**Sample Task** **Solutions**

**Hints and Solutions** **MathonGo**

**Q1**

$\lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let, $x = \frac{\pi}{2} + y$

$\Rightarrow \lim_{y \to 0} = \frac{\tan\left(\frac{-y}{2}\right)(1-\cos y)}{(-2y)^3}$

$= \lim_{y \to 0} \frac{-\tan\frac{y}{2} \cdot 2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y \to 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2$

$= \lim_{y \to 0} \frac{1}{32} \times \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left(\lim_{y \to 0} \frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2$

$\because \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$

$\Rightarrow \frac{1}{32} \times 1 \times 1^2$

$= \frac{1}{32}$

**Q2**

Let, $\sin x = t$

$\Rightarrow \lim_{t \to 0} \frac{1-\cos^3 t}{t \sin t \cos t}$

$\Rightarrow \lim_{t \to 0} \frac{(1-\cos t)}{t^2} \times \left(\frac{t}{\sin t}\right) \times \frac{(1+\cos t+\cos^2 t)}{\cos t}$

$\Rightarrow \frac{1}{2} \times 1 \times \frac{(1+1+1)}{1} = \frac{3}{2}$

**Q3**

$\lim_{x \to -\infty} \frac{x \frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}}{-x\sqrt{4-\frac{1}{x}+\frac{1}{x^2}}} = \frac{1}{-\sqrt{4}} = -\frac{1}{2}$

**Q4**

Let, $L = \lim\limits_{x \to 0} \frac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3\sqrt{x}}$

$= \lim\limits_{x \to 0} \frac{\left(\tan x\right)^{\frac{3}{2}}\left[1-\left(\cos x\right)^{\frac{3}{2}}\right]}{x^{3/2} \cdot x^2}$

$= 1^{\frac{3}{2}} \cdot \lim\limits_{x \to 0} \frac{1-\cos^3 x}{x^2} \cdot \frac{1}{1+\left(\cos\ x\right)^{\frac{3}{2}}}$ (Rationalizing)

$= \lim\limits_{x \to 0} \frac{1-\cos x}{x^2} \cdot \left(1+\cos x+\cos^2 x\right) \cdot \frac{1}{1+\left(\cos x\right)^{\frac{3}{2}}}$

$= \frac{1}{2} \cdot \frac{1}{2}(1+1+1) = \frac{3}{4}.$

**Q5**

$\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} \cdot \frac{(3+\cos 2x)}{1} \cdot \frac{2x}{\tan 2x} \cdot \frac{1}{2}$

$= \frac{1}{2} \cdot \left(4\right) \cdot \frac{1}{2} = 1$

**Q6**

$\lim_{x\to0} \frac{1-\cos^3 x}{x \sin x \cos x}$

$= \lim_{x\to0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x \sin x \cos x}$

$= \lim_{x\to0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x\cdot 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \times \frac{(1+\cos x+\cos^2 x)}{\cos x}$

$= \lim_{x\to0} \frac{\sin\left(\frac{x}{2}\right)}{2\left(\frac{x}{2}\right)} \times \frac{1+\cos x+\cos^2 x}{\cos\left(\frac{x}{2}\right)\cos x} = \frac{1}{2}\times 3 = \frac{3}{2}$

**Q7**

$\lim_{x\to0}\left(\frac{\sin 3x}{x^3}+\frac{a}{x^2}+b\right)= \lim_{x\to0}\frac{\sin 3x+ax+bx^3}{x^3}$

$= \lim_{x\to0}\frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$

limit exists if

$3+a=0$

or $a=-3$

$\therefore\ L=\lim_{x\to0}\frac{\sin 3x-3x+bx^3}{x^3}=27\left(\lim_{t\to0}\frac{\sin t-t}{t^3}+b\right)=0\ \left(\text{Putting } 3x=t\right)=-\frac{27}{6}+b=0$

or $b=\frac{9}{2}$

**Q8**

$\lim_{x\to0}\frac{f(x^2+x+1)-f(1)}{f(x^4-x^2+2x+4)-f(4)}\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to0}\frac{(2x+1)f^1(x^2+x+1)}{(4x^3-2x+2)f^1(x^4-x^2+2x+4)}$

