Q1

$\log_{(|x|-1)}\left(x^2+4x+4\right)\geq 0$

Case 1: $0<|x|-1<1$

i.e., $1<|x|<2$, then

$0<x^2+4x+4 \leq 1 \Rightarrow \quad x^2+4x+3 \leq 0$ & $(x+2)^2>0$

$\Rightarrow -3 \leq x \leq -1$ & $x \neq -2$

So, $x \in(-2,-1)$

Case 2: $|x|-1>1$

i.e., $|x|>2$

$x^2+4x+4 \geq 1 \Rightarrow (x+1)(x+3) \geq 0 \Rightarrow x \in(-\infty,-3] \cup[-1, \infty)$

$\Rightarrow x \in(-\infty, -3] \cup(2, \infty)$

Hence, domain is $(-\infty,-3] \cup(-2,-1) \cup(2, \infty)$

Q2

$1 \leq \log_2\left(x^2+3x-2\right) \leq 3 \Rightarrow 2 \leq\left(x^2+3x-2\right) \leq 8$

On solving them, we get $-5 \leq x \leq -4$ and $1 \leq x \leq 2$

Q3

$\because\ x^2 + 4x \geq 0$ for the first term

and $0 \leq x^2 + 4x + 1 \leq 1$ for the second term

$\therefore$ by both the results, there is only one possibility

$x^2 + 4x = 0$

$\Rightarrow x = 0, -4$

$\therefore\ f(x) = \tan^{-1}(0) + \sin^{-1}(1) = \frac{\pi}{2}$

Hence, $f(x)$ contains only two elements in its domain

Q4

$x^2 - [x^2] = \{x^2\}$

$\therefore\ f(x) = \frac{\{x^2\}}{1+\{x^2\}} = \frac{1+\{x^2\}-1}{1+\{x^2\}} = 1 - \frac{1}{1+\{x^2\}}$

$\because\ 0 \leq \{x^2\} < 1 \Rightarrow 1 \leq \{x^2\} + 1 < 2$

$\frac{1}{2} < \frac{1}{\{x^2\}+1} \leq 1 \Rightarrow -\frac{1}{2} > \frac{-1}{1+\{x^2\}} \geq -1$

$\frac{1}{2} > 1 + \frac{(-1)}{1+\{x^2\}} \geq 0 \Rightarrow$ Range of $f(x) \in \left[0, \frac{1}{2}\right)$

Q5

Let $f(x) = \sin^4 x - \sin x \cos x + \cos^4 x$

$f\left(x\right) = 1 - \frac{1}{2}\sin 2x - \frac{1}{2}\sin^2 2x$

$\Rightarrow f\left(x\right) = g\left(t\right) = 1 - \frac{1}{2}t - \frac{1}{2}t^2 \ \ t \in \left[-1, 1\right]$

Hence $f\left(x\right) \in \left[0, \frac{9}{8}\right]$

Hints and Solutions **MathonGo**

Q6

$\sin^{-1}(3x)+\frac{\pi}{3}\geqslant 0$ and $3x\in[-1,1]$

$\sin^{-1}(3x)\geqslant -\frac{\pi}{3}$

$1\geqslant 3x\geqslant -\frac{\sqrt{3}}{2}$

$\frac{1}{3}\geqslant x\geqslant \frac{-\sqrt{3}}{6}$

domain is $\left[\frac{-1}{2\sqrt{3}},\frac{1}{3}\right]\quad\therefore a=\frac{-1}{2}\ \&\ b=1.$

For Range put $x=\frac{-1}{2\sqrt{3}}$ and $x=\frac{1}{3}$

at $x=\frac{-1}{2\sqrt{3}}\quad b\left(\frac{-1}{2\sqrt{3}}\right)=0$

at $x=\frac{1}{3}\quad b\left(\frac{1}{3}\right)=\sqrt{\frac{5\pi}{6}}$

So $C=0$ and $d=\frac{1}{\sqrt{6}}$

$2a+b+c+6d=\sqrt{6}$

Q7

Since $|a+b|=|a|+|b|\Rightarrow ab\geq 0$

So, $\log_4(2x^2-x)\geq 0\Rightarrow 2x^2-x\geq 1$

$\Rightarrow\left(2x+1\right)\left(x-1\right)\geq 0\Rightarrow x\leq -\frac{1}{2}$ or $x\geq 1$

and $\log_2(2-x^2)\geq 0\Rightarrow 2-x^2\geq 1$

$\Rightarrow x^2\leq 1\Rightarrow -1\leq x\leq 1$

Hence, $x\in\left[-1,\ \frac{-1}{2}\right]\cup\{1\}$

i.e. 2 integral values.

