Q1

$\because x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$

Put $t=\tan\theta$ in both the equations,

we get $x=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$

and $y=\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$...(ii)

On differentiating both the eqs.(i) and (ii), we get $\frac{dx}{d\theta}=-2\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$

Therefore, $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=-\frac{\cos 2\theta}{\sin 2\theta}=-\frac{x}{y}$

Q2

Given, $\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2$

$\Rightarrow\left(\frac{x^3-y^3}{x^3+y^3}\right)=(10)^2=100$

Applying componendo and dividendo, we get, $\frac{x^3}{y^3}=-\frac{101}{99}\Rightarrow\frac{x}{y}=\left(-\frac{101}{99}\right)^{\frac{1}{3}}\ldots\ldots$(i)

Hence, $y=\left(-\frac{99}{101}\right)^{\frac{1}{3}}x$

Differentiating, we get,

$\frac{dy}{dx}=\left(-\frac{99}{101}\right)^{\frac{1}{3}}=\frac{y}{x}$ (from (i))

Q3

$y=a\cos(\log x)+b\sin(\log x)$ On differentiating w.r.t. x, we get $y'=\frac{-a\sin(\log x)}{x}+\frac{b\cos(\log x)}{x}$

$\Rightarrow xy'=-a\sin(\log x)+b\cos(\log x)$

Again, on differentiating w.r.t. x, we get

$xy''+y'=-a\cos(\log x)\frac{1}{x}-b\sin(\log x)\frac{1}{x}$

$\Rightarrow x^2y''+y'x=-[a\cos(\log x)+b\sin(\log x)]$

$\Rightarrow x^2y''+xy'=-y$

Q4

We have,

$\phi(x)=\log_8 \log_3 x=\log_8\left(\frac{\log x}{\log 3}\right)$
$=\log_8(\log x)-\log_8(\log 3)$
$=\frac{\log(\log x)}{\log 8}-\log_8(\log 3)$
$\phi'(x)=\frac{1}{\log 8}\cdot\frac{1}{\log x}\cdot\frac{1}{x}-0$
$\therefore \phi'(e)=\frac{1}{\log 8}\cdot\frac{1}{\log e}\cdot\frac{1}{e}=\frac{1}{e\log 8}$

Q5

$y=\log_{10} x+\log_x 10+\log_x x+\log_{10} 10$

$=\log_{10} x+\frac{\log_e 10}{\log_e x}+1+1$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x}\log_{10} e-\frac{\log_e 10}{x(\log_e x)^2}$

Q6

$f\left(x^3\right)=x^5$

On differentiating with respect to x

$f'\left(x^3\right)\cdot 3x^2=5\cdot x^4$
$f'\left(x^3\right)=\frac{5}{3}x^2$

Putting $x=3$, we get,

$f'(27)=\frac{5}{3}(9)=15$

Q7

Given equation can be rewritten as

$y=2+\sqrt{\sin x+y}$
$\Rightarrow(y-2)^2=\sin x+y$
$\Rightarrow y^2-4y+4=\sin x+y\ldots$(i)

Putting $x=0$ in the equation (i), we get,

$y^2-4y+4=0+y \Rightarrow y^2-5y+4=0$

$\Rightarrow (y-1)(y-4)=0$

$y=1$ or $y=4$

$\because y>2 \Rightarrow y=4$

Now differentiating equation (i) with respect to x, we get, $\frac{dy}{dx}=\frac{\cos x}{2y-5}$