$=\frac{f'(1)}{2f'(4)}=4$ (Applying $L$' Hospital's Rule)

**Q9**

$lim_{x\to\infty}\left(\sqrt{x^2+x+2}-ax-b\right)=2$

$a=1; b=\frac{-3}{2}$ required equation is $(x-1)^2+(y+3)^2=1$

**Q10**

$\lim_{x\to\infty} x^3\ \ln\left(1+\frac{1}{x}\right)+\frac{x}{2}-x^2$

$x=\frac{1}{t}$

$\lim_{t\to0}\left(\frac{\ln(1+t)}{t^3}+\frac{1}{2t}-\frac{1}{t^2}\right)=\lim_{t\to0}\frac{2\ln(1+t)+t^2-2t}{2t^3}$

$= \lim_{t\to 0} \frac{2\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\ldots\right)+t^2-2t}{2t^3}$

$= \lim_{t\to 0}\left(\frac{1}{3}-\frac{t}{4}+\frac{t^2}{5}\ldots\ldots\right)=\frac{1}{3}=\frac{1}{m}\Rightarrow m=3.$

**Q11**

$\lim_{n\to\infty} \frac{3.2^{n+1}-4.5^{n+1}}{5.2^n+7.5^n}$

$= \lim_{n\to\infty} \frac{5^n\left(6.\left(\frac{2}{5}\right)^n-20\right)}{5^n\left(5.\left(\frac{2}{5}\right)^n+7\right)} = -\frac{20}{7}\left(\because \lim_{n\to\infty}\left(\frac{2}{5}\right)^n=0\right)$

**Q12**

$\lim_{x\to 0} \frac{\ln(2-\cos 15x)}{\ln^2(\sin 3x+1)}$

$= \lim_{x\to 0} \frac{\ln\{1+(1-\cos 15x)\}}{\ln^2(1+\sin 3x)}$

$= \lim_{x\to 0} \frac{1-\cos 15x}{(\sin 3x)^2}$ (Applying$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$)

$= \lim_{x\to 0} \frac{(15x)^2}{2(3x)^2}$ (using standard limit)

$= \frac{1(225)}{2\times 9} = 12.5$

**Q13**

Given,

$f(x) = a\sin\left(\frac{\pi[x]}{2}\right)+[2-x], a\in\mathbb{R}$

Now given $\lim_{x\to -1} f(x)$ exists,

So $\lim_{x\to -1^+} a\sin\left(\pi\frac{[x]}{2}\right)+[2-x] = -a+2$

And $\lim_{x\to -1^-} a\sin\left(\pi\frac{[x]}{2}\right)+[2-x] = 0+3=3$

So, $\lim_{x\to -1} f(x)$ exist when $-a+2=3\Rightarrow a=-1$

Now,

$\int_0^4 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$

$\Rightarrow \int_0^4 f(x)dx = \int_0^1 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx + \int_1^2 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx + \int_2^3 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx + \int_3^4 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx$

$= \int_0^1(0+1)dx + \int_1^2(-1+0)dx + \int_2^3(0-1)dx + \int_3^4(1-2)dx$

$$= 1 - 1 - 1 - 1 = -2$$

**Q14**

$\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$= \frac{1}{x^n}\left(\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-1\right)\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\right.\right.$$
$$\left.\left.-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots\right)\right)\right)$$
$$= \frac{1}{x^n}\left[\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\left(-x-x^2-\frac{x^3}{3!}\ldots\right)\right]$$
$$= \frac{-1}{x^{n-3}}\left[\left(-\frac{1}{2!}+\frac{x^2}{4!}-\ldots\right)\left(1+x+\frac{x^2}{3!}\ldots\right)\right]$$

$\therefore$ For $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ to exist as a nonzero number we must have $n-3=0 \Rightarrow n=3$.

