarity. But not preferred, following Susskind's arguments.

2. **Fuzzballs:** Looks good with SUGRA and monogamy paradox, since one no longer has the multiple entanglement partners. But seemingly against EFT (?)

4.2 AdS/CFT and BHIP

We will now turn our attention to the AdS/CFT picture of BHIP. This is a very interesting version of the information paradox, with the added intricacies of AdS/CFT. One can make sense of the usual properties of black holes in AdS in almost the same way, except that when needed there are some brane-y things. For instance, black hole evaporation in AdS is complicated and requires coupling the bulk gravitating region to a "bath". Most of what one does with islands and things requires doing so with branes in the geometry, but this is will be slightly put aside in the discussion. For instance, the temperature for an eternal AdS black brane is [16]

$$T = \frac{\hbar c \mathbf{d}}{4\pi z_H k_B} \ . \tag{86}$$

We will start by reviewing bulk fields and CFT operators outside and inside an AdS black hole, where the interpolation of operators in regions I and III (see fig. 8) are used to find operators in region II.

4.2.1 Raju-Papadodimas

The point of this subsection is to recap quickly an interesting result from Suvrat Raju and Kyriakos Papadodimas [16, 17, 18], regarding reconstruction of bulk fields in the interior of AdS black holes. We are, of course, missing a lot of details on the nature of partner operators. Ideally, we should have started with the splitting into coarse- and fine-grained Hilbert spaces, but the final result that we wanted is that the horizon interior is the same for all pure states. The infalling observer does not find anything special, and this is a contradiction to the fuzzballs proposal. We will once again be starting from solutions of the scalar field equation $(\Box - m^2)\Phi = 0$,

$$\xi_{\omega,k}(t, x, z) \ . \tag{87}$$

These can be expanded like

$$e^{-kx - i\omega t}\psi_{\omega,k}(z) \ , \tag{88}$$

where $\psi_{\omega,k}$ have a unique normalizable solution once we fix ω and k. We then identify three regions of interest in the spacetime – see fig. 8.

Fields in, say, region I of this geometry can be transformed into fields in region II by operating ϕ_I with a CPT-conjugation operator Θ_{CPT}:

$$\Theta^\dagger \phi_I \Theta = \phi_{III} \ , \tag{89}$$

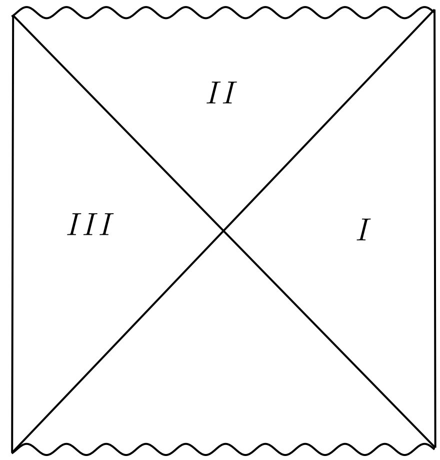

Figure 8: The three regions in the AdS black hole geometry. Bulk fields in region I are nicely make sense of from usual bulk reconstruction.

where we have $\phi_I(r,t,\Omega)$ and $\phi_{III}(t,-t,\Omega)$. Decompose the respective Hamiltonians into right and left terms H_R and H_L. The basis of H_R eigenstates relate to those of H_L like

$$|i^*\rangle_L = \Theta^\dagger |i\rangle_R \ . \tag{90}$$

Now the usual quantization process applies, which is not very non-trivial. One has the usual expansion of a bulk field in region I in terms of creation and annihilation modes $a_{\omega,k}$ and $a^\dagger_{\omega,k}$:

$$\phi(t,x,z) = \int \frac{d\omega d^{D-1}k}{(2\pi)^D\sqrt{2\omega}} \, \left(a_{\omega,k}\tilde{\xi}_{\omega,k}(t,x,z) + \text{ Hermitian conjugates}\right) \ . \tag{91}$$

The modes $a_{\omega,k}$ and $a^\dagger_{\omega,k}$ satisfy the usual commutator rule. Similarly, one has modes in region III, which we will call $\tilde{a}_{\omega,k}$ and $\tilde{a}^\dagger_{\omega,k}$. Then, the pair of modes $a_{\omega,k}$ and $a^\dagger_{\omega,k}$, and $\tilde{a}_{\omega,k}$ and $\tilde{a}^\dagger_{\omega,k}$ together are used to construct region II.

Now, notice that in region I, we can define a CFT operator[8],

$$\phi^I_{CFT} = \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \, \left(\mathcal{O}_{\omega,k}\xi_{\omega,k} + \mathcal{O}^\dagger_{\omega,k}\xi^*_{\omega,k}\right) \ . \tag{92}$$

One then defines operators $\tilde{\mathcal{O}}$, with the Fourier modes

$$\tilde{\mathcal{O}}_{\omega,k} = \int dt d^{D-1}x \; e^{-ikx+i\omega t}\tilde{\mathcal{O}}(t,x) \ . \tag{93}$$

[8]Throughout, for these integrals we take $\omega > 0$, although we do not explicitly mention it for convenience.

With this, we can write an operator in region III like

$$\phi^{III}_{CFT} = \frac{d\omega d^{D-1}k}{(2\pi)^D} \left(\tilde{\mathcal{O}}_{\omega,k}\xi_{\omega,k} + \tilde{\mathcal{O}}^{\dagger}\xi^{*}_{\omega,k}\right) . \tag{94}$$

Here, the idea is that the partner operators are the Tilde-d ones in the thermofield double Hilbert space. Effectively, this allows us to write a nice description for the black hole interior. The point of all this is to similarly be able to write an expansion in region II. Take solutions to the Klein-Gordon equation,

$$g^{(1,2)}_{\omega,k} = e^{ikx-i\omega t}\chi^{(1,2)}(z) , \tag{95}$$

with the properties that will be discussed below. Then, we can write an operator in region II like

$$\phi^{II}_{CFT} = \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \left(\mathcal{O}_{\omega,k}g^{1}_{\omega,k} + \tilde{\mathcal{O}}_{\omega,k}g^{2}_{\omega,k} + \text{ Hermitian conjugates}\right) . \tag{96}$$

One can explicitly compute the correlator

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{\Psi} \tag{97}$$

in a pure CFT state $|\Psi\rangle$, telling us what an infalling observer would experience when going from region I to II. What Raju-Papadodimas tells us is that as Alice falls through the horizon (see fig. 9), she experiences natural physics, essentially augmenting **Postulate 4** of black hole complementarity. Clearly, this suggests that the argument of fuzzballs (or firewalls, although fuzzballs are more natural a description in comparison to firewalls) seems in contradiction with what one should and *does* expect of black hole horizons. There are some more arguments about what these correlators constitute and what the entire proposal on a complete scale would provide, in comparison to fuzzballs, but we will defer a discussion of those.