Putting $x=0, y=4$

$\frac{dy}{dx}=\frac{\cos(0)}{2(4)-5}=\frac{1}{3}$

Q8

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{5\cos t}{-3\sin t}=\frac{-5}{3}\cot(t)$$

$$\frac{d^2y}{dx^2}=\frac{-5}{3}\cdot\left(-\operatorname{cosec}^2 t\right)\cdot\frac{dt}{dx}=\frac{5}{3}\left(\frac{1}{\sin^2 t}\right)\left(\frac{1}{-3\sin t}\right)$$

$$=\frac{5}{9}(8)$$

Hence, $9\frac{d^2y}{dx^2}=40$

Q9

$\mathrm{y}=\mathrm{x}^2+\frac{1}{\mathrm{y}}$

$\mathrm{y}^2=\mathrm{x}^2\mathrm{y}+1$

$2\mathrm{y}\frac{\mathrm{dy}}{\mathrm{dx}}=\mathrm{y}.2\mathrm{x}+\mathrm{x}^2\frac{\mathrm{dy}}{\mathrm{dx}}$

$\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{2\,\mathrm{xy}}{2\mathrm{y}-\mathrm{x}^2}$

Q10

By taking logarithm on both sides,

$\log(x) = (1+x)\left[\log(2+x)-\log(1+x)\right]$

On differentiating, we get,

$\frac{1}{f(x)} \cdot f'(x) = \frac{(1+x)}{(2+x)} + \log(2+x) - \frac{(1+x)}{(1+x)} - \log(1+x)$

Now, put $x = 0$

$f'(0) = f(0)\left\{\frac{1+\log 2}{2} - 1\right\} = f(0)\left[\log 2 - \frac{1}{2}\right]$

$= 2\log 2 - 1(\because f(0) = 2)$

Q11

$x^3 + y^3 = t + \frac{4}{t}$

$x^6 + y^6 + 2x^3y^3 = t^2 + \frac{16}{t^2} + 8$

... [By squaring both the sides]

$\Rightarrow \left(t^2 + \frac{16}{t^2}\right) + 2x^3y^3 = t^2 + \frac{16}{t^2} + 8$

$\Rightarrow x^3y^3 = 4 \quad \ldots(i)$

Differentiating with respect to x we get

$x^3\left(3y^2\frac{dy}{dx}\right) + y^3 \cdot (3x^2) = 0$

$\Rightarrow 3x^3y^2\frac{dy}{dx} = -3x^2y^3$

$\Rightarrow x^3y^2\frac{dy}{dx} = -x^2y^3$

$\Rightarrow x^4y^2\frac{dy}{dx} = -x^3y^3$

$\Rightarrow x^4y^2\frac{dy}{dx} = -4$ ...[From (i)]

Q12

Let $f(x)=a(x-3)^3+b(x-3)^2+c(x-3)+d$

$f(3)=21\Rightarrow d=21$

$f'(3)=30\Rightarrow c=30$

$f''(3)=22\Rightarrow b=11$

$f'''(3)=6\Rightarrow a=1$

$f(x)=(x-3)^3+11(x-3)^2+30(x-39)+21$

$\Rightarrow f'(x)=3(x-3)^2+22(x-3)+30$

$\Rightarrow f'(2)=11$

Q13

$y=fgh$

$\frac{dy}{dx}=f'gh+fg'h+fgh'$

$=\frac{1}{2}(2f'gh+2fg'h+2fgh')$

$=\frac{1}{2}(h(f'g+g'f)+g(f'h+fh')+f(g'h+gh))$

$=\frac{1}{2}\left[h\cdot(fg)'+g\cdot(fh)'+f\cdot(gh)'\right]$

$(fgh)'\left(0\right)=\frac{1}{2}\left[h(0)(fg)'(0)+g(0)(fh)'(0)+f(0)(gh)'(0)\right]$

