# Distribution of charges on conductors and Thomson's theorem

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*A simple numerical method of multidimensional minimisation is applied in order to find the equilibrium configuration of N point charges confined to a given volume or surface. The method is applied to different geometric forms with both Coulomb and non-harmonic interacting potentials. The technique presented in this paper is very useful from a didactic point of view as it allows the approximate configuration of charges on conductors to be visualised and stresses the importance of the inverse square law for the charge distribution. For this purpose, different three-dimensional and two-dimensional examples are presented.*

In textbooks on electromagnetism, Gauss's law is used to demonstrate that an excess of charge on a conductor resides on its surface. However, the charge distribution within the volume of a conductor that follows from any interaction law other than the inverse square law is usually not discussed. The exactness of the inverse square law (Coulomb's law) is a fundamental question in electrostatics and Cavendish's study of the charge distribution between concentric conducting shells was one of the first verifications of it. In an interesting paper, Spencer[1] also studied the charge distribution within a conductive sphere for laws different from Coulomb's, but not much attention has been paid to this study in courses on electrostatics due to the complexity of the analytical solutions.

When an excess of charge is given to a conductor, the charges move away from one another until they reach the conductor surface and redistribute themselves in such a way that both the charge and the field inside the conductor will vanish. A common student misconception is to think that the charges move to the surface due only to the repulsive force that exists among the charges, irrespective of the type of interaction. It is considered that the interaction would force the charges to locate on the surface, at the furthest possible distances from one another, even though a simple analysis shows that a distribution of the charges on the surface does not imply maximum separation between them.

With respect to the exact configuration of charges on a conductor, Thomson showed that in a state of equilibrium the electrostatic energy of the charge distribution has a minimum value. Thomson himself studied the equilibrium distribution of $N$ discrete point charges ($1 < N < 100$) on a spherical surface[2] (Thomson's problem) and used approximations to minimise the energy of the charges, distributed as rings over the sphere. Minimisation of energy shows that, in general, the confinement of charge to the surface is a direct consequence of the inverse square law (or its equivalent, Gauss's law). Several authors[3–5] have investigated Thomson's problem using numerical and topological methods and also similar problems related to the maximisation of the minimum distance among $N$ points over a spherical surface (Tamm's problem)[6], with very interesting applications to biology, chemistry and mathematics.

In this paper, a simple numerical method of multidimensional minimisation is applied in order to find the equilibrium configuration of $N$ point charges confined to a given volume. The interest of this work, from a didactic point of view, is to show numerically that the confinement of charges on the surface depends critically on the inverse square law. The simulation can be easily implemented for different geometric forms and types of interaction (Coulomb and non-Coulomb forces). The student can visualise some interesting characteristics of the distribution of charges on conductors, such as its concentration on the edges and vertices. Some examples involving a geometry having an infinite extent along one direction are considered and it is shown that a $1/r$ interaction law is required in this case to make the charge reside on the conductor

