Q1

$x^2-3>0, 6x-10>0 \Rightarrow x>\sqrt{3}$

Also $\log_2\left(\frac{x^2-3}{6x-10}\right)=-1 \Rightarrow \frac{x^2-3}{6x-10}=\frac{1}{2}$

$\Rightarrow \quad x^2-3=3x-5$

$\Rightarrow \quad x^2-3x+2=0 \Rightarrow x=1,2$

Thus, $\quad x=2$

Q2

The given expression is

$\log_{10}(2^x+1)+x=\log_{10}(6)+x\log_{10}(5)$

We know that $\log_m(x)+\log_m(y)=\log_m(xy)$ & $\log_m(x)+\log_m(y)=\log_m\left(\frac{x}{y}\right)$

$\Rightarrow \log_{10}(2^x+1)=x(\log_{10}(5)-\log_{10}(10))+\log_{10}(6)$

$\Rightarrow \log_{10}(2^x+1)=-\log_{10}(2)^x+\log_{10}(6)$

$\Rightarrow \log_{10}(2^x+1)+\log_{10}(2^x)=\log_{10}(6)$

$\Rightarrow \log_{10}[(2^x)(2^x+1)]=\log_{10}(6)$

Taking antilog on both sides, we get

$\Rightarrow (2^x)(2^x+1)=6$

$\Rightarrow (2^x)^2+2^x-6=0$

$\Rightarrow (2^x-2)(2^x+3)=0$

$\Rightarrow 2^x=2 \Rightarrow x=1$ Hence, $x=1$

Q3

As we know that $\log_a(b) > 0$, if $0 < a < 1 \& 0 < b < 1$.

Given, $\log_{\frac{1}{3}}\left(x^2 + 2x\right) > 0$

$\Rightarrow 0 < x^2 + 2x < 1$

Breaking into two cases:

Case I : $x^2 + 2x > 0$

$\Rightarrow x(x+2) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (0, \infty) \quad \ldots(1)$

Case II : $x^2 + 2\text{x} < 1$

$\Rightarrow x^2 + 2x - 1 < 0$

$\Rightarrow \left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right) < 0$

$\Rightarrow -1 - \sqrt{2} < x < -1 + \sqrt{2} \quad \ldots(2)$

From equation (1) and (2),we get

$x \in \left(-1 - \sqrt{2}, -2\right) \cup \left(0, \sqrt{2} - 1\right)$

Thus, $\left(-1 - \sqrt{2}, -2\right) \cup \left(0, \sqrt{2} - 1\right)$ is correct option.

Q4

$\log_{175} 5x = \log_{343} 7x = k$

$\Rightarrow \frac{5}{7} = \left(\frac{175}{343}\right)^k \Rightarrow k = \frac{1}{2} \Rightarrow x = \sqrt{7}.$

Q5

We have,

$\log_3(x-3) = \log_9(\mathrm{x}-1)$

$\Rightarrow \frac{\log(x-3)}{\log 3} = \frac{\log(\mathrm{x}-1)}{\log 9}$

$\Rightarrow \frac{\log(x-3)}{\log 3} = \frac{\log(\mathrm{x}-1)}{\log 3^2}$

$\Rightarrow \frac{\log(x-3)}{\log 3} = \frac{\log(\mathrm{x}-1)}{2\log 3}$

$\Rightarrow 2\log(x-3) = \log(\mathrm{x}-1)$

$\Rightarrow \log(x-3)^2 = \log(\mathrm{x}-1)$

$\Rightarrow (\mathrm{x}-3)^2 = (\mathrm{x}-1)$

$\Rightarrow \mathrm{x}^2 - 7\mathrm{x} + 10 = 0$

$\Rightarrow \mathrm{x} = 2,\ 5$

$\mathrm{x} = 2$ is not possible as $\log_3(x-3)$ is not defined for $x = 2$.

Therefore, $x = 5$.

