Q1

We have,

$\operatorname{Re}\left(\frac{iz+1}{iz-1}\right)=2$

Let $z=x+iy$, then

$\operatorname{Re}\left(\frac{i(x+iy)+1}{i(x+iy)-1}\right)=2$

$\Rightarrow \operatorname{Re}\left(\frac{ix+i^2y+1}{ix+i^2y-1}\right)=2$

$\Rightarrow \operatorname{Re}\left(\frac{1-y+ix}{-1-y+ix}\right)=2$

$\Rightarrow \operatorname{Re}\left\{\left(\frac{1-y+ix}{-1-y+ix}\right)\times\left(\frac{-1-y-ix}{-1-y-ix}\right)\right\}=2$

$\Rightarrow \operatorname{Re}\left\{\frac{(1-y+ix)((-1-y)-ix)}{(-1-y)^2+x^2}\right\}=2$

$\Rightarrow \left\{\frac{(1-y)(-1-y)+x^2}{(-1-y)^2+x^2}\right\}=2$

$\Rightarrow y^2-1+x^2=2y^2+2x^2+2+4y$

$\Rightarrow x^2+y^2+4y+3=0$

Hence, the locus is a circle.

Q2

Let $\mathrm{z}=\mathrm{x}+\mathrm{iy}$

$\frac{\mathrm{z}-\mathrm{i}}{\mathrm{z}+\mathrm{i}}$ is a purely imaginary number

$\Rightarrow \frac{\mathrm{x}+\mathrm{i}(\mathrm{y}-1)}{\mathrm{x}+\mathrm{i}(\mathrm{y}+1)}\times\frac{\mathrm{x}-\mathrm{i}(\mathrm{y}+1)}{\mathrm{x}-\mathrm{i}(\mathrm{y}+1)}$ is a purely imaginary

$\Rightarrow \frac{(\mathrm{x}^2+\mathrm{y}^2-1)-\mathrm{i}(2\mathrm{x})}{\mathrm{x}^2+(\mathrm{y}+1)^2}$ is purely imaginary

$\Rightarrow \mathrm{x}^2+\mathrm{y}^2-1=0 \Rightarrow \mathrm{x}^2+\mathrm{y}^2=1 \ldots(\mathrm{i})$

$\mathrm{z}+\frac{1}{\mathrm{z}}=\mathrm{x}+\mathrm{iy}+\frac{1}{\mathrm{x}+\mathrm{iy}}$

$=(\mathrm{x}+\mathrm{iy})+\frac{1}{(\mathrm{x}+\mathrm{iy})} \times \frac{(\mathrm{x}-\mathrm{iy})}{(\mathrm{x}-\mathrm{iy})}$

$=(\mathrm{x}+\mathrm{iy})+\frac{(\mathrm{x}-\mathrm{iy})}{\mathrm{x}^2+\mathrm{y}^2}=2\mathrm{x}$

$y \neq \pm 1$ so $\mathrm{x} \neq 0$ (from (i) and since, z won't be an imaginary number)

Q3

$u=\frac{2(x+iy)+i}{(x+iy)-ki}=\frac{2x+(2y+1)i}{x+(y-k)i} \times \frac{x-(y-k)i}{x-(y-k)i}$

Real part of $u=Re(u)=\frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2}$

Imaginary part of $u=\operatorname{Im}(u)\frac{x(2y+1)-2x(y-k)}{x^2+(y-k)^2}$

Now $\operatorname{Re}(u)+\operatorname{Im}(u)=1$

$\frac{2x^2+(2y+1)(y-k)+x(2y+1)-2x(y-k)}{x^2+(y-k)^2}=1$

for y-axis put $x=0$

$\Rightarrow \frac{(2y+1)(y-k)}{(y-k)^2}=1$

$\Rightarrow (y-k)(y+1+k)=0$

$y=k,\ -(1+k)$

Now point $P(0,\ k),\ Q(0,\ -(1+k))$

$PQ=|2K+1|=5$

$2k+1=\pm 5$

$2k=4,-6$

$k=2,-3$

Hence, $k=2\quad(k>0)$.

