AlgebraicDynamics

vinay.pdx

February 2026

1 Introduction

Coarsened Representation of Phase-Change Dynamics via Nakajima Quivers Your Name February 7, 2026

1. Quiver Region Representation

Let a region of interest be represented as a quiver:

$$Q_R = (V_R, E_R),$$

where

$$V_R = C \cup P, \quad |C| = 200, \ |P| = 800,$$

and

$$E_R = C_p \cup P_p, \quad |C_p| = 300, \ |P_p| = 2700.$$

Here, C denotes central nodes with high-weight dynamics, and P denotes peripheral nodes with lower-weight dynamics. Paths C_p correspond to central high-impact paths, while P_p represent dense low-weight peripheral paths.

2. Path Algebra

The path algebra over a field k is

$$kQ_R = \mathrm{span}_k\{\text{all paths in } Q_R\}.$$

A representation of Q_R assigns:

$$M : V_R \to \mathrm{Vect}_k, \quad E_R \to \text{linear maps}.$$

Each node v_i carries a vector space representing local probability or dynamic state, while each edge corresponds to the linear map encoding flow along the path.

3. Maximal Ideal of Peripheral Paths

Let $I_{\text{discard}} \subset kQ_R$ denote the two-sided ideal generated by low-weight peripheral paths:

$$I_{\text{discard}} = \langle p \in P_p : w(p) < w_{\min} \rangle.$$

This ideal is *maximal* in the sense that the quotient algebra

$$A_{\text{coarse}} = kQ_R / I_{\text{discard}}$$

captures the essential dynamics of the central nodes and paths, while peripheral noise is factored out.

4. Spectral / Zeta Filtering

Define the weighted adjacency matrix A of the quiver:

$$A_{ij} = \sum_{p: v_i \to v_j} w(p),$$

where $w(p)$ is the path weight (including dynamic modulation by Δ_1, Δ_2, and wavelet intensity).

Let $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ be the eigenvalues of A. Then the *spectral zeta function* is:

$$\zeta_Q(s) = \sum_{i=1}^{n} \lambda_i^{-s}.$$

Peripheral paths contributing mostly to small eigenvalues ($\lambda_i < \lambda_{\min}$) are included in the maximal ideal I_{discard}.

5. Coarsened Representation

The coarsened representation algebra is

$$A_{\text{coarse}} = kQ_R / I_{\text{discard}}.$$

- Nodes: central C and a subset of peripheral nodes connected to C. - Paths: central C_p and essential peripheral paths interacting with C. - Linear maps encode probability flow and dynamic modulation:

$$\phi_p = w(p) \cdot f(\Delta_1, \Delta_2, \text{wavelet})$$

for each surviving path p.

6. Wavelet and Dynamic Labeling

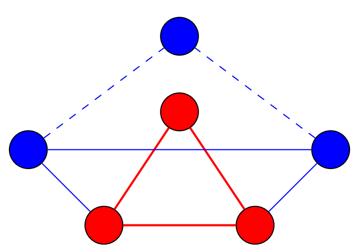
Assign a wavelet-based intensity to each surviving path:

$$I_p(t) = a_p \sin(w_1 t) + b_p \cos(w_2 t),$$

where a_p, b_p correspond to Δ_1, Δ_2 for path p, and w_1, w_2 are frequency modes.

These labels allow for **temporal detection of phase-change events** along the coarsened representation.

7. Visualization (Conceptual)



Red nodes and paths: essential central dynamics
Blue nodes: peripheral nodes
Dashed blue edges: paths included in the maximal ideal I_{discard}

8. Summary

- **Quiver + Path Algebra:** Models region dynamics. - **Maximal Ideal:** Removes low-impact peripheral paths. - **Spectral Filtering:** Selects ideal based on adjacency eigenvalues (zeta-inspired). - **Coarsened Representation:** Captures essential central dynamics while allowing peripheral influence. - **Wavelet/12 Labels:** Encode dynamic modulation for phase-change detection.

