Sample Task **Questions**

Questions with Answer Keys **MathonGo**

Q1

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[1-\tan \left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan \left(\frac{x}{2}\right)\right][\pi-2x]^3}$$ is equal to

(1) $\frac{1}{8}$

(2) 0

(3) $\frac{1}{32}$

(4) ∞

Q2

The value of $\lim_{x \rightarrow 0} \frac{1-\cos^3(\sin x)}{\sin x \sin(\sin x) \cos(\sin x)}$ is equal to

(1) $\frac{3}{2}$

(2) 1

(3) 0

(4) 2

Q3

The value of $\lim_{x \rightarrow -\infty} \frac{x^2 \tan\left(\frac{1}{x}\right)}{\sqrt{4x^2-x+1}}$ is equal to

(1) 1

(2) $\frac{1}{2}$

(3) -1

(4) $-\frac{1}{2}$

Q4

$$\lim_{x \to 0} \frac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3\sqrt{x}} \text{ equals}$$

(1) $\frac{1}{4}$

(2) $\frac{3}{4}$

(3) $\frac{1}{2}$

(4) 1

Q5

$$\lim_{x \to 0} \frac{(1 - \cos x)(3 + \cos 2x)}{x \cdot \tan 2x} =$$

(1) 0

(2) 1

(3) $\frac{1}{2}$

(4) -1

Q6

The value of $\lim_{x \to 0} \frac{1 - \cos^3 x}{x \sin x \cos x}$ is

(1) $\frac{2}{5}$

(2) $\frac{3}{5}$

(3) $\frac{3}{2}$

(4) $\frac{3}{4}$

Q7

If $\lim_{x \to 0} \left(x^{-3} \sin 3x + ax^{-2} + b\right)$ exists and is equal to 0, then

(1) a = - 3 and b = 9/2

(2) $a = 3$ and $b = 9/2$

(3) $a = -3$ and $b = -9/2$

(4) $a = 3$ and $b = -9/2$

Q8

If $f(x)$ is a differentiable function such that $f'(1) = 4$ and $f'(4) = \frac{1}{2}$, , then value of $\lim_{x \to 0} \frac{f\left(x^2+x+1\right)-f(1)}{f\left(x^4-x^2+2x+4\right)-f(4)}$ is :-

(1) 8

(2) 16

(3) 4

(4) Does not exist

Q9

If $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 2} - ax - b\right) = 2$, then equation of circle whose centre is $(a, 2b)$ and radius 1 unit is

(1) $x^2 + y^2 + 2x + 6y + 9 = 0$

(2) $x^2 + y^2 - 2x + 6y + 1 = 0$

(3) $x^2 + y^2 - 2x + 6y + 9 = 0$

(4) none of these

Q10

For a positive integer m, if $\lim_{x \to \infty} \left(x^3 \ln\left(\frac{x+1}{x}\right) + \frac{x}{2} - x^2\right) = \frac{1}{m}$. Then the value of m is

(1) 1

(2) 2

(3) 3

(4) 4

Q11

The value of $\lim_{n \to \infty} \frac{3.2^{n+1} - 4.5^{n+1}}{5.2^n + 7.5^n}$ is equal to

(1) $\frac{3}{5}$

(2) $-\frac{4}{7}$

(3) $-\frac{20}{7}$

(4) 0

Q12

The value of $\lim_{x \to 0} \frac{\ln(2 - \cos 15x)}{\ln^2(\sin 3x + 1)}$ is equal to

Q13

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = a\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]$, $a \in \mathbb{R}$, where $[t]$ is the greatest integer less than or equal to t. If $\lim_{x \to -1} f(x)$ exists, then the value of $\int_0^4 f(x)dx$ is equal to

(1) -1

(2) -2

(3) 1

(4) 2

Q14

The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)\left(\cos x - e^x\right)}{x^n}$ is a finite nonzero number is

(1) 1

(2) 2

(3) 3

(4) 4

Q15

If the largest value of the $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{\frac{x}{b}}$ where $a,\ b$ lies in the interval $\left[\frac{1}{5},\ 403\right]$ is e^{λ}, then λ equals