Q4

Let $z=x+iy$

Given, $|z|-z=1+2i$

$\Rightarrow\quad\sqrt{x^2+y^2}-(x+iy)=1+2i$

$\Rightarrow\quad\sqrt{x^2+y^2}-x=1,\quad y=-2$

$\Rightarrow\quad\sqrt{x^2+4}-x=1$

$\Rightarrow\quad x^2+4=(1+x)^2$

$\Rightarrow\quad 2x=3\Rightarrow\quad x=\frac{3}{2}$

$\therefore\quad z=\frac{3}{2}-2i$

Q5

$Z^2+|Z|=\left(\overline{Z}\right)^2\ \ldots(1)$

Taking conjugate,

$\left(\overline{Z}\right)^2+|Z|=Z^2\ \ldots(2)$

Adding (1) & (2), we get,

$\Rightarrow 2|Z|=0 \Rightarrow Z=0$

Q6

$\sqrt{2}\,|z-1|=-(i+z)$

$\Leftrightarrow\ \sqrt{2}\sqrt{(x-1)^2+y^2}=-[x+i(y+1)]\quad\ldots(i)$

$L.\ H.\ S.\ \geq 0 \Rightarrow R.\ H.\ S.$ must be real

$\Rightarrow y=-1$

(i) reduces to $\sqrt{2}\ \sqrt{(x-1)^2+1}=-x\qquad\ldots(ii)$

$L.\ H.\ S.$ of $(ii)\geq 0 \Rightarrow x\leq 0\qquad\ldots(iii)$

Squaring (ii), we get $2\left[x^2-2x+2\right]=x^2$

$\Leftrightarrow x^2-4x+4=0$

$\Leftrightarrow x=2$

(rejected) $\ldots$[From (iii)]

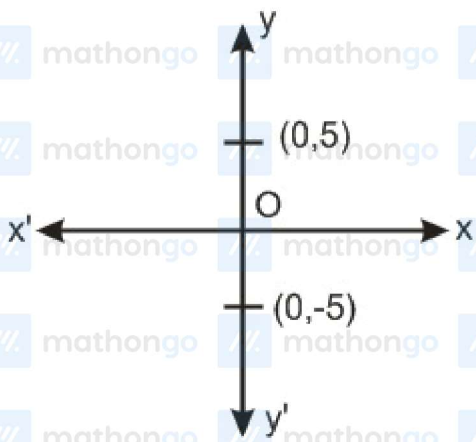
$\Rightarrow$ There is no z that satisfies the given equation. Thus, there are 0 such complex numbers.

Q7

Given, $\left|\frac{z-5i}{z+5i}\right|=1$

$\Rightarrow|z-5i|=|z+5i|$

(if $|z-z_1|=|z-z_2|$, then it is a perpendicular bisector of the line segment joining points z_1 and z_2)



and perpendicular bisector of z_1 $(0,\ 5)$ and z_2 $(0,\ \ -5)$ is $x-axis$,

Therefore, z will lie on the $x-$ axis.

Q8

$\left(Z^2-Z\right)=Z(Z-1)$

$=(\cos\phi+i\sin\phi)(\cos\phi-1+i\sin\phi)$

$=(\cos\phi+i\sin\phi)\left(-2\sin^2\frac{\phi}{2}+i2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$

$=2i\sin\frac{\phi}{2}\left(\cos\frac{3\phi}{2}+i\sin\frac{3\phi}{2}\right)$

$=2\sin\frac{\phi}{2}\left(\cos\frac{3\phi+\pi}{2}+i\sin\frac{3\phi+\pi}{2}\right)$

Now, $3\phi\in(\pi,3\pi)\Rightarrow\frac{3\phi+\pi}{2}\in(\pi,2\pi)$

But, the argument lies in $(-\pi,\pi]$, hence

$\arg\left(Z^2-Z\right)=\frac{3\phi+\pi}{2}-2\pi=\frac{3}{2}(\phi-\pi)$

Q9

$\arg\left(\frac{z}{w}\right) = \frac{\pi}{2} \Rightarrow \frac{z}{w} = \left|\frac{z}{w}\right| e^{i\pi/2}$

$\Rightarrow \frac{z}{w} = \left|\frac{z}{w}\right| i$

$\Rightarrow w = z\frac{|w|}{|z|}(-i)$

$\Rightarrow w\bar{z} = \frac{\bar{z}z|w|}{|z|}(-i)$

$\Rightarrow 5i\bar{z}w = 5\left(-i^2\right)\frac{|z|^2|w|}{|z|}$

$= 5(1)|z||w|$

$= 5$

Q10

Let, $z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

$Im\ \left(z^5\right) = r^5 \sin 5\theta$

and $(Im\ z)^5 = r^5 \sin^5\theta$

$\frac{Im\ \left(z^5\right)}{(Im\ z)^5} = \frac{\sin 5\theta}{\sin^5\theta} = \text{A (Let)}$