We will now turn to a second interesting aspect of AdS black holes, which has to do with coupling a bath to the bulk AdS and entanglement wedges.

4.2.2 Entanglement Wedges

We will now discuss an intriguing aspect of entanglement wedges in AdS black hole spacetimes. Evaporation in AdS black hole spacetimes is weird and usually requires coupling to a *bath* [16, 19, 26]. Take the gravitating bulk AdS_D with the matter content a holographic CFT itself. This would have a dual CFT in $D-1$. We then couple this to a CFT_D in a flat background, and this is referred to as the AdS/CFT+bath system. For now, our interest would be in AdS/CFT with the following total action:

$$S(g^{(2)}_{\mu\nu}, \phi, \chi) = S_{grav}(g^{(2)}_{\mu\nu}, \phi) + S_{CFT}(g^{(2)}_{\mu\nu}, \chi) , \tag{98}$$

where ϕ is the dilaton field. The matter fields are taken to constitute a holographic CFT_2, and we locate the bath CFT as some CFT_2 (see upper fig. 10). One can then imagine

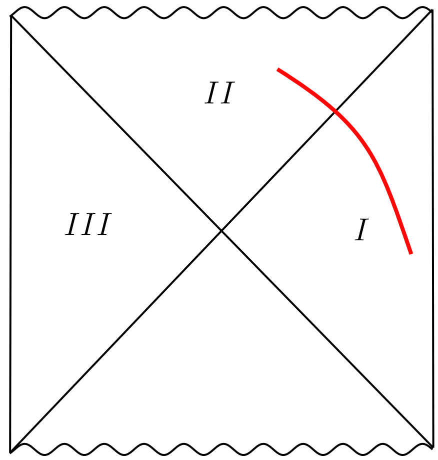

Figure 9: Infalling observer in the AdS black hole setup.

the CFT_2 as living on the boundary of some AdS_3 theory, with the 2D dilaton theory description being on a Planck brane, as shown in lower fig. 10.

Let $\mathfrak{y}$ be a point in the 2D theory. The generalized entropy would be

$$S_{gen}(\mathfrak{y}) = \frac{\phi(\mathfrak{y})}{4G_N} + S_{bulk}(\mathbb{R}_{\mathfrak{y}}) \ , \tag{99}$$

where $\mathbb{R}$ is some interval to a weakly coupled region in the theory, and S_{bulk} measures the bulk von Neumann entropy of $\mathbb{R}_{\mathfrak{y}}$. For convenience, ignoring fluctuations of ϕ and the metric, we will write $S_{gen}(\mathfrak{y})$ as

$$S_{gen}(\mathfrak{y}) \sim \frac{\phi(\mathfrak{y})}{4G_N} + \frac{A(\mathcal{X}_{\mathfrak{y}})}{4G_N} \ . \tag{100}$$

The entanglement wedge corresponding to $\mathcal{X}_{\mathfrak{y}}$ can then be found out, which looks something like fig. 11.

What is interesting about this construction, among many things, is that this links to islands and entanglement wedges in an interesting fashion. Taking the HRT prescription into account with islands, one can formulate, in a quantum min-ext formalism, the entropy of some A in terms of the area of the boundary of an island + corrections:

$$S(A) = \min \text{ext} \left(\frac{A(\partial I)}{4G_N} + S(A \cup I) \right) \ , \tag{101}$$

which also leads to an interesting Bousso-like bound for the generalized entropy of the entanglement wedge and a region A [19]:

$$S_{gen}(\mathcal{E}_W, A) < \frac{\text{Area of } (A)}{4G_N} \ . \tag{102}$$

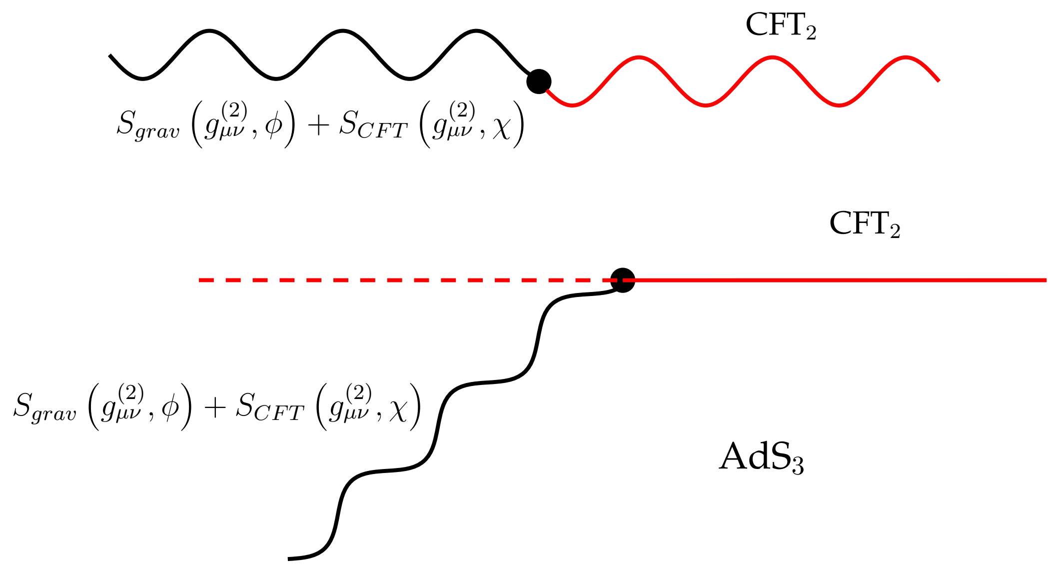

Figure 10: (Top) A CFT$_2$ bath (thick red wavey line) coupled to the 2D dilaton system. (Bottom) The 2D dilaton theory is on a Planck brane, with CFT$_2$ the dual to an AdS$_3$ bulk.