Q6

Let

Hints and Solutions **MathonGo**

$$D=\begin{vmatrix} 5\sqrt{\log_5 3} & 5\sqrt{\log_5 3} & 5\sqrt{\log_5 3} \\ 3^{-\log_{1/3}(4)} & (0.1)^{\log_{0.01}(4)} & 7^{\log_7(3)} \\ 7 & 3 & 5 \end{vmatrix}$$

$C_2 \leftrightarrow C_2 - C_1$ and $C_3 \leftrightarrow C_3 - C_1$

$$\Rightarrow D=\begin{vmatrix} 5\sqrt{\log_5 3} & \left[5\sqrt{\log_5 3}-5\sqrt{\log_5 3}\right] & \left[5\sqrt{\log_5 3}-5\sqrt{\log_5 3}\right] \\ 3^{-\log_{1/3}(4)} & \left[\left(0.1\right)^{\log_{0.01}(4)}-3^{-\log_{1/3}(4)}\right] & \left[7^{\log_7(3)}-3^{-\log_{1/3}(4)}\right] \\ 7 & -4 & -2 \end{vmatrix}$$

$$\Rightarrow D=\begin{vmatrix} 5\sqrt{\log_5 3} & 0 & 0 \\ 3^{-\log_{1/3}(4)} & \left[\left(0.1\right)^{\log_{0.01}(4)}-3^{-\log_{1/3}(4)}\right] & \left[7^{\log_7(3)}-3^{-\log_{1/3}(4)}\right] \\ 7 & -4 & -2 \end{vmatrix}$$

$$\Rightarrow D=\begin{vmatrix} 5\sqrt{\log_5 3} & 0 & 0 \\ 3^{-\log_{1/3}(4)} & \left[(4)^{\log_{0.01}(0.1)}-(4)^{-\log_{1/3}(3)}\right] & \left[3^{\log_7(7)}-(4)^{-\log_{1/3}(3)}\right] \\ 7 & -4 & -2 \end{vmatrix}$$

$\left[\because c^{\log_a(b)}=b^{\log_a(c)}\right]$

$$\Rightarrow D=\begin{vmatrix} 5\sqrt{\log_5 3} & 0 & 0 \\ 3^{-\log_{1/3}(4)} & \left[(4)^{\frac{1}{2}}-(4)^1\right] & \left[3^1-(4)^1\right] \\ 7 & -4 & -2 \end{vmatrix}$$

$\left[\because \log_a(b)=\frac{1}{\log_b(a)}\right]$

$$\Rightarrow D\begin{vmatrix} 5\sqrt{\log_5 3} & 0 & 0 \\ 4 & -2 & -1 \\ 7 & -4 & -2 \end{vmatrix}=0$$

Q7

For $x=1$, both parts of the equation vanish, consequently $x=1$ is root of the equation. For $x \neq 1$

$$1=\frac{1}{\log_5 x}+\frac{1}{\log_3 x}+\frac{1}{\log_4 x}=\log_x 5+\log_x 3+\log_x 4$$
$$=\log_x 60$$

$\Rightarrow x=60$. Thus the required set is $\{1,60\}$.

Q8

The given expression is equal to

$$\log_n 2+\log_n 3+\ldots+\log_n 53$$
$$=\log_n(2.3\ldots 53)=\log_n 53!=\frac{1}{\log_{53!} n}$$

Q9

$\log_2 \log x$ is meaningful if $x>1$

Since $4^{\log_2 \log x}=2^{2\log_2 \log x}=\left(2^{\log_2 \log x}\right)^2=(\log x)^2$

$\left[a^{\log_a x}=x, a>0, a \neq 1\right]$

So the given equation reduces to

$$2(\log x)^2-\log x-1=0$$
$\Rightarrow \log x=1, \log x=-1/2.$ But for $x>1$

$\log x>0$ so $\log x=1$ i.e. $x=e$

Q10

Let $X=x^{\frac{1}{\ln y}+\frac{1}{\ln z}} \cdot y^{\frac{1}{\ln z}+\frac{1}{\ln x}} \cdot z^{\frac{1}{\ln x}+\frac{1}{\ln y}}$

$$\Rightarrow \quad \ln x=\ln x\left(\frac{1}{\ln y}+\frac{1}{\ln z}\right)+(\ln y)\left(\frac{1}{\ln z}+\frac{1}{\ln x}\right)+\left(\frac{1}{\ln x}+\frac{1}{\ln y}\right)(\ln z)$$

Now given $\ln x+\ln y+\ln z=0$

$$\therefore \quad \frac{\ln x}{\ln y}+\frac{\ln z}{\ln y}=-1$$

Similarly $\frac{\ln y}{\ln x}+\frac{\ln z}{\ln x}=-1$ and

$\frac{\ln x}{\ln z}+\frac{\ln y}{\ln z}=-1$

$\therefore \quad R.H.S. \ =-3$

$\therefore \quad \ln X=-3$

$X=e^{-3}$

Q11

$x \in(2n+1)\pi/2, n\pi$ where $n \in \mathbf{I}$. The given inequality can be written as $\frac{\log_2\left(x^2-8x+23\right)}{\log_2|\sin x|}>\frac{3}{\log_2|\sin x|}$