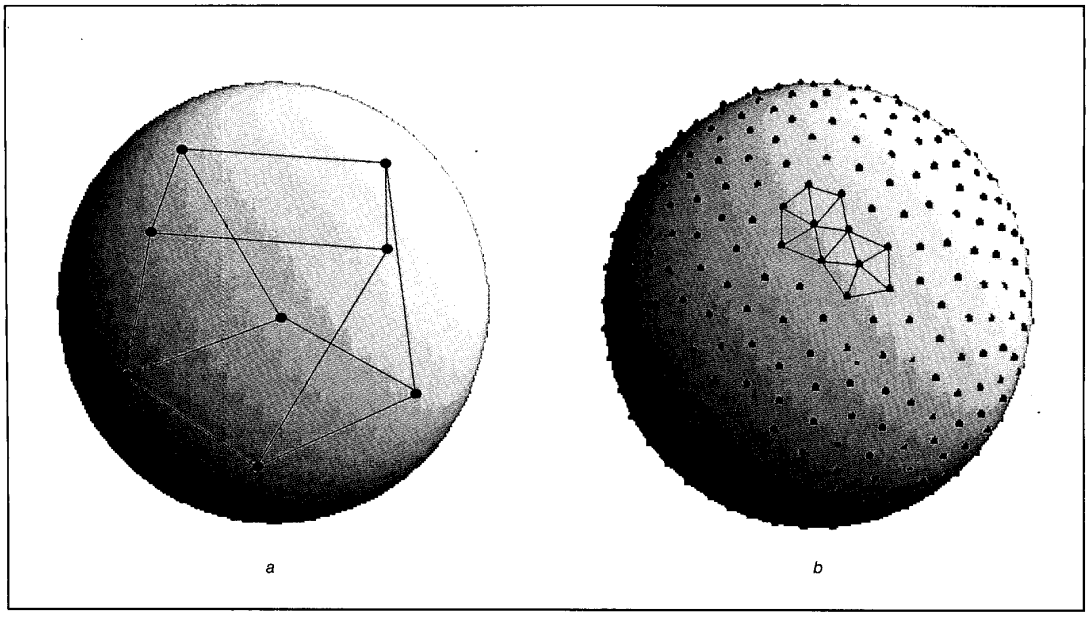


**Fig. 1 Equilibrium distribution of *N* identical point charges on a sphere for the case of Coulomb's law: (*a*) twisted parallelepiped (*N* = 8): (*b*) non-regular polyhedron (*N* = 600), showing vertices at the intersection of 6 triangles and at the intersection of only 5 triangles**

surface, corresponding to a two-dimensional harmonic potential.

## Monte Carlo Metropolis simulation

The distribution of $N$ discrete point charges over a conductor surface may be simulated numerically by analysing the evolution of the charge positions as they try to reach the minimum energy locations. From an initial arbitrary configuration of the point charges within the conductor, the charges must move towards the surface and in the state of equilibrium the net tangential force acting on each of them due to the other charges must be zero. Also, in the equilibrium state the net normal component of the interaction force is compensated by the electrical force that prevents the charges from leaving the surface. In this situation, the electrostatic energy potential has reached its minimum value. The goal is, therefore, to minimise the energy of a set of $N$ point charges given by:

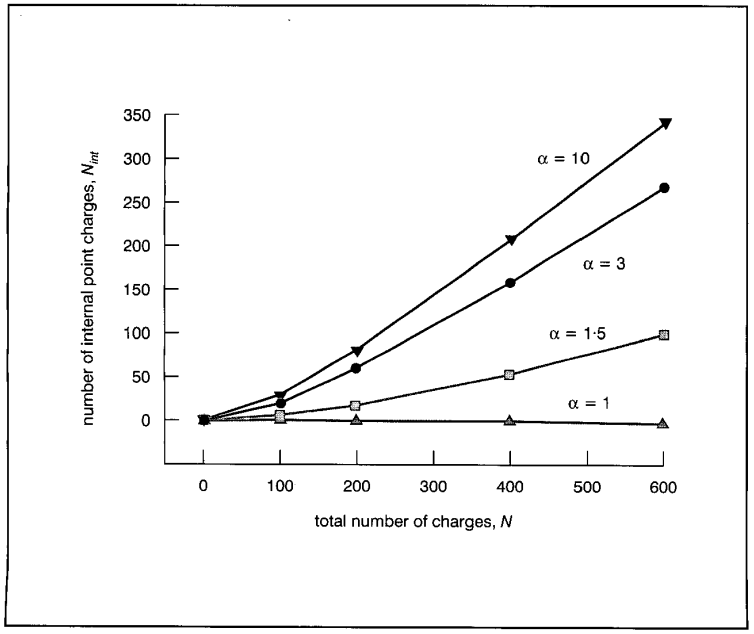


**Fig. 2 Number of internal point charges in a sphere in the equilibrium configuration as a function of the total number of charges *N*, for a $1/r^{\alpha}$ potential and for different values of $\alpha$**

$$U = \tfrac{1}{2}\sum_{i=1}^{N} q_i \Phi_i \qquad (1)$$

where $\Phi_i$ is the electrostatic potential at $\boldsymbol{r}_i$, the position of the charge $q_i$, due to the rest of the charges and is given by:

$$\Phi_i = \sum_{i \neq j=1}^{N} \frac{q_j}{4\pi\varepsilon_0 |\boldsymbol{r}_i - \boldsymbol{r}_j|} \qquad (2)$$

This is a multidimensional and nonlinear minimisation problem, where the space of energy variables has $3N$ dimensions.