4.2.3 An ER=EPR Paradox?

One remarkable thing about the eternal AdS black hole is that one can take the left and right boundary CFTs in the thermofield double state $|TFD\rangle$, which leads to an interesting paradox[9]. Start by noting that the TFD state is

$$|TFD\rangle = \mathcal{Z}_\beta^{-\frac{1}{2}} \exp -\frac{\beta E}{2} |E,\ E\rangle \ , \tag{103}$$

where $|E,\ E\rangle$ are eigenstates of the left and right Hamiltonians H_L and H_R. It is a very well-known thing that the state $|TFD\rangle$ is dual to an eternal AdS black hole. So now, one could say that operating on either CFT does not affect the other due to the nature of the TFD state. However, this leads to a paradox, which can be visualised by our usual protagonists Alice and Bob again. Let us say that Alice jumps in from the right wedge (an excitation on the right CFT), whereas Bob jumps in from the left wedge (an excitation on the left CFT). Since both fall into the black hole, where Alice and Bob meet is clearly affected by each other [20]. See fig. 12.

The above thought experiment can be made more precise mathematically by identifying a state $|W\rangle$ and $|TFD\rangle$ as in Marolf and Wall's paper. The unitaries become

$$e^{iA} \quad \text{and} \quad e^{iB} \tag{104}$$

for Alice and Bob respectively. Then, the probabilities of meeting behind the black hole horizon is non-zero with Bob, that is,

$$\langle e^{-i(A+B)} \mathcal{P} e^{i(A+B)} \rangle_W \sim 1 \ , \tag{105}$$

[9] I thank Aayush Verma for pointing out this paradox for inclusion in these notes.

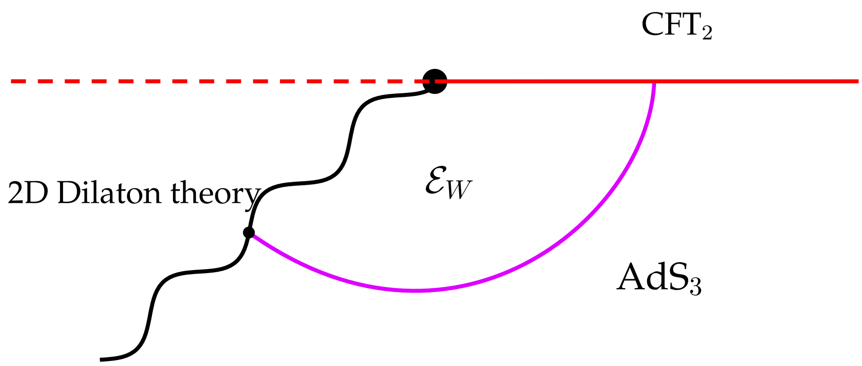

Figure 11: The entanglement wedge associated to $\mathcal{X}_\mathfrak{y}$.

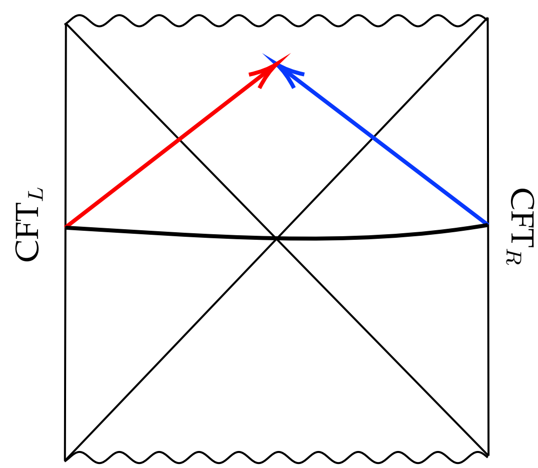

Figure 12: Alice (thick blue line) and Bob (thick red line) meeting behind the horizon.

whereas ~ 0 without Bob. A little more precision can be adopted by the use of superselection in the language [20], but we will not do so here.

4.2.4 Firewalls In Our Time

Let us reverse the locations of Alice and Bob, partly because Bob never does anything other than hover over a horizon while Alice does most of the work, and let us say that she now hates Bob. Alice could affect the boundary conditions on the left boundary CFT, so that it generates a firewall (with decay), which basically fries anyone it comes into contact with. One could ask if Bob experiences the firewall, which clearly is dependent on whether Alice shoots the shockwave in the first place [22].

If you imagine the two black holes to be very distant in one space without entanglement, there does not exist an ER bridge and all is good. However, suppose that both were prepared in an entangled state. Then, the ER bridge joining them is not traversable (in agreement with non-locality). However, what one *could* do, is to jump inside the black holes and meet behind the horizon in the entangled state, due to which communication would be possible. Of course, this seems impractical, but on a more cautious note, Alice may very well send in shockwaves, whereas Bob would jump in and get fried waiting for Alice.

It is, however, possible that there are traversable wormholes in this fashion of ER=EPR, as Jafferis, Gao and Wall showed [21], in which case there are some interesting observations linking to the above discussion. However, one still has to maintain that in the Alice-hates-Bob scenario, the systems are non-interacting, whereas in [21], this is not so. You can read the paper and find the consequences of the present situation for yourself.

