Q1

$\alpha+\beta=-\mathrm{a}$ and $\alpha\beta=1$

Let S and P be the sum and product of the roots of the required equation. Then,

$\mathrm{S}=-\alpha-\frac{1}{\beta}-\frac{1}{\alpha}-\beta=-\left(\alpha+\beta\right)-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

$=-\left(\alpha+\beta\right)-\left(\frac{\alpha+\beta}{\alpha\beta}\right)=-\left(-\mathrm{a}\right)-\left(\frac{-\mathrm{a}}{1}\right)=2\mathrm{a}$

$\mathrm{P}=-\left(\alpha+\frac{1}{\beta}\right)\left(-\left(\frac{1}{\alpha}+\beta\right)\right)$

$=1+\alpha\beta+\frac{1}{\alpha\beta}+1=1+1+1+1=4$

So, the required equation is

$\mathrm{x}^2-\mathrm{Sx}+\mathrm{P}=0$

i.e. $\mathrm{x}^2-2\mathrm{ax}+4=0$

Q2

We have $\frac{k+1}{k}+\frac{k+2}{k+1}=\frac{-b}{a}\ \ldots.(i)$

and $\frac{k+1}{k}\cdot\frac{k+2}{k+1}=\frac{c}{a}$

$\Rightarrow\ \frac{k+2}{k}=\frac{c}{a}$

or $\frac{2}{k}=\frac{c}{a}-1=\frac{c-a}{a}$

or $\mathrm{k}=\frac{2\mathrm{a}}{\mathrm{c}-\mathrm{a}}$ .....(ii)

Now eliminate k putting the value of k in 1^{st} relation, we get

$\frac{c+a}{2a}+\frac{2c}{c+a}=\frac{-b}{a}$

$\Rightarrow\ \left(c+a\right)^2+4ac=-2b\left(a+c\right)$

$\Rightarrow\ \left(a+c\right)^2+2b\left(a+c\right)=-4ac$

Adding b^2 on both sides,

$\left(a+b+c\right)^2=b^2-4ac$

Q3

$D=\left(2n-1\right)^2-4n\left(n-1\right)=4n^2+1-4n-4n^2+4n=1>0$

Product of roots $=\frac{n-1}{n}<0$

$\Rightarrow n\left(n-1\right)<0$

$\Rightarrow n\in\left(0,1\right)$

Q4

$\mathrm{x}^2+2\mathrm{x}-\mathrm{n}=0;\ \mathrm{n}\in[5,\ 100]$

D will be a perfect square,

$\mathrm{D}=4+4\mathrm{n}=4(1+\mathrm{n})$

$\Rightarrow 1+\mathrm{n}$ is a perfect square

$\Rightarrow 1+\mathrm{n}=9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100$

$\Rightarrow \mathrm{n}=8,\ 15,\ 24,\ 35,\ 48,\ 63,\ 80,\ 99$

Therefore, 8 values are possible.

Q5

Given equation has more than two roots if it is an identity

$\Rightarrow \cos 3\theta+1=0$; $2\cos 2\theta-1=0$ and $1-2\cos\theta=0$

$\Rightarrow \cos 3\theta=-1\Rightarrow\theta=\pm\frac{\pi}{3}$ which does not satisfy $2\cos 2\theta-1=0$

Hence, no value possible

Q6

$1, \alpha+\beta, \alpha\beta$ are in A.P. $\Rightarrow 1, \frac{-b}{a}, \frac{c}{a}$ are in A.P.

$\Rightarrow 1+\frac{c}{a}=\frac{-2b}{a} \Rightarrow a+c+2b=0 \ldots(1)$

$\frac{1}{\alpha}, \frac{1}{2}, \frac{1}{\beta}$ are in A.P. $\Rightarrow \frac{1}{\alpha}+\frac{1}{\beta}=1 \Rightarrow \alpha+\beta=\alpha\beta$

$\Rightarrow \frac{-b}{a}=\frac{c}{a} \Rightarrow b+c=0 \ldots(2)$

From (1) & (2) we get,

$a=-b=c$

$\Rightarrow \alpha, \beta$ are roots of equation $x^2-x+1=0$

Now, $\frac{\alpha^2+\beta^2-2\alpha^2\beta^2}{2(\alpha^2+\beta^2)}=\frac{1}{2}-\frac{(\alpha\beta)^2}{(\alpha+\beta)^2-2\alpha\beta}$

$=\frac{1}{2}-\frac{(1)^2}{(1)^2-2(1)}=\frac{1}{2}+1=1.5$

Q7

$\alpha+\beta=\alpha^2+\beta^2$ & $\alpha\beta=\alpha^2\beta^2 \Rightarrow \alpha\beta=0$ or 1.

