Abstract

In this paper, we propose economic model predictive control with guaranteed closed-loop properties for supply chain optimization. We propose a new multiobjective stage cost that captures economics as well as risk at a node, using a weighted sum of an economic cost and a tracking stage cost. We also demonstrate integration of scheduling with control using a supply chain example. We integrate a scheduling model for a multiproduct batch plant with a control model for inventory control in a supply chain. We show recursive feasibility of such integrated control problems by developing simple terminal conditions.

Economic model predictive control for inventory management in supply chains

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1. Introduction

Control theory for inventory management in supply chains is an important area of research [23, 19]. Over the past decade, rolling horizon optimization for inventory management has been studied [20, 17, 4, 24, 7, 2, 13, 14]. An important consideration in process control is the stability of the closed-loop under the proposed control. In [12, 28, 6], the closed-loop stability for classical control theory applied to inventory management was studied. However, the stability of rolling horizon optimization methods for inventory management has not been studied extensively. Model Predictive Control (MPC) is a rolling horizon optimization based control algorithm with guaranteed stability properties. Although, stability theory for MPC is a fairly mature field [22], most implementations of MPC for supply chain do not consider stability. In [27], we proposed centralized and cooperative MPC with stability and asymptotic convergence guarantees for a supply chain tracking inventories to its setpoint. Our focus in this paper is to use recent developments in Economic MPC [1, 3] to develop a closed-loop stable model predictive controller for supply chains in which we directly optimize the economic cost of operating the supply chain.

This paper is organized as follows. In Section 2, we briefly describe the main results of Economic MPC. In Section 3, we derive the state space model for a two-node supply chain. In Section 3.1, we show the results for economic MPC on a simple two-node supply chain. In Section 3.2, we propose a novel multiobjective MPC formulation that accounts for supply chain economics as well as risk. In Section 4 we implement MPC for a multiechelon, multiproduct supply chain example. In Section 5, we use periodic terminal condition ideas from [26] to integrate scheduling with inventory control. Finally, we present our conclusions in Section 6.

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2. Economic MPC

Note. We only state important Economic MPC stability results in this section. The interested reader can refer to [1, 3] for details.

Model.. We consider the following linear model

$$x^+ = Ax + Bu + B_d d \tag{1}$$

in which $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the manipulated input and $d \in \mathbb{R}^d$ is the disturbance to the system. We develop economic MPC theory for the nominal disturbance, denoted by d_s. We assume that the system (A, B) is stabilizable.

Constraints.. The states and inputs are constrained as follows:

$$x \in \mathbb{X} \qquad u \in \mathbb{U} \tag{2}$$

Stage cost.. The economic cost for implementing input u from state x is given by $\ell(x, u)$.

Optimal steady state.. We define the steady-state problem for the nominal demand d_s as follows:

$$\min_{x,u} \ell(x, u) \qquad \text{s.t. } x = Ax + Bu + B_d d_s, \quad x \in \mathbb{X}, u \in \mathbb{U} \tag{3}$$

The optimal steady state is denoted by $(x_s, u_s; d_s)$

We make the following assumptions:

Assumption 1. The constraint set $\mathbb{X}$ is convex and closed. The constraint set $\mathbb{U}$ is convex and compact. The optimal steady state $(x_s, u_s; d_s)$ is such that $x_s \in \mathbb{X}$ and $u_s \in \mathbb{U}$

Assumption 2. There exists $(x_s, u_s; d_s)$ and λ_s so that

(a) $(x_s, u_s; d_s)$ is a unique solution of (3).

(b) The multiplier λ_s is such that $(x_s, u_s; d_s)$ uniquely solves (4)

$$\min_{x,u} \ell(x, u) + \lambda_s'[x - (Ax + Bu + B_d d_s)] \quad \text{s.t. } x \in \mathbb{X}, u \in \mathbb{U} \tag{4}$$

(c) The system $x^+ = Ax + Bu + B_d d_s$ is strictly dissipative with respect to the supply rate $s(x, u) = \ell(x, u) - \ell(x_s, u_s)$ and storage function $\lambda(x) = \lambda_s' x$. That is, there exists a positive definite function $\rho(\cdot)$ such that for all $(x, u) \in \mathbb{X} \times \mathbb{U}$

$$\lambda_s'(Ax + Bu + B_d d_s - x) \leq -\rho(x - x_s) + s(x, u) \tag{5}$$

Assumption 3 (Basic stability assumption)**.** There exists a convex, compact terminal region $\mathbb{X}_f \subseteq \mathbb{X}$, containing the point x_s and a control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ and a function $V_f : \mathbb{X}_f \to \mathbb{R}$ such that the following holds for all $x \in \mathbb{X}_f$

$$V_f(Ax + B\kappa_f(x) + B_d d_s) \leq V_f(x) - \ell(x, \kappa_f(x)) + \ell(x_s, u_s) \tag{6}$$
$$Ax + B\kappa_f(x) + B_d d_s \in \mathbb{X}_f \tag{7}$$

We first define the terminal constraint MPC problem

$$\begin{aligned}
\mathbb{P}_N(x; d_s) : \min_{\mathbf{u}} \; & V_N(\mathbf{u}; x) \\
& \text{s.t. } x(0) = x \\
& x(j+1) = Ax(j) + Bu(j) + B_d d_s, && j \in \mathbb{I}_{0:N-1} \\
& x(j) \in \mathbb{X} && j \in \mathbb{I}_{0:N-1} \\
& u(j) \in \mathbb{U} && j \in \mathbb{I}_{0:N-1} \\
& x(N) = x_s
\end{aligned} \tag{8}$$

in which the cost function $V_N(\mathbf{u}; x)$ is the sum of stage costs

$$V_N(\mathbf{u}; x) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) \tag{9}$$

The control horizon is denoted by N, the input sequence $\mathbf{u} = (u(0), u(1), \ldots, u(N-1))$ and the symbol $\mathbb{I}_{k:l}$ stands for the set $\{k, k+1, \ldots, l\}$.

The control law $\kappa(x)$ is the first input in the optimal solution $\mathbf{u}^0(x)$ to optimization problem (8). The admissible region $\mathcal{X}_N$ is given by

$$\mathcal{X}_N := \left\{ x \in \mathbb{X} \mid \exists \mathbf{u} \in \mathbb{U}^N, \text{ s.t. (8) is feasible} \right\}$$

The following is the exponential stability theorem for economic MPC that solves problem (8) online [3].