$\Rightarrow \frac{dA}{d\theta} = \frac{5\sin^5\theta\cos 5\theta - 5\sin 5\theta\sin^4\theta\cos\theta}{\left(\sin^5\theta\right)^2}$

$\Rightarrow \frac{dA}{d\theta} = \frac{5\sin^4\theta\left(\sin\theta\cos 5\theta - \sin 5\theta\cos\theta\right)}{\sin^{10}\theta}$

$\Rightarrow \frac{dA}{d\theta} = \frac{5}{\sin^6\theta}[\sin(-4\theta)] = 0$

$\Rightarrow \theta = \frac{\pi}{4}$

The minimum value of A will be at $\theta = \frac{\pi}{4}$.

$\Rightarrow \frac{\sin 5\frac{\pi}{4}}{\left(\sin\frac{\pi}{4}\right)^5}$

$= \frac{\frac{-1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^5}$

$= -\left(\sqrt{2}\right)^4 = -4$

Q11

Given $\overline{z} + i\overline{w} = 0 \Rightarrow \overline{z} = -i\overline{w}$ or $z = iw$ or $\frac{z}{w} = i$

amp(z)−amp(w)= amp $i = \frac{\pi}{2}$. . . .(i)

also amp(zw)= π

amp(z)+amp(w)= π . . . . .(ii)

Adding (i) & (ii), we get,

2amp(z)= $\frac{3\pi}{2}$

$\Rightarrow$ amp(z)= $\frac{3\pi}{4}$

Also, amp(w)= $\frac{\pi}{4}$

Q12

$\mathrm{x}^2 + \mathrm{x} + 1 = 0$

$\Rightarrow \mathrm{x} = \omega, \omega^2$

$\Rightarrow \alpha = \omega$ and $\Rightarrow \beta = \omega^2$

$\Rightarrow \alpha^{2020} = \omega^{2020} = \left(\omega^3\right)^{673} . \omega = \omega$

$\Rightarrow \beta^{2020}=\left(\omega^2\right)^{2020}=\left(\omega^3\right)^{2\times 673}\times\omega^2=\omega^2$

$\Rightarrow$ The required equation is $\mathrm{x}^2+\mathrm{x}+1=0$

Q13

$z=\frac{-1}{2}i\left(1+i\sqrt{3}\right)=i\omega^2$

$z^{101}=i\omega$

$\left(z^{101}+i^{109}\right)^{106}=(i\omega+i)^{106}=\left(i\left(-\omega^2\right)\right)^{106}=-\omega^2$

as given that $\left(z^{101}+i^{109}\right)^{106}=z^n$

$\therefore -\omega^2=\left(i\omega^2\right)^n=i^n\omega^{2n}$

$\omega^{2n-2}i^n=-1$

this is possible only when $n=4r+2$ and $2n-2$ is a multiple of 3 i.e.,

$2(4r+2)-2$ is a multiple of 3

i.e., $8r+2$ is a multiple of $3\Rightarrow r=2$

$\therefore n=10 \quad \therefore \frac{2}{5}k=4$

Q14

Let, $S=\sum_{n=0}^{100}(i)^{n!}$ (where $i=\sqrt{-1}$)

$= (i)^{0!} + (i)^{1!} + (i)^{2!} + (i)^{3!} + (i)^{4!} + \ldots + (i)^{100!}$

$= (i)^{1} + (i)^{1} + (i)^{2} + (i)^{6} + (i)^{24} + \ldots . + (i)^{100!}$

(since, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$, where $n \in N$)

From i^{24} term onwards, every term is of the form i^{4n} which equals 1,

Simplifying above expression, we get

$= i + i - 1 - 1 + 1 + 1 + \ldots . + 1$

$= 2i - 2 + 97 = 2i + 95$

Q15

We have, $|a\omega + b|^2 = 1$

$\Rightarrow (a\omega + b)\left(a\bar{\omega} + b\right) = 1$

$\Rightarrow a^2 + ab\left(\omega + \bar{\omega}\right) + b^2 = 1$

$\Rightarrow a^2 - ab + b^2 = 1$

$\Rightarrow (a-b)^2 + ab = 1 \ldots .(\text{i})$

$\left(\text{As, } 1 + \omega + \omega^2 = 0\right)$

When $(a-b)^2 = 0$ and $ab = 1$ then $(1,1);(-1,-1)$

When $(a-b)^2 = 1$ and $ab = 0$ then $(0,1);(1,0);(0,-1);(-1,0)$

Hence, $(0,1);(1,0);(0,-1);(-1,0);(1,1);(-1,-1)$ i.e., 6 ordered pairs.