If $\alpha\beta=0$, then let, $\alpha=0 \Rightarrow \beta=0$ or 1.

If $\beta=\frac{1}{\alpha}$,

$\alpha+\frac{1}{\alpha}=\alpha^2+\frac{1}{\alpha^2}=\left(\alpha+\frac{1}{\alpha}\right)^2-2$

$\Rightarrow \left(\alpha+\frac{1}{\alpha}\right)^2-\left(\alpha+\frac{1}{\alpha}\right)-2=0 \Rightarrow \alpha+\frac{1}{\alpha}=2$ or $-1 \Rightarrow \alpha=1$ or ω, ω^2

Hence number of such equations are four $(0, 0), (0, 1), (1, 1)$ & $\left(\omega, \omega^2\right)$

Q8

Given, $x^2+5\sqrt{2}x+10=0$

and $P_n=\alpha^n-\beta^n$

Now $\frac{P_{17}P_{20}+5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19}+5\sqrt{2}P_{18}^2}=\frac{P_{17}\left(P_{20}+5\sqrt{2}P_{19}\right)}{P_{18}\left(P_{19}+5\sqrt{2}P_{18}\right)}$

$\frac{P_{17}\left(\alpha^{20}-\beta^{20}+5\sqrt{2}(\alpha^{19}-\beta^{19})\right)}{P_{18}\left(\alpha^{19}-\beta^{19}+5\sqrt{2}(\alpha^{18}-\beta^{18})\right)}$

$\frac{P_{17}\left(\alpha^{19}(\alpha+5\sqrt{2})-\beta^{19}(\beta+5\sqrt{2})\right)}{P_{18}\left(\alpha^{18}(\alpha+5\sqrt{2})-\beta^{18}(\beta+5\sqrt{2})\right)}$

Since $\alpha+5\sqrt{2}=-10/\alpha$ ... (1)

and $\beta+5\sqrt{2}=-10/\beta$ ... (2)

Now put there values in above expression

$\frac{P_{17}\left(\alpha^{19}(\alpha+5\sqrt{2})-\beta^{19}(\beta+5\sqrt{2})\right)}{P_{18}\left(\alpha^{18}(\alpha+5\sqrt{2})-\beta^{18}(\beta+5\sqrt{2})\right)}=-\frac{10P_{17}P_{18}}{-10P_{18}P_{17}}=1$

Q9

Given α,β are the roots of the quadratic equation $2x^2-5x+1=0$

Let us find an equation with roots α^2 and β^2, let $y=x^2$, so $x=\sqrt{y}$

$2y-5\sqrt{y}+1=0$

$\Rightarrow 2y+1=5\sqrt{y}$

$\Rightarrow 4y^2+4y+1=25y$

$\Rightarrow 4y^2-21y+1=0 \left\langle{c \atop d}\right.$

Put $\alpha^2=c$ and $\beta^2=d$

Now, $S_n=(c)^n+(d)^n$

Consider

$4S_{2021}+S_{2019}=4\left(c^{2021}+d^{2021}\right)+c^{2019}+d^{2019}$
$=c^{2019}\left(4c^2+1\right)+d^{2019}\left(4d^2+1\right)$
$=c^{2019}(21c)+d^{2019}(21d)$

$=21S_{2020}$

Hence, $\frac{4S_{2021}+S_{2019}}{S_{2020}}=21$

Q10

Consider $x^2-47x+k=0$

For real roots, $47^2-4k\geq 0\Rightarrow k\leq 552$

$\therefore\quad k=1,2,3\ldots.,552$

Product of real roots $=1\times 2\times 3\times 4\times\ldots..\times 552=552!$

Q11

$-3<\frac{x^2-\lambda x-2}{x^2+x+1}<2$
$\Rightarrow -3x^2-3x-3<x^2-\lambda x-2<2x^2+2x+2$ $\left(\text{since } x^2+x+1>0,\ \forall x\in R\right)$