Theorem 4 (Lyapunov function with terminal constraint)**.** *Let the system (A, B) be stabilizable. Let Assumptions 1 and 2 hold. Then the steady-state solution of the closed-loop system $x^+ = Ax + B\kappa(x) + B_d d_s$ is asymptotically stable with $\mathcal{X}_N$ as the region of attraction. The Lyapunov function is*

$$\tilde{V}(x) := V_N^0(x) + \lambda_s'[x - x_s] - N\ell(x_s, u_s)$$

in which $V_N^0(x)$ is the optimal cost function of (8)

We also define the terminal region/penalty MPC problem as follows:

$$\begin{aligned}
\mathbb{P}_N(x; d_s) : \min_{\mathbf{u}} \; & V_N(\mathbf{u}; x) \\
& \text{s.t. } x(j+1) = Ax(j) + Bu(j) + B_d d_s, && j \in \mathbb{I}_{0:N-1} \\
& x(j) \in \mathbb{X} && j \in \mathbb{I}_{0:N-1} \\
& u(j) \in \mathbb{U} && j \in \mathbb{I}_{0:N-1} \\
& x(N) \in \mathbb{X}_f
\end{aligned} \tag{10}$$

in which the cost function $V_N(\mathbf{u}; x)$ is

$$V_N(\mathbf{u}; x) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N)) \tag{11}$$

Note that in contrast to objective function (9), we modify the cost function in the terminal penalty formulation by adding a terminal penalty $V_f(x(N))$. Similarly, in the optimization problem (10), the terminal constraint is replaced by the terminal region. The terminal region $\mathbb{X}_f$ and the terminal penalty $V_f(\cdot)$ are chosen to satisfy Assumption 3. We use $\kappa(x)$ to denote the first input in the optimal input sequence to problem (10). The admissible region is defined as the set of states for which (10) admits a feasible solution.

The following is the theorem for exponential stability of economic MPC using the terminal region/penalty formulation [1].

Theorem 5 (Lyapunov function with terminal penalty)**.** *Let the system* (A, B) *be stabilizable. Let Assumptions 1, 2 and, 3 hold. Then the steady-state solution of the closed-loop system* $x^+ = Ax + B\kappa(x) + B_d d_s$ *is asymptotically stable with* $\mathcal{X}_N$ *as the region of attraction. The Lyapunov function is*

$$\tilde{V}(x) := V_N^0(x) - N\ell(x_s, u_s) - \lambda_s'(x_s) - V_f(x_s)$$

in which $V_N^0(x)$ *is the optimal value function of* (10)

3. Two-node, single-product supply chain

Figure 1 shows a two-stage, single-product supply chain with a retailer and a manufacturer. The manufacturing delay as well as the shipment delay is 2 time units. The retailer responds to the customer demand Dm (nominal demand d_s) by shipping S_1 units to the customer and ordering O_1 units to the manufacturer. These decisions are based on the retailer "states": the inventory at the retailer, Iv_1, and the backorder at the retailer, BO_1. The inventory and backorder balance equations are the dynamics of the retailer states and can be written as:

$$\mathrm{Iv}_1(k+1) = \mathrm{Iv}_1(k) + S_2(k-2) - S_1(k)$$

$$\mathrm{BO}_1(k+1) = \mathrm{BO}_1(k) - S_1(k) + \mathrm{Dm}(k)$$

in which $S_2(k-2)$ is the shipment made by the manufacturer two time periods ago.

Similarly, the manufacturer responds to retailer orders O_1 by making shipments S_2 and production O_2. The dynamics for the manufacturer is:

$$\mathrm{Iv}_2(k+1) = \mathrm{Iv}_2(k) + O_2(k-2) - S_2(k)$$

$$\mathrm{BO}_1(k+1) = \mathrm{BO}_1(k) + O_1(k) - S_2(k)$$

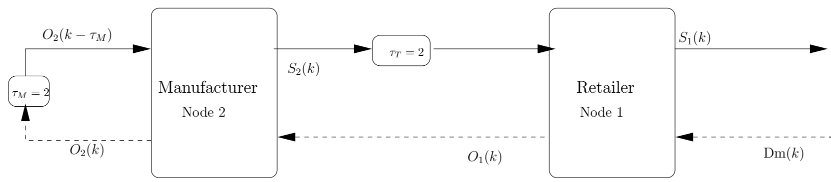

Figure 1: Two-stage supply chain.

Denoting the state of the supply chain as $x = \begin{bmatrix} \mathrm{Iv}_1 & \mathrm{BO}_1 & \mathrm{Iv}_2 & \mathrm{BO}_2 \end{bmatrix}'$ and the input as $u = \begin{bmatrix} S_1 & O_1 & S_2 & O_2 \end{bmatrix}'$, the state space model for the two-node supply chain can be written as:

$$\underbrace{\begin{bmatrix} \mathrm{Iv}_1 \\ \mathrm{BO}_1 \\ \mathrm{Iv}_2 \\ \mathrm{BO}_2 \end{bmatrix}_{k+1}}_{x(k+1)} = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mathrm{Iv}_1 \\ \mathrm{BO}_1 \\ \mathrm{Iv}_2 \\ \mathrm{BO}_2 \end{bmatrix}_{k}}_{x(k)} + \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} S_1 \\ O_1 \\ S_2 \\ O_2 \end{bmatrix}_{k}}_{u(k)} + \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{B^{(2)}} \underbrace{\begin{bmatrix} S_1 \\ O_1 \\ S_2 \\ O_2 \end{bmatrix}_{k-2}}_{u(k-2)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{B_d} \underbrace{\begin{bmatrix} \mathrm{Dm} \end{bmatrix}_{k}}_{d(k)} \tag{12}$$

The equation (12) can be easily written in the form (1) by redefining the state $x(k) \leftarrow \begin{bmatrix} x(k) & u(k-1) & u(k-2) \end{bmatrix}'$. We use the following economic stage cost denoted by ℓ_E (to distinguish from the tracking stage cost ℓ_T that is introduced subsequently) as:

$$\ell_E(x, u) = q'x + r'u$$

The vector q consists of the inventory holding cost and lost sales penalty, while the vector r consists of shipping and ordering/production costs. The input constraints consist of the non-negativity of the shipments and orders and the maximum production/shipping between nodes. It is represented as:

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leq Iu \leq \begin{bmatrix} \bar{S}_1 \\ \bar{O}_1 \\ \bar{S}_2 \\ \bar{O}_2 \end{bmatrix} = \bar{u}$$

in which I is the identity matrix. Similarly, the state constraint consists of the non-negativity constraints and the maximum inventory capacity. It is represented as:

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leq Ix \leq \begin{bmatrix} \bar{\mathrm{Iv}}_1 \\ \bar{\mathrm{BO}}_1 \\ \bar{\mathrm{Iv}}_2 \\ \bar{\mathrm{BO}}_2 \end{bmatrix} = \bar{x}$$

With this model definition and constraint sets, the steady-state optimization problem (3) decomposes into two separate problems (because the state matrix in (12) is the identity matrix) as:

$$\min_x q'x \qquad \text{s.t. } 0 \leq Ix \leq \bar{x} \tag{13}$$

and

$$\min_u r'u \qquad \text{s.t. } B^{(2)}u + Bu + B_d d_s = 0, 0 \leq Iu \leq \bar{u} \tag{14}$$

For non-negative costs in the vector q, the solution to the linear program (13) is 0, i.e., the economically optimal operating point is to hold no inventories and backorders. Corresponding to this steady state $x_s = 0$, the economic shipping quantities (in response to nominal demands) is such that $B^{(2)}u_s + Bu_s = -B_d d_s$, i.e., all the flows in the supply chain are balanced (and for this 2 node example, it means that all the flows are equal to the customer demand). The economic cost of operating at the steady state is $r'u_s$.

