State Space Analysis and Controller Design

7.1 Aim

Identify the state spece model of the designed system and design the full state feedback Controller.

7.2 List of Equipment

Table 7.1: List of Equipment

S.No.	Equipment Name
1	Arduino UNO
2	Project 1 Components

7.3 Theory

The state-space model of a dynamic system can be identified using the System Identification Toolbox in MATLAB. This involves collecting input-output data from the system and applying identification algorithms to estimate the system matrices (A, B, C, D).

State-Space Model

Given the input $u(t)$ and output $y(t)$ data, the standard state-space representation of a linear time-invariant (LTI) system is given by :

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t)$ is the state vector, $u(t)$ is the input, and $y(t)$ is the output. The matrices A, B, C, and D define the system dynamics.

7.3.1 System Identification

Use MATLAB system identification toolbox to identify the 2nd order state space model of the system. The identified state space model is given by,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where, A, B, C and D are system matrices. for example, in Project 1, the identified second order state space model of the DC motor system is,

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

State Feedback Controller Design

Once the state-space model is identified, a full state feedback controller can be designed using pole placement or optimal control methods. The control law is:

$$u(t) = -K_1x(t) + K_2r(t)$$

The K_1 is designed to place the closed-loop poles at desired locations for stability and performance, while K_2 is a reference gain to ensure zero steady state error. The controller configured system is shown in figure 7.1.

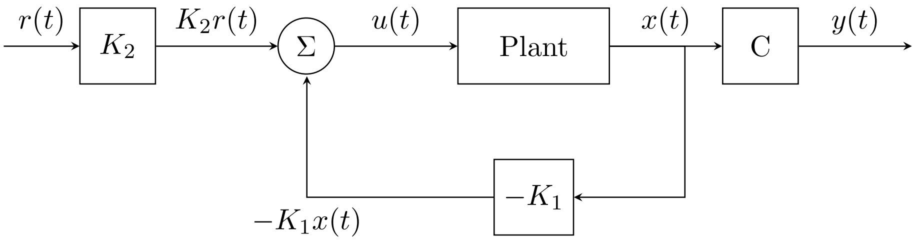

Figure 7.1: Block diagram of K

The Simulink model of the state feedback controller is shown in figure 7.2.

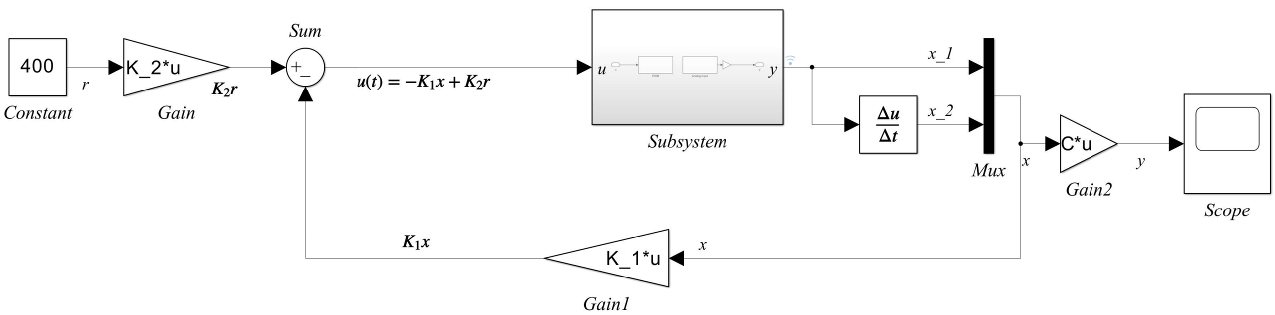

Figure 7.2: Simulink model of state feedback controller

7.3.2 Using MATLAB

To place the closed-loop poles at desired locations, use the `place` function in MATLAB:

$$K_1 = \text{place}(A, B, p)$$

where p is a vector of desired pole locations.

An example MATLAB code snippet to design the state feedback controller is given below:

```
K_1 = place(A, B, [p1 p2]);
A_m = A - B*K_1; % new A matrix with controller
K_2 = -1/(C*inv(A_m)*B);
B_m = B*K_2; % new B matrix with reference gain
sys_cl = ss(A_m, B_m, C, D); % new closed-loop system
```

7.3.3 Using numerical method

The characteristic equation of the closed-loop system is given by,

$$\det(sI - (A - BK)) = 0 \tag{7.1}$$

The desired characteristic equation is given by,

$$(s+p_1)(s+p_2) = s^2 + a_1 s + a_0 \tag{7.2}$$

where, p_1 and p_2 are desired pole locations. By comparing (7.1) and (7.2), we can find the value of K.

Example: Suppose the desired closed-loop poles are at $p_1 = -2$ and $p_2 = -4$. The desired characteristic equation is:

$$(s+2)(s+4) = s^2 + 6s + 8$$

For the identified system:

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The characteristic equation of the closed-loop system is:

$$\det(sI - (A - BK)) = 0$$

Let $K = [k_1 \quad k_2]$. Then,

$$A - BK = \begin{bmatrix} 0 & 1 \\ -0.5 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} [K_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ -0.5 - 3k_1 & -3 - 3k_2 \end{bmatrix}$$

The characteristic equation is:

$$s^2 + (3 + 3k_2)s + (0.5 + 3k_1)$$

Equate to desired equation:

$$s^2 + 6s + 8$$

So,

$$3 + 3k_2 = 6 \implies k_2 = 10.5 + 3k_1 = 8 \implies k_1 = \frac{8 - 0.5}{3} = 2.5$$

Thus, the state feedback gain is:

$$K = [2.5 \quad 1]$$

Reference Gain K_2: To ensure zero steady-state error for a step reference input, calculate K_2 as:

$$K_2 = -\frac{1}{C(A - BK)^{-1}B}$$

Calculating K_2:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -0.5 - 3(2.5) & -3 - 3(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

The inverse is:

$$(A - BK)^{-1} = \frac{1}{\det(A - BK)} \begin{bmatrix} -6 & -1 \\ 8 & 0 \end{bmatrix}$$

where $\det(A - BK) = (0)(-6) - (1)(-8) = 0 + 8 = 8$. Therefore:

$$(A - BK)^{-1} = \frac{1}{8} \begin{bmatrix} -6 & -1 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -0.75 & -0.125 \\ 1 & 0 \end{bmatrix}$$

Then,

$$C(A - BK)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -0.75 & -0.125 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.75 & -0.125 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = -0.375$$

Thus,

$$K_2 = -\frac{1}{-0.375} = \frac{1}{0.375} = \frac{8}{3} \approx 2.67$$

Final Control Law:

The control law is:

$$u(t) = -K_1 x(t) + K_2 r(t)$$

where:

$$K_1 = \begin{bmatrix} 2.5 & 1 \end{bmatrix}, \quad K_2 = 2.67$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Substituting the values:

$$u(t) = -\begin{bmatrix} 2.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2.67r(t) = -2.5x_1(t) - x_2(t) + 2.67r(t)$$

7.4 Observations

Design the state feedback controller using both MATLAB and numerical methods, and verify the closed-loop system's performance through simulation and experimental results.