Mathematics -2
Real Life Application of Differential Equations -
The Lotka Volterra Model

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1 Introduction

The Lotka-Volterra Model is a pair of first order nonlinear differential equations that seek to give a simple representation of the relationship of predator and prey in an environment. The basic concept is that, When there is no predator, prey can flourish. When there are predators, prey will die When there are prey, predators can live, but when there is no prey, predators will die. Here we see a tug of war between both the parties, where predator needs prey to live but consequently consumes the very thing it needs to survive. Let us assume that prey can spontaneously generate at rate $\alpha.x$, and predators gradually die out at rate $\gamma.y$ where

- X represents the population of prey
- Y represents the population of predators

Hence we can model the equations like this -

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

Here β and δ are variables that represent how much each group interacts with each other. Also note that all the variables here are positive.

2 History

The Lotka-Volterra equations were developed independently by Alfred J. Lotka and Vito Volterra in the early 20th century. Lotka, an American mathematician, formulated the equations in 1920 to describe autocatalytic chemical reactions. Volterra, an Italian mathematician, derived similar equations in 1926 to model fish populations in the Adriatic Sea.

Lotka was interested in understanding how the concentrations of reactants and products change over time in systems where the presence of one substance catalyzes the production of another. He developed a set of differential equations to describe the rates of change of these substances based on their interactions.

Meanwhile, Volterra was investigating the dynamics of fish populations in the Adriatic Sea. He independently derived similar equations in 1926 to model the interactions between predators (such as fish) and their prey (such as smaller fish or plankton) in ecological systems. Volterra observed cyclic fluctuations in fish populations and sought to understand the underlying mechanisms driving these oscillations.

Although Lotka and Volterra approached the problem from different perspectives—Lotka from the realm of chemistry and Volterra from ecology—the equations they developed shared a common structure and mathematical form. Lotka's work laid the foundation for understanding feedback mechanisms in chemical systems, while Volterra's work provided insights into the dynamics of predator-prey relationships in ecological systems. Together, their contributions formed the basis of what is now known as the Lotka-Volterra equations, which have since become a cornerstone of mathematical ecology and population dynamics.

3 Assumptions Made By Equation

There are a number of assumptions that we make for this model, namely -

- The prey has ample food at all times
- The prey can spontaneously reproduce
- Rate of change of either population is proportional to its size
- The environment does not favor either species, both are equally likely to survive.
- The entire population is identical and can be described by a single variable. For example, there is no age, gender or natural cause of death.

In pure absence of predator, the prey grows as -

$$\frac{dx}{dt} = \alpha x$$

And in absence of prey, predator is affected as -

$$\frac{dy}{dt} = -\gamma y$$

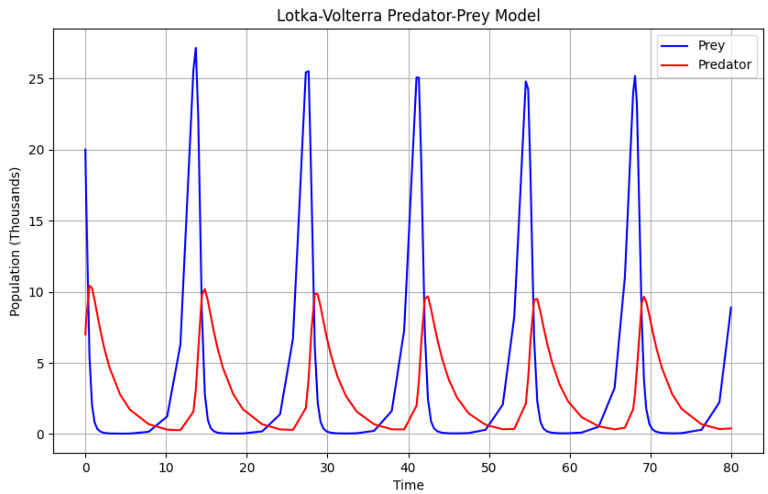

Figure 1: Enter Caption

Their interaction causes promotion of predator and inhibition of prey but then predators die out due to lack of prey, and this cycle reverses yet again.

4 Plots and Inferences

4.1 Plots of Functions

Let us plot a population vs time graph of our equation. The values of variables are taken as follows -

$$\alpha = 1.1$$

$$\beta = 0.1$$

$$\delta = 0.1$$

$$\gamma = 0.4$$

And the initial values of predator and prey are given by -

$$x_0 = 20$$

$$y_0 = 7$$

(Values are assumed to be in the thousands)