3.1. Results

First, we show that simple rolling horizon optimization without terminal constraints can lead to unfavorable closed-loop solutions. We choose the stage cost $\ell(x, u) = q'x + r'u$ in which the costs q, r are chosen as follows:

$$q = (1, 1, 1, 0.01)' \qquad r = (10, 0.1, 10, 1)'$$

Note that the shipping cost is 10, and it is much greater than the backorder cost (1,0.01). In this simulation study,the maximum production/shipping between the nodes is 20 units every time period.

The optimization problem solved is one without terminal constraint and is given by (15)

$$\begin{aligned} \mathbb{P}_N(x; d_s) : \min_{\mathbf{u}} & \; V_N(\mathbf{u}; x) \\ \text{s.t. } & x(j+1) = Ax(j) + Bu(j) + B_d d_s, && j \in \mathbb{I}_{0:N-1} \\ & x(j) \in \mathbb{X} && j \in \mathbb{I}_{0:N-1} \\ & u(j) \in \mathbb{U} && j \in \mathbb{I}_{0:N-1} \end{aligned} \tag{15}$$

in which $V_N(\mathbf{u}; x)$ is given by (9). The stage cost used is $\ell_E(x, u)$. In Figure 2, we show the closed-loop response to the rolling horizon implementation in which (15) is solved online at each sampling time. The prediction horizon N is 15. The system started at the economic steady state $(x_s, u_s; d_s)$. We observe that despite implementing the optimal input at each sampling time, the backorder increases with time, indicating that the demands are not being met (for this choice of costs and control horizon). Hence, the closed-loop economic cost is increasing with time, while there exists a cheaper solution of staying on the steady state and bearing a cost of $r'u_s$ at each time.

In Figure 3, we show the closed-loop response to the MPC implementation in which optimization problem (8) is solved at each sampling instance. We start

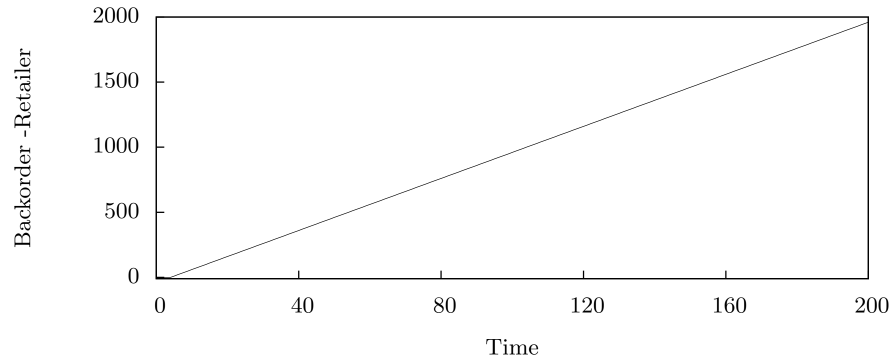

Figure 2: Backorder in the retailer for rolling horizon optimization without stability constraints.

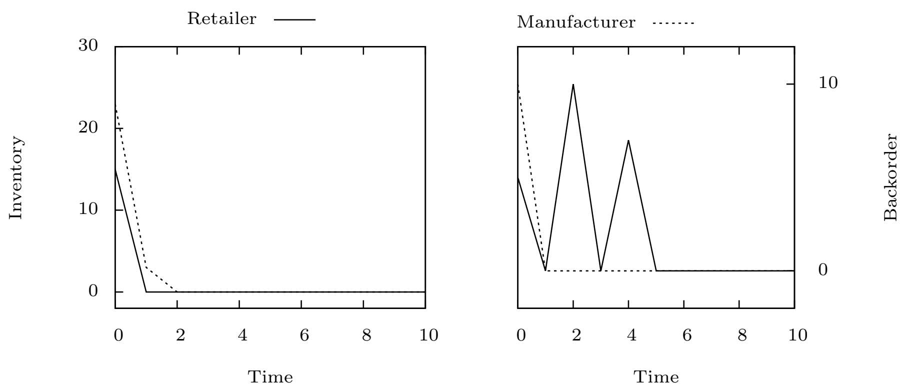

Figure 3: Closed loop evolution using stabilizing MPC.

from an initial condition that is not the steady state. Notice that the closed-loop response quickly settles to the optimal economic steady state. Hence, by using economic MPC theory, we could stabilize a supply chain, even with a pathological cost vector.

For this simple two-node supply chain, we show that simple reoptimization can lead to unfavorable solutions. Although, we use an unfavorable cost function, for larger, complex networks, it can be hard to a priori judge if simple reoptimizations can lead to favorable solutions. On the other hand, using economic MPC theory, we have guarantees on the closed-loop performance.

3.2. Multiobjective stage cost

Supply chain managers wish to balance profit maximization with risk minimization. One way to minimize risk is to hold some buffer stock so that the supply chain can respond to rush orders or demand spikes. As seen in the previous section, the unique economic steady state (for a non-negative cost vector

q) is 0. That is, the economic optimum is to carry no stock. In this section, we introduce a multiobjective stage cost, which is a weighted sum of the economic stage cost discussed in the previous section and a tracking stage cost. The multiobjective stage cost is of the form:

$$\ell(x,u) = \frac{\omega}{s_E}\ell_E(x,u) + \frac{(1-\omega)}{s_T}\ell_T(x,u;z_t) \tag{16}$$

in which the parameter $\omega \in [0,1]$ is a relative weighting given to the economic costs and the tracking cost. The function $\ell_T(x,u;z_t)$ is the tracking stage cost, which penalizes the deviations from a safety stock (chosen to be a steady state of (12)) $z_t = (x_t, u_t)$. The tracking stage cost is defined as:

$$\ell_T(x,u;z_t) = 1/2(x-x_t)'Q(x-x_t) + 1/2(u-u_t)'R(u-u_t) \tag{17}$$

The parameters s_T, s_E are scaling parameters. To obtain the scaling parameters, we consider the utopia and nadir points of the individual stage costs $\ell_E(x,u)$ and $\ell_T(x,u)$ [8]. Denote $z = (x,u)$. We solve

$$z_E = \arg\min_{z\in\mathbb{X}\times\mathbb{U}} \ell_E(x,u), \quad z_T = \arg\min_{z\in\mathbb{X}\times\mathbb{U}} \ell_T(x,u;z_t)$$