$= \frac{1}{2}(3 \times 6 + 2 \times 5 + 1 \times 4)$

$= \frac{1}{2}(18 + 10 + 4) = \frac{32}{2} = 16$

Q14

Given, $f(x+y) = 2^x f(y) + 4^y (f(x)$

Put $y = 2$ we get,

$f(x+2) = 2^x \times 3 + 16 f(x)$

$f'(x+2) = 16 f'(x) + 3 \times 2^x \ln 2$

Now put $x = 2$ we get,

$f'(4) = 16 f'(2) + 12 \ln 2 \quad \ldots(\text{i})$

Similarly, $f(y+2) = 4 f(y) + 3 \times 4^y$

$f'(4) = 4 f'(y) + 3 \times 4^y \ln 4$

$f'(4) = 4 f'(2) + 96 \ln 2 \quad \cdots(\text{ii})$

solving eq. (i) and (ii), we get

$f'(2) = 7 \ln 2$

from equation (i), we get

$f'(4) = 124 \ln 2$

Now $\Rightarrow 14 \times \frac{f'(4)}{f'(2)}$

$14 \times \frac{124 \ln 2}{7 \ln 2} = 248$

Q15

$f^2(4) + g^2(4) = ?$

$\frac{d}{dx}\left[f^2(x) + g^2(x)\right] = 2f\left(x\right)f'\left(x\right) + 2g\left(x\right)g'\left(x\right)$

$= 2\ f(x)\ g(x) + 2\ g(x)(-f(x))$

$= 0$

$\therefore f^2\left(x\right) + g^2\left(x\right) = \text{constant}$

$\therefore f^2\left(4\right) + g^2\left(4\right) = f^2\left(2\right) + g^2\left(2\right)$

$= 4^2 + (f'(2))^2$

$= 4^2 + 4^2 = 32$

Q16

$y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x}\right)$

$= \tan^{-1}\left(\frac{2\left(3x^{\frac{3}{2}}\right)}{1-\left(3x^{\frac{3}{2}}\right)^2}\right) = 2\tan^{-1}\left(3x^{\frac{3}{2}}\right)$

$\frac{dy}{dx} = \frac{2\cdot\left(3x^{\frac{1}{2}}\right)\left(\frac{3}{2}\right)}{1+9x^2} = \frac{9}{1+9x^2}\sqrt{x}$

$\therefore \quad g(x) = \frac{9}{1+9x^3}$

Q17

$y = e^{nx} \Rightarrow \text{In}\, y = nx$

Now, $\frac{d^2y}{dx^2} = n^2e^{nx} = n^2y$

and $\frac{d^2x}{dy^2} = \frac{1}{ny^2}$

$\therefore \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = \frac{n}{y} = -ne^{-nx}$

Q18

We have, $\phi(x)=f^{-1}(x)$

$\Rightarrow x=f[\phi(x)]$

On differentiating both sides w.r.t. x, we get $1=f'[\phi(x)]\cdot\phi'(x)\Rightarrow\phi'(x)=\frac{1}{f'[\phi(x)]}\cdots$ (i)

Since, $f'(x)=\frac{1}{1+x^5}$ (given)

$f'(\phi(x))=\frac{1}{1+[\phi(x)]^5}$

From Equation (i),

$\phi'(x)=\frac{1}{f'[\phi(x)]}=1+[\phi(x)]^5$

Q19

$m\log x+n\log y=(m+n)\log(x+y)$

$\Rightarrow\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{y}$

$\Rightarrow\frac{nx-my}{y(x+y)}\frac{dy}{dx}=\frac{nx-my}{(x+y)x}$

$\Rightarrow\frac{dy}{dx}=\frac{y}{x}$

Q20

$\because f(x),g(x)$ are inverse function of each other

$\therefore g'(f(x))=\frac{1}{f'(x)}$

Now put $x=1;g'(f(1))=\frac{1}{f'(1)}$

Hence, $g'(5)=\frac{1}{(3x^2+3)_{x=1}}=\frac{1}{6}$

Q21

$g'(x)=2[f(2f(x)+2)]f'(2f(x)+2)\,(2f'(x))$

$\Rightarrow g'(0)=2[f(2f(0)+2)]f'(2f(0)+2)\,(2f'(0))$

$=4(f(0))f'(0)^2=4(-1)(1)^2=-4$

Q22

$f''(x) - g''(x) = 0 \Rightarrow f'(x) - g'(x) = \text{constant.}$

Putting $x = 1$, we get $f'(1) - g'(1) = C \Rightarrow C = 2g'(1) - g'(1)$

$= g'(1) = 2$

So $f'(x) - g'(x) = 2 \Rightarrow f(x) - g(x) = 2x + C'$.