In the grand scheme of theoretical high energy physics, there are few things more fascinating than the black hole information problem. In this discussion, we covered a few interesting things, like the flat space discussion of the BHIP in a pedagogical manner, particularly discussing some aspects of Hawking radiation, the Page curve, the generalized second law, complementarity and a monogamy paradox, with firewalls as a solution. We have not covered firewalls and fuzzballs with a lot of details, but as they stand, the discussion is sufficient for a motivation to the BHIP. In the AdS/CFT section, we discussed an argument on black hole interior reconstruction [18]. We then remarked on entanglement wedges and the coupling of a bath for the evaporation of an AdS black hole [19]. We then briefly discussed an ER=EPR paradox, and finally commented on traversability of wormholes and ER=EPR [20, 21]. While this discussion has not been very sophisticated with a lot of mathematics or lengthy elaboration, in the next Part, we will try to be a little more elaborate with some recent developments in the particular direction of Alheiri, Engelhardt, Marolf and Maxfield's work [27], along with Pennington's work on entanglement wedge reconstruction and BHIP [6].

5 de Sitter Quantum Gravity

Note: *This section is based off a review I had written with Aayush Verma, which can be read from* https://vkalvakotamath.github.io/files/dSnote.pdf*, also accessible from* https://aayushayh.github.io/dSnote.pdf*.*

de Sitter holography is naturally motivated from AdS/CFT, since anti-de Sitter space and de Sitter space are related via double Wick rotations. Another natural motivation is that our universe is approximately de Sitter, as would be any suitable universe via the cosmic No Hair theorem. Therefore, a holographic description of de Sitter quantum gravity will do much to make sense of holographic proposals in our cosmology. In this section, we will talk about the notion of holography in the global de Sitter and static patch settings, and discuss the corresponding entanglement entropy proposals and some relevant aspects.

5.1 dS/CFT Correspondence

The dS/CFT correspondence tells us that in the global de Sitter coordinates, the dual CFT to the bulk dS in D dimensions is the $D-1$ boundary at $\mathcal{I}^{\pm,}$. In global coordinates, from $t=0$ to some t_∞, we start from the half-sphere with Ψ_{dS}, given by

$$\Psi_{\text{dS}}[g_0, \Phi_0] = \int \mathcal{D}g\mathcal{D}\Phi \; e^{iS[g,\Phi]} \;, \tag{106}$$

which satisfies the WDW equation and the boundary condition for the metric,

$$g = g_0|_{t_\infty} \;.$$

We then evolve this hemisphere into the full Lorentzian geometry, so that the boundary CFT at t_∞ describes the CFT correlators dual to the bulk. These can be used to compute the dual n-point functions of bulk fields in a similar way to that of AdS/CFT. So, placing $|\Psi\rangle$ on the half-sphere, one can compute usefully the two-point correlator $\langle \Phi(x_1\Phi(x_2))\rangle_\Psi$ from the full path integral. The overall dictionary then looks like

$$\Psi_{dS}[g,\Phi] \sim Z[g,\Phi] \;. \tag{107}$$

In this dS/CFT framework, one can similarly make sense of the extrapolate dictionary, but with a subtlety; there are in fact *two* dictionaries depending on whether one approaches the picture from only one boundary or both. The former is referred to as the *differentiate* dictionary as used by Maldacena, and the latter is referred to as the *extrapolate* dictionary. For this, one can reproduce the HKLL construction and work with bulk reconstruction taking the fields to the asymptotic limit of the boundary. However, this discrepancy is also partly attributable to the complex-valued conformal weight, which looks something like

$$\Delta_\pm = \frac{1}{2}\left[(D-1)^2 \pm \sqrt{(D-1)^2 - 4(ml_{\text{dS}})^2}\right] \;. \tag{108}$$

One can also do *static patch holography*, which has the stretched horizons as the holographic duals to the pode-antipode systems in the static patch. This has had an interesting development involving operator algebras (particularly type II_1 algebras) by Chandrasekharan, Longo, Pennington and Witten. However, for this thesis, we will exclude a discussion of those. The interested reader is directed to `https://vkalvakotamath.github.io/files/dSnote.pdf`.

5.2 Timelike Entanglement Entropy

In timelike entanglement entropy calculations, we consider a timelike geodesics rather than spacelike ones to find the entanglement entropy. Let A be a subsystem with some spacelike and timelike components X, T. Then, the entanglement entropy is

$$S[R(X,T)] = \frac{c}{3} \log \frac{\sqrt{X^2 - T^2}}{\epsilon} \ , \tag{109}$$

which gives the RT formula when $T = 0$. One can instead set $X = 0$ to get timelike entanglement entropy, which becomes

$$S[R(T)] = \frac{c}{3} \log \frac{T}{\epsilon} + \frac{ic\pi}{6} \ . \tag{110}$$

To illustrate this a little more explicitly, consider some scalar field theory setting. For this, the partition function is

$$\mathcal{Z}[\phi] = \int \mathcal{D}\phi e^{iS[\phi]}, \tag{111}$$

where the field ϕ has the usual Lagrangian of a scalar field without mass. We now perform Wick rotations to the coordinates so that the t coordinate plays the role of space, and x plays the role of Wick rotated time. By further imposing some "real" time coordinate T condition, $x = iT$, the action looks like

$$S = \frac{i}{2} \int dT dt \left[(\partial_T \phi)^2 + (\partial_t \phi)^2 \right] \ , \tag{112}$$

and from this one gets the Hamiltonian, which can be written as

$$H = \frac{1}{2} \int dt \left[\pi^2 + (\partial_t \phi)^2 \right] \ , \tag{113}$$

where $\pi = i\partial_x \phi$ is the conjugate canonical momentum to ϕ. Then, the partition function becomes $\mathcal{Z}[\phi] = \operatorname{Tr} e^{i\beta H}$ (where β is the periodicity). However, the density matrix attributed here is no longer a hermitian density matrix but one that can give complex valued entanglement entropy, and is therefore non-hermitian. The entanglement entropy corresponding to this would be the timelike entanglement entropy (110). A slightly more sophisticated version of this situation can be arrived at by instead taking the Renyi approach with the replica trick. As usual, we are considering a subsystem A with endpoints $A(T_a, X_a)$ and $B(T_b, X_b)$, and taking the analytic continuation of the limit of $N \to 1$. With this, we arrive back at the previous form (109),

$$S_R^{N\to 1} = \frac{c}{3} \log \frac{\sqrt{(X_b - X_a)^2 + (T_b - T_a)^2}}{\epsilon} \,, \tag{114}$$

and setting the spacelike component to zero, we get back the timelike entanglement entropy expression,

$$S_R^{N\to 1} = \frac{c}{3} \log \frac{T}{\epsilon} + \frac{ic\pi}{6} \,.$$

Since this has a non-hermitian density matrix, timelike entanglement entropy can be given the interpretation of pseudo holographic entanglement entropy, which suitably describes entanglement entropy in global de Sitter. There are two versions of making sense of entanglement entropy in de Sitter; one in global dS and the other in static patch dS.