The utopia point is the best possible costs that can be attained for both the cost functions:

$$J^U = (\ell_E(z_E), \ell_T(z_T;z_t)) \in \mathbb{R}^2$$

The nadir point is the cost attained by one stage cost at the optimal solution of the other stage cost.

$$J^N = (\ell_E(z_T), \ell_T(z_E;z_t)) \in \mathbb{R}^2$$

The parameters s_T, s_E are then defined as:

$$(s_E, s_T) = J^N - J^U$$

Also, we chose $u_t = u_s$, the economic steady state, because as mentioned earlier, the steady-state inputs are independent of the steady-state inventories and only depend on the nominal demand. The steady-state problem now can be written as:

$$\begin{aligned}&\min_{x,u}\left(\frac{\omega}{s_E}(q'x + r'u) + \frac{(1-\omega)}{s_T}((x-x_t)'Q(x-x_t) + (u-u_s)'R(u-u_s))\right)\\ &\text{s.t. } x = Ax + Bu + B_d d_s, u \in \mathbb{U}, x \in \mathbb{X}\end{aligned} \tag{18}$$

Note that the steady state obtained is a function of the parameter ω. When $\omega = 0$, the steady state is (x_t, u_s). On the other hand, when $\omega = 1$, the steady state is $(0, u_s)$. Hence, the parameter ω captures the manager's relative importance of profit and risk. For $\omega = 0$, the manager is risk averse, while $\omega = 1$ implies that the manager is risk seeking. In Figure 4, we plot the inventory steady state as a function of ω. We chose the backorder targets

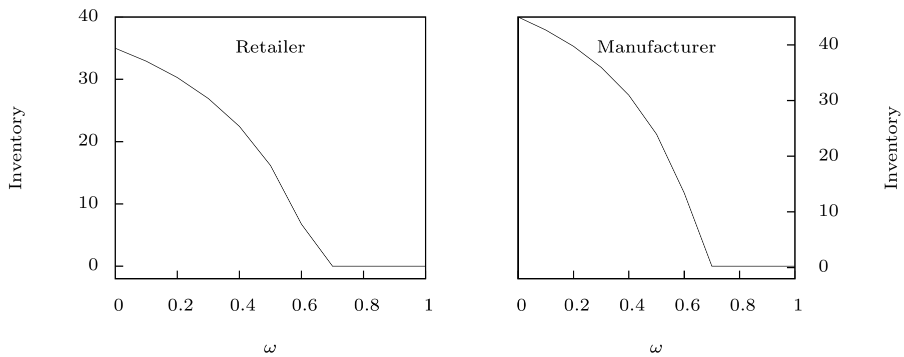

Figure 4: Steady state as a function of the relative weighting between tracking and economics

$\text{BO}_{1,t} = \text{BO}_{2,t} = 0$. Therefore, irrespective of ω, the backorder steady state is 0. The parameters used are $Q = 10\text{diag}(1,1,1,1), R = 10^{-5}\text{diag}(1,1,1,1)$, $x_t = (35,0,45,0)$. The economic costs are $q = (10,10,10,1), r = (10,0.1,10,100)$. The nominal demand is $d_s = 10$.

In this section we use the terminal penalty formulation, (10) as the centralized MPC optimization problem. We can also formulate the on-line MPC problem using optimization problem (8),

To obtain the terminal region and penalty, we first have to find the terminal controller $\kappa_f(x)$. For the supply chain model, we choose the controller $\kappa_f(x) = Kx$. Since the optimal steady state for a fixed value of ω implies that the backorders at both the nodes are zero, i.e., some state constraints are active at the steady state, we use the method outlined in [21] to obtain the feedback gain. It is important to note that when the steady-state inventories are also zero (for example, when $\omega = 1$), then we can use only the terminal constraint problem (8), because the theory in [21] reduces to an equality constraint for this case. Given the choice of feedback gain K, we have that $A_K := A + BK$ is stable (i.e., all eigenvalues of A_K are strictly inside the unit circle). Defining $Q_K = Q + K'RK$ and $q_k = q + K'r$, we can choose the terminal penalty to be:

$$V_f(x; x_s) = (x - x_s)'P(x - x_s) + p'(x - x_s) \tag{19}$$

in which the positive definite matrix P is the solution to the Lyapunov equation

$$A_K' P A_K - P = -Q_K$$

and p is the solution to

$$(A_K - I)'p = -q_K$$

In order to satisfy the basic stability assumption, we require that

$$(x - x_s)'Q(x_s - x_t) \leq 0 \tag{20}$$

Having found $\kappa_f(x) = Kx, V_f(x) = 1/2x'Px + p'x$, the terminal region can be found by using condition (20) along with algorithms to find the maximal output admissible set (see [5, 9] for algorithms).

In Figure 5, we plot the closed-loop response for three different values of ω. The online MPC problem solved is (10) with objective function given by (11). The stage cost is given by (16) while the terminal function is chosen using the above mentioned method. Alongside, we also plot the response for pure-tracking ($\omega = 0$) to the same steady state. That is, we use the stage cost as $\ell_T(x, u; z_s)$ in which $z_s = (x_s, u_s)$ is the steady state of the multiobjective formulation.

In Table 1, we compare the economic cost incurred in using the three controllers: (i) Multiobjective MPC $\ell(x, u; z_t)$, (ii) Economic MPC $\ell_E(x, u)$ and (iii) Tracking MPC to the steady state of the multiobjective MPC $\ell_T(x, u, z_s)$.

Table 1: Economic cost of implementing MPC

ω	Multiobjective$_{\times 10^4}$	Tracking$_{\times 10^4}$	Economic$_{\times 10^4}$
0	4.43	4.43	infeasible
0.2	4.02	4.06	infeasible
0.4	3.56	3.63	infeasible
0.8	2.27	2.27	2.27
1.0	2.27	2.27	2.27

From Figure 5 and Table 1, we can observe that when economic information is available to the controller, it follows an economically attractive transient while stabilizing the steady sate. Hence, the advantage of the multiobjective formulation as compared to a pure tracking formulation (risk averse) is that the steady state can be stabilized via a cost effective transient. The advantage of the multiobjective formulation as compared to a pure economic formulation (risk seeking) is that the multiobjective formulation allows a larger region of attraction. For instance, if we wished to stabilize a set-point $z_p = (x_p, u_p)$ using economic MPC, one way to do it would be to change the constraint set $x_p \leq Ix \leq \bar{x}$, so that the solution to (13) is x_p. However, as observed in Table 1, by using the multiobjective formulation, we could stabilize a larger space of initial states.