[12pt]article amsmath,amssymb,amsfonts,amsthm tikz geometry margin=1in hyperref

Phase-Change Detection in High-Dimensional Probability Networks via Nakajima Quivers and Perverse Sheaves Your Name February 7, 2026

Abstract

We present a mathematical framework for modeling high-dimensional probabilistic interactions and phase-change phenomena using Nakajima quiver varieties enriched with perverse sheaves. Our approach integrates path algebra, maximal ideals, spectral zeta-inspired filtering, Koszul cochains, Rees algebra, and wavelet dynamics to coarsen complex regions while preserving essential interaction dynamics. Derived functorial and geometric invariants such as Chern classes and Picard numbers are employed to detect global colimit failures and singularities. This framework enables efficient computational analysis of millions of nodes and paths while capturing violent phase transitions and local-to-global interactions.

2 Introduction

Phase-change phenomena in high-dimensional probability networks arise in contexts where interactions among nodes lead to abrupt changes in collective dynamics. We model these systems using Nakajima quiver varieties where:

- Nodes represent probability states.
- Edges correspond to probability flows, modulated by dynamics Δ_1, Δ_2 and wavelet intensities.
- Multiple quivers intersect to form interaction planes, and their intersections capture emergent varieties.

To detect local and global phase changes, we employ tools from representation theory, derived algebraic geometry, and perverse sheaf theory.

3 Quiver Representation and Path Algebra

A region is represented as a quiver:

$$Q_R = (V_R, E_R), \quad V_R = C \cup P, \quad E_R = C_p \cup P_p$$

with central nodes C and peripheral nodes P, and corresponding central and peripheral paths.

The path algebra kQ_R is generated by all linear combinations of paths. A representation M assigns:

$$M : V_R \to \mathrm{Vect}_k, \quad E_R \to \text{linear maps.}$$

3.1 Maximal Ideals and Coarsening

Low-weight peripheral paths are included in a maximal two-sided ideal:

$$I_{\text{discard}} = \langle p \in P_p : w(p) < w_{\min} \rangle$$

and the coarsened representation algebra is

$$A_{\text{coarse}} = kQ_R / I_{\text{discard}}.$$

3.2 Spectral / Zeta Filtering

Define weighted adjacency A:

$$A_{ij} = \sum_{p:i \to j} w(p) \cdot f(\Delta_1, \Delta_2, \text{wavelet})$$

with eigenvalues $\{\lambda_1 \geq \cdots \geq \lambda_n\}$ and spectral zeta function:

$$\zeta_Q(s) = \sum_{i=1}^{n} \lambda_i^{-s}.$$

Peripheral paths with small spectral contribution ($\lambda_i < \lambda_{\min}$) are incorporated into I_{discard}.

4 Perverse Sheaves and Local-to-Global Dynamics

We attach perverse sheaves $\mathcal{P}$ to the Nakajima quiver, supported on singular loci, encoding:

- Local cohomology along nodes and paths
- Wavelet-modulated intensity of interactions
- Dynamic inversion at boundaries for phase-change detection

The pushforward and restriction maps of perverse sheaves allow local-to-global propagation of information across 71 regions, detecting violent transitions and snapping of syzygies.

5 Derived Structures and Koszul Cohains

Koszul cochains $K^\bullet$ are associated with each node/path:

$$K^i = \bigwedge^i V_R^* \otimes kQ_R$$

tracking cohomological interactions and local syzygies.

Rees algebra constructions encode the graded dynamics along 1, 2:

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$$

allowing us to track the growth of interactions and identify colimit/gluing failures.

Violent transitions are captured by blow-ups along singularities of $K^\bullet$ and blow-downs of dissipative stable structures.

6 DAG and Principal Interaction Plane

The 71-region network is organized into a Directed Acyclic Graph (DAG) where each node is a region and edges represent inter-region influence.

We define a Principal Interaction Plane (PIP) as the intersection of high-intensity Nakajima quivers:

$$f(\Delta_1, \Delta_2) = 0$$

where f encodes the intersection of varieties. Wavelet-labeled paths allow us to detect rhythmic coordination among isolated active sectors.

7 Spectral Prolate Operator and Phase Detection

Prolate operators acting on the adjacency matrix spectrum allow the identification of dominant eigenmodes corresponding to active interaction regions. Changes in eigenvectors indicate phase transitions or centrality shifts in the quiver dynamics.

8 Global Invariants: Chern Classes and Picard Numbers

Global geometry is monitored via:

- Chern classes $c_i(E)$ of vector bundles over quivers, detecting curvature/phase-change accumulation.
- Picard numbers ρ capturing the rank of divisor class groups, reflecting independence of cohomological interaction sectors.