(1) 2015

(2) 2016

(3) 2017

(4) 2018

Q16

$\lim_{n \to \infty} \left(\frac{2n^2 - 3}{2n^2 - n + 1}\right)^{\frac{n^2 - 1}{n}}$ is equal to

(1) $\frac{1}{\sqrt{e}}$

(2) $\sqrt{e}$

(3) e

(4) $\frac{1}{e}$

Q17

$\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{-\frac{x}{2}} - 3^{1-x}}$ is equal to

Q18

The value of $\lim_{x \to 0} \left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{2}{\sin x}}$ is equal to

(1) 0

(2) 1

(3) -1

(4) None of these

Q19

The value of $\lim_{x \to 0^+} ((x\cot x) + (x\ln x))$ is equal to

(1) 1

(2) 2

(3) 3

(4) 0

Q20

The value of $\lim_{x \to \frac{\pi}{2}} \frac{\left[\frac{x}{3}\right]}{\ln(\sin x)}$ (where, [.] denotes the greaest integer function)

(1) does not exist

(2) is equal to 1

(3) is equal to 0

(4) is equal to - 1

Q21

The $\lim_{x \to 0} x^8 \left[\frac{1}{x^3}\right]$ (where $[x]$ is greatest integer function) is (Mark incorrect option)

(1) a nonzero real number

(2) a rational number

(3) an integer

(4) zero

Q22

If $\lim\limits_{x \to 0} \frac{\sin 2x - a\sin x}{x^3}$ exists finitely, then the value of a is

(1) 0

(2) 2

(3) 1

(4) 4

Q23

The value of $\lim\limits_{x \to 0} \frac{\sin x}{3} \left[\frac{5}{x}\right]$ is equal to

(where, [.] represents the greatest integer function)

(1) $\frac{1}{3}$

(2) 0

(3) $\frac{5}{3}$

(4) 1

Q24

The value of the limit

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)\ldots\ldots\infty}}}\right\}$$ is

(1) $\frac{1}{2}$

(2) -2

(3) 2

(4) $-\frac{1}{2}$

Q25

The value of $\lim_{n\to\infty} \frac{[x]+\left[2^2x\right]+\left[3^2x\right]+\ldots\ldots+\left[n^2x\right]}{1^2+2^2+3^2+\ldots\ldots+n^2}$ is equal to (where $[x]$ represents the greatest integer part of x)

(1) x

(2) $2x$

(3) $\frac{x}{2}$

(4) $\frac{x}{6}$

Q26

Let $\alpha,\ \beta \in R$ be such that $\lim_{x \rightarrow 0} \frac{x^2 \tan\ (ax)}{\beta x - \tan\ (2x)} = 1$, then the value of $5\beta + 3\alpha$ is :

Q27

$\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to

(1) $\frac{1}{3}$

(2) $\frac{1}{6}$

(3) $\frac{1}{4}$

(4) $\frac{1}{12}$

Q28

The value of $\lim_{n \rightarrow \infty} \frac{[r] + [2r] + \ldots + [nr]}{n^2}$, where r is non-zero real number and $[r]$ denotes the greatest integer less than or equal to r, is equal to :

(1) $\frac{r}{2}$

(2) r

(3) $2r$

(4) 0

Q29

The value of $\lim_{x \rightarrow 0} \frac{e^{-\frac{x^2}{2}} - \cos x}{x^3 \tan x}$ is equal to

(1) $\frac{1}{4}$

(2) $\frac{1}{8}$

(3) $\frac{1}{12}$

(4) $\frac{1}{16}$

Q30

If $\lim_{x \to \infty} \frac{ae^x + b\cos x + c + dx}{x\sin^2 x} = 3$, then the value of $272\frac{abd}{c^3}$ is equal to