4. Multiechelon, multiproduct supply chain

We follow the design procedure presented in the previous section to develop a stabilizing MPC for an 8-node supply chain. The supply chain studied is shown in Figure 6. It consists of a manufacturing facility that can produce 2 products – A and B. The manufacturing facility is node $M1$. Node $M1$ supplies products to nodes $D1$ and $D2$, which are distribution centers. Node $M1$ also sells directly to retailer $R5$. Distributor $D1$ sells products to retailers $R1$ and $R2$, while $D2$ serves retailers $R3$ and $R4$.

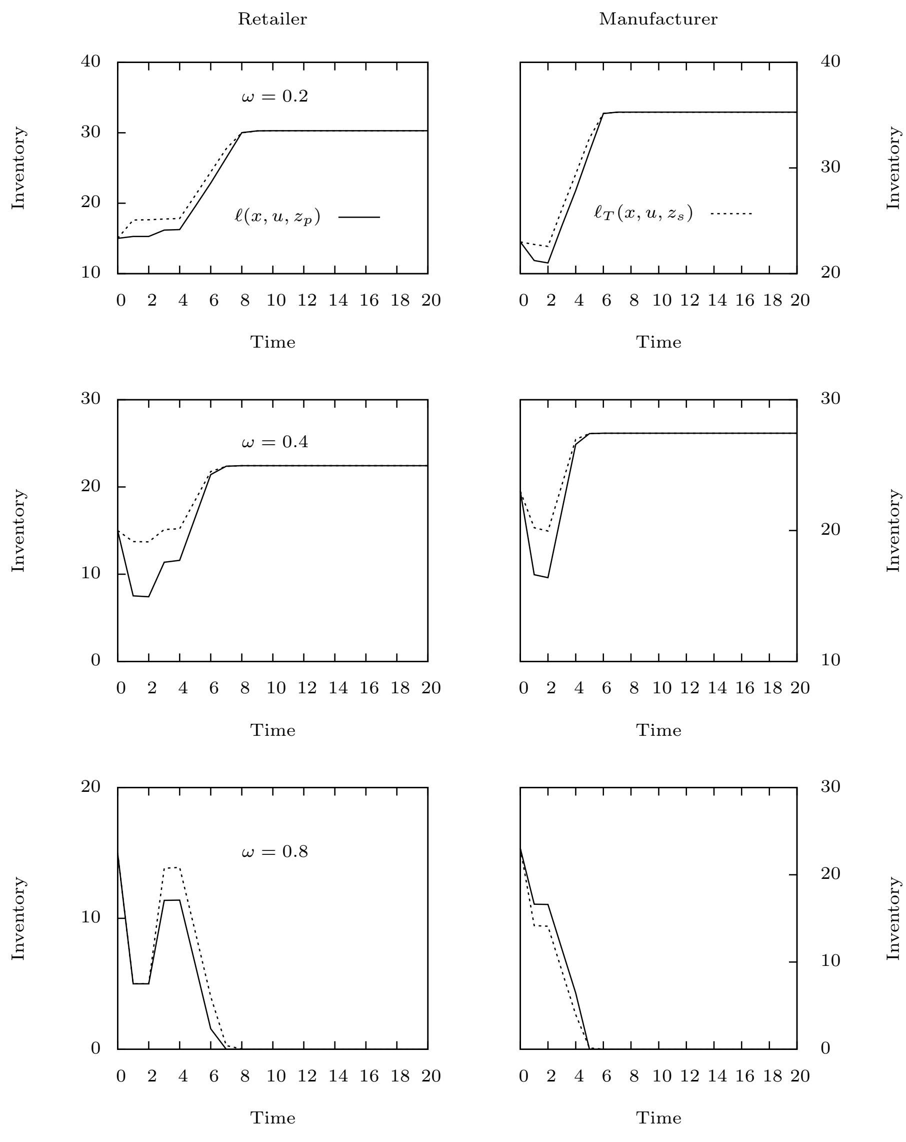

Figure 5: Comparison of Closed-loop response using a pure tracking stage cost and a multi-objective stage cost for different values of ω

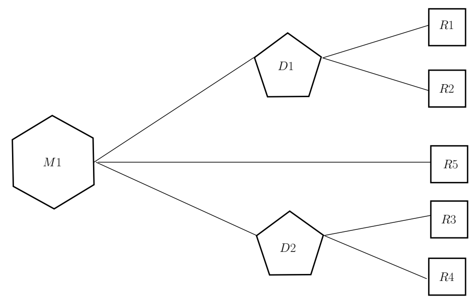

Figure 6: Multiproduct, Multiechelon supply chain studied

Table 2: Transportation lead times

	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
$M1$	2	1					4
$D1$			1	1			
$D2$					2	1	

The production time for product A is 2 time units while that for product B is 3 time units. We assume that the facility $M1$ can start producing either batch at every sampling time.

In Table 2, we list the shipment delay between the nodes. We assume that the customer demand is normally distributed around the nominal demand. In Table 3, we list the nominal demand and the variance of the demand for each product at the retailer nodes. The first entry in Table 3 is the nominal demand while the second entry is the variance of the demand. In the simulation examples, we assumed zero demand if the demand realization is negative.

The target inventory is chosen as the amount of product to be maintained to meet nominal customer demands for the longest delay in the supply chain. In this example, because the longest delay is four time units (between $M1$ and $R5$), the target inventory at each retailer node was four times the nominal demand. For the upstream nodes, the target inventory is four times the sum of the nominal demands at all the retailers served by that node. The target inventories are listed in the Table 4. The target backorder at all nodes are zero. At each node, we also assign a capacity constraint. The capacity constraint is the total amount of product that can be stored in the node. The capacity constraint is listed in Table 5.