Q16

Let, $z = re^{i\theta}$

$\Rightarrow r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$

$\Rightarrow r^2 e^{i5\theta} = -3$

$\Rightarrow r^2 = 3$ and $e^{i5\theta} = -1$

$\Rightarrow r = \sqrt{3}$ and $\theta = \frac{\pi}{5} + \frac{2k}{5}$ where, $k = 0, 1, 2, 3, 4$

$\Rightarrow 5$ solutions

Q17

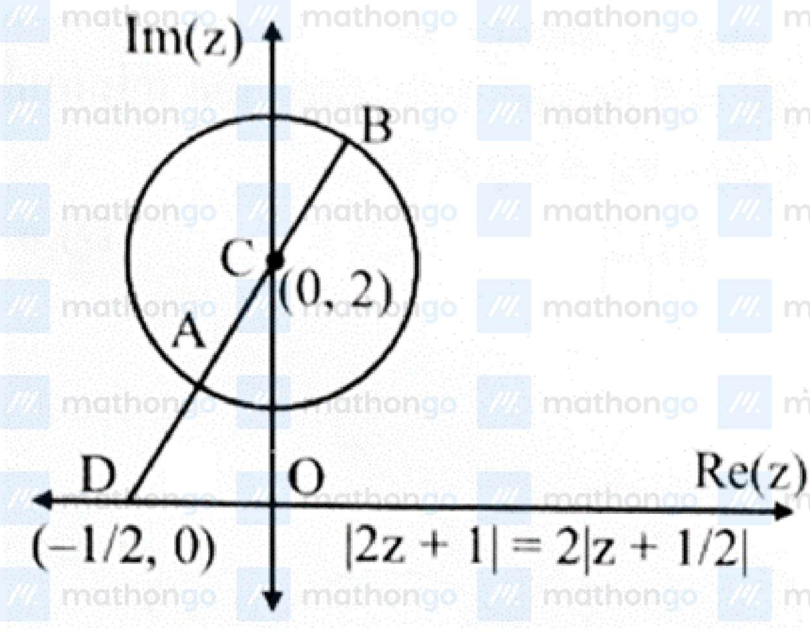
From triangle inequality we know that $|z_1 + z_2| \geq ||z_1| - |z_2||$

hence $\left|z + \frac{1}{z}\right| = \left|z - \left(-\frac{1}{z}\right)\right| \geq |z| - \left|-\frac{1}{z}\right| \geq 3 - \frac{1}{3} = \frac{8}{3}$

Hence $\frac{8}{3}$ is the correct answer.

Q18

$z - 2i = 1$ represents a circle with centre at $(0, 2)$ and radius 1 unit



$|2z + 1| = 2\left|z - \left(\frac{-1}{2}\right)\right|$ represents twice the distance from the point $\left(\frac{-1}{2}, 0\right)$

Hence from the diagram,

$m=2AD=2(CD-AC)$

$m=2\left(\frac{\sqrt{17}}{2}-1\right)=\sqrt{17}-2$

$M=2BD=2(CD+BC)$

$M=2\left(\frac{\sqrt{17}}{2}+1\right)=\sqrt{17}+2$

$\Rightarrow (M-m)^2=4^2=16$

Q19

Given, $|Z_1|=|Z_2|=|Z_3|=1$

Now, $|Z_1+Z_2+Z_3|\leq|Z_1|+|Z_2|+|Z_3|$

$\Rightarrow|-a|\leq 1+1+1$

$\Rightarrow|a|\leq 3$

Also, $|Z_1Z_2Z_3|=|Z_1|\times|Z_2|\times|Z_3|$

$\Rightarrow|-c|=1\times1\times1$

$\Rightarrow|c|=1$

Q20

Given, $\mathrm{zw}=|\mathrm{z}|^2\Rightarrow \mathrm{zw}=\mathrm{z}\,\overline{\mathrm{z}}$