Answer Key

Q1 (3)	**Q2** (1)	**Q3** (4)	**Q4** (2)
Q5 (2)	**Q6** (3)	**Q7** (1)	**Q8** (3)
Q9 (3)	**Q10** (3)	**Q11** (3)	**Q12** (12.5)
Q13 (2)	**Q14** (3)	**Q15** (1)	**Q16** (2)
Q17 (36)	**Q18** (2)	**Q19** (1)	**Q20** (3)
Q21 (1)	**Q22** (2)	**Q23** (3)	**Q24** (1)
Q25 (1)	**Q26** (2.00)	**Q27** (2)	**Q28** (1)
Q29 (3)	**Q30** (34)		

Sample Task **Solutions**

Hints and Solutions **MathonGo**

Q1

$\lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let, $x = \frac{\pi}{2} + y$

$\Rightarrow \lim_{y \to 0} = \frac{\tan\left(\frac{-y}{2}\right)(1-\cos y)}{(-2y)^3}$

$= \lim_{y \to 0} \frac{-\tan\frac{y}{2} \cdot 2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y \to 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2$

$= \lim_{y \to 0} \frac{1}{32} \times \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left(\lim_{y \to 0} \frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2$

$\because \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$

$\Rightarrow \frac{1}{32} \times 1 \times 1^2$

$= \frac{1}{32}$

Q2

Let, $\sin x = t$

$\Rightarrow \lim_{t \to 0} \frac{1-\cos^3 t}{t \sin t \cos t}$

$\Rightarrow \lim_{t \to 0} \frac{(1-\cos t)}{t^2} \times \left(\frac{t}{\sin t}\right) \times \frac{(1+\cos t+\cos^2 t)}{\cos t}$

$\Rightarrow \frac{1}{2} \times 1 \times \frac{(1+1+1)}{1} = \frac{3}{2}$

Q3

$\lim_{x \to -\infty} \frac{x\frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}}{-x\sqrt{4-\frac{1}{x}+\frac{1}{x^2}}} = \frac{1}{-\sqrt{4}} = -\frac{1}{2}$

Q4

Let, $L = \lim\limits_{x \to 0} \frac{\tan x\sqrt{\tan x}-\sin x\sqrt{\sin x}}{x^3\sqrt{x}}$

$= \lim\limits_{x \to 0} \frac{\left(\tan x\right)^{\frac{3}{2}}\left[1-\left(\cos x\right)^{\frac{3}{2}}\right]}{x^{3/2} \cdot x^2}$

$= 1^{\frac{3}{2}} \cdot \lim\limits_{x \to 0} \frac{1-\cos^3 x}{x^2} \cdot \frac{1}{1+\left(\cos\ x\right)^{\frac{3}{2}}}$ (Rationalizing)

$= \lim\limits_{x \to 0} \frac{1-\cos x}{x^2} \cdot \left(1+\cos x+\cos^2 x\right) \cdot \frac{1}{1+\left(\cos x\right)^{\frac{3}{2}}}$

$= \frac{1}{2} \cdot \frac{1}{2}(1+1+1) = \frac{3}{4}.$

Q5

$\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} \cdot \frac{(3+\cos 2x)}{1} \cdot \frac{2x}{\tan 2x} \cdot \frac{1}{2}$

$= \frac{1}{2} \cdot \left(4\right) \cdot \frac{1}{2} = 1$

Sample Task **Solutions**

Hints and Solutions **MathonGo**

Q6

$\lim_{x \to 0} \frac{1-\cos^3 x}{x \sin x \cos x}$

$= \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x \sin x \cos x}$

$= \lim_{x \to 0} \frac{2 \sin^2\left(\frac{x}{2}\right)}{x \cdot 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)} \times \frac{(1+\cos x+\cos^2 x)}{\cos x}$

$= \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{2\left(\frac{x}{2}\right)} \times \frac{1+\cos x+\cos^2 x}{\cos\left(\frac{x}{2}\right)\cos x} = \frac{1}{2} \times 3 = \frac{3}{2}$