As $\log_2|\sin x|<0$, we get

$\log_2\left(x^2-8x+23\right)<3$

$\Rightarrow x^2-8x+23<2^3=8$

$\Rightarrow x^2-8x+15<0$

$\Rightarrow(x-5)(x-3)<0 \Rightarrow 3<x<5$

For $x \in(3,5), x \in \pi, \frac{\pi}{2}, \frac{3\pi}{2}$. Hence

$x \in(3, \pi) \cup\left(\pi, \frac{3\pi}{2}\right) \cup\left(\frac{3\pi}{2}, 5\right)$

Q12

Taking log of both the sides with base 3, we have

$\left(\log_3 x^2+\left(\log_3 x\right)^2-10\right)\left(\log_3 x\right)=-2 \log_3 x$

This equation is equivalent to

$\log_3 x=0$ or $2 \log_3 x+\left(\log_3 x\right)^2-8=0$

$\Rightarrow \quad x=1, \log_3 x=-1 \pm 3$ i.e. $\log_3 x=2, \log_3 x=-4$

Hence $x=1,3^2,3^{-4}=1,9,1/81$

Q13

We use the fact that $\log_b a=\frac{\log_c a}{\log_c b}$ to simply both the sides.

$\frac{6}{5} a^{\frac{\log x}{\log a} \cdot \frac{\log a}{\log 10} \cdot \frac{\log 5}{\log a}}=3^{\log_{10} x-1}+9^{\frac{1}{\log 10 x+\frac{1}{2}}}$ ...(1)

Consider the term on the left side:

$a^{\frac{\log x}{\log a} \cdot \frac{\log 5}{\log_{10} 0}}=a^{\log_u x \cdot \log_{10} 5}=x^{\log_{10} 5}=5^{\log_{10} x} \quad$ (how?)

Using this in (1), along with the substitution $\log_{10} x=t$, we have

$$\frac{6}{5} 5^t=3^{t-1}+9^{\frac{t+1}{2}}=3^{t-1}+3^{t+1}=3^t\left(3+\frac{1}{3}\right)=10.3^{t-1}$$

$\Rightarrow 5^{t-2}=3^{t-2} \Rightarrow t=2 \Rightarrow x=100$

Thus, the correct option is (D).

Q14

For (1) to hold, we must have

$x>0, x \neq 1$ and $2 x^2+x-1>0$

$\Rightarrow x>0, x \neq 1$ and $(2 x-1)(x+1)>0$

$\Rightarrow x>1/2, x \neq 1$

We can write (1) as

$\log_x\left(\frac{2 x^2+x-1}{2}\right)>-1$ (2)

For $1/2<x<1$, (2) can be written as

$\frac{2 x^2+x-1}{2}<\frac{1}{x}$

$\Rightarrow \quad 2 x^3+x^2-x<2$

$\Rightarrow \quad 2\left(x^3-1\right)+x(x-1)<0$

$\Rightarrow \quad(x-1)\left(2 x^2+3 x+2\right)<0$

$\Rightarrow \quad x<1 \quad\left[\because 2 x^2+3 x+2>0 \forall x>0\right]$

For $x>1$, (2) can be written as

$$\frac{2x^2+x-1}{2} > \frac{1}{x}$$

$\Rightarrow \quad (x-1)\left(2x^2+3x+2\right) > 0$

This is true for each $x > 1$.

Thus, (1) holds for $1/2 < x < 1, x > 1$.

Q15

$\log_{\sqrt{2}\sin x}\left(1+\cos x\right) = 2$

$\sqrt{2}\sin x \neq 1, \quad \sqrt{2}\sin x > 0, \quad 1+\cos x > 0$

$\Leftrightarrow \sin x \neq \frac{1}{\sqrt{2}}, \sin x > 0$ and

$x \neq$ odd multiple of $\pi \Rightarrow x \in \left(0, \pi\right) - \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (feasible region)

(i) $\Leftrightarrow \left(\sqrt{2}\sin x\right)^2 = 1+\cos x$

$\Leftrightarrow 2\sin^2 x = 1+\cos x$

$\Leftrightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Leftrightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2} \quad \ldots \left[\cos x + 1 > 0\right]$