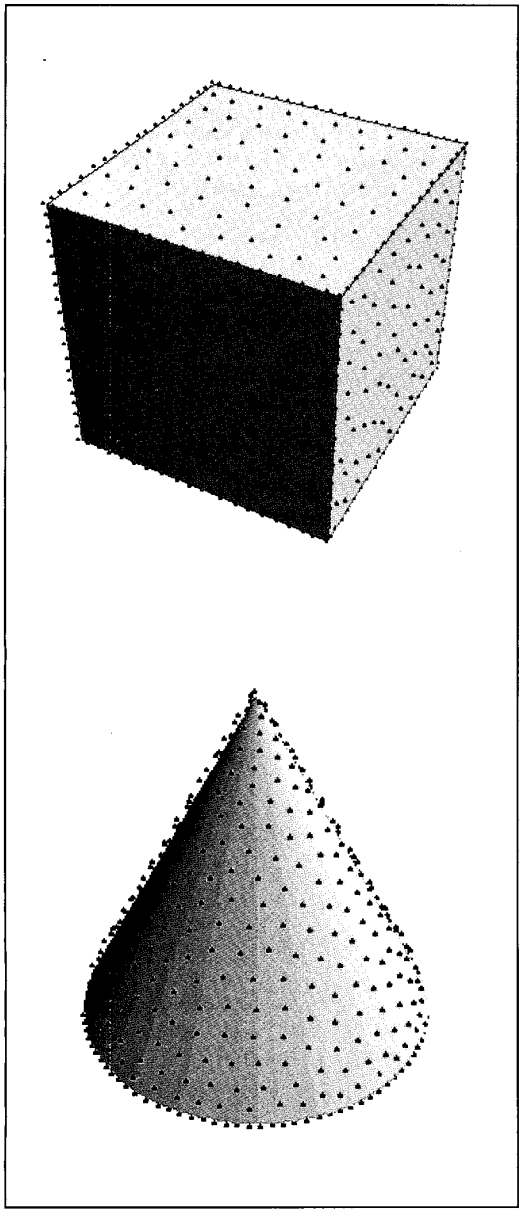

**Fig. 3 Equilibrium distribution of $N = 600$ identical point charges on the surface of a cube and a cone having a height of twice its radius when Coulomb's law applies**

There are different numerical techniques available to solve this type of optimisation problems. Among the Monte Carlo methods, the Metropolis technique is very efficient and is therefore extensively used in the analysis of the evolution of multiparticle systems[7]. The technique can be summarised in the following steps:

- An arbitrary initial distribution of $N$ point charges is placed within a conductor.
- A small arbitrary displacement both in length (between 0 and a maximum value $\delta$) and direction (between 0 and $2\pi$) is allowed for each charge $q_i$.
- For each charge $q_i$, the energy is calculated before and after the displacement. If the energy variation is negative ($\Delta E_i < 0$), then the new position is accepted. If the energy variation is positive ($\Delta E_i > 0$), then the value of the expression $\exp(-\Delta E_i/kT)$ is calculated and compared with a random value $R$ ($0 \leq R \leq 1$) arbitrarily generated. If the value of the function is higher than the value of $R$, the shift is considered to be a valid one. If the value of the function is lower than the value of $R$ the shift is disregarded and the charges are kept in their original positions. The process is repeated for every charge, from $i = 1$ to $N$.

  This characteristic of the Metropolis technique results in the charges not only occupying positions of lower energy but also occupying positions of higher energy, the probability being given by the Boltzmann exponential function. This enables potential barriers to be overcome with the aim of preventing the set of charges getting trapped in a configuration corresponding to a local minimum. The 'temperature' parameter, $T$, is adjusted by examining the convergence of the method in previous trial simulations.
- The full process is repeated until a stationary value for the total system energy is achieved.

It is not possible to assure that for a finite number of charges there is a unique configuration for minimum energy. There can be different local minima besides one or several absolute energy minima. In fact, it has been shown that in the case of a sphere and a large number of charges, $N$, the number of configurations of minimum energy increases exponentially with the value of $N$. However, the differences between the values of the energy minima get smaller every time and any of these discrete configurations may be considered as a good approximation to the theoretical continuous distribution on the conductor surface.

## Simulation examples

The distribution of a varying number of discrete charges on different two-dimensional and three-dimensional geometric forms has been analysed. This analysis also shows the versatility of the Metropolis simulation. Figs. 1 and 2 correspond to the three-dimensional analysis of the distribution of $N$ identical point charges on a sphere for Coulomb's law and for

$1/r^{\alpha}$ ($\alpha > 1$) non-harmonic potential interactions, respectively. Starting from a random distribution of the charges inside the sphere, a numerical solution may easily be obtained for the distribution of the point charges over the surface of the sphere. However, the problem of finding the optimal energy configuration for an arbitrary value of $N$ remains unsolved.