Q7

$\lim_{x \to 0}\left(\frac{\sin 3x}{x^3} + \frac{a}{x^2} + b\right) = \lim_{x \to 0} \frac{\sin 3x+ax+bx^3}{x^3}$
$= \lim_{x \to 0} \frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$

limit exists if

$3+a=0$

or $a=-3$

$\therefore \quad L = \lim_{x \to 0} \frac{\sin 3x-3x+bx^3}{x^3} = 27\left(\lim_{t \to 0} \frac{\sin t-t}{t^3} + b\right) = 0 \quad \left(\text{Putting } 3x=t\right) = -\frac{27}{6} + b = 0$

or $b=\frac{9}{2}$

Q8

$\lim_{x \to 0} \frac{f(x^2+x+1)-f(1)}{f(x^4-x^2+2x+4)-f(4)} \quad \left(\frac{0}{0} \text{ form}\right)$

$\lim_{x \to 0} \frac{(2x+1)f^1(x^2+x+1)}{(4x^3-2x+2)f^1(x^4-x^2+2x+4)}$

$= \frac{f'(1)}{2f'(4)} = 4$ (Applying L' Hospital's Rule)

Q9

$\lim_{x \to \infty}\left(\sqrt{x^2+x+2} - ax - b\right) = 2$

$a=1; b=\frac{-3}{2}$ required equation is $(x-1)^2 + (y+3)^2 = 1$

Q10

$\lim_{x \to \infty} x^3 \ln\left(1+\frac{1}{x}\right) + \frac{x}{2} - x^2$

$x=\frac{1}{t}$

$\lim_{t \to 0}\left(\frac{\ln(1+t)}{t^3} + \frac{1}{2t} - \frac{1}{t^2}\right) = \lim_{t \to 0} \frac{2\ln(1+t)+t^2-2t}{2t^3}$

$= \lim_{t \to 0} \frac{2\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots\right) + t^2 - 2t}{2t^3}$

$= \lim_{t \to 0}\left(\frac{1}{3} - \frac{t}{4} + \frac{t^2}{5} \ldots \ldots\right) = \frac{1}{3} = \frac{1}{m} \Rightarrow m = 3.$

Q11

$\lim_{n \to \infty} \frac{3.2^{n+1} - 4.5^{n+1}}{5.2^n + 7.5^n}$

$= \lim_{n \to \infty} \frac{5^n\left(6.\left(\frac{2}{5}\right)^n - 20\right)}{5^n\left(5.\left(\frac{2}{5}\right)^n + 7\right)} = -\frac{20}{7}\left(\because \lim_{n \to \infty}\left(\frac{2}{5}\right)^n = 0\right)$

Q12

$\lim_{x \to 0} \frac{\ln(2 - \cos 15x)}{\ln^2(\sin 3x + 1)}$

$= \lim_{x \to 0} \frac{\ln\{1 + (1 - \cos 15x)\}}{\ln^2(1 + \sin 3x)}$

$= \lim_{x \to 0} \frac{1 - \cos 15x}{(\sin 3x)^2}$ (Applying $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$)

$= \lim_{x \to 0} \frac{(15x)^2}{2(3x)^2}$ (using standard limit)

$= \frac{1(225)}{2 \times 9} = 12.5$

Q13

Given,

$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2 - x], a \in \mathbb{R}$

Now given $\lim_{x \to -1} f(x)$ exists,

So $\lim_{x \to -1^+} a \sin\left(\pi \frac{[x]}{2}\right) + [2 - x] = -a + 2$

And $\lim_{x \to -1^-} a\sin\left(\pi \frac{[x]}{2}\right) + [2 - x] = 0 + 3 = 3$

So, $\lim_{x \to -1} f(x)$ exist when $-a + 2 = 3 \Rightarrow a = -1$

Now,

$\int_0^4 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$

$\Rightarrow \int_0^4 f(x)dx = \int_0^1 -\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]dx + \int_1^2 -\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]dx + \int_2^3 -\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]dx + \int_3^4 -\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]dx$

$= \int_0^1 (0 + 1)dx + \int_1^2 (-1 + 0)dx + \int_2^3 (0 - 1)dx + \int_3^4 (1 - 2)dx$

$= 1 - 1 - 1 - 1 = -2$

Q14

$\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$= \frac{1}{x^n}\left(\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-1\right)\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots\right)\right)\right)$$

$$= \frac{1}{x^n}\left[\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\left(-x-x^2-\frac{x^3}{3!}\ldots\right)\right]$$

$$= \frac{-1}{x^{n-3}}\left[\left(-\frac{1}{2!}+\frac{x^2}{4!}-\ldots\right)\left(1+x+\frac{x^2}{3!}\ldots\right)\right]$$

$\therefore$ For $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ to exist as a nonzero number we must have $n-3=0 \Rightarrow n=3$.