$\Rightarrow x = \frac{\pi}{3}$

$\Rightarrow p = 1, q = 3$

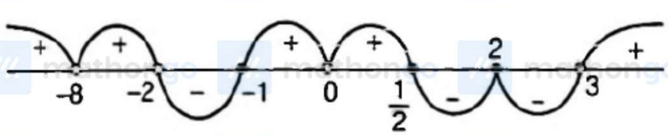
$\Rightarrow p^2 + q^2 = 10$

Q16

We have, $\frac{(x-2)^{10000}(x+1)^{253}\left(x-\frac{1}{2}\right)^{971}(x+8)^4}{x^{500}(x-3)^{75}(x+2)^{93}} \geq 0$

The critical points are $(-8)\cdot(-2), (-1), 0, \frac{1}{2}, 2, 3$

$[\because x \neq -2, 0, 3]$



Hence, $x \in (-\infty, -8] \cup [-8, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

or $x \in (-\infty, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

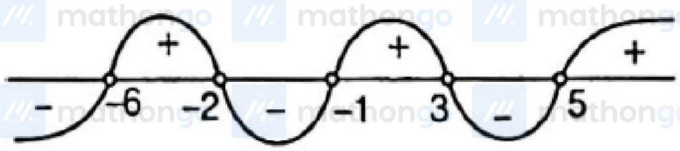
$\{2\}$ also satisfy the given inequality.

Hence, answer is option 4.

Q17

We have, $f(x) = \frac{(x-3)(x+2)(x+6)}{(x+1)(x-5)}$

The critical points are $(-6), (-2), (-1), 3, 5$



For $f(x) > 0, \forall x \in (-6, -2) \cup (-1, 3) \cup (5, \infty)$

For $f(x) < 0, \forall x \in (-\infty, -6) \cup (-2, -1) \cup (3, 5)$

Q18

This equation has the form $|f(x)| = -f(x)$

when, $f(x) = \frac{x^2-8x+12}{x^2-10x+21}$

such an equation is equivalent to the collection of systems

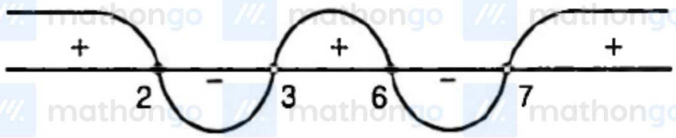
$$\begin{cases} f(x) = -f(x), \text{ if } f(x) \geq 0 \\ f(x) = f(x), \text{ if } f(x) < 0 \end{cases}$$

The first system is equivalent to $f(x) = 0$ and the second system is equivalent to $f(x) < 0$ the combining both systems, we get

$$f(x) \leq 0$$

$$\therefore \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \leq 0$$

$$\Rightarrow \frac{(x-2)(x-6)}{(x-3)(x-7)} \leq 0$$



Hence, by Wavy curve method,

$x \in [2,3) \cup [6,7)$

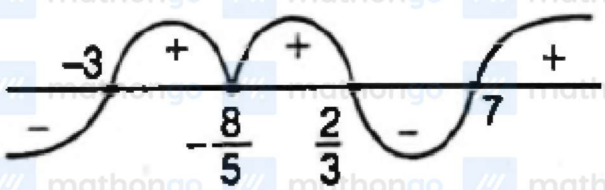
Q19

We have, $(x+3)(3x-2)^5(7-x)^3(5x+8)^2 \geq 0$

$\Rightarrow \quad -(x+3)(3x-2)^5(x-7)^3(5x+8)^2 \geq 0$

$\Rightarrow (x+3)(3x-2)^5(x-7)^3(5x+8)^2 \leq 0$

[take before $x, +$ ve sign in all brackets]



The critical points are $(-3), \left(-\frac{8}{5}\right), \frac{2}{3}, 7$

Hence, $x \in (-\infty, -3] \cup \left[\frac{2}{3}, 7\right] \cup \left\{-\frac{8}{5}\right\}$

Q20

$\frac{x|x|}{x+2} \leq 1$

$\frac{x|x|-x-2}{x+2} \leq 0$

Case I $x \in [0, \infty)$

$\frac{x^2-x-2}{x+2} \leq 0$

$\Rightarrow \quad \frac{(x-2)\ (x+1)}{x+2} \leq 0$

$\Rightarrow \quad x \leq 2$

$\Rightarrow$ integral values 0, 1, 2

Case II $x \in(-\infty,\ 0)$

$\frac{-x^2-x-2}{x+2} \leq 0$

$\Rightarrow \quad x > -2$

$\Rightarrow \quad x = -1$

So 4 integral values

Q21

We have, $\sqrt{(-x^2+4x-3)} > 6-2x$

This inequation is equivalent to the collection of two systems, of inequations

$$\text{i.e. } \begin{cases} 6-2x \geq 0 \\ -x^2+4x-3 > (6-2x)^2 \end{cases} \text{ and } \begin{cases} 6-2x < 0 \\ -x^2+4x-3 \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \leq 3 \\ (x-3)(5x-13) < 0 \end{cases} \text{ and } \begin{cases} x > 3 \\ (x-1)(x-3) \leq 0 \end{cases}$$