Table 3: Nominal demand and variance of demand

	$R1$	$R2$	$R3$	$R4$	$R5$
A	(3.0,1.1)	(4.5,1.3)	(5.0,1.1)	(2.0,1.2)	(4.0,1.4)
B	(4.2,1.3)	(3.1,1.4)	(1.4,1.1)	(2.5,1.1)	(4.2,1.4)

Table 4: Target inventories

	$M1$	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
A	70	30	24	12	18	20	8	16
B	61	29.2	15.6	16.8	12.4	5.6	10	16.8

Table 5: Capacity constraints

	$M1$	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
$\mathrm{Iv}_A + \mathrm{Iv}_B$	140	80	50	40	40	30	25	45

Table 6: State economic costs (per unit)

	$M1$	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
Inventory holding	1	1	1	1	1	1	1	1
Back-order	10	10	10	10	10	10	10	10

The economic cost at a node i is:

$$\ell_E(x_i, u_i) = \sum_{j \in \{A,B\}} h_{i,j}\mathrm{Iv}_{i,j} + g_{i,j}\mathrm{BO}_{i,j} + s_{i,j}S_{i,j} + o_{i,j}O_{i,j}$$

in which $h_{i,j}$ is the inventory holding cost, $g_{i,j}$ is the backorder penalty, $s_{i,j}$ is the shipping cost and $o_{i,j}$ is the ordering cost at node i for product j. The variable $\mathrm{BO}_{i,j}$ is the total backorder at the node. For the manufacturing node $o_{i,j}$ represents the production cost. The economic cost parameters are listed in Tables 6 and Table 7 (the first entry is for product A, while the second entry is for product B). The ordering cost is 1 per unit except at $R5$ where it is 0.5 per unit. The production cost of A is 10 per unit while that for B is 4 per unit.

The tracking objective function is the sum of squares of the deviation from the target. That is:

$$\ell_T(x_i, u_i) = (x_i - x_{i,t})' Q_i (x_i - x_{i,t}) + (u_i - u_{i,t})' R_i (u_i - u_{i,t})$$

The matrix R_i was $0.1I$, I being the identity matrix. The matrix Q_i was $Q_i = \mathrm{diag}(1, 10, 1, 10)$. Recall that the definition of state for node i is $\begin{bmatrix} \mathrm{Iv}_{i,A} & \mathrm{BO}_{i,A} & \mathrm{Iv}_{i,B} & \mathrm{BO}_{i,B} \end{bmatrix}'$.

Table 7: Input economic costs (per unit)

	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
$M1$	(4,2)	(1,2)					(5,4)
$D1$			(1,1)	(1,1)			
$D2$					(2,2)	(1.5,1.5)	

Table 8: Initial inventories

	$M1$	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
A	63	14	24	0	2.1	3.1	3.1	0
B	40	12	7	0	1.2	1.2	0	5.2

We choose $\omega = 0.4$ in the stage cost $\ell(x, u)$ given by (16) with the economic and tracking functions defined using the parameters listed in the tables. The steady-state problem (3) is solved for the nominal demand. The online MPC problem is given by (21). The prediction horizon is 15 days. We simulate the supply chain for 50 days using a stochastic demand signal. For making the predictions, we make a demand forecast $\mathbf{d}$. We assume that we have perfect demand information for three days. For the remainder of the horizon, the demand forecast is set to the nominal demand. The initial inventories of the nodes is given in Table 8. All the backorders are zero, and the inputs are at their steady state at the beginning of the simulation.

$$\begin{aligned}
\mathbb{P}_N(x; \mathbf{d}) : \min_{\mathbf{u}} \; & V_N(\mathbf{u}; x) \\
\text{s.t. } & x(j+1) = Ax(j) + Bu(j) + B_d d(j), && j \in \mathbb{I}_{0:N-1} \\
& x(j) \in \mathbb{X} && j \in \mathbb{I}_{0:N-1} \\
& u(j) \in \mathbb{U} && j \in \mathbb{I}_{0:N-1} \\
& x(N) = x_s
\end{aligned} \tag{21}$$

In Figure 7, we plot the variance of the orders placed by a node and compare it with the variance of the orders arriving at the node. It has been found that classical control methods can often increase the variance of orders placed by the node when compared to the variance of the incoming orders as we move upstream in the supply chain. This effect is called the bullwhip effect [10, 11]. From Figure 7, it is clear that the centralized model predictive controller does not show a bullwhip effect. As has been noted in previous studies [18],the knowledge of the entire supply chain dynamics along with better forecasts results in the better solution (no bullwhip effect).

In Figure 8, we plot the orders placed by the MPC controller and the corresponding inventory and backorder profile in response to demands of product A at Retailer $R3$. Note that the steady state for product A is 7.93 units.

In Table 9, we list the average inventory at each node for products A. and the steady-state inventory for A. The data in the table highlights the inherent robustness of MPC. Although the MPC is designed for the nominal demand (the steady state and the terminal condition are calculated based on the nominal demand); we observe that the controller was able to reject deviations around the nominal demand.

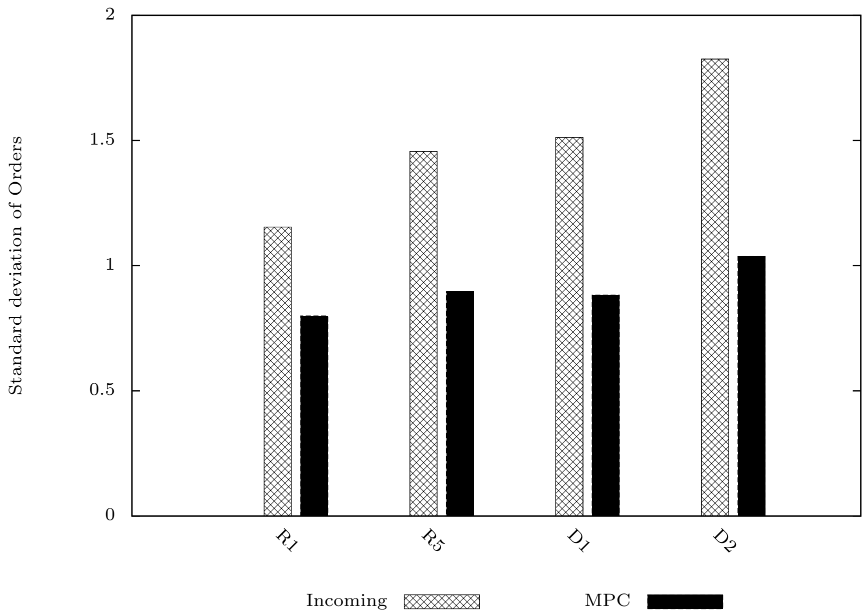

Figure 7: Bullwhip effect using MPC

Figure 8: State and Input profile at $R3$ using MPC

Table 9: Average inventory for Product-A

	$M1$	$D1$	$D2$	$R1$	$R2$	$R3$	$R4$	$R5$
MPC	58.0	17.5	12.4	0.34	5.43	7.71	0.26	3.44
Steady state	57.9	17.9	11.9	0	5.93	7.93	0	3.93

5. Integration of scheduling and control

In the previous section, the manufacturing facility could start a batch of either product at every time. In this section, we use the following model for the manufacturing facility: There is one unit in the facility that can carry out both the tasks of producing A and B from raw-materials. The batch time to make A is two sampling times while that to make B is 3 sampling times. In addition, there is a 1 period changeover time whenever a change of batch from that of A to B has to be made.

