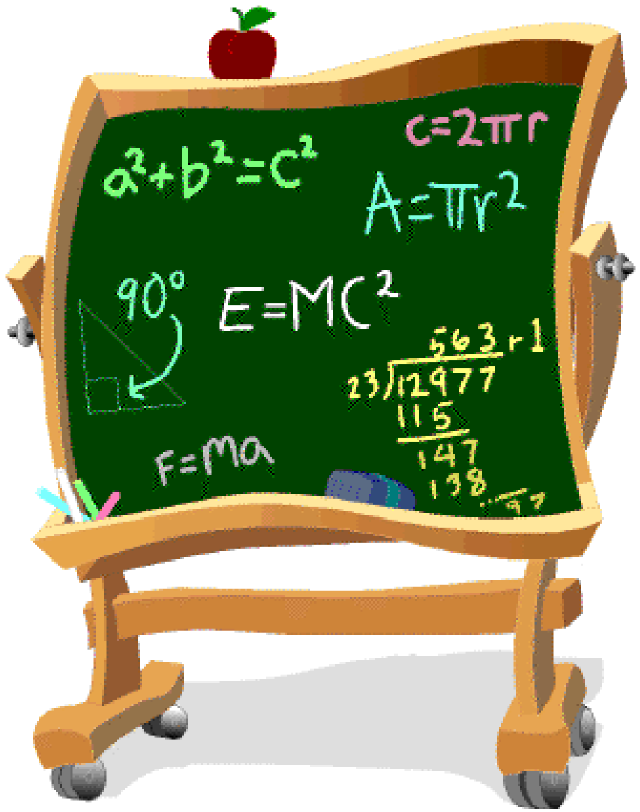
Math Review Towards Fourier Transform

- Linear Spaces
- Change of Basis
- Cosine and Sine
- Complex Numbers