$$\Rightarrow \begin{cases} x > 3 \\ \frac{13}{5} < x < 3 \end{cases} \text{ and } \begin{cases} x > 3 \\ 1 \leq x < 3 \end{cases}$$

The second system has no solution and the first system has solution in the interval $\left(\frac{13}{5} < x < 3\right\}$

Hence, $x \in \left(\frac{13}{5}, 3\right)$ is the set of solution of the original inequation.

Q22

We have, $y=2[x]+3=3[x-2] \ldots$ (i)

$\Rightarrow \quad 2[x]+3=3([x]-2) \quad$ [from property (i)]

$\Rightarrow \quad 2[x]+3=3[x]-6$

$\Rightarrow \quad [x]=9$

From Eq. (i), $y=2 \times 9+3=21$

$\therefore \quad [x+y]=[x+21]=[x]+21=9+21=30$

Hence, the value of $[x+y]$ is 30

Q23

$[x]+\sum_{r=1}^{2000} \frac{\{x+r\}}{2000}=[x]+\sum_{r=1}^{2000} \frac{\{x\}}{2000} \quad$ [from property (i)]

$=[x]+\frac{\{x\}}{2000} \sum_{r=1}^{2000} 1=[x]+\frac{\{x\}}{2000} \times 2000=[x]+\{x\}=x$

Q24

This equation is equivalent to the collection of systems

$$\begin{cases}|x-(4-x)|-2x=4, \text{ if } 4-x \geq 0 \\ |x+(4-x)|-2x=4, \text{ if } 4-x<0\end{cases}$$

$$\Rightarrow \begin{cases}|2x-4|-2x=4, & \text{if } x \leq 4 \\ 4-2x=4, & \text{if } x>4\end{cases} \quad \ldots \text{(i)}$$

The second system of this collection

gives $x=0$

but $x>4$

Hence, second system has no solution.

The first system of collection Eq. (i) is equivalent to the system of collection

$$\begin{cases}2x-4-2x=4, \text{ if } 2x \geq 4 \\ -2x+4-2x=4, \text{ if } 2x<4\end{cases}$$

$$\Rightarrow \begin{cases}-4=4, \text{ if } x \geq 2 \\ -4x=0, \text{ if } x<2\end{cases}$$

The first system is failed and second system gives $x=0$.

Hence, $x=0$ is unique solution of the given equation.

Q25

As the minimum value of $|x-1|+|x-2|+|x-3|+|x-4|$ is 4.

Hence number of solutions $=0$

Q26

We have, $\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$

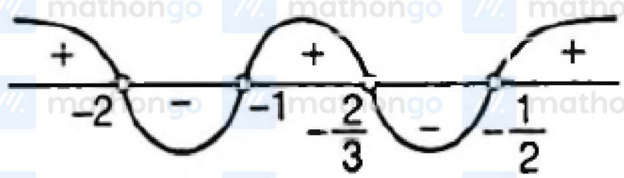
$\Rightarrow \quad \frac{2x}{(x+2)(2x+1)} - \frac{1}{(x+1)} > 0$

$\Rightarrow \quad \frac{(2x^2+2x)-(2x^2+5x+2)}{(x+2)(x+1)(2x+1)} > 0$

$\Rightarrow \quad -\frac{(3x+2)}{(x+2)(x+1)(2x+1)} > 0$

or $\quad \frac{(3x+2)}{(x+2)(x+1)(2x+1)} < 0$

The critical points are $(-2), (-1), \left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)$



Hence, $x \in (-2,-1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

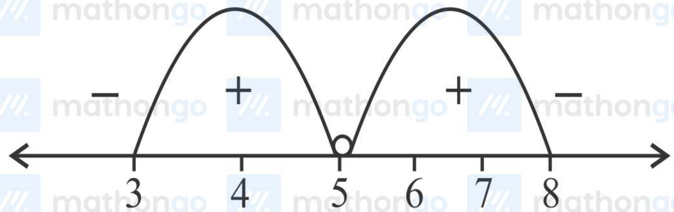
Q27

$\frac{(x^2-2x+8)(e^x+2)(x-3)(x-8)}{(\log_2(x^2+3))(x-5)^2} \leq 0$

x^2-2x+8, e^x+2 and $\log_2(x^2+3)$ are positive quantities

Next we have to find condition for $(x-3)$,$(x-5)$ and $(x-8)$

At $x=5$, the denominator $=0$. So $x=5$ is not a solution. Therefore, number of integral solutions will be between 3 and 8 excluding 5 (using wavy curve method)



Thus, we have 5 integral values possible.