$\Rightarrow \mathrm{w}=\overline{\mathrm{z}}\ \{\mathrm{z}\neq 0\}$

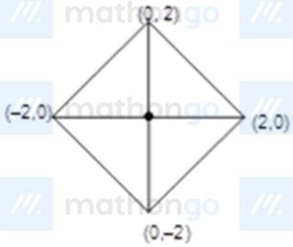
Now, $\left|\mathrm{z}-\overline{\mathrm{z}}\right|+\left|\mathrm{w}+\overline{\mathrm{w}}\right|=4$

$\Rightarrow\left|\mathrm{z}-\overline{\mathrm{z}}\right|+\left|\mathrm{z}+\overline{\mathrm{z}}\right|=4$

Let, $\mathrm{z}=\mathrm{x}+i\mathrm{y}$, then we get,

$|\mathrm{x}|+|\mathrm{y}|=2$

which represents a square of side length equal to $2\sqrt{2}$



$\Rightarrow$ The perimeter of the locus is $8\sqrt{2}$ units

Q21

Let $z = x + iy$

$\Rightarrow (1 + 2i)z + (2i - 1)\bar{z} = 10i$

$\Rightarrow (1 + 2i)(x + iy) + (2i - 1)(x - iy) = 10i$

$\Rightarrow (x - 2y) + i(2x + y) + (-x + 2y) + i(2x + y) = 10i$

$\Rightarrow 2i(2x + y) = 10i$

Or $2x + y = 5$

For interception on imaginary axis

Put $x = 0$

So, we get $y = 5$

Intercept on imaginary axis $= 5$

Q22

Let us assume that z lies on a circle with centre z_0 (fixed point) and radius $\frac{1}{2}$ units.

$\Rightarrow \quad |z - z_0| = \frac{1}{2}$

Now, $\omega = -1 + 4z \Rightarrow \omega + 1 = 4z$

$\Rightarrow \quad \omega + 1 - 4z_0 = 4z - 4z_0$

Now, taking modulus on both sides, we get,

$|\omega + 1 - 4z_0| = 4|z - z_0| \Rightarrow |\omega + 1 - 4z_0| = 2$

Locus of ω represents the circle having centre $(-1 + 4z_0)$ and radius 2 units.

Q23

Given, $|z - i| = |z + 2i|$

$\Rightarrow x^2 + (y-1)^2 = x^2 + (y+2)^2$

$\Rightarrow -2y + 1 = 4y + 4$

$\Rightarrow 6y = -3 \Rightarrow y = -\frac{1}{2}$

From, $|z| = \frac{5}{2}$, we get,

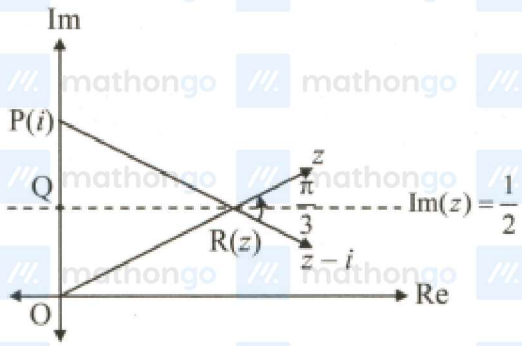
$x^2 + y^2 = \frac{25}{4} \Rightarrow x^2 = \frac{24}{4} = 6$

$\Rightarrow z = \pm\sqrt{6} - \frac{i}{2}$

Hence, $|z + 3i| = \sqrt{6 + \frac{25}{4}} = \sqrt{\frac{49}{4}}$

$|z + 3i| = \frac{7}{2}$

Q24



$|z|=|z-i|$

$\Leftrightarrow |PR|=|OR|$

$arg\ \frac{z}{z-i}=\frac{\pi}{3}$

$\Leftrightarrow \angle PRO=\frac{\pi}{3}$

$\Rightarrow \Delta POR$ is an equilateral triangle.

$\Rightarrow QR$ is angle bisector as well as median.