$\Rightarrow 4x^2+x(3-\lambda)+1>0, x^2+x(2+\lambda)+4>0$
(i) $4x^2-x(\lambda-3)+1>0$
$\Rightarrow D<0\Rightarrow(\lambda-3)^2-4\times 4\times 1<0$

$\Rightarrow(\lambda-3+4)(\lambda-3-4)<0$

$\Rightarrow(\lambda+1)(\lambda-7)<0\Rightarrow\lambda\in(-1,7)$

(ii) $x^2+x(\lambda+2)+4>0$

$\Rightarrow D<0\Rightarrow(\lambda+2)^2-4\times 4<0$

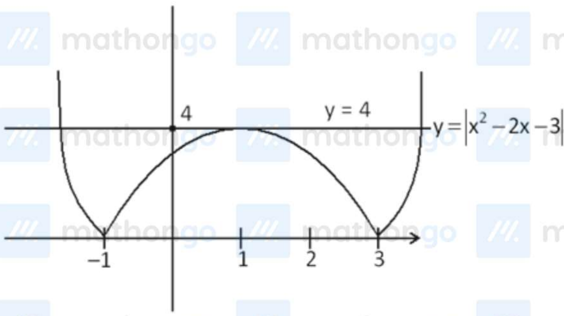
$(\lambda+2-4)(\lambda+2+4)< 0$

$(\lambda-2)(\lambda+6)< 0 \Rightarrow \lambda \in(-6,2)$

Taking the intersection of the solutions of (i) and (ii), we get,

$\lambda \in(-1,2)$

Q12



$b<0 \rightarrow$ no solution

$b=0 \rightarrow$ two solutions

$0<b<4 \rightarrow$ four solutions

$b=4 \rightarrow$ three solutions

Q13

$D=25b^2-4\times 3a\times 7c$

$=25(-a-c)^2-84ac$

$=25\left(a^2+c^2+2ac\right)-84ac$

$=25\left(a^2+c^2\right)-34ac$

$=17\left(a^2+c^2-2ac\right)+8\left(a^2+c^2\right)$

$=17(a-c)^2+8\left(a^2+c^2\right)$

$D>0 \Rightarrow$ Roots are real and distinct

Q14

$4a + 4b + 4c - 9c > 0$

$4a + 4b - 5c > 0$

$f(x) = ax^2 + 2bx - 5c, \ D < 0$

$f(2) = 4a + 4b - 5c > 0$

$a > 0$ and

$$\begin{aligned} &-5c > 0 \\ &\Rightarrow c < 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} a > 0 \\ c < 0 \end{aligned}}$$

Q15

$$\begin{aligned} D &= B^2 - 4AC \\ &= (2(a+b-2c))^2 - 4(a-b)^2 \\ &= 4\left\{(a+b-2c)^2 - (a-b)^2\right\} \\ &= 4(a+b-2c-a+b)(a+b-2c+a-b) \\ &= 4(2b-2c)(2a-2c) \\ &= 16(b-c)(a-c) \\ &= 16(c-b)(c-a) \end{aligned}$$

If c lies between a and b, then D is negative. Hence, the roots will be imaginary and the graph will be entirely above the x-axis as the coefficient of x^2 is positive.

Q16

Let, α be the common root and the other roots of the equations be 4β and 3β respectively. Then,

$\alpha + 4\beta = 6,\ \alpha(4\beta) = a$

$\alpha + 3\beta = c,\ \alpha(3\beta) = 6 \Rightarrow \frac{4}{3} = \frac{a}{6} \Rightarrow \alpha = 8$

The first equation is $x^2 - 6x + 8 = 0$

Whose roots are 2 and 4

If $\alpha = 2 \Rightarrow \beta = 1$

So, roots of first equation is 2, 4 and that of second equation is 2, 3

If $\alpha = 4 \Rightarrow \beta = \frac{1}{2} \Rightarrow 3\beta = \frac{3}{2}, 4\beta = 2$

Here the roots are not integers $\Rightarrow \alpha = 2$

Q17

Let, $f(x) = 4x^2 - 20kx + (25k^2 + 15k - 66) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

Let the roots of $f(x) = 0$ be $\alpha,\ \beta$

Since $\alpha,\ \beta$ are real.