Q8

$f\left(x\right)=\left[x\right]+\sum_{r=1}^{2020}\frac{\{x+r\}}{2020}$

$=\left[x\right]+\sum_{r=1}^{2020}\frac{\{x\}+r}{2020}$

$=\left[x\right]+\frac{2020}{2020}\left\{x\right\}+\frac{1+2+3+...+2020}{2020}$

$=\left[x\right]+\left\{x\right\}+\frac{2020\times2021}{2\times2020}$

$=x+\frac{2021}{2}$

Q9

$f(-x)=\sec\left(\log\left(-x+\sqrt{1+x^2}\right)\right)$

$=\sec\left(\log\left(x+\sqrt{1+x^2}\right)^{-1}\right)$

$\left(\because\sqrt{1+x^2}-x=\frac{1}{\sqrt{1+x^2}+x}\right)$

$=\sec\left(-\log\left(x+\sqrt{1+x^2}\right)\right)$

$=\sec\left(\log\left(x+\sqrt{1+x^2}\right)\right)$ $(\because\sec(-\theta)=\sec(\theta))$

$=f(x)$

Hence $f(x)$ is an even function.

Q10

A function whose graph is symmetrical about the origin must be odd.

$\left(2^x+2^{-x}\right)$ is an even function.

Since, $\log\left(x+\sqrt{1+x^2}\right)$ is an odd function,

$\therefore\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$ is an even function.

If $f(x+y)=f(x)+f(y)\forall x,y\in R$, then

Put $x=y=0\Rightarrow f(0)=0$

Now, put $y=-x\Rightarrow f(x)+f(-x)=0$

$\therefore f(x)$ is an odd function

Q11

$f(x)$ is symmetric about the line $x=2$

$\therefore f(2+x)=f(2-x)$

$a(2+x)^3+(2+x)^2+b(2+x)+c=a(2-x)^3+(2-x)^2+b(2-x)+c$

$a\left\{(2+x)^3-(2-x)^3\right\}+\left\{(2+x)^2-(2-x)^2\right\}+b\{(2+x)-(2-x)\}=0$

$a\left\{\left(8+12x+6x^2+x^3\right)-\left(8-12x+6x^2-x^3\right)\right\}+2(4x)+b(2x)=0$

$a\left(24x+2x^3\right)+8x+2bx=0$

$2ax^3+(24a+2b+8)x=0$

Which must be true $\forall x\in R$

$\therefore$ it is an identity

$\therefore 2a=0,\ 24a+2b+8=0$

$a=0,\ 2b+8=0$

$b=-4$

Q12

$f\left(\frac{\pi}{2}+x\right)=\left|\sin\left(\frac{\pi}{2}+x\right)\right|+\left|\cos\left(\frac{\pi}{2}+x\right)\right|$

$=|\cos\ x|+|\sin\ x|$ for all x.

Hence, $f(x)$ is periodic with period $\frac{\pi}{2}$.

Q13

$f(x)=\frac{1}{2}\left(\frac{|\sin x|}{\cos x}+\frac{\sin x}{|\cos x|}\right)$

$=\frac{1}{2}\left(\frac{\sin x}{\cos x}+\frac{\sin x}{\cos x}\right),\ 0\le x<\frac{\pi}{2}$

$=\frac{1}{2}\left(\frac{\sin x}{\cos x}-\frac{\sin x}{\cos x}\right),\ \frac{\pi}{2}\le x<\pi$

$=\frac{1}{2}\left(\frac{-\sin x}{\cos x}-\frac{\sin x}{\cos x}\right),\ \pi\le x<\frac{3\pi}{2}$

$=\frac{1}{2}\left(\frac{-\sin x}{\cos x}+\frac{\sin x}{\cos x}\right),\ \frac{3\pi}{2}\le x<2\pi$