5.3 Pseudo Entropy

Before approaching de Sitter holographic entanglement entropy and pseudo entropy, we will first make sense of *what* pseudo entropy is. Let $|\psi\rangle$ and $\langle\phi|$ be two pure states and the inner product be nonzero. We define the transition matrix,

$$\mathcal{T} \equiv \mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} \,, \tag{115}$$

for which the trace is normalized to unity. The trace of the N^{th} power of the transition matrix also follows this trace property since one can see that $(\mathcal{T})^N = \mathcal{T}$, and as a result the traces must also be equal to unity. Defining a bipartitioned Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, we can define the reduced transition matrices for each subsystem. Choosing say A, we then trace out $\mathcal{T}$ w.r.t B to define

$$\mathcal{T}_A = \mathrm{Tr}_B \left[\frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} \right] \,. \tag{116}$$

For this one can define the N-th Renyi entropy:

$$\mathcal{S}^N(\mathcal{T}_A) = \frac{1}{1-N} \log \mathrm{Tr} \left(\mathcal{T}_A^N \right) \,. \tag{117}$$

Since the matrix $\mathcal{T}$ is not Hermitian in particular, the entropy defined by this matrix is allowed to take complex values. For the moment, we will assume that this necessarily has a non-hermitian nature. Then, due to the non-hermitian nature of this matrix (and $\mathcal{T}_A$ in general), the complex values of (117) interpret this Renyi entropy as a “pseudo-entropic” quantity. Setting $N \geq 2$ and positive natural number values, one can establish that the N-th Rényi entropy can be expressed in terms of the eigenvalues of $\mathcal{T}_A$ as

$$\mathcal{S}^N(\mathcal{T}_A) = \frac{1}{1-N} \log \left(\sum_a \lambda_a \left(\mathcal{T}_A\right)^N \right) . \tag{118}$$

Since we do not take $\mathcal{T}$ to be hermitian, diagonalization is a non-trivial issue. As stated in [28], one can go around this by Jordan decomposition of $\mathcal{T}_A$ to arrive at (118), which we will not discuss here. Taking the the $N \to 1$ limit of the N-th Renyi entropy, one gets the pseudo von Neumann entropy, which is interpreted as the pseudo-entanglement entropy defined from the transition matrix $\mathcal{T}_A$:

$$S_{vN}\left(\mathcal{T}_A\right) = \lim_{N\to 1} \mathcal{S}^N(\mathcal{T}_A) \Longrightarrow -\sum_a \lambda_a\left(\mathcal{T}_A\right) \log\left(\lambda_a\left(\mathcal{T}_A\right)\right) . \tag{119}$$

In the sense of QFTs, one can compute the pseudo entanglement entropy using the replica trick, using which one can find the N-th Renyi entropy in the path integral formulation. This is done by taking the path integral over manifolds for some Euclidean action $S[\varphi]$ (where φ is the field configuration), and considering the products $|\psi\rangle\langle\phi|$ and $\langle\phi|\psi\rangle$ as the path integrals giving the transition matrix $\mathcal{T}$.

Pseudo entropy has some basic features, essentially amounting to the following conditions choosing positive real values of N: if a state has no entanglement, the corresponding N-th Renyi entropy is equal to zero. Next, the N-th Renyi entropy computed w.r.t the reduced transition matrix $\mathcal{T}_A$ and the N-th Renyi entropy computed w.r.t. the reduced transition matrix $\mathcal{T}_B$ are equal to each other. Finally, choosing positive real values of N except unity, the N-th Renyi entropy satisfies $\mathcal{S}^N(\mathcal{T}_A) = \mathcal{S}^N(\mathcal{T}_A)^*$.

5.4 Global dS/CFT

As seen previously, the global dS case can be highlighted similarly to that of the AdS/CFT case by looking at the Gubser-Klebanov-Polyakov-Witten dictionary,

$$\Psi_{\text{dS}}[\varphi] = \mathcal{Z}[\varphi] , \tag{120}$$

where φ acts as the generating functional on the boundary and gives the boundary conditions for fields:

$$\Psi_{\text{dS}}[\varphi_0] = \int_{\varphi_0 \equiv \infty} \mathcal{D}\varphi \exp\left(iS_{\text{dS}}[\varphi]\right) \Psi_0 , \tag{121}$$

where Ψ_0 is an initial state at $t = 0$ and we have the boundary condition φ_0 at the future asymptotic boundary. Then, one can see that the partition function $\mathcal{Z}[g, \varphi]$ takes complex values. In order to give this a geometric interpretation, start by considering the $D + 1$ dS in global coordinates,

$$ds^2 = l_{\text{dS}}^2 \left(-dt^2 + \cosh^2 t d\Omega_D^2\right) . \tag{122}$$

We will now perform a Wick rotation by $t \to i\tau$ for considering the Hartle-Hawking state, which gives us the Euclidean de Sitter space,

$$ds^2 = l_{\text{dS}}^2 \left(d\tau^2 + \cos^2 \tau d\Omega_D^2 \right) \ . \tag{123}$$