$\therefore\ \Delta \geq 0$

$\Rightarrow 400k^2 - 4.4(25k^2 + 15k - 66) \geq 0$

$\Rightarrow\ -15k + 66 \geq 0 \quad \Rightarrow \quad k \leq \frac{22}{5} \ldots\ldots\ldots\ldots (ii)$

We have $\alpha,\ \beta < 2$

$\therefore \quad \alpha+\beta<4$

$\Rightarrow \quad -\frac{(-20k)}{4}<4 \quad \Rightarrow k<\frac{4}{5} \ldots\ldots\ldots\ldots\ldots(iii)$

$f(x)=4(x-\alpha)(x-\beta)$

$\therefore \quad f(2)=4(2-\alpha)(2-\beta)=4(+)(+)=+\text{ve}$

$\therefore \quad f(2)=16-40k+\left(25k^2+15k-66\right)>0$

$\Rightarrow 25k^2-25k-50>0 \quad \Rightarrow \quad k^2-k-2>0$

$\Rightarrow \quad (k+1)(k-2)>0 \quad \Rightarrow \quad k<-1 \text{ or k} > 2 \ldots\ldots\ldots\ldots(\text{iv})$

Combining (ii), (iii) & (iv), we get $\text{k} \in(-\infty,-1)$

Q18

Clearly, $\text{f}(-1)>0$, $\text{f}(2)<0$

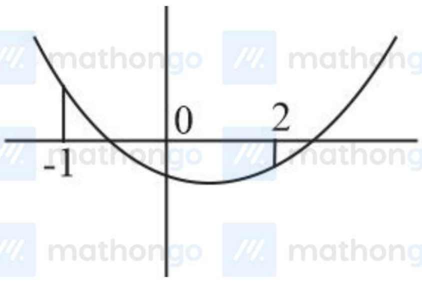
since, $\text{f}(0)=-4<0$

$\Rightarrow 1-\text{a}-4>0 \Rightarrow -\text{a}-3>0 \Rightarrow -\text{a}>3$

or $\text{a}<-3$ and $4+2\text{a}-4<0$

$\Rightarrow \text{a}<0$

$\Rightarrow$ a $\in$ $(-\infty, -3)$.



Q19

The roots of $f(x) - x = 0$ are 1, 2 and 3.

So, we get,

$f(x) - x = (x-1)(x-2)(x-3)(x-a)$

For $x = -1$, we get,

$f(-1) + 1 = (-2)(-3)(-4)(-1-a) = 24(1+a)$

For $x = 5$, we get,

$f(5) - 5 = 4 \cdot 3 \cdot 2(5-a) = 24(5-a)$

$f(-1) + f(5) = (23 + 24a) + (125 - 24a) = 148$

For $x = 0$, we get,

$f(0) - 0 = (-1)(-2)(-3)(-a) = 6a$

For $x = 4$, we get,

$f(4) - 4 = 3 \cdot 2 \cdot 1 \cdot (4 - a) = 24 - 6a$

$f(0) + f(4) = 28$

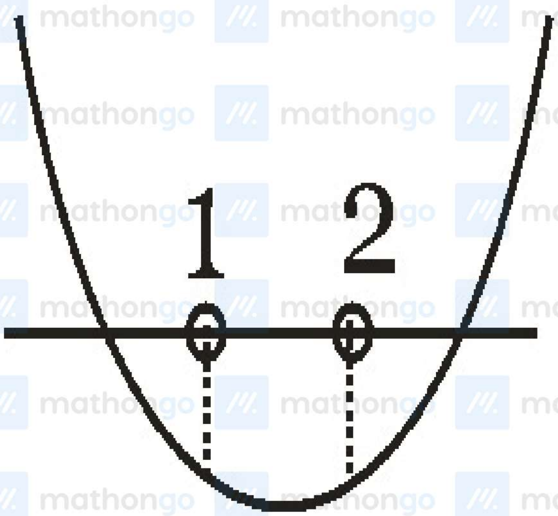
$\left[\frac{f(-1)+f(5)}{f(0)+f(4)}\right] = \left[\frac{148}{28}\right]$