**Q15**

$\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{\frac{x}{b}} = e^{\lim\limits_{x\to\infty}\frac{x}{b}\left(1+\frac{a}{x}-1\right)}$

$[\because 1^\infty$ form $]$

$= e^{\frac{a}{b}} = e^{\frac{403\times 5}{1}} = e^{2015} \equiv e^\lambda \Rightarrow \lambda = 2015.$

**Q16**

$\mathrm{L} = e^l$

$l = \lim\limits_{n\to\infty}\frac{n^2-1}{n}\left[\frac{2n^2-3-2n^2+n-1]}{2n^2-n+1}\right]$

$= \lim\limits_{n\to\infty}\frac{n^2\left(1-\frac{1}{n^2}\right)-n\left(1-\frac{4}{n}\right)}{n\cdot n^2\left(2-\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{1}{2}$

$\therefore\ \mathrm{L} = \sqrt{e}.$

**Q17**

Let, $3^{\frac{x}{2}} = t, x \to 2 \Rightarrow t \to 3$

$\lim_{t\to 3}\frac{t^{2\frac{2}{t^2}}-12}{\frac{1}{t}-\frac{3}{t^2}} = \lim_{t\to 3}\frac{t^4+27-12t^2}{t-3}$

$\lim_{t\to 3}\frac{(t^2-3)(t+3)(t-3)}{(t-3)} = 6\times 6 = 36$

**Q18**

Let, $f(x) = \frac{1+\tan x}{1+\sin x}$

and $g(x) = \frac{2}{\sin x}$

Clearly, $f(x) \rightarrow 1$ and $g(x) \rightarrow \infty$ as $x \rightarrow 0$

$\therefore \lim_{x\rightarrow 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{2}{\sin x}} = e^{\lim_{x\rightarrow 0}\frac{2}{\sin x}\left(\frac{1+\tan x}{1+\sin x}-1\right)}$

$\left\{\text{ using } \lim_{x\rightarrow a}[f(x)]^{g(x)} = e^{\lim_{x\rightarrow a} g(x)[f(x)-1]} \text{ for } 1^{\infty} \text{ form }\right\}$

$= e^{\lim_{x\rightarrow 0}\frac{2}{\sin x}\left(\frac{\tan x-\sin x}{1+\sin x}\right)} = e^{\lim_{x\rightarrow 0}\frac{2(1-\cos x)}{\cos x(1+\sin x)}}$

$= e^{0} = 1$

**Q19**

$\lim\limits_{x\rightarrow 0^{+}} x\cot x + \lim\limits_{x\rightarrow 0^{+}} x\ln x$

$= \lim\limits_{x\rightarrow 0^{+}}\frac{x}{\tan x} + \lim\limits_{x\rightarrow 0^{+}}\frac{\ln x}{\left(\frac{1}{x}\right)}$

$= \lim\limits_{x\rightarrow 0^{+}}\frac{x}{\tan x} + \lim\limits_{x\rightarrow 0^{+}}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^{2}}\right)} = 1 + \lim\limits_{x\rightarrow 0^{+}}(-x)$

$= 1 + 0 = 1$

**Q20**

$\because \frac{\pi}{6} < 1,$

$\therefore \left[\frac{\pi}{6}\right] = 0$

$\therefore \lim\limits_{\mathrm{x}\rightarrow\frac{\pi}{2}}\frac{\left[\frac{\mathrm{x}}{2}\right]}{\ln(\sin \mathrm{x})} = 0$

**Q21**

Since $x-1 \leq [x] \leq x$ for all $x \in \mathbf{R}$ so

$$\frac{1}{x^{3}} - 1 \leq \left[\frac{1}{x^{3}}\right] \leq \frac{1}{x^{3}}$$

$\Rightarrow \quad x^{8}\left(\frac{1}{x^{3}}-1\right) \leq x^{5}\left[\frac{1}{x^{3}}\right] \leq x^{5}$ for all $x$

But $\lim_{x\rightarrow 0} x^{5} = 0 = \lim_{x\rightarrow 0}\left(x^{5}-x^{8}\right)$ so

$\lim_{x\rightarrow 0} x^{8}\left[\frac{1}{x^{3}}\right] = 0 \in \mathbf{I} \subseteq \mathbf{Q}$