In general, the central charge of the D-CFT has a complex value in the dS/CFT framework. To see this better, one can consider the relation between the AdS and dS length scales, given by

$$l_{\text{AdS}} \longrightarrow -il_{\text{dS}} \ , \tag{124}$$

from which one gets the central charge of the CFT in dS. This can be seen from the Brown-Henneaux central charge formula, from which we see that the central charge is $c_{\text{AdS}} = \frac{3l_{\text{AdS}}}{2G_N}$, which gives the central charge for dS_3/CFT_2,

$$c_{\text{dS}} = \frac{3R_{\text{dS}}}{2G_N} \ , \tag{125}$$

and we have $c = -ic_{\text{dS}}$. Overall, it is clear that in the dS/CFT framework, the dual CFT is non-unitary. Indeed, one can arrive at the timelike entanglement entropy in dS by a double Wick rotation, defined by $R_{\text{AdS}} = -iR_{\text{dS}}$, $z = -i\eta$, $t = -ix$

$$ds^2 = R_{\text{dS}^2} \left(\frac{-d\eta^2 + d\tau^2 + dx^2}{\eta^2} \right) \ . \tag{126}$$

With this in mind, one gets the timelike entanglement entropy in de Sitter space as:

$$S(A) = -i\frac{c_{\text{dS}}}{3} \log \frac{x}{\epsilon} + \frac{\pi c_{\text{dS}}}{6} \ . \tag{127}$$

One can easily see that this entropy has the exact interpretation as that of pseudo entropy. We could now be interested in the higher-dimensional notions of timelike entanglement entropy, which we will discuss below.

In the Poincaré dS_{D+1} coordinates, which would be given by the metric

$$ds^2 = R_{\text{dS}}^2 \left(\frac{-d\eta^2 + dt_E^2 + d\mathbf{x}^2 + dy^2}{\eta^2} \right) \ , \tag{128}$$

where t_E is related to t as $t \longrightarrow -t_E$. As considered in [29], we will consider the subsystem R on a $y = 0$ slice with radius $T/2$ and introduce a radial coordinate $\mathbf{r} = \sqrt{t_E^2 + \mathbf{x}^2}$. We then consider a timelike surface given by η, $\mathbf{r}$ and T values, and the area of this surface (including the real part) by $-\eta^2 + \mathbf{r}^2 = T^2/4$ (and $\eta^2 - \mathbf{r}^2 = T^2/4$ respectively) generates the pseudo entropy, which comes in the case-specific values of even or odd D:

$$S(A) = \frac{R^{D-1}}{4G_N^{D+1}} \left[\left(\text{vol}(\mathbb{S}^{D-2}) \right) \frac{\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)}{2\Gamma\left(\frac{D}{2}\right)} + i\mathcal{D} \right] \ , \tag{129}$$

where $\mathcal{D}$ is given by

$$\begin{cases} \sum_{a=0}^{\frac{D-3}{2}} \binom{\frac{D-3}{2}}{a} \frac{1}{D-2a-2} \left(\frac{T}{2\epsilon}\right)^{D-2a-2} & \text{odd } D \, , \\ \sum_{a=0}^{\frac{D-3}{2}} \binom{\frac{D-3}{2}}{a} \frac{1}{D-2a-2} \left(\frac{T}{2\epsilon}\right)^{D-2a-2} + \frac{\Gamma\left(\frac{D-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{D}{2}\right)} \log \frac{T}{2\epsilon} & \text{even } D \, . \end{cases} \tag{130}$$

In the sense of the Wick rotations considered previously to arrive at timelike entanglement entropy, one can start by the length scale transformations and the T transformations, and define the geometric rotation

$$\mathrm{vol}\left(\mathbb{S}^{D-2}\right) \longrightarrow (-i)^{D-2}\mathrm{vol}\left(\mathbb{H}^{D-2}\right) \, , \tag{131}$$

from which one gets the higher dimensional case in EAdS/CFT as discussed previously.

5.5 Static Patch Entanglement Entropy

Static patch holography offers a slightly neater way of approaching entanglement entropy. One can start by taking the usual RT prescription and note that the extremal surface corresponding to the entanglement entropy of the pode-antipode system lies between the two stretched horizons, which are the boundaries for the system. So one can make sense of the de Sitter Ryu-Takayanagi formula by taking this extremal surface to describe the entanglement entropy.The proposal then is to take a minimal surface $\mathcal{X}_{\mathrm{dSRT}}$ that is homologous to the boundary corresponding to either component. The entanglement entropy is then given by

$$\frac{\text{Area of } \mathcal{X}_{\mathrm{dSRT}}}{4G_N} \, . \tag{132}$$

However, clearly this has an issue: one could "flow" this surface to either side and obtain a zero area surface, which is clearly not correct. In order to go around this, Susskind and Shaghoulian argued that this has to necessarily lie *in between* the stretched horizon degrees of freedom. This way, we have

$$S = \frac{\text{Area of dS horizon}}{4G_N} \, , \tag{133}$$

which gives us the appropriate Gibbons-Hawking entropy.

As shown by Susskind and Shaghoulian, one can make this formalism more sophisticated by adopting the bit threads formalism, which is described as follows.

In the bit threads formalism to entanglement entropy, one considers *bit threads* of fixed width $1/4G_N$ as sourced from the boundary. In the AdS/CFT sense, this can be described as follows: take a boundary subregion ∂A, and let it source bit threads (the origin of bit threads is from the sense of a max-flow-min-cut theorem in Riemannian geometry concerning the area-minimizing representative of some $m(\partial A)$ (the "bottleneck") and flow-maximizers). Then, the Ryu-Takayanagi formula can be re-derived by by considering

$\max \#$ of bit threads leaving the boundary subregion ∂A and minimizing the area of a surface homologous to ∂A anchored to the boundary:

$$S(\partial A) = \frac{1}{4G_N} \max_{\#\text{threads}} \leftrightarrow \min_{\mathcal{X} \sim \partial A} \text{Area of } \mathcal{X} \ , \tag{134}$$

By applying the max-flow-min-cut theorem, the minimizing term is equivalent to maximizing the flows v originating from ∂A, becoming redundant and reducing to

$$\max_{\#\text{threads}} \int_{\partial R} v \ . \tag{135}$$

One can then re=derive the HRT prescription in terms of bit threads in the form of a covariant bit threads formula. The HRT in maximin terms is of the form

$$S(\partial A) = \max_{\partial A \subset \Sigma} \min_{\mathcal{X} \sim \partial A} \text{Area of } \mathcal{X} \ . \tag{136}$$