Q28

If $|a+b+c|=|a|+|b|+|c|$ then a, b, c have same sign

$\left|1-\log_{1/6} x\right|+\left|-\log_2 x\right|+|2|=\left|3-\log_{1/6} x-\log_2 x\right|$

$\therefore 1-\log_{1/6} x \geqslant 0$

$\frac{1}{6} \leqslant x$

$-\log_2 x \geqslant 0$

$x \leqslant 2$

$\therefore x \in\left[\frac{1}{6}, 2\right], a=2$ and $b=12$

$\frac{a+b}{2}=7$

Q29

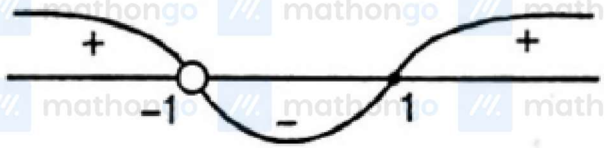
The given inequation is equivalent to the collection of systems

$$\left\{\begin{array}{l}\left|1-\frac{x}{1+x}\right| \geq \frac{1}{2}, \text{ if } x \geq 0 \\ \left|1+\frac{x}{1-x}\right| \geq \frac{1}{2}, \text{ if } x<0\end{array} \Rightarrow\left\{\begin{array}{l}\frac{1}{|1+x|} \geq \frac{1}{2}, \text{ if } x \geq 0 \\ \frac{1}{|1-x|} \geq \frac{1}{2}, \text{ if } x<0\end{array}\right.\right.$$

$$\Rightarrow\left\{\begin{array}{l}\frac{1}{1+x} \geq \frac{1}{2}, \text{ if } x \geq 0 \\ \frac{1}{1-x} \geq \frac{1}{2}, \text{ if } x<0\end{array} \Rightarrow\left\{\begin{array}{l}\frac{1-x}{1+x} \geq 0, \text{ if } x \geq 0 \\ \frac{1+x}{1-x} \geq 0, \text{ if } x<0\end{array}\right.\right.$$

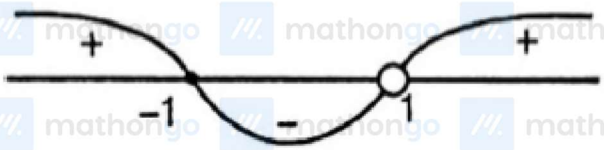
$$\Rightarrow\left\{\begin{array}{l}\frac{x-1}{x+1} \leq 0, \text{ if } x \geq 0 \\ \frac{x+1}{x-1} \leq 0, \text{ if } x<0\end{array}\right.$$

For $\frac{x-1}{x+1} \leq 0$, if $x \geq 0$



$\therefore \quad 0 \leq x \leq 1 \quad ...(i)$

For $\frac{x+1}{x-1} \leq 0$, if $x < 0$



$\therefore \quad -1 \leq x < 0$ ...(ii)

Hence, from Eqs. (i) and (ii), the solution of the given equation is $x \in [-1, 1]$

Aliter

$\left|1 - \frac{|x|}{1+|x|}\right| \geq \frac{1}{2} \Rightarrow \left|\frac{1}{1+|x|}\right| \geq \frac{1}{2}$

$\Rightarrow \quad \frac{1}{1+|x|} \geq \frac{1}{2} \Rightarrow 1 + |x| \leq 2$ or $|x| \leq 1$

$\therefore \quad -1 \leq x \leq 1$ or $x \Rightarrow [-1, 1]$

Q30

$[2x] - [x+1] = 2x \quad \ldots(1)$

$-[x+1] = \{2x\},\ 0 \leq \{2x\} < 1,$

$-[x+1] = 0,\ [x+1] = 0$

$-1 \leq x < 0,\ -2 \leq 2x < 0$

$[2x] = -2, -1$

from equation (1)

Hints and Solutions

$[2x]-0=2x,\ 2x=-2,\ -1$

$x=-1,\ -\frac{1}{2}$