$\left[\frac{f(-1)+f(5)}{f(0)+f(4)}\right] = 5$

Q20

Let, $f(x) = x^2 + ax + a^2 + 6a$

$\therefore \; f(1) \leq 0$



$\Rightarrow \quad a^2 + 7a + 1 < 0$

Hints and Solutions **MathonGo**

or $\frac{-7-3\sqrt{5}}{2} < a < \frac{-7+3\sqrt{5}}{2}$ ....(i)

$f(2) \leq 0$

$\Rightarrow \quad a^2 + 8a + 4 < 0$

or $-4 - 2\sqrt{3} < a < -4 + 2\sqrt{3}$ ....(ii)

and $D > 0$

$\Rightarrow a^2 - 4 \cdot 1\left(a^2 + 6a\right) > 0$

$\Rightarrow a^2 + 8a < 0$

or $-8 < a < 0$ ....(iii)

From Eqs. (i), (ii) and (iii), we get,

$\frac{-7-3\sqrt{5}}{2} \leq a \leq -4 + 2\sqrt{3}$

Hence, integral values of a are $-6, -5, -4, -3, -2, -1$

Required Sum

$= (-6)^2 + (-5)^2 + (-4)^2 + (-3)^2 + (-2)^2 + (-1)^2$

$= 91.$

Q21

Let α be the common root

$\Rightarrow k\alpha^2 + \alpha + k = 0$

$k\alpha^2 + k\alpha + 1 = 0$

On solving, we get,

$\frac{\alpha^2}{1-k^2} = \frac{\alpha}{k^2-k} = \frac{1}{k^2-k}$

$\frac{\alpha^2}{\alpha} = \frac{1-k^2}{k^2-k}$ and $\frac{\alpha}{1} = \frac{k^2-k}{k^2-k}$

$\Rightarrow \alpha = \frac{1-k^2}{k^2-k} = 1 \Rightarrow k^2 - k = 1 - k^2$

$\Rightarrow 2k^2 - k - 1 = 0 \Rightarrow k = -\frac{1}{2}, 1$

For $k = 1$, equations are identical, thus not possible

Hence, $k = -\frac{1}{2}$

Q22

$(6k+2)x^2 + rx + 3k - 1 = 0$

$(12k+4)x^2 + px + 6k - 2 = 0$ have

both roots common.

So,

$\frac{6k+2}{12k+4} = \frac{r}{p} = \frac{3k-1}{6k-2}$

$\Rightarrow \frac{r}{p} = \frac{1}{2} \Rightarrow 2r - p = 0$

Q23

Let, the roots be $\alpha, \beta, \alpha+2$.

$S_1=\alpha+\beta+\alpha+2=2\alpha+\beta+2=13 \Rightarrow 2\alpha+\beta=11 \Rightarrow \beta=11-2\alpha$

$S_2=\alpha\beta+\beta(\alpha+2)+(\alpha+2)\alpha=15$

$\Rightarrow \beta(\alpha+\alpha+2)+\alpha(\alpha+2)=15$

$\Rightarrow(11-2\alpha)(2\alpha+2)+\alpha(\alpha+2)=15$

$\Rightarrow 22\alpha+22-4\alpha^2-4\alpha+\alpha^2+2\alpha=15$

$\Rightarrow 3\alpha^2-20\alpha-7=0 \Rightarrow(\alpha-7)(3\alpha+1)=0$

$\Rightarrow \alpha=7$ or $-\frac{1}{3}$.

$\alpha=7, \beta=11-2\alpha=11-14=-3, \gamma=\alpha+2=9$

$\alpha=-\frac{1}{3}, \beta=11-2\alpha=11+\frac{2}{3}=\frac{35}{3}, \gamma=\alpha+2=\frac{5}{3}$.

Since, $\alpha\beta\gamma=-189$, hence we will take the first case.