We then apply the Lorentzian max-flow-min-cut theorem by maximizing the flows v on Σ:

$$\max_{\Sigma \sim \partial\partial A} \text{vol}(\Sigma) = \min_{\text{flows } v} \int_{\partial A} n \cdot v \ , \tag{137}$$

where by $\partial\partial A$ we mean the boundary of the boundary subregion ∂A. Then,

$$S(\partial A) = \max_{\partial\partial A \subset \Sigma} \max_{v \text{ in } \Sigma} \int_{\mathcal{E}(\partial A)} n \cdot v \ , \tag{138}$$

where $\mathcal{E}(\partial A)$ is the union of Σ and the domain of dependence $D(A) = D^+(A) \cup D^-(A)$. Similar to the vector field v, where $\nabla \cdot v = 0$ on the spacelike slice, we generalize this to the full Lorentzian manifold by introducing a vector field V with same properties as v – therefore, $\nabla \cdot V = 0$. On $I^{\pm}(\partial\partial A)$, we will introduce some boundary conditions so that the role of Σ is replaced by a "smearing" field ϕ. Then, with the boundary conditions so that

$$\phi_{D^-(\partial\partial A)} = 0 \quad \text{and} \quad \phi|_{D^+(\partial\partial A)} = 1 \ ,$$

we have the bit threads formulation of HRT formula:

$$S(\partial A) = \max_{V,\phi} \int_{D(\partial A)} n \cdot V \ . \tag{139}$$

We can then apply the bit threads formalism to static patch de Sitter by identifying bit threads sourced from either component. However, one could source these bit threads in a monolayer or bilayered fashion, where the bit threads are sourced from both sides of the component. In the monolayer proposal, these bit threads originate only towards one component and do not cross the horizon. Applying the usual bit threads formulation for the we take points on the horizons, "anchor" them so as to connect them via some codimension 1 surface. The bit threads then span the space between these anchor points. The bottleneck for this system would be at the horizon, giving us the Gibbons-Hawking entropy as before. In the bilayer proposal, we have bit threads going towards the horizon and towards the other component. We consider the largest components as the only sources of bit threads, so that cosmic horizons are the only sources of bit threads. Therefore, we take the horizon and source bit threads towards the antipode (or pode) and also in the usual sense of the monolayer proposal. Since the bit threads emitted towards the "bulge" end up at a bottleneck of vanishing area, there is no difference between the monolayer and the bilayer proposals, and boils down to the usual Gibbons-Hawking entropy.

6 Canonical Quantum Gravity

An interesting aspect of AdS/CFT is that the holographic description is also somewhat implicit from the earliest formalism of quantum gravity, formulated by Wheeler and DeWitt. In this section, we will highlight some important aspects of making sense of holographic quantum gravity. We will primarily talk about how doing canonical quantum gravity at asymptotics motivates holography the way we understand it. The starting point is from the importance of *asymptotic quantization* schemes.

In a holographic theory, the operators dual to a bulk field live in a boundary CFT and the CFT functional $\mathcal{Z}[g,\Phi]$ lives on this boundary. This way, the holographic description looks something like

$$\Psi[g,\Phi] \sim \mathcal{Z}[g,\Phi] \ , \tag{140}$$

where the Wheeler-DeWitt (WDW) state $\Psi[g,\Phi]$ satisfies the Hamiltonian and momentum constraints of canonical quantum gravity:

$$\mathcal{H}\Psi[g,\Phi] = 0 \ , \mathcal{D}_a\Psi[g,\Psi] = 0 \ . \tag{141}$$

The Hamiltonian constraint is the WDW equation, which is a very complicated thing to solve. One could ask the following: if you rescale the metric

$$g \longrightarrow \tilde{g} \tag{142}$$

so that the $\tilde{g} \to 0$ limit denotes the asymptotic bulk limit, what would be the nature of the duality (140)? This was done in the AdS/CFT case by Friedel, who showed that in such asymptotic limit, the dictionary looks something like

$$\lim_{\tilde{g}\to 0} \Psi[\tilde{g},\tilde{\Phi}] \sim \mathcal{Z}[\tilde{g},\tilde{\Phi}] \ , \tag{143}$$

where we also suitably rescale the matter fields $\Phi \to \tilde{\Phi}$. In this case, we could ask if the reverse would have been true; that is, could one arrive at the usual holographic dictionary from this asymptotic quantization process? Clearly, if one starts from asymptotic quantization, we have something like

$$\lim_{\tilde{g}\to 0} \Psi[\tilde{g},\tilde{\Phi}] = e^{iS[\tilde{g},\tilde{\Phi}]}\mathcal{Z}^{+}[\tilde{g},\tilde{\Phi}] + e^{-iS[\tilde{g},\tilde{\Phi}]}\mathcal{Z}^{-}[\tilde{g},\tilde{\Phi}] \ , \tag{144}$$

where the functionals $\mathcal{Z}^{\pm}[\tilde{g},\tilde{\Phi}]$ have some interesting features, in that they are CFT-like functionals. To illustrate this in a de Sitter context, we will also discuss the result on the Hilbert space of de Sitter quantum gravity by Suvrat et al, 2023.