$\Rightarrow f(x)=\tan x,\ \ 0\le x<\frac{\pi}{2}$

$=0\ ,\ \frac{\pi}{2}\le x<\pi$

$=-\tan x,\ \pi\le x<\frac{3\pi}{2}$

$=0,\ \frac{3\pi}{2}\le x<2\pi$

$\Rightarrow f$ is periodic with fundamental period $2\pi\Rightarrow m=2$

Q14

$\because\ x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$\therefore\ 4\left(x^2+x+1\right)\ge 3$

$\Rightarrow\sqrt{4(x^2+x+1)}\ge\sqrt{3}$

$\Rightarrow\tan^{-1}\left(\sqrt{4(x^2+x+1)}\right)\ge\tan^{-1}\left(\sqrt{3}\right)$

$\Rightarrow f\left(x\right)\ge\frac{\pi}{3}$

$\therefore$ Range of $f(x)$ *is* $\left[\frac{\pi}{3},\frac{\pi}{2}\right)$

Q15

$f(x)= x^4+2x^3-x^2+1$ is polynomial of even degree hence its range can't be R. Hence not surjective.

$f(x)= x^3+x+1,\ f'(x)= 3x^2+1 \Rightarrow$ monotonic hence bijective.

$f(x)= \sqrt{1+x^2}$, neither surjective not injective

$f(x)= x^3+2x^2-x+1,\ f'(x)= 3x^2+4x-1 \Rightarrow D=16+12>0$

Hence $f(x)$ is non-monotonic cubic polynomial hence surjective but not injective.

Q16

Let us check for invertibility of $f(x)$

(A) one-one: we have, $f(x)= \frac{e^{2x}-e^{-2x}}{2}$

$\Rightarrow f'(x)= \frac{e^{4x}+1}{e^{2x}}$, which is strictly increasing as $e^{4x}>0$ for all x.

Thus, f is one-one

(B) Onto: Let $y=f(x)$

$\Rightarrow \frac{dy}{dx}=e^{2x}+e^{-2x}$, where y is strictly monotonic

Hence, the range of $f(x)$=$(f(-\infty), f(\infty))$

$\Rightarrow$ range of $f(x)$=$(-\infty, \infty)$

So, the range of $f(x)$= co-domain

Hence, $f(x)$ is one-one and onto

(C) To find $f^{-1}: y=\frac{e^{4x}-1}{2e^{2x}}$

$\Rightarrow e^{4x}-2e^{2x}y-1=0$

$\Rightarrow e^{2x}=\frac{2y\pm\sqrt{4y^2+4}}{2}$

$\Rightarrow 2x=\log\left(y\pm\sqrt{y^2+1}\right)$

$\Rightarrow f^{-1}(y)= \frac{\log\left(y\pm\sqrt{y^2+1}\right)}{2}$

Since, $e^{f^{-1}(x)}$ is always positive, so neglecting negative sign.

Hence, $f^{-1}(x)=\frac{\log\left(x+\sqrt{x^2+1}\right)}{2}$

Q17

$f(x)=x^3+x^2+4x+\sin x$

$\Rightarrow f'(x)=3x^2+2x+4+\cos x$

$f'(x)=3\left[\left(x+\frac{1}{3}\right)^2+\frac{11}{9}\right]-(-\cos x)>0$ as $3\left[\left(x+\frac{1}{3}\right)^2+\frac{11}{9}\right]_{\text{min}}=\frac{11}{3}$

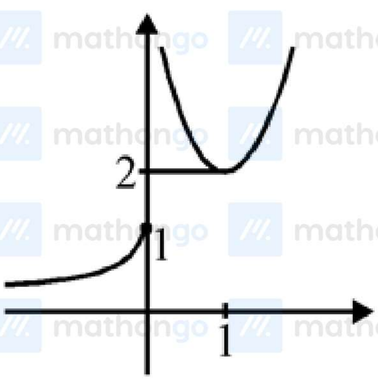
and $-\cos x$ has the maximum value 1.

$\Rightarrow f(x)$ is strictly increasing and hence it is one-one

Also, $\lim_{x\to\infty} f(x)\Rightarrow\infty$ and $\lim_{x\to-\infty} f(x)\Rightarrow-\infty$.

Thus, the range of $f(x)$ is R, hence it is onto.