$|\alpha|+|\beta|+|\gamma|=|7|+|-3|+|9|=19$

Q24

$x^3+3x^2+5x+3=0$ has one root $x=-1$

$\therefore x^3+3x^2+5x+3=(x+1)\left(x^2+2x+3\right)$

$\Rightarrow a=2,\ b=3$

Q25

Given equation,

$4\left(x^2+\frac{1}{x^2}\right)+16\left(x+\frac{1}{x}\right)-57=0$

Let, $x+\frac{1}{x}=y\ ;\ x^2+\frac{1}{x^2}=y^2-2$

$\Rightarrow 4y^2+16y-65=0$

$\Rightarrow y=-\frac{13}{2}$ or $\frac{5}{2}$

When, $y=\frac{5}{2}$

$x+\frac{1}{x}=\frac{5}{2} \Rightarrow x=2$ or $\frac{1}{2}$

When, $y=-\frac{13}{2}$

$\Rightarrow x+\frac{1}{x}=-13/2$

$\Rightarrow 2x^2+13x+2=0$

$\Rightarrow x=\frac{-13\pm\sqrt{153}}{4}$

Since x is rational, $x=2$ or $\frac{1}{2}$

Q26

The given equation will be true in 3 cases.

Case 1: when $x^2-5x+5=1$

$\Rightarrow x^2-5x+4=0$

$\Rightarrow x=1,\ 4$

Case 2: when $x^2+4x-60=0$

$\Rightarrow x=6,\ -10$

Case 3: when $x^2-5x+5=-1$ and $x^2+4x-60\in$ even integers

Now, $x^2-5x+5=-1$

$\Rightarrow x=2,\ 3$

Only $x=2$ satisfies the given condition, Hence, sum of all real values of x is $1+4+6-10+2=3$

Q27

Let $\log_x 10 = t$

$\therefore t^3 - t^2 - 6t = 0$

$\Rightarrow t\left(t^2 - t - 6\right) = 0$

$\Rightarrow t = 0, -2, 3$

$\Rightarrow \log_x 10 = 0, \ -2, \ 3$

$\Rightarrow 10 = x^0, x^{-2}, x^3$

$\Rightarrow x = 10^{-\frac{1}{2}}, 10^{\frac{1}{3}}$

Let $\alpha = 10^{-\frac{1}{2}}$ and $\beta = 10^{\frac{1}{3}}$

Now, $\left|\frac{1}{\log_{10} \alpha\beta}\right| = \left|\frac{1}{\log_{10} 10^{-\frac{1}{6}}}\right|$

$\Rightarrow \left|\frac{1}{\log_{10} \alpha\beta}\right| = \left|\frac{-6}{\log_{10} 10}\right| = 6$

Q28

Let, $\mathrm{t} = 2^{11\mathrm{x}}$

$\Rightarrow \frac{\left(2^{11x}\right)^3}{2^2} + 2^{11x} . 2^2 = \left(2^{11x}\right)^2 . 2 + 1$

$\Rightarrow \frac{t^3}{4} + 4t = 2t^2 + 1$

$\Rightarrow t^3 - 8t^2 + 16t - 4 = 0$

Cubic in t has roots t_1, t_2, t_3

i.e. $t_1t_2t_3 = 4 \Rightarrow 2^{11x_1}.2^{11x_2}.2^{11x_3} = 4$

$\Rightarrow 2^{11(x_1+x_2+x_3)} = 2^2$

$\Rightarrow 11(x_1+x_2+x_3) = 2 \Rightarrow x_1+x_2+x_3 = \frac{2}{11}$

Q29

We have, $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$

Let $e^x = t$

$t^4 - t^3 - 4t^2 - t + 1 = 0$

$\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$

$\Rightarrow \alpha^2 - \alpha - 6 = 0, \alpha = t + \frac{1}{t} \geq 2$

$\Rightarrow \alpha = 3, -2$ (reject)

$\Rightarrow t + \frac{1}{t} = 3$

$\Rightarrow$ The number of real roots $= 2$.

Q30

Let $\frac{\pi}{2^{\cos^{-1}x}} = t \quad \Rightarrow \quad t \geq 2$

equation becomes $t^2 - \left(a + \frac{1}{2}\right)t - a^2 = 0$

has one roots 2 or greater than 2 and other root less than 2, $f(2) \leq 0$

$\Rightarrow 4 - \left(a + \frac{1}{2}\right)2 - a^2 \leq 0$

$a^2 + 2a - 3 \geq 0$

$(a+3)(a-1) \geq 0$

$a \leq -3$ or $a \geq 1$