It was shown that the WDW states in de Sitter quantum gravity on "late-time" slices (in that the volume acts as a clock, implying asymptotic quantization) take the form

$$\Psi[\tilde{g},\tilde{\Phi}] = e^{iS[\tilde{g},\tilde{\Phi}]}\mathcal{Z}[\tilde{g},\tilde{\Phi}] \ . \tag{145}$$

Here, $e^{iS[\tilde{g},\tilde{\Phi}]}$ is a universal phase factor, which in the AdS/CFT counterpart act as a set of counterterms. This is also observed in the context of Cauchy slice holography, where we deform the theory so that the holographic dictionary lives on Cauchy slices, which will be discussed later below. The functional $\mathcal{Z}[\tilde{g},\tilde{\Phi}]$ is of interest, since it obeys Diff$\times$Weyl properties the way one would expect a CFT partition-like functional to behave. It obeys diffeomorphism invariance and Weyl transformation with invariance upto an anomaly, so that for instance in $D = 2$, we have something like

$$\left(2\tilde{g}_{ab}\frac{\delta}{\delta\tilde{g}_{ab}} - \Delta\Phi\frac{\delta}{\delta\Phi}\right)\mathcal{Z}[\tilde{g},\tilde{\Phi}] = \mathcal{A}_2\mathcal{Z}[\tilde{g},\tilde{\Phi}] \;, \quad \mathcal{A}_2 = \frac{1}{16\pi G_N} i\sqrt{\tilde{g}}\mathcal{R} \;. \tag{146}$$

One can now look at this in the context of deformations, by interpreting the $\tilde{g} \to 0$ limit as being the limit in letting a deformation parameter $\mu \to 0$.

To interpret this in the AdS framework, the deformation parameter μ tells us how the boundary ∂M moves into the bulk. Denoting the final deformed boundary by ∂M^μ, the $\mu \to 0$ limit restores the deformed boundary to i^0. In this fashion, the general deformed theory is the limit

$$\lim_{\mu\to\epsilon} \Psi^\epsilon[\tilde{g},\tilde{\Phi}] = e^{C(\epsilon)}\mathcal{Z}^\epsilon[\tilde{g},\tilde{\Phi}] \;. \tag{147}$$

In the de Sitter context above, by turning off these deformations, we arrive at the usual dS/CFT correspondence, where the functional $\mathcal{Z}[\tilde{g},\tilde{\Phi}]$ becomes the generating functional $Z[g,\Phi]$ for the CFT correlators and $\Psi[g,\Phi]$ tells us about bulk correlators, so as to arrive at

$$\langle\Phi_1 \ldots \Phi_n\rangle_\Psi \sim \langle\mathcal{O}_1^\Phi \ldots \mathcal{O}_n^\Phi\rangle_Z \;. \tag{148}$$

This is the motivation towards Cauchy slice holography, where one takes such deformations (more accurately, $T\overline{T}$-deformations, which are referred to as T^2 deformations in $D > 2$) to find a holographic duality between Ψ^Σ and $\mathcal{Z}^\Sigma$ on Cauchy slices Σ. For that matter, this is not the only way to arrive at holography from the purely canonical approach. Holography in AdS/CFT can be expressed as the global extension of *holography of information*, where the information on a Cauchy slice is localized onto the boundary of the Cauchy slice. This way, holography of information tells us that $\partial\Sigma$ contains all information from Σ, which at first sight seems to be strange, since the split property tells us that the information outside a bounded region on Σ could be unavailable to an observer in the region. That is, given a bounded region $\mathcal{U} \subset \Sigma$, in the lattice regularization, we define a collar around $\mathcal{U}$, denoted by ε, so that the density matrix for $\mathcal{U} \cup \varepsilon$ can be prepared independently to that of the complement, yielding a type I factor for the full Hilbert space. However, holography of information tells us otherwise, and in this sense the split property is not fully applicable to the non-gravitational holographic picture. To this end, the notion of split property in curved spacetime has not found its way into the mainstream picture of holography, and involves refining some aspects of perturbative algebraic quantum field theory such as locally covariant QFTs. See some comments on this in [10] and references therein.

In CSH, one turns on deformations from the T^2 operator, which labelled by a deformation parameter tells us how the partition function moves into the bulk, and the corresponding notion of RG flow. In the $\mu \to \epsilon$ limit of these deformations, the partition function on the Cauchy slice will be labelled by $\mathcal{Z}^{\Sigma}$, and is given by

$$Z^{\Sigma}[g, \Phi] = e^{\mathrm{CT}} \left(\mathcal{P} \exp \int_0^{\lambda} \frac{d\lambda}{\lambda} O(\lambda) \right) Z^{\Sigma}_{\mathrm{CFT}}[g, \Phi] \;, \tag{149}$$

There are many other aspects involving making sense of holography from the WDW equation, but the cutoff ϵ for the PSI application is getting close, and I must conclude this thesis here.

7 Conclusion

The point of this thesis was two-fold. One, to discuss some very elementary aspects of the very rapidly changing landscape of holographic quantum gravity, and two, to show that the present status of high energy physics theory is at the peak of its evolution, where one is able to make sense of many operator algebraic aspects of things like entanglement entropy, modular flows and type II_∞ algebras in the $1/N$ limit, states in de Sitter static patch, the Hilbert space of de Sitter quantum gravity, etc. We have discussed in very pedagogical ways some of the aspects of AdS/CFT and de Sitter quantum gravity.

We started by discussing some aspects of light-sheets and entropy bounds, which motivated the inclusion of light-sheets in holographic entanglement entropy. We discussed singularity theorems from entropy, semiclassical quasilocal mass inequalities and a basic notion of holographic entanglement entropy. We then discussed some information theoretic aspects of AdS/CFT. We started from RT and HRT, and then we discussed some aspects of bulk reconstruction and subregion-subregion duality, where we reviewed the extrapolate dictionary, HKLL reconstruction, subregion duality, modular flows in the sense of Faulkner and Lewkowycz, relative entropy and the JLMS formula, and some aspects of operator algebras. We then discussed some results in the black hole information problem, where we first elaborated on the BHIP in usual flat space. Here, we discussed the Page curve, generalized second law, complementarity and the notion of firewalls from monogamy. We then discussed some brief aspects of BHIP in AdS/CFT, where we focused on the Raju-Papadodimas proposal for an infalling observer, entanglement wedges, an ER=EPR paradox and outlined some aspects of firewalls. We then discussed de Sitter quantum gravity, where we reviewed the dS/CFT correspondence, timelike entanglement entropy, pseudo entropy as the appropriate form of holographic entanglement entropy in dS/CFT, and static patch entanglement entropy from bit threads. We then commented on canonical quantum gravity and holography.

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