Q15

$\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{\frac{x}{b}} = e^{\lim\limits_{x\to\infty}\frac{x}{b}\left(1+\frac{a}{x}-1\right)}$

$[\because 1^\infty \text{ form }]$

$= e^{\frac{a}{b}} = e^{\frac{403\times 5}{1}} = e^{2015} \equiv e^\lambda \Rightarrow \lambda = 2015.$

Q16

$\mathrm{L} = e^l$

$l = \lim\limits_{n\to\infty}\frac{n^2-1}{n}\left[\frac{2n^2-3-2n^2+n-1]}{2n^2-n+1}\right]$

$= \lim\limits_{n\to\infty}\frac{n^2\left(1-\frac{1}{n^2}\right)-n\left(1-\frac{4}{n}\right)}{n\cdot n^2\left(2-\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{1}{2}$

$\therefore\ \mathrm{L} = \sqrt{e}.$

Q17

Let, $3^{\frac{x}{2}} = t, x \to 2 \Rightarrow t \to 3$

$\lim_{t\to 3}\frac{t^{2\frac{2}{t^2}}-12}{\frac{1}{t}-\frac{3}{t^2}} = \lim_{t\to 3}\frac{t^4+27-12t^2}{t-3}$

$\lim_{t\to 3}\frac{(t^2-3)(t+3)(t-3)}{(t-3)} = 6\times 6 = 36$

Q18

Let, $f(x) = \frac{1+\tan x}{1+\sin x}$

and $g(x) = \frac{2}{\sin x}$

Clearly, $f(x) \to 1$ and $g(x) \to \infty$ as $x \to 0$

$\therefore \lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{2}{\sin x}} = e^{\lim_{x\to 0}\frac{2}{\sin x}\left(\frac{1+\tan x}{1+\sin x}-1\right)}$

$\{$ using $\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$ for 1^{∞} form $\}$

$= e^{\lim_{x\to 0}\frac{2}{\sin x}\left(\frac{\tan x-\sin x}{1+\sin x}\right)} = e^{\lim_{x\to 0}\frac{2(1-\cos x)}{\cos x(1+\sin x)}}$

$= e^{0} = 1$

Q19

$\lim\limits_{x\to 0^{+}} x\cot x + \lim\limits_{x\to 0^{+}} x\ln x$

$= \lim\limits_{x\to 0^{+}}\frac{x}{\tan x} + \lim\limits_{x\to 0^{+}}\frac{\ln x}{\left(\frac{1}{x}\right)}$

$= \lim\limits_{x\to 0^{+}}\frac{x}{\tan x} + \lim\limits_{x\to 0^{+}}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^{2}}\right)} = 1 + \lim\limits_{x\to 0^{+}}(-x)$

$= 1 + 0 = 1$

Q20

$\because \frac{\pi}{6} < 1,$

$\therefore \left[\frac{\pi}{6}\right] = 0$

$\therefore \lim\limits_{\mathrm{x}\to\frac{\pi}{2}}\frac{\left[\frac{\mathrm{x}}{2}\right]}{\ln(\sin \mathrm{x})} = 0$

Q21

Since $x - 1 \leq [x] \leq x$ for all $x \in \mathbf{R}$ so

$$\frac{1}{x^{3}} - 1 \leq \left[\frac{1}{x^{3}}\right] \leq \frac{1}{x^{3}}$$

$\Rightarrow \quad x^{8}\left(\frac{1}{x^{3}} - 1\right) \leq x^{5}\left[\frac{1}{x^{3}}\right] \leq x^{5}$ for all x