Violent transitions often correspond to sudden jumps in these invariants, signaling breaking of colimits or loss of gluing.

9 Computational Strategy

For large networks (millions of nodes and paths):

1. Threshold low-weight edges and cluster similar nodes to reduce dimensionality.
2. Compute adjacency spectra and identify maximal ideals via spectral filtering.
3. Form quotient path algebra and coarsened Nakajima representation.
4. Attach perverse sheaves and wavelet labels for phase-change detection.
5. Track Koszul cochains and Rees algebra growth for syzygy snaps.
6. Aggregate over DAG and PIP to identify active sectors and rhythmic coordination.

10 Conclusion

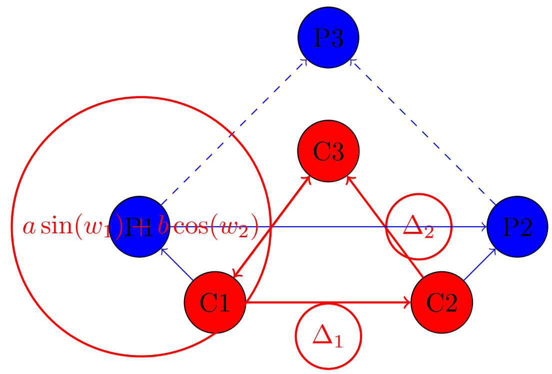
This framework integrates quiver representation theory, perverse sheaf theory, derived structures, spectral analysis, and cohomological invariants to detect local and global phase changes in high-dimensional probability networks. Coarsening via maximal ideals and spectral filtering allows computationally feasible analysis while preserving essential dynamics.

Acknowledgements

Your acknowledgements here.

11 Visualization

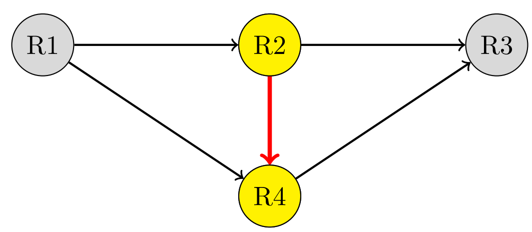
11.1 Region Quiver with Central and Peripheral Paths



Legend:

- Red nodes/edges: Central nodes and paths (high weight)
- Blue nodes/edges: Peripheral nodes and paths (low weight)
- Dashed blue edges: Paths in maximal ideal (discarded)
- Wavelet-labeled edges: Dynamic modulation for 1/2

11.2 DAG of Regions



Legend:

- Gray nodes: Regions in the DAG
- Yellow nodes: Active regions in Principal Interaction Plane (PIP)
- Red thick edges: High-intensity interaction between PIP regions

11.3 Integration of Perverse Sheaves and Wavelets

Each edge in the quiver can be labeled with a **wavelet-modulated perverse sheaf support**:

$$\phi_p(t) = w(p) \cdot f(\Delta_1, \Delta_2) \cdot (a\sin(w_1 t) + b\cos(w_2 t))$$

- Supports of perverse sheaves correspond to **singular loci** (where syzygies snap) - Wavelet modulation tracks **temporal intensity and phase-change dynamics** - Allows visualization of **local-to-global propagation of interactions**

Together, these diagrams illustrate:

1. How central dynamics are preserved after maximal ideal coarsening
2. Where violent phase transitions may occur (edges with high Δ_1/Δ_2 change)
3. DAG structure of 71 regions and coordination among active PIP regions

12 Perverse Sheaves, Contravariant Functors, and Derived Category Framework

12.1 Perverse Sheaves over Nakajima Quivers

Let $Q_R = (V_R, E_R)$ be a Nakajima quiver representing a probabilistic region [8]. We attach a perverse sheaf $\mathcal{P}$ over Q_R to encode local-to-global interactions along singular loci:

$$\mathcal{P} : V_R \cup E_R \to D^b(\mathrm{Vect}_k),$$

where $D^b(\mathrm{Vect}_k)$ is the bounded derived category of vector spaces over k.
Interpretation:

- Each node $v \in V_R$ carries a stalk $\mathcal{P}_v$ representing local cohomology of probability flow.
- Each edge $e \in E_R$ carries a costalk $\mathcal{P}_e$ encoding interactions along paths.
- Singular loci (where $\det(A_{ij}) < 0$ or phase-change occurs) support the perverse sheaf.