It is interesting to observe that the configuration of minimal energy is not, in general, the arrangement of greatest symmetry. For example, for $N = 8$, the configuration for minimum energy is not a cube but a twisted parallelepiped, and, in general, the Platonic solids* do not necessarily lead to the minimum energy configurations. It must be noted that the spherical symmetry condition of the geometry does not demand that a particular solution be spherically symmetrical but refers to the whole set of possible solutions. That is to say, if you have one equilibrium distribution, any other distribution obtained from this one by rotation and/or reflection will also be a solution, which is obviously true in our case. A detailed discussion on this topic can be found in the work of Shaw[8]. Connecting the nearest neighbours by an edge leads to non-regular polyhedra made up of a number of triangles, as shown in Fig. 1*b* for $N = 600$. Careful examination reveals that most of the vertices are at the intersection of 6 triangles, but some vertices are placed where only 5 triangles intersect.

To show that the confinement of charges on the surface depends critically on an inverse square law interaction, a non-harmonic potential interaction $1/r^{\alpha}$ ($\alpha > 1$) has been analysed. Fig. 2 shows the number of charges that remain within the sphere, $N_{int}$, as a function of the total number of charges, $N$. For a non-harmonic potential and a small number of point charges, an equilibrium configuration may be reached where all the charges lie on the surface of the sphere for all values of $\alpha$. However, for large values of $N$ it is found that, in the equilibrium condition, a number of particles remain in positions inside the sphere. As can be observed in Fig. 2, the number of internal charges depends on the value of $\alpha$. This number is very significant for large values of $\alpha$ as the energy becomes dominated by the interactions with the nearest neighbours.

For solids with edges and vertices, where charges mainly accumulate, the method can also be easily applied. Fig. 3 shows two examples of the distribution of $N = 600$ identical point charges on the surface of a cube and a cone for the case of Coulomb's law.

The Metropolis simulation has also proved to be a very useful tool for analysing the difference between two-dimensional and three-dimensional problems with similar geometry. Fig. 4*a* shows the final minimum energy configuration for $N$ identical charges that have been placed inside a circle and that obey a $1/r$ potential law. This configuration is an approximation to the continuous distribution of charge that is found on a conducting circular disc at a constant potential. From the point of view of the two-dimensional Laplace equation, the presence of charges in positions other

*The five regular solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

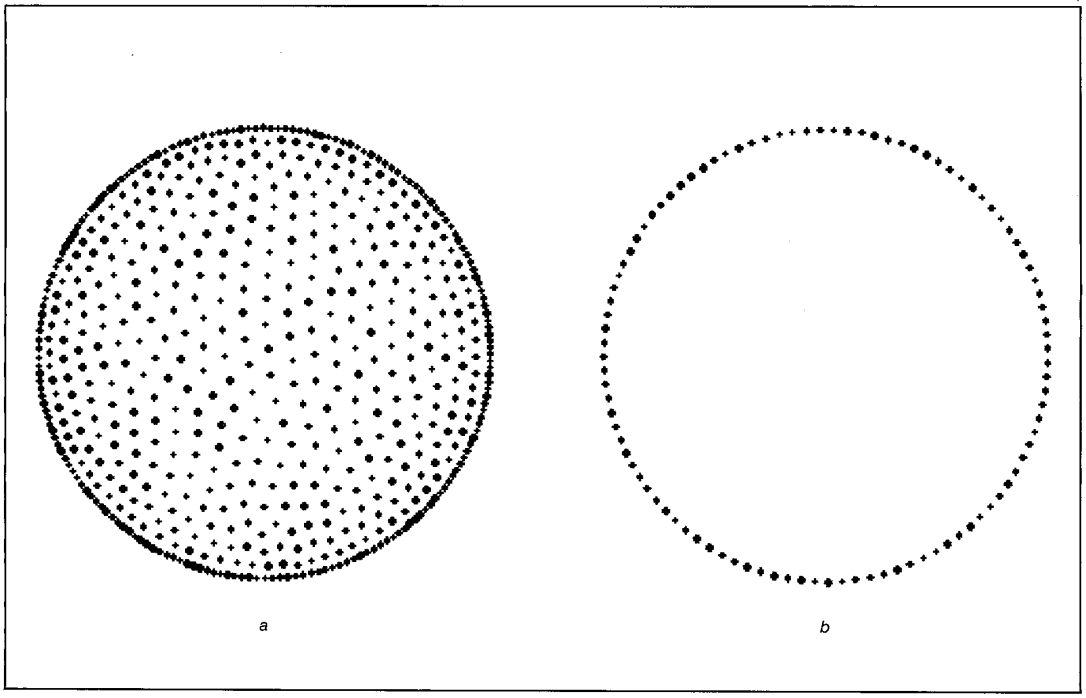


**Fig. 4 Equilibrium distribution of *N* identical point charges over a circle: (*a*) minimum energy configuration for a $1/r$ potential ($N = 600$); (*b*) equilibrium configuration for a logarithmic potential, $-\ln r$ ($N = 100$)**