**Q22**

$\lim\limits_{x\rightarrow 0}\frac{\sin x(2\cos x-a)}{x\cdot x^{2}} = \lim\limits_{x\rightarrow 0}\left(\frac{\sin x}{x}\right)\left(\frac{2\cos x-a}{x^{2}}\right)$

For this limit to exists finitely,

$\lim\limits_{x\rightarrow 0}\frac{2\cos x-a}{x^{2}} = \text{finite}$

$\therefore$ It must be $\frac{0}{0}$ form

$\therefore\ 2\cos(0)-a = 0 \Rightarrow a = 2$

**Q23**

$\frac{5}{x} - 1 < \left[\frac{5}{x}\right] \leq \frac{5}{x}$

$\underbrace{\frac{\sin x}{3}\left(\frac{5}{x}-1\right)}_{h(x)} < \underbrace{\frac{\sin x}{3}\left[\frac{5}{x}\right]}_{f(x)} \leq \underbrace{\frac{\sin x}{3}\left(\frac{5}{x}\right)}_{g(x)}$

by sandwich theorem

$\because \lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = \frac{5}{3}$

$\therefore \lim_{x \to 0} f(x) = \frac{5}{3}$

**Q24**

Let the given expression be $y$.

Then, $y = \lim_{n \to \infty} n^2$

$$\left\{ \sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)}}} \right.$$

On putting $\frac{1}{n} = \theta \ldots\ldots \infty$

So that, $n \to \infty \Rightarrow \theta \to 0$

Thus,

$$\mathrm{y} = \lim_{\theta \to 0} \frac{1}{\theta^2}\left(1 - \cos\theta\right)^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \text{to } \infty}$$

$$= \lim_{\theta \to 0}\left(\frac{1 - \cos\,\theta}{\theta^2}\right)$$

$$\left\{\because\ \frac{1}{2} + \frac{1}{2^2} + \ldots\infty = 1\right\}$$

$$= \lim_{\theta \to \infty} \frac{2\sin^2\,\theta/2}{\theta^2}$$

$$= \lim_{\theta \to 0} 2\left(\frac{\sin\,\theta/2}{\theta/2}\right)^2 \times \frac{1}{4}$$

$$= 2 \cdot 1^2 \cdot \frac{1}{4} = \frac{1}{2}$$

**Q25**

Let $f(x) = \frac{[x] + [2^2 x] + [3^2 x] + \ldots + [n^2 x]}{1^2 + 2^2 + 3^2 + \ldots + n^2}$

Now, we have,

$$f(x) \leq \frac{x + 2^2 x + 3^2 x + \ldots + n^2 x}{1^2 + 2^2 + 3^2 + \ldots + n^2} = x$$

and, $f(x) > \frac{(x-1) + (2^2 x - 1) + (3^2 x - 1) + \ldots + (n^2 x - 1)}{1^2 + 2^2 + 3^2 + \ldots + n^2}$

$$= \frac{x\Sigma n^2 - n}{\Sigma n^2} = x - \frac{6}{(n+1)(2n+1)}\ (\because x - 1 \leq [x] < x,\ \forall x \in R)$$

Thus, we have,

$x-\frac{6}{(n+1)(2n+1)}<f(x)\leq x$

Now, we have,

$\lim_{n\to\infty} x-\frac{6}{(n+1)(2n+1)}=x \ \& \ \lim_{n\to\infty} x=x$

Hence, by Sandwich Theorem, we have

$\lim_{n\to\infty} f(x)=x$

**Q26**

$\lim_{x\to 0}\frac{x^2\tan(\alpha x)}{\beta x-\tan(2x)}=1\to\lim_{x\to 0}\frac{x^2\tan(\alpha x)}{\beta x-\left\{2x+\frac{(2x)^3}{3}+\frac{2(2x)^5}{15}\right\}}=1$

$\to\beta=2$ and $\frac{3\alpha}{8}=-1,$

So, $5\beta+3\alpha=2$

**Q27**

Consider $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}\ \left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\cdot\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$=\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{\sin x+x}{2}\right)\times\left(\frac{x-\sin x}{2}\right)\times\frac{1}{x^4}$