We model the manufacturing facility using the State Task Network approach[25], though the same approach can be followed using any discrete-time scheduling model [16, 15]. A single unit U in $M1$ can perform two tasks TA, TB that produce final products A, B respectively. We denote the set $\mathbf{I} = \{\mathrm{TA}, \mathrm{TB}\}$, the production lead times by the set $\tau_P(i)$ and the changeover times from task $i \in \mathbf{I}$ to task $i' \in \mathbf{I}, i' \neq i$ by $\tau_C(i, i')$. There is an economic cost associated with starting a batch as well as changeovers. For ease of presentation we use four binary variables $W_{i,t}, Z_{i,i',t}, Y_{i,t}, X_{i,t}$, to model that (i) only one task can be running in the unit at any given time, and (ii) if a changeover from one task to another has to take place then we need to wait for $\tau_C(i, i')$ time period. The binary variable $W_{i,t} = 1$ denotes that a batch of i has started at time t. The binary variable $Z(i, i', t) = 1$ denotes that a changeover has been made at time t from $i \to i'$. The binary variable $Y_{i,t} = 1$ if the task i is being carried out at time t, while the binary variable $X_{i,t}$ is 1 if the last task to be carried out in the unit was i. The scheduling constraints are enforced by the following inequalities:

$$
\begin{aligned}
&\sum_{i\in\mathbf{I}}\sum_{t'=t-\tau_i+1}^{t} W_{i,t'} + \sum_{\substack{i'\in\mathbf{I}\\ i'\neq i}}\sum_{t'=t-\tau_C(i,i')+1}^{t} Z_{i,i',t'} \leq 1 && \forall t\\
&\sum_{t'=t-\tau_i+1}^{t} W_{i,t'} = Y_{i,t} && \forall t, \forall i \in \mathbf{I}\\
&X_{i,t} \geq Y_{i,t} && \forall t, \forall i \in \mathbf{I}\\
&\sum_{i\in\mathbf{I}} X_{i,t} = 1 && \forall t\\
&Z_{i,i',t} \leq X_{i,t-1} && \forall t, \forall i \in \mathbf{I}, i' \in \mathbf{I}, i' \neq i\\
&Z_{i,i',t} \leq X_{i',t} && \forall t, \forall i \in \mathbf{I}, i' \in \mathbf{I}, i' \neq i\\
&Z_{i,i',t} \geq X_{i,t-1} + X_{i',t} - 1 && \forall t, \forall i \in \mathbf{I}, i' \in \mathbf{I}, i' \neq i
\end{aligned}
\tag{22}
$$

The batch size is given by $B_{i,t}$ and is constrained by

$$W_{i,t}\underline{\mathrm{B}}_i \leq B_{i,t} \leq W_{i,t}\bar{B}_i \tag{23}$$

in which $\underline{\mathrm{B}}_i, \bar{B}_i$ are the minimum and maximum batchsizes.

The dynamics of the manufacturing node for the inventory of A is now

$$\mathrm{Iv}_{A,\mathrm{M1},t+1} = \mathrm{Iv}_{A,\mathrm{M1},t} + B_{A,t-\tau_P(\mathrm{TA})} - \sum_{i\in\{D1,D2,R5\}} S_{A,i,t} \tag{24}$$

Similarly, the dynamics for inventory of B at the node also changes. The dynamic equations for the rest of the supply chain remains the same.

Following the procedure outlined in [26], the supply chain dynamics with the integrated scheduling model can be converted into the state space form (1). The terminal conditions used in (8) and (10) ensure that the supply chain is able to respond to the nominal demand at the end of the optimization horizon indefinitely. For example, the steady state is chosen so that the supply chain can meet the customer demands and stay at the same state. Hence, the terminal conditions help us identify a suboptimal infinite horizon control sequence. When the system moves to the successor state after implementing the first input move in the suboptimal infinite horizon control sequence, we have a readily available candidate input sequence for the successor state. Thus, the terminal conditions ensure recursive feasibility. In the case of the scheduling problem, we find a periodic steady state. We define N_p as the period, and solve the following optimization problem (25), in which we ensure that the system returns to the same state every N_p sampling times. The periodic schedule is another example of an infinite horizon suboptimal input sequence. For this problem, many choices of the period N_p exists. We choose $N_p = 24$. For a general scheduling model cast in the state space form, a periodic schedule may not exist. An avenue of future work is to find suitable terminal constraints for such cases.

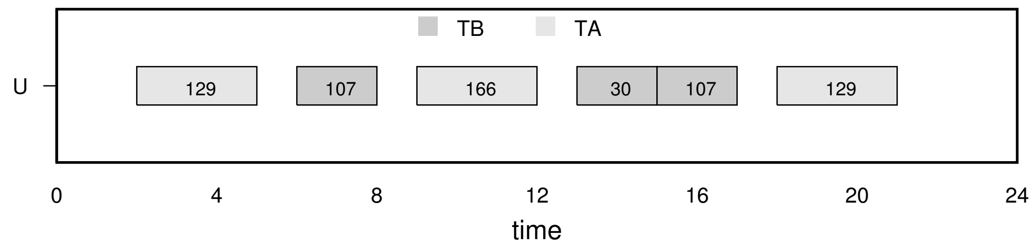

Figure 9: Periodic production schedule to respond to nominal demands. The numbers indicate batch size

We solve the integrated scheduling and control problem only for the economic objective. The cost associated with starting a batch of TA as 10, while that of TB was 6. The cost of changeover from TA to TB was 10 while that of TB to TA was 5.

$$\begin{aligned}\mathbb{P}_p : \min_{\mathbf{u}, x(0)} & \sum_{i=0}^{N_p-1} \ell_E(x(i), u(i), d_s(i)) \\ & \text{s.t. } x(i+1) = Ax(i) + Bu(i) + B_d d_s(i), i = \mathbb{I}_{0:N_p-1} \\ & (x(i), u(i)) \in \mathbb{Z} \qquad i = \mathbb{I}_{0:N_p-1} \\ & x(0) = x(N_p)\end{aligned} \tag{25}$$

in which the set $\mathbb{Z}$ is the combined state-input constraints that consists of the supply chain constraints from the previous section and the assignment constraints (22) and (23).