12.2 Contravariant Functorial Perspective

The perverse sheaf is a **contravariant functor**:

$$F : Q_R^{op} \to D^b(\mathrm{Vect}_k),$$

which assigns to each node/edge a complex of vector spaces and to each morphism $f : x \to y$ a pullback morphism:

$$F(f) : \mathcal{P}_y \to \mathcal{P}_x.$$

Why contravariant? - Pullback along a morphism naturally captures **flow of influence backward** from singular or critical nodes to source nodes. - Violent phase changes correspond to **snapping syzygies**, which are detected via the failure of gluing conditions:

$$\mathrm{colim}\ F \ncong \mathcal{P}_{\mathrm{global}}.$$

12.3 Derived Category Formulation

Using the derived category $D^b(\mathrm{Vect}_k)$ allows us to:

- Track complexes of interactions along multiple levels of paths (chains of probability flows)
- Model cohomological obstructions to smooth phase transitions
- Compute higher Ext groups $\mathrm{Ext}^i(\mathcal{P}_x, \mathcal{P}_y)$ representing secondary interactions

Functorial diagram:

$$\begin{tikzcd}
\mathcal{P}_v \arrow[r, "F(f)"] & \mathcal{P}_u \\
\mathcal{P}_w \arrow[u, "F(g)"] \arrow[r, "F(h)"'] & \mathcal{P}_x \arrow[u, "F(k)"']
\end{tikzcd}$$

- Vertical arrows: contravariant pullbacks along DAG edges - Horizontal arrows: interactions along quiver paths - Commutativity fails at **phase-change singularities**, marking violent transitions.

12.4 Integration with Maximal Ideals and Wavelets

- Only nodes/paths **retained after maximal ideal coarsening** are assigned nontrivial stalks. - Wavelet modulation along edges is incorporated in the chain complex:

$$\phi_p(t) = w(p) \cdot f(\Delta_1, \Delta_2) \cdot (a \sin(w_1 t) + b \cos(w_2 t)) \in \mathcal{P}_e.$$

- Pullbacks detect propagation of phase-change events from boundary/peripheral nodes to central nodes.

12.5 Summary

1. The perverse sheaf $\mathcal{P}$ encodes **local cohomology along singular loci**[3]. 2. Contravariant functors capture **pullback dynamics** from affected nodes to sources.[6] 3. Derived categories allow tracking **higher-order interactions and obstructions**, such as snapped syzygies.[10] 4. Coupling with maximal ideal coarsening and wavelet-labeled paths enables **computational detection of phase changes** and rhythmic coordination among isolated sectors.

13 Perverse Sheaves, Contravariant Functors, and Derived Category Framework

13.1 Perverse Sheaves over Nakajima Quivers

Let $Q_R = (V_R, E_R)$ be a Nakajima quiver representing a probabilistic region. We attach a perverse sheaf $\mathcal{P}$ over Q_R to encode local-to-global interactions along singular loci:

$$\mathcal{P} : V_R \cup E_R \to D^b(\mathrm{Vect}_k),$$

where $D^b(\mathrm{Vect}_k)$ is the bounded derived category of vector spaces over k.

Interpretation:

- Each node $v \in V_R$ carries a stalk $\mathcal{P}_v$ representing local cohomology of probability flow.
- Each edge $e \in E_R$ carries a costalk $\mathcal{P}_e$ encoding interactions along paths.
- Singular loci (where $\det(A_{ij}) < 0$ or phase-change occurs) support the perverse sheaf.

13.2 Contravariant Functorial Perspective

The perverse sheaf is a **contravariant functor**:

$$F : Q_R^{op} \to D^b(\mathrm{Vect}_k),$$

which assigns to each node/edge a complex of vector spaces and to each morphism $f : x \to y$ a pullback morphism:

$$F(f) : \mathcal{P}_y \to \mathcal{P}_x.$$

Why contravariant? - Pullback along a morphism naturally captures **flow of influence backward** from singular or critical nodes to source nodes. - Violent phase changes correspond to **snapping syzygies**, which are detected via the failure of gluing conditions:

$$\mathrm{colim}\, F \ncong \mathcal{P}_{\mathrm{global}}.$$

13.3 Derived Category Formulation

Using the derived category $D^b(\mathrm{Vect}_k)$ allows us to[7]:

- Track complexes of interactions along multiple levels of paths (chains of probability flows)
- Model cohomological obstructions to smooth phase transitions
- Compute higher Ext groups $\mathrm{Ext}^i(\mathcal{P}_x, \mathcal{P}_y)$ representing secondary interactions

Functorial diagram:

$$\begin{tikzcd}
\mathcal{P}_v \arrow[r, "F(f)"] & \mathcal{P}_u \\
\mathcal{P}_w \arrow[u, "F(g)"] \arrow[r, "F(h)"'] & \mathcal{P}_x \arrow[u, "F(k)"']
\end{tikzcd}$$

- Vertical arrows: contravariant pullbacks along DAG edges - Horizontal arrows: interactions along quiver paths - Commutativity fails at **phase-change singularities**, marking violent transitions.

13.4 Integration with Maximal Ideals and Wavelets

- Only nodes/paths **retained after maximal ideal coarsening** are assigned nontrivial stalks. - Wavelet modulation along edges is incorporated in the chain complex:

$$\phi_p(t) = w(p) \cdot f(\Delta_1, \Delta_2) \cdot (a \sin(w_1 t) + b \cos(w_2 t)) \in \mathcal{P}_e.$$

- Pullbacks detect propagation of phase-change events from boundary/peripheral nodes to central nodes.[1]

13.5 Summary

1. The perverse sheaf $\mathcal{P}$ encodes **local cohomology along singular loci**. 2. Contravariant functors capture **pullback dynamics** from affected nodes to sources. 3. Derived categories allow tracking **higher-order interactions and obstructions**, such as snapped syzygies. 4. Coupling with maximal ideal coarsening and wavelet-labeled paths enables **computational detection of phase changes** and rhythmic coordination among isolated sectors.

14 Singular Loci, Colimit Failure, and Topological Indicators

14.1 Singular Loci as Syzygies

In our framework, the *singular loci* are precisely the points in the Nakajima quiver where syzygies fail:

$$\mathrm{Sing}(Q_R) = \{v \in V_R \mid \ker(d^i)/\mathrm{im}(d^{i-1}) \neq 0\}.$$

Interpretation: - These are the nodes or sets of nodes where the relations between relations break. - They indicate that the local algebraic structure of the path algebra is failing, signaling a potential *violent phase change* in probability flow.

—

14.2 Failure of Colimits

Let $\mathcal{P}$ be a perverse sheaf over Q_R evolving over time. The gluing morphisms along the quiver satisfy the usual cocycle condition for smooth sheaves:

$$f_{uv} \circ f_{vw} = f_{uw}, \quad u, v, w \in Q_R.$$

During violent phase changes, these gluing maps fail:

$$f_{uv} \circ f_{vw} \neq f_{uw},$$

so the colimit

$$\varinjlim \mathcal{P}$$

ceases to exist in the category of smooth sheaves.

Interpretation: - The failure of colimits reflects the breakdown of global integration of local probability dynamics. - The quiver's perverse sheaf stalks can no longer be coherently glued; this signals a non-gentle, abrupt transition.[5]

—

14.3 Chern Numbers as Indicators

Characteristic classes detect these topological changes. Let $c_1(\mathcal{P})$ denote the first Chern class of the perverse sheaf viewed as a vector bundle over the quiver:

$$c_1(\mathcal{P}) = \sum_i \deg(L_i),$$

where L_i are line bundle components of $\mathcal{P}$.

- A **jump in c_1** indicates a twist or topological obstruction forming due to colimit failure. - In the context of a C*-algebra (optional), the breakdown of the algebraic structure corresponds to a **persistent "hole" in representation**; the Chern number measures this hole.

Interpretation: - c_1 jumps are **global indicators of phase change**. - This can complement local indicators such as snapping syzygies.

—

14.4 Homological Collapse and Betti Numbers

Let $H^i(Q_R)$ denote the cohomology of the quiver (or Koszul cochain complex).

- Violent phase changes manifest as **shifts in Betti numbers**:

$$\beta_i = \dim H^i(Q_R).$$

- Example: - Loss of integration in an executive γ-network $\Rightarrow$ drop in β_1 (collapsed hole). - Emergence of a new localized resonance in the θ-band $\Rightarrow$ increase in β_1 (new hole forming).