Q18



Q19

$\because\ k \in \text{odd integer}$

$f(k) = k + 1$ which is even

$\therefore f(f(k)) = \frac{k+1}{2}$

If $\frac{k+1}{2}$ is odd $\Rightarrow 33 = \frac{k+1}{2} + 1 \Rightarrow k = 63$ not possible (because $\frac{k+1}{2}$ is even)

If $\frac{k+1}{2}$ is even $\Rightarrow 33 = \frac{\frac{k+1}{2}}{2} \Rightarrow 33 = \frac{k+1}{4} \Rightarrow k = 131$

Sum of digits $= 5$

Q20

$If\ f(2) = 3 \Rightarrow f(4) = 2,\ f(3) = 2$

$If\ f(2) = 4\ \Rightarrow f(4) = 2\ ,\ f(3) = 3$

or

$f(4) = 3,\ f(3) = 2$

Hence there are 3 such functions

Q21

There is no loss of generality in considering the order of bi's as $b_1 < b_2 < b_3 \cdots < b_n$ also given that $f(a_1) \leq f(a_2) \cdot f(a_{2n})$. Now suppose number of preimages of every b_i are x_i in numbers. Therefore $x_1 + x_2 + \cdots + x_n = 2n$ where $1 \leq x_i \leq n+1 \rightarrow (1)$

Number of solutions of (1) is ${}^{2n-1}C_{n-1}$ or ${}^{2n-1}C_n$

Q22

Given the set $A = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$.

And, also a function $f : A \rightarrow A$ satisfying $f(1) + f(2) = 3 - f(3)$

$\Rightarrow\ f(1) + f(2) + f(3) = 3$

Since, the range of the function is also A, hence the only possibility satisfying the given condition is:

$0 + 1 + 2 = 3$

We know that, the number of arrangements of n objects at n places is $n!$.

Since, the given function is bijective i.e. one-one and onto, hence, the elements $1,\ 2,\ 3$ in the domain can be mapped with only $0,\ 1,\ 2$ in the co-domain in 3! ways and the remaining 5 elements $0,\ 4,\ 5,\ 6,\ 7$ in the domain can be mapped with any of the remaining 5 elements $3,\ 4,\ 5,\ 6,\ 7$ in 5! ways.

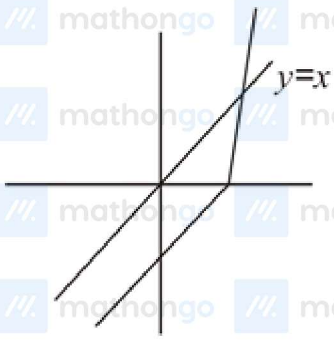
So, the number of bijective functions are $= 3! \times 5! = 6 \times 120 = 720$.

Q23

$f(x) = 2x - 1 + \left|x - \frac{1}{2}\right|$

$$= \begin{cases} x - \frac{1}{2} & x < \frac{1}{2} \\ 3x - \frac{3}{2} & x \geq \frac{1}{2} \end{cases}$$

As $f(x)$ is increasing, solution of equation $f(x) = f^{-1}(x)$ is same as $f(x) = x$.

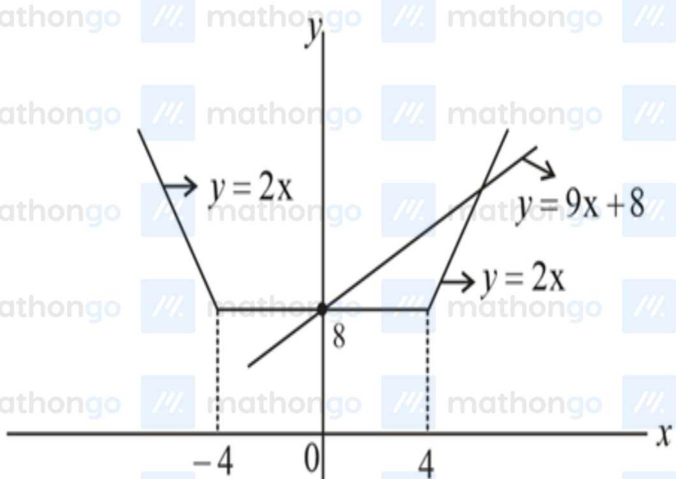


$\therefore$ No. of solutions of $f(x)= f^{-1}(x)$ is one.

