Binet's formula for the Fibonacci sequence

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1 Introduction

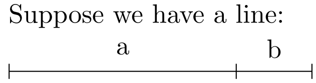
Okay, so this is just my take on Binet's formula that derives the closed-form Fibonacci number expression. This pretty much copies the proof given on page 30 of "Elementary number theory in nine chapters" by J. Tattersall, except that I try to justify the intuition why those steps were taken because I had some trouble doing it on my own.

2 Proof

2.1 Golden ratio

To begin, let us remind ourselves of how the golden ratio is defined.

Suppose we have a line:



$\frac{(a+b)}{a} = \frac{a}{b} = \phi$

Multiplying by ϕ we get the following quadratic equation: $\phi^2 = \phi + 1$

Solving it with respect to ϕ gives us the golden ratio $\phi = \frac{(1+\sqrt{5})}{2}$ and its conjugate $\hat{\phi} = \frac{(1-\sqrt{5})}{2}$.

If we further multiply this quadratic expression by ϕ^n we will get

$$\phi^{n+2} = \phi^{n+1} + \phi^n$$

And likewise, the same will hold for the conjugate:

$$\hat{\phi}^{n+2} = \hat{\phi}^{n+1} + \hat{\phi}^n$$

2.2 The great reveal

Reminds you of something?

Like, maybe that Fibonacci sequence we wanted to represent with the golden ratio?

$$F_{n+2} = F_{n+1} + F_n$$

I wonder if F can be represented as a linear combination of ϕ and $\hat{\phi}$... *wink wink nudge nudge*

$$F_n = C_1\phi^n + C_2\hat{\phi}^n$$

Well, since $F_1 = 1$ and $F_2 = 2$, we can just substitute the numbers and solve the system easily.

Result?

$$C_1 = \frac{1}{\sqrt{5}}, C_2 = -\frac{1}{\sqrt{5}}$$

thus

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \tag{1}$$

3 Consecutive quotient limit

As a bonus, consider the following limit:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

By using the formula above we get:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\phi^{n+1} + \hat{\phi}^{n+1}}{\phi^n + \hat{\phi}^n} = \lim_{n\to\infty} \frac{\phi + \hat{\phi}(\frac{\hat{\phi}}{\phi})^n}{1 + (\frac{\hat{\phi}}{\phi})^n}$$

and since

$$\lim_{n\to\infty} (\frac{\hat{\phi}}{\phi})^n = \lim_{n\to\infty} (\frac{1-\sqrt{5}}{1+\sqrt{5}})^n = 0$$

we finally get

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \tag{2}$$

In other words, the ratio of two consecutive Fibonacci numbers approaches the golden ratio as the numbers grow larger! This was first shown by Kepler back in the XVII century.

References

[1] James Tattersall *Elementary number theory in nine chapters* page 30.

[2] Cormen, Leiserson, Rivest *Introduction to algorithms* p.60, but they invite the reader to prove it by induction which again isn't very conducive to getting that "aha!" moment.