But $\lim_{x\to 0} x^{5} = 0 = \lim_{x\to 0}\left(x^{5} - x^{8}\right)$ so

$\lim_{x\to 0} x^{8}\left[\frac{1}{x^{3}}\right] = 0 \in \mathbf{I} \subseteq \mathbf{Q}$

Q22

$\lim\limits_{x\to 0}\frac{\sin x(2\cos x-a)}{x\cdot x^{2}} = \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{2\cos x-a}{x^{2}}\right)$

For this limit to exists finitely,

$\lim\limits_{x\to 0}\frac{2\cos x-a}{x^{2}} =$ finite

$\therefore$ It must be $\frac{0}{0}$ form

$\therefore\ 2\cos(0) - a = 0 \Rightarrow a = 2$

Q23

$\frac{5}{x} - 1 < \left[\frac{5}{x}\right] \leq \frac{5}{x}$

$\underbrace{\frac{\sin x}{3}\left(\frac{5}{x} - 1\right)}_{h(x)} < \underbrace{\frac{\sin x}{3}\left[\frac{5}{x}\right]}_{f(x)} \leq \underbrace{\frac{\sin x}{3}\left(\frac{5}{x}\right)}_{g(x)}$

by sandwich theorem

$\because \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = \frac{5}{3}$

$\therefore \lim_{x\to 0} f(x) = \frac{5}{3}$

Q24

Let the given expression be y.

Then, $y = \lim_{n\to\infty} n^2$

$$\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)}}}\right.$$

On putting $\frac{1}{n} = \theta \ldots\ldots \infty$

So that, $n \to \infty \Rightarrow \theta \to 0$

Thus,

$$\text{y} = \lim_{\theta\to 0} \frac{1}{\theta^2}\left(1-\cos\theta\right)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\text{.to } \infty}$$

$$= \lim_{\theta\to 0}\left(\frac{1-\cos\,\theta}{\theta^2}\right)$$

$$\left\{\because\ \frac{1}{2}+\frac{1}{2^2}+\ldots\infty = 1\right\}$$

$$= \lim_{\theta\to\infty} \frac{2\sin^2\,\theta/2}{\theta^2}$$

$$= \lim_{\theta\to 0} 2\left(\frac{\sin\,\theta/2}{\theta/2}\right)^2 \times \frac{1}{4}$$

$$= 2\cdot 1^2\cdot\frac{1}{4} = \frac{1}{2}$$

Q25

Let $f(x) = \frac{[x]+[2^2x]+[3^2x]+\ldots+[n^2x]}{1^2+2^2+3^2+\ldots+n^2}$

Now, we have,

$f(x) \le \frac{x+2^2x+3^2x+\ldots+n^2x}{1^2+2^2+3^2+\ldots+n^2} = x$

and, $f(x) > \frac{(x-1)+(2^2x-1)+(3^2x-1)+\ldots+(n^2x-1)}{1^2+2^2+3^2+\ldots+n^2}$

$= \frac{x\Sigma n^2 - n}{\Sigma n^2} = x - \frac{6}{(n+1)(2n+1)}\ (\because x-1 \le [x] < x,\ \forall x \in R)$

Thus, we have,

$x-\frac{6}{(n+1)(2n+1)}<f(x)\leq x$

Now, we have,

$\lim_{n\to\infty} x-\frac{6}{(n+1)(2n+1)}=x\ \&\ \lim_{n\to\infty} x=x$

Hence, by Sandwich Theorem, we have

$\lim_{n\to\infty} f(x)=x$

Q26

$\lim_{x\to 0}\frac{x^2\tan(\alpha x)}{\beta x-\tan(2x)}=1\to\lim_{x\to 0}\frac{x^2\tan(\alpha x)}{\beta x-\left\{2x+\frac{(2x)^3}{3}+\frac{2(2x)^5}{15}\right\}}=1$

$\to\beta=2$ and $\frac{3\alpha}{8}=-1,$

So, $5\beta+3\alpha=2$

Q27

Consider $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\cdot\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$=\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{\sin x+x}{2}\right)\times\left(\frac{x-\sin x}{2}\right)\times\frac{1}{x^4}$