$\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)$

$\lim_{x\to 0}2\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)\ \left(\frac{0}{0}\text{ form}\right)\quad\left(\because\lim_{t\to 0}\frac{\sin t}{t}=1\right)$

Applying L'Hospital's Rule,

$\lim_{x\to 0}2\times\left(\frac{2x-2\sin x\cos x}{4\cdot 4x^3}\right)=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{8x^3}\right)\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\left(\frac{2-2\cos 2x}{24x^2}\right)\ \left(\frac{0}{0}\text{ form}\right)$

$\lim_{x \to 0}\left(\frac{4 \sin 2x}{48x}\right) = \frac{1}{6}\lim_{x \to 0}\left(\frac{\sin 2x}{2x}\right) = \frac{1}{6}$

**Q28**

We know that $r \leq [r] < r + 1$

$2r \leq [2r] < 2r + 1$

$3r \leq [3r] < 3r + 1$

$\vdots \ \vdots \ \vdots$

$nr \leq [nr] < nr + 1$

On adding all the above inequalities, we get

$r + 2r + \ldots + nr \leq [r] + [2r] + \ldots + [nr] < r + 2r + \ldots nr + n$

Using $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}, \ n \in N,$

$\frac{n(n+1)}{2} \cdot r \leq [r] + [2r] + \ldots + [nr] < \frac{n(n+1)}{2} \cdot r + n$

$\Rightarrow \frac{\frac{n(n+1)}{2} \cdot r}{n^2} \leq \frac{[r] + [2r] + \ldots + [nr]}{n^2} < \frac{\frac{n(n+1)}{2} \cdot r + n}{n^2}$

Now, $\lim_{n \to \infty} \frac{n(n+1) \cdot r}{2 \cdot n^2} = \lim_{n \to \infty} \frac{n^2\left(1 + \frac{1}{n}\right) \cdot r}{2 \cdot n^2} = \frac{r}{2}$

and $\lim_{n \to \infty} \frac{\frac{n(n+1)}{2} \cdot r + n}{n^2} = \lim_{n \to \infty} \frac{n^2\left\{\left(1 + \frac{1}{n}\right) \cdot r + \frac{1}{n}\right\}}{2 \cdot n^2} = \frac{r}{2}$

So, by Sandwich Theorem, we can conclude that

$\lim_{n \to \infty} \frac{[r] + [2r] + \ldots + [nr]}{n^2} = \frac{r}{2}.$

**Q29**

Using expansion,

$\lim_{x \to 0} \frac{\left(1 + \left(-\frac{x^2}{2}\right) + \frac{\left(\frac{x^4}{4}\right)}{2!} + \ldots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^2}{4!} - \ldots\right)}{x^3\left(x + \frac{x^3}{3} + \ldots\right)}$

$\lim_{x \to 0} \frac{(1-1) + \left(-\frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^4}{8} - \frac{x^4}{24}\right) + \ldots}{x^3\left(x + \frac{x^3}{3} + \ldots\right)}$

$\Rightarrow \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$

**Q30**

Using expansions, we get,

$$\lim_{x\to0}\frac{a\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..\right)+b\left(1-\frac{x^2}{2!}+...\right)+c+dx}{x\left(x-\frac{x^3}{3!}+.....\right)^2}=3$$

$$\lim_{x\to0}\frac{(a+b+c)+(a+d)x+\left(\frac{a-b}{2}\right)x^2+\frac{a}{6}x^3+...}{x^3\left(1-\frac{x^2}{3!}+.....\right)^2}=3$$

$\because$ in the denominator lowest power of x is 3

For the limit to be finite, the numerator should also have the least power of x as 3

$\therefore a+b+c=0 \quad ...(1)$

$a+d=0...(2)$

$\frac{a-b}{2}=0...(3)$

Now, $\frac{\left(\frac{a}{6}\right)}{1}=3\Rightarrow a=18$

From $(1),(2),(3)$, we get,

$a=18,\ b=18,\ c=-36,\ d=-18$

$\frac{abd}{c^3}=\frac{-(18)^3}{-8(18)^3}=\frac{1}{8}$