Interpretation: - These shifts correspond to **topological reorganization** of the network: holes disappear, new holes emerge, signaling violent rearrangements in dynamics.[4]

—

14.5 Integration with the Existing Framework

1. **Syzygies** locate the singular loci (local algebraic failure). 2. **Failure of colimits** encodes global breakdown of coherent probability flow. 3. **Chern numbers** quantify the twist/topological obstruction created by these singularities. 4. **Homological collapse / Betti numbers** track redistribution of holes in the network, complementing spectral analysis. 5. Combined with **prolate operator eigenmodes, wavelet modulation, and Mittag-Leffler tower planes**, we have a **multi-scale, topologically-informed detection of violent phase changes**[2].

—

Remark:

Even without invoking C*-algebras, this scheme provides **local (syzygies), global (colimit failure), and topological (Chern, Betti) indicators**, giving a robust, multi-layered detection framework. The optional C*-algebra formalism could formalize spectra and global representation gaps more rigorously if needed.

15 Maximal Prime Ideals in Weighted Path Algebras

[Maximal-Prime-Envelope Operator with Phase-Change Heartbeat] Let Q be a weighted Nakajima quiver with path algebra kQ and relations encoded by a Koszul complex $K^\bullet$. Let $A(t) \in \mathbb{C}^{N \times N}$ denote the weighted adjacency matrix encoding dynamic interactions along edges, and let P_{syz} be the orthogonal projector onto the syzygy-consistent subspace (kernel of $d^\dagger d$).

Define the *maximal-prime-envelope operator*:

$$\mathcal{O}(t) = P_{\text{syz}}\, A(t)\, P_{\text{syz}} + \alpha A(t), \quad \alpha \geq 0.$$

Let $P_{\text{principal}}(t)$ be the orthogonal projector onto the principal eigenspace of $\mathcal{O}(t)$, and let $h(t)$ be a smooth, time-dependent heartbeat function encoding the amplitude and timing of phase-change events.

Then the *heartbeat-modulated operator*:

$$\mathcal{O}_h(t) = \mathcal{O}(t) + h(t)\, P_{\text{principal}}(t)$$

satisfies the following[9]:

1. The principal eigenspace of $\mathcal{O}_h(t)$ corresponds to the *complement of maximal prime ideals* of $R(t)$, identifying the *minimal subalgebra supporting dominant flow*.

2. Peripheral nodes and paths, contained in the maximal prime ideal, remain in the low-eigenvalue subspace, and thus are dynamically suppressed.

3. Temporal modulation via $h(t)$ produces a *heartbeat of phase change*, highlighting the rise and fall of dominant interactions.

4. Toric surface movements, wavelet edge dynamics, and syzygy relations are naturally encoded in $\mathcal{O}_h(t)$, allowing global detection of phase transitions in the quiver dynamics.

References

[1] E. Arbarello and G. Saccà. Singularities of moduli spaces and quiver varieties. *International Mathematics Research Notices*, (13):4011–4055, 2019.

[2] A. Beilinson, J. Bernstein, P. Deligne, and O. Gabber. *Faisceaux Pervers*, volume 100 of *Astérisque*. Société Mathématique de France, Paris, 1982. English translation in "Perverse Sheaves and Applications".

[3] D. Ben-Zvi and D. Nadler. The character theory of a complex group. *arXiv preprint*, 2009.

[4] R. Bezrukavnikov and M. Kapranov. Microlocal sheaves and quiver varieties. *Astérisque*, pages 87–145, 2015. No. 366.

[5] G. D'Alesio. Derived representation schemes and nakajima quiver varieties. *arXiv preprint*, 2020.

[6] L. Göttsche and D. Huybrechts. Hilbert schemes and moduli spaces of sheaves: survey. *Clay Mathematics Proceedings*, 16:357–422, 2012.

[7] B. Keller and S. Scherotzke. Graded quiver varieties and derived categories. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 696:1–47, 2014.

[8] Hiraku Nakajima. Quiver varieties and kac–moody algebras. *Duke Mathematical Journal*, 91(3):515–560, 1998.

[9] Hiraku Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *Journal of the American Mathematical Society*, 14(1):145–238, 2001.

[10] B. Webster. Koszul duality between p–canonical bases and representations of quiver hecke algebras. *Representation Theory*, 19:13–40, 2015.