Putting $x = 2$ we have $f(2) - g(2) = 4 + C \Rightarrow 4+$

$C' = 3g(2) - g(2) = 2g(2) = 6 \Rightarrow C' = 6 - 4 = 2$

Hence $f(x) - g(x) = 2x + 2$. At $x = \frac{3}{2}$

$f\left(\frac{3}{2}\right) - g\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2} + 2 = 5$

Q23

Putting $x = 1$ in $x^{2x} - 2x^x \cot y - 1 = 0$

we get $1 - 2\cot y - 1 = 0$

$\Rightarrow \cot y = 0 \Rightarrow y = \pi/2$

Using the formula,

$\frac{d}{dx}\left[f(x)^{g(x)}\right] = g(x) f(x)^{g(x)-1} f'(x) + (g'(x) \log$

$f(x)) f(x)^{g(x)}$, we have

$2x \cdot x^{2x-1}(1) + (2\log x)x^{2x} - 2\left[x \cdot x^{x-1}(1) + (\log x)\right.$

$\left.x^x\right] \cot y + 2x^x \left(\operatorname{cosec}^2 y\right) y'(x) = 0$

Putting $x = 1, y = \pi/2$, we get

$2 + 0 - 2(1)(0) + 2\operatorname{cosec}^2(\pi/2) y'(1) = 0$

$\Rightarrow y'(1) = -1$

Q24

$2y = \left(\cot^{-1} \frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)^2$

$= \left(\cot^{-1}\left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right)\right)^2$

$= \left(\cot^{-1} \tan\left(\frac{\pi}{3} + x\right)\right)^2$

$= \left(\cot^{-1} \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{3} + x\right)\right)\right)^2$

$\Rightarrow 2y = \begin{cases} \left(\frac{\pi}{6} - x\right)^2, \ x \in \left(0, \frac{\pi}{6}\right) \\ \left(\pi + \frac{\pi}{6} - x\right)^2, \ x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \end{cases}$

$\Rightarrow \ \frac{2dy}{dx} = 2\left(\frac{\pi}{6} - x\right).(-1) \Rightarrow \frac{dy}{dx} = x - \frac{\pi}{6}, \ x \in \left(0, \frac{\pi}{6}\right)$

And $\frac{dy}{dx} = x - \frac{7\pi}{6}, \ x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

Left Hand and Right Hand Derivatives are not same so function is non derivable at $x = \frac{\pi}{6}$.

Hence, $\frac{dy}{dx}$ does not exist for all values in the given interval.

Q25

Given:

$\log_e(x + y) = 4xy$

When $x = 0$, then $y = 1$

$\log_e(x + y) = 4xy$

$\Rightarrow x + y = e^{4xy}$

Now differentiate w.r.t. x

$1 + y' = e^{4xy}(4y + 4xy') \quad \ldots \text{(i)}$

At $(0, 1) \Rightarrow y'(0) + 1 = 4 \Rightarrow y'(0) = 3$

Now, again differentiate equation (i), we get

$$y'' = e^{4xy}\left(4y + 4xy'\right)^2 + e^{4xy}\left(4y' + 4y' + 4xy''\right)$$

At $(0, 1)$

$$y''(0) = 1\left(4 \times 1 + 0\right)^2 + 1\left(4 \times 3 + 4 \times 3 + 0\right)$$

$$\Rightarrow y''(0) = 16 + 24 = 40$$

$$\Rightarrow y''(0) = 40$$

Q26

Let, $f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow f'(x) = 3x^2 + 2ax + b$