$\Rightarrow \angle PRQ=\frac{\pi}{6},\ |PQ|=\frac{1}{2}$

$\Rightarrow Im\ (z)=\frac{1}{2},\ Re(z)=|QR|$

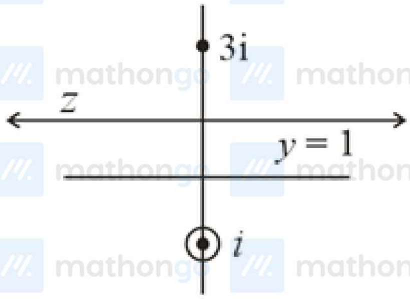
$=|PQ|\ cot\ \frac{\pi}{6}$

$=\frac{1}{2}\left(\sqrt{3}\right)$

$z=\frac{\sqrt{3}}{2}+\frac{i}{2}\Rightarrow 2z-i=\sqrt{3}$

$\Rightarrow\ [Re\ (2z-i)]=[1.73]=1$

Q25



$\omega = z\bar{z} - 2z + 2$

$\left|\frac{z+i}{z-3i}\right| = 1$

$\Rightarrow |z+i| = |z-3i|$

$\Rightarrow z = x + i,\ x \in R$

$\omega = (x+i)(x-i) - 2(x+i) + 2$

$= x^2 + 1 - 2x - 2i + 2$

$\text{Re}(\omega) = x^2 - 2x + 3$

For min $(\text{Re}(\omega)), x = 1$

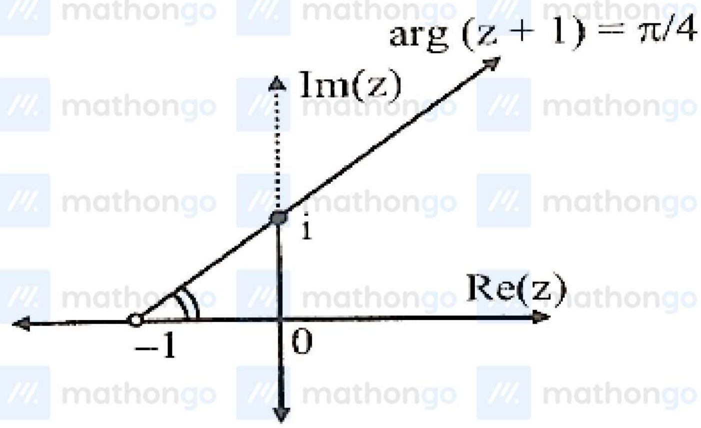
$\Rightarrow \omega = 2 - 2i = 2(1-i) = 2\sqrt{2}e^{-i\frac{\pi}{4}}$

$\omega^n = \left(2\sqrt{2}\right)^n e^{-i\frac{n\pi}{4}}$

For real and minimum value of n,

$n = 4$

Q26



$z^3 = \overline{z} \Rightarrow |z| = 0$ or 1

$|z| = 0 \Rightarrow z = 0$

$|z| = 1 \Rightarrow z^4 = 1$

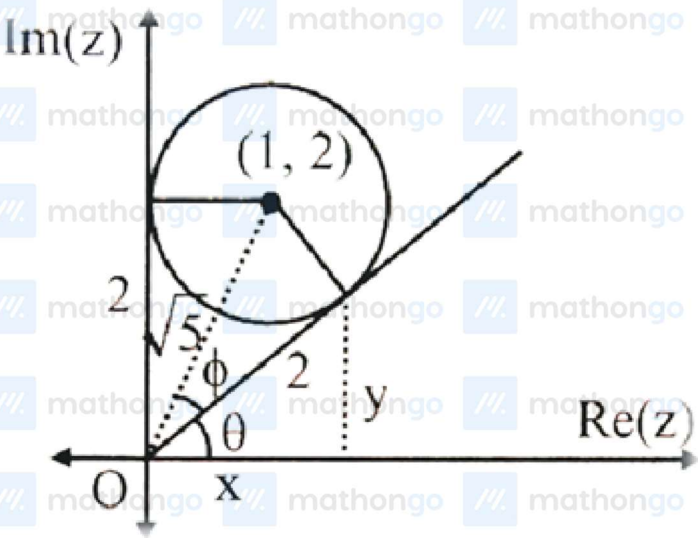
$\Rightarrow z = \pm 1, \pm 1$

Only $z = i$ satisfies $\arg(z + 1) = \frac{\pi}{4}$

Q27

Here, $|z - 1 - 2i| = 1$ represents a circle with centre $(1, 2)$ and radius 1 unit.

The complex number $z = \mathrm{x} + i\mathrm{y}$ satisfying the given inequality and having the least positive argument is the point of contact of the tangent from the origin to the circle with the least positive slope.