$\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)$

$\lim_{x\to 0}2\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)\ \left(\frac{0}{0}\text{ form}\right)\quad\left(\because\lim_{t\to 0}\frac{\sin t}{t}=1\right)$

Applying L'Hospital's Rule,

$\lim_{x\to 0}2\times\left(\frac{2x-2\sin x\cos x}{4\cdot 4x^3}\right)=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{8x^3}\right)\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\left(\frac{2-2\cos 2x}{24x^2}\right)\ \left(\frac{0}{0}\text{ form}\right)$

$$\lim_{x\to 0}\left(\frac{4\sin 2x}{48x}\right)=\frac{1}{6}\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)=\frac{1}{6}$$

Q28

We know that $r \leq [r] < r+1$

$2r \leq [2r] < 2r+1$

$3r \leq [3r] < 3r+1$

$\vdots \quad \vdots \quad \vdots$

$nr \leq [nr] < nr+1$

On adding all the above inequalities, we get

$r+2r+\ldots+nr \leq [r]+[2r]+\ldots+[nr] < r+2r+\ldots nr+n$

Using $1+2+3+\ldots+n=\frac{n(n+1)}{2},\ n\in N,$

$\frac{n(n+1)}{2}\cdot r \leq [r]+[2r]+\ldots+[nr] < \frac{n(n+1)}{2}\cdot r+n$

$\Rightarrow \frac{\frac{n(n+1)}{2}\cdot r}{n^2} \leq \frac{[r]+[2r]+\ldots+[nr]}{n^2} < \frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}$

Now, $\lim_{n\to\infty}\frac{n(n+1)\cdot r}{2\cdot n^2}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)\cdot r}{2\cdot n^2}=\frac{r}{2}$

and $\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}=\lim_{n\to\infty}\frac{n^2\left\{\left(1+\frac{1}{n}\right)\cdot r+\frac{1}{n}\right\}}{2\cdot n^2}=\frac{r}{2}$

So, by Sandwich Theorem, we can conclude that

$\lim_{n\to\infty}\frac{[r]+[2r]+\ldots+[nr]}{n^2}=\frac{r}{2}.$

Q29

Using expansion,

$$\lim_{x\to 0}\frac{\left(1+\left(-\frac{x^2}{2}\right)+\frac{\left(\frac{x^4}{4}\right)}{2!}+\ldots\right)-\left(1-\frac{x^2}{2!}+\frac{x^2}{4!}-\ldots\right)}{x^3\left(x+\frac{x^3}{3}+\ldots\right)}$$

$$\lim_{x\to 0}\frac{(1-1)+\left(-\frac{x^2}{2}+\frac{x^2}{2}\right)+\left(\frac{x^4}{8}-\frac{x^4}{24}\right)+\ldots}{x^3\left(x+\frac{x^3}{3}+\ldots\right)}$$

$\Rightarrow \frac{1}{8}-\frac{1}{24}=\frac{1}{12}$

Q30

Using expansions, we get,

$$\lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..\right)+b\left(1-\frac{x^2}{2!}+...\right)+c+dx}{x\left(x-\frac{x^3}{3!}+.....\right)^2}=3$$

$$\lim_{x\to 0}\frac{(a+b+c)+(a+d)x+\left(\frac{a-b}{2}\right)x^2+\frac{a}{6}x^3+...}{x^3\left(1-\frac{x^2}{3!}+.....\right)^2}=3$$

$\because$ in the denominator lowest power of x is 3

For the limit to be finite, the numerator should also have the least power of x as 3

$\therefore a+b+c=0 \;\;...(1)$

$a+d=0...(2)$

$\frac{a-b}{2}=0...(3)$

Now, $\frac{\left(\frac{a}{6}\right)}{1}=3 \Rightarrow a=18$

From (1), (2), (3), we get,

$a=18,\ b=18,\ c=-36,\ d=-18$

$\frac{abd}{c^3}=\frac{-(18)^3}{-8(18)^3}=\frac{1}{8}$