We denote the solution to (25) by $(\mathbf{u}_p^0, x(0)_p^0)$. The solution to (25) gives us the periodic state-profile

$$\mathbb{X}_p = \left\{x_p^0(0), x(1; x_p^0(0), \mathbf{u}_p^0, \mathbf{d}_s), \ldots, x(N_p; x_p^0(0), \mathbf{u}_p^0, \mathbf{d}_s)\right\} \tag{26}$$

In Figure 5, we show the Gantt chart for the periodic schedule with $N_p = 24$.

Corresponding to Problem (15) and (8), we define the online MPC optimization problem without and with a periodic terminal constraint, respectively, as follows:

$$\begin{aligned}\mathbb{P}_N(x; \mathbf{d}) : \min_{\mathbf{u}} & V_N(\mathbf{u}; x) \\ & \text{s.t. } x(j+1) = Ax(j) + Bu(j) + B_d d(j), \qquad j \in \mathbb{I}_{0:N-1} \\ & (x(j), u(j)) \in \mathbb{Z} \qquad j \in \mathbb{I}_{0:N-1}\end{aligned} \tag{27}$$

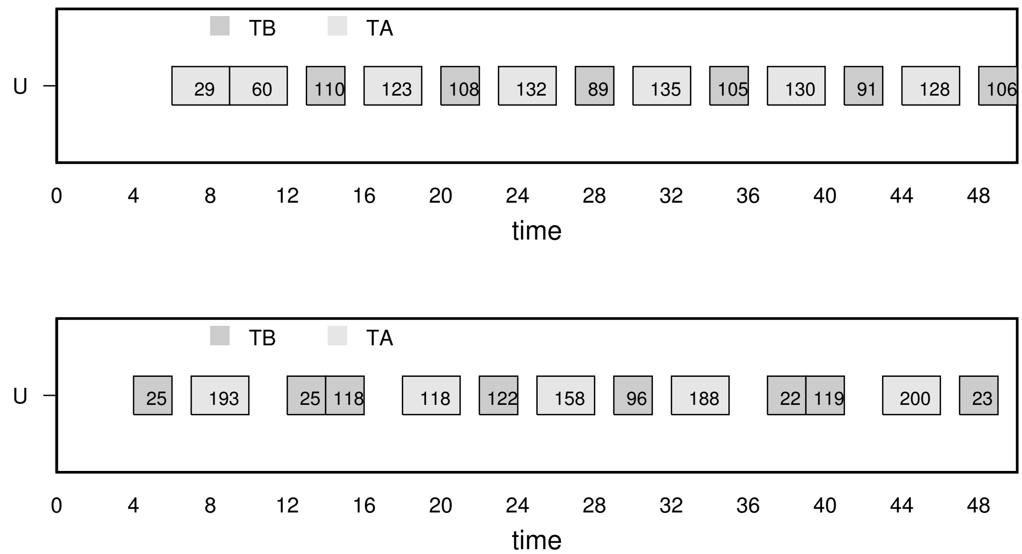

Figure 10: Production schedule for the MPC without terminal constraints that optimized (27) (Top) compared with production schedule for the MPC with terminal constraints that optimized (28). Note how larger batches are made for the problem with terminal constraints

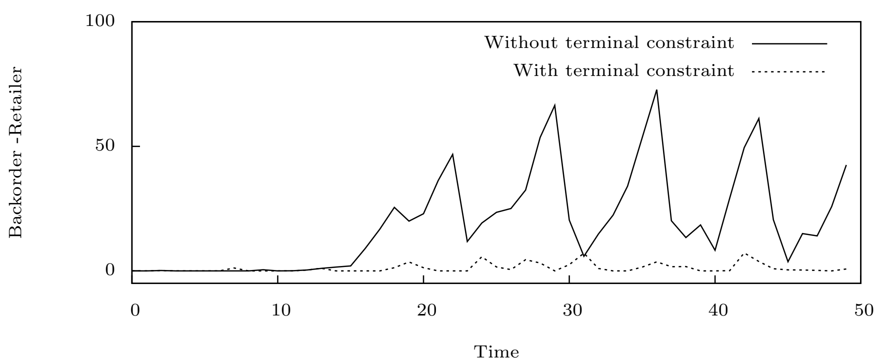

Figure 11: Combined backorder at all the retailer nodes

$$
\begin{aligned}
\mathbb{P}_N(x;\mathbf{d}) : \min_{\mathbf{u}} & V_N(\mathbf{u};x) \\
& \text{s.t. } x(j+1) = Ax(j) + Bu(j) + B_d d(j), \qquad && j \in \mathbb{I}_{0:N-1} \\
& (x(j), u(j)) \in \mathbb{Z} && j \in \mathbb{I}_{0:N-1} \\
& x(N) \in \mathbb{X}_P
\end{aligned}
\tag{28}
$$

in which $V_N(\mathbf{u}; x)$ is the sum of the N stages of economic cost $\ell_E(x, u)$.

As shown in Figure 11, the solution to the simple re-optimization of the N period economic costs leads to greater backorders at the retailers. The backorders are larger for the MPC without periodic constraint, (27), because the optimizer has no information about the demands occurring after the planning period N. As a result, it starts smaller batches (see Figure 5) during the planning period. The cumulative effect of these decisions is that there is not enough inventory when new demands are observed in the subsequent optimization problems. On the other hand, in the MPC with periodic constraint, (28), the terminal constraint $x(N) \in \mathbb{X}_P$ ensures that the decisions are made such that the supply chain can respond to the nominal demand at the end of the planning horizon. Hence, although, larger batches are started (see Figure 5), more inventory is available to meet demands. Figure 11 also highlights the inherent robustness of the formulation (28). It should be noted that we have shown only convergence and not shown Lyapunov stability for the MPC problem that solves (28).

6. Conclusions

The main contribution of this paper is to advance the research in rolling horizon optimization framework for supply chain management by demonstrating the design of model predictive control algorithms with guaranteed closed-loop properties. We used recent developments in economic MPC to design algorithms with guaranteed properties that directly minimized the supply chain economics. We also proposed a multiobjective cost function that captured the economics as well as risk in the supply chain. The multiobjective cost function captures the risk seeking/averse nature of the manager using a single parameter. Finally, we demonstrated the integration of scheduling with control on a multiechelon, multiproduct supply chain. We showed how a properly chosen terminal constraint can add robustness to the rolling horizon optimization framework.

Among the directions for future research include: (a) developing new methods to find terminal conditions for the integrated scheduling and control problem; (b) applying existing and/or developing new theory for stability and convergence properties for the hybrid control problem, and; (c) using demand forecasts and designing terminal conditions for a robust demand scenario instead of a nominal demand scenario.

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