Q24

We have,

$$|x+4|+|x-4|=\begin{cases} -2x & x<-4 \\ 8 & -4 \le x \le 4 \\ 2x & x>4 \end{cases}$$



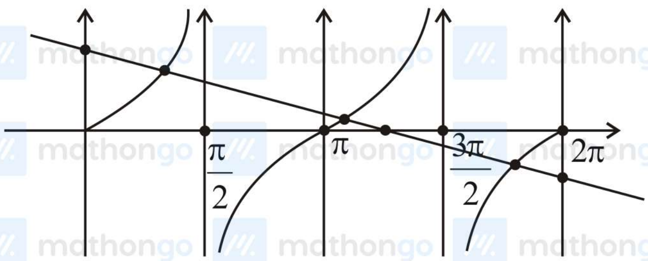
It is clear from the above straight line $y = ax + 8$ cuts the curve $y = |x+4|+|x-4|$ at exactly one point of $a \in (-\infty, -2] \cup [2, \infty)$ exactly two points of $a \in (-2, 0) \cup (0, 2)$ more than two points if

$a=0$. Hence, option (C) is correct.

Q25

$2x+3\tan x=\frac{5\pi}{2}\Rightarrow \tan x=-\frac{2}{3}x+\frac{5\pi}{6}$

$y=\tan x$ and $y=\frac{5\pi}{6}-\frac{2x}{3}$



Both the graphs meet exactly three times in $[0,2\pi]$.

Thus, there are 3 solutions.

Q26

From given functional equation, $2f(xy)=(f(x))^{y}+(f(y))^{x},\ \forall x,y\in R$

putting $y=1$,

$2f(x)=f(x)+(f(1))^{x}$

$f(x)=3^{x}$

$\therefore\quad \sum_{r=1}^{10}f(r)=\sum_{r=1}^{10}3^{r}=\frac{3\left(3^{10}-1\right)}{3-1}=\frac{3}{2}\left(3^{10}-1\right)$

Q27

Replacing x by $\frac{1}{(1-x)}$, we obtain $f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=-2x+\frac{2}{x}$

Again, replacing x by $1-\frac{1}{x}$ and solve

Q28

$\because f(x+2)+f(x-2)=f(x)$ ...(i)

Replace x by $x+2$: $f(x+4)+f(x)=f(x+2)$ ...(ii)

from eq (i) and (ii) : $f(x-2)+f(x+4)=0$

Replace x by $x+2$: $f(x)+f(x+6)=0$ ...(iii)

Replace x by x_{-2} in eq (i): $f(x)+f(x-4)=f(x-2)$ ...(iv)

from eq (i) and eq (iv): $f(x+2)+f(x-4)=0$

Replace x by $x-2$: $f(x)+f(x-6)=0$ ...(v)

from (iii) and (v) : $f(x+6)=f(x-6)$

$\therefore f(x)$ is periodic with period 12.

$\therefore \sum_{r=0}^{15} f(1+12r)=f(1)+f(13)+f(25)+\ldots\ldots$

$=16\times f(1)$

$= 16 \times 3$

$= 48$

Q29

Given, $f(x+y) = f(x) + f(y) - xy - 1, \forall\, x,\ y \in R$

$\therefore\ f(x+1) = f(x) + f(1) - x - 1$ [putting $y = 1$]

$\Rightarrow f(x+1) = f(x) - x\ \ [\because\ f(1) = 1]$

$\therefore\ f(n+1) = f(n) - n < f(n)$

$\Rightarrow f(n+1) < f(n)$

So, $f(n) < f(n-1) < f(n-2) < \ldots < f(3) < f(2) < f(1) = 1$

$\therefore\ f(n) = n$ holds only for $n = 1$

Q30

In given equation, putting $x \rightarrow x+3$

$f(x+9) - f(x+6) + f(x+3) = 0$

On adding both equations, we get,

$f(x+9) + f(x) = 0$ …(i)

now, putting $x \rightarrow x+9$

$f(x+18) + f(x+9) = 0$ …(ii)

Equation (ii) – Equation (i)

$f(x+18)=f(x)$

$\therefore \text{Period}=18$