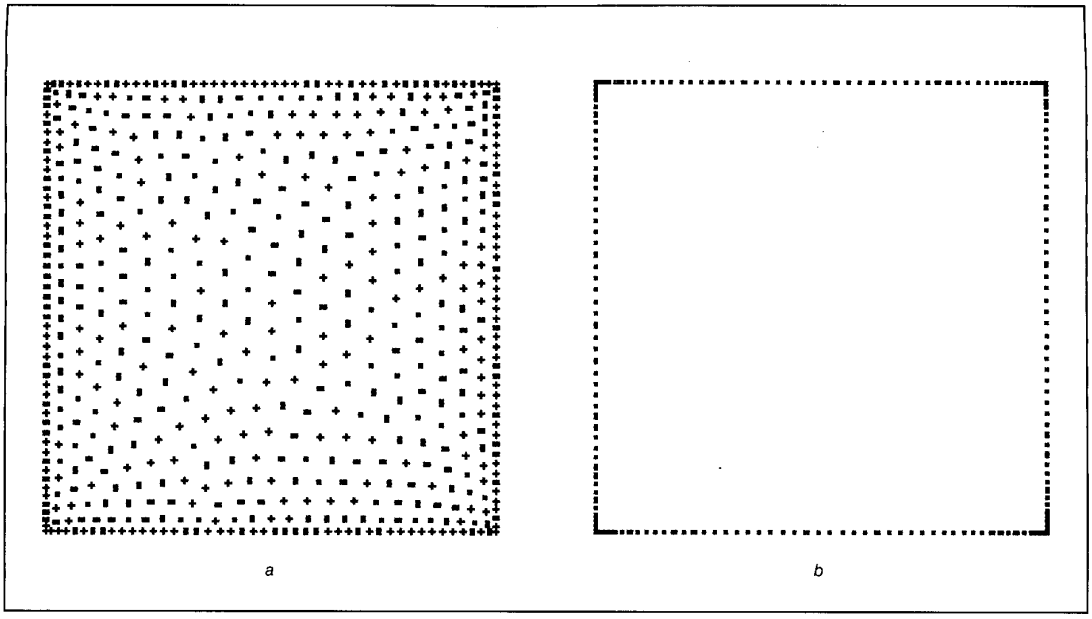


**Fig. 5 Distribution of $N$ identical point charges over a square: (*a*) minimum energy configuration for the $1/r$ potential ($N = 600$); (*b*) equilibrium configuration for a logarithmic potential, $-\ln r$ ($N = 200$)**

than the circumference of the circle is related to the non-harmonic character of the potential $1/r$. However, when a logarithmic potential ($-\ln r$) is used, it is found that the equilibrium configurations always correspond to those charges uniformly distributed along the circumference.

This last configuration is shown in Fig. 4*b* and is an approximate representation of the distribution of charge on the surface of an infinitely long conducting circular cylinder when viewed in cross-section.

Finally, Figs. 5*a* and 5*b* show similar cases for the equilibrium distributions of $N$ identical point charges placed initially inside a square and interacting via $1/r$ and $-\ln r$ potentials, respectively. Fig. 5*a* corresponds to the charge distribution of a square plate, whereas Fig. 5*b* approximates the distribution of charge per unit length on the surface of an infinite cylinder of square section. The accumulation of charge near the corners can be observed.

## Conclusions

The charge distribution within a conductor depends critically on the type of interaction potential, and it is only in the case of the inverse law potential that all the charges remain on the surface. Also, in a state of equilibrium, the geometric arrangement of the charge distribution corresponds to the minimum value of electrostatic energy. These two fundamental issues have been analysed numerically in this paper, for both three-dimensional and two-dimensional electrostatic problems, using a Monte Carlo simulation. This technique has been shown to be very useful, not only for calculating the charge distribution but also for visualising its characteristics for geometric forms having edges and vertices. Once the charge distribution has been determined, the results can be used for finding approximate values of the field or for finding the electrical characteristics, such as capacitance, of conductors with geometric forms that would otherwise require very complex mathematical calculations.

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