$\Rightarrow f''(x) = 6x + 2a$

$\Rightarrow f'''(x) = 6$

According to the question,

$a = f'(1) = 3 + 2a + b \Rightarrow a + b = -3 \ldots (i)$

$b = f''(2) = 12 + 2a \Rightarrow 2a - b = -12 \ldots (ii)$

$c = f'''(3) \Rightarrow c = 6$

Solving equations (i) & (ii), we get, $a = -5$ & $b = 2$

$\Rightarrow f(x) = x^3 - 5x^2 + 2x + 6$

$\therefore\ f(2) = 8 - 20 + 4 + 6 = -2.$

Q27

$8f(x) + 6f\left(\frac{1}{x}\right) - x = 5$ (i)

Replace x by $\frac{1}{x}$, we have $8f\left(\frac{1}{x}\right) + 6f(x) - \frac{1}{x} = 5$ (ii)

Solving (i) and (ii) for $f(x)$ and $f(1/x)$, we have

$14f(x) = 5 - \frac{3}{x} + 4x$

$\Rightarrow 14f'(x) = \frac{3}{x^2} + 4$

So $f(-1) = \frac{1}{14}[5 + 3 - 4] = \frac{2}{7}$

and $f'(-1) = \frac{1}{14}[3 + 4] = \frac{1}{2}$

Differentiating $y = x^2 f(x)$, we have $\frac{dy}{dx} = 2xf(x) +$ $x^2 f'(x)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = -2f(-1) + f'(-1) = -\frac{4}{7} + \frac{1}{2} = -\frac{1}{14}$

Q28

Given, $(2x)^{2y} = 4.\,e^{2x-2y}$

Taking natural logarithm on both sides, we get

$2y\log_e 2x = \log_e 4 + (2x - 2y)$

$\Rightarrow 2y = \left(\frac{\log_e 4 + 2x}{1 + \log_e 2x}\right)$

Differentiating both sides with respect to x, we get

$\frac{2dy}{dx} = \frac{(1+\log_e 2x).2 - (\log_e 4 + 2x)\frac{1}{x}}{(1+\log_e 2x)^2}$ (Using quotient rule)

$\Rightarrow (1 + \log_e 2x)^2 \frac{dy}{dx} = \left(\frac{x.\log_e 2x - \log_e 2}{x}\right).$

Q29

let $b^{-1}(x)=g(x)$

$g(b(x))=x$

$g'(f(x))b'(x)=1$

$g'(f(x))=\frac{1}{b'}(x)$

$g''(f(x))f'(x)=\frac{-b''(x)}{(f'(x))^2}$

If $f(x)=5, x=1$

$\therefore g''(5)=-\frac{f''(1)}{(f'(1))^3}=\frac{-4}{8}=\frac{-1}{2}$

$|2g''(5)|=1$

Q30

Given,

$y(x)=(x^x)^x$

Taking $\log_e$ both side

$\ln y(x)=x^2\cdot\ln x$

Now differentiating both side w.r.t x we get,

$\frac{1}{y(x)}\cdot y'(x)=\frac{x^2}{x}+2x\cdot\ln x$

$y'(x)=y(x)[x+2x\ \ln x]$ ....(i)

Given $y(1)=1$, so $y'(1)=1$

Now rewriting equation (i) again we get,

$\frac{dx}{dy}=\frac{1}{x^{x^2+1}(1+2\ln x)}$

Now $\frac{d^2x}{dy^2}=\frac{d}{dx}\left(\left(x^{x^2+1}(1+2\ln x)\right)^{-1}\right)\frac{dx}{dy}$

$\Rightarrow \frac{d^2x}{dy^2}=\frac{-x^{x^2}(1+2\ln x)(x^2+3+2x^2\ln x)}{\left(x^{x^2}(1+2\ln x)\right)^3}\times 1$

$\Rightarrow \left(\frac{d^2x}{dy^2}\right)_{x=1}=-4$

So, $\left(\frac{d^2x}{dy^2}\right)_{x=1}+20=-4+20=16$