From the diagram,

$\tan \phi=\frac{1}{2}$

$\therefore \tan 2 \phi=\frac{2 \tan \phi}{1-\tan ^2 \phi}=\frac{4}{3}$

For the least positive argument, $\arg (z)=\theta$ (let)

$\tan \theta=\tan \left(\frac{\pi}{2}-2 \phi\right)=\cot 2 \phi=\frac{3}{4}$

Also, from the diagram,

$x^2+y^2=4$ and $\tan \theta=\frac{y}{x}=\frac{3}{4}$

i.e. $y=\frac{3}{4} x$

$\Rightarrow x^2+\frac{9 x^2}{16}=4 \Rightarrow x=\frac{8}{5}$

Hence, for the least positive argument, the real part of z is equal to $\frac{8}{5}$

Q28

If $Z_1,\ Z_2,\ Z_3$ form an equilateral triangle, then we know that,

$Z_1^2 + Z_2^2 + Z_3^2 = Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1$

$\Rightarrow a^2 - 1 + 2ai + 1 - b^2 + 2bi - a + b - i - abi = 0$

$\Rightarrow (a-b)(a+b-1) + (2a+2b-ab-1)i = 0$

Case-I:

$a = b\ \&\ 2a + 2b - ab - 1 = 0$

$\Rightarrow a = b\ \&\ a^2 - 4a + 1 = 0 \Rightarrow a = b = 2 - \sqrt{3}$

Case-II:

$a + b - 1 = 0\ \&\ 2a + 2b - ab - 1 = 0$

$\Rightarrow ab = 1$ (not possible because $a, b \in (0,1)$)

$\Rightarrow a = b = 2 - \sqrt{3}$ is the only solution

Q29

Given, $\arg\left(\frac{z+i}{z-i}\right) = \frac{2\pi}{3}$

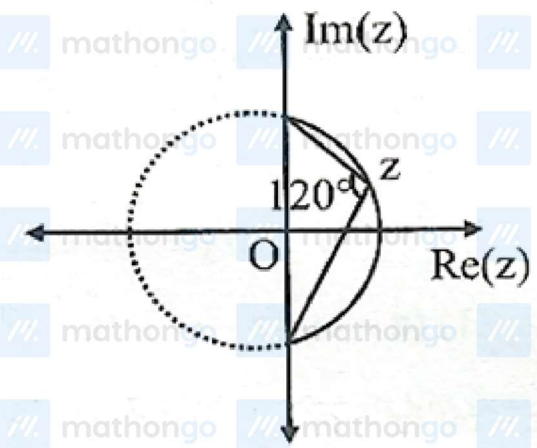
Let, $z = x + iy$

$\Rightarrow \frac{x+i(y+1)}{x+i(y-1)} = \frac{x^2+(y^2-1)+2ix}{x^2+(y-1)^2}$

$\Rightarrow \tan^{-1}\frac{2x}{x^2+y^2-1} = \frac{2\pi}{3}$

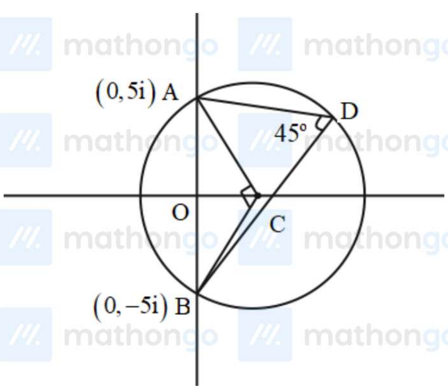
$\Rightarrow x^2 + y^2 + \frac{2}{\sqrt{3}}x - 1 = 0$

Hence, the given locus is a circle with centre $\left(-\frac{1}{\sqrt{3}}, 0\right)$ and radius $\frac{2}{\sqrt{3}}$ units



$\Rightarrow$ Length of the arc of the circle is $\frac{2\pi}{3} \times \left(\frac{2}{\sqrt{3}}\right) = \frac{4\pi}{3\sqrt{3}}$ units

Q30



$r^2 + r^2 = 10^2 \Rightarrow r = 5\sqrt{2}$

max.$(OP) = OC +$ radius $= 5 + 5\sqrt{2}$

and min.$(OP) = OA = 5$

Required value $= \frac{5+5+5\sqrt{2}}{2} = 5 + \frac{5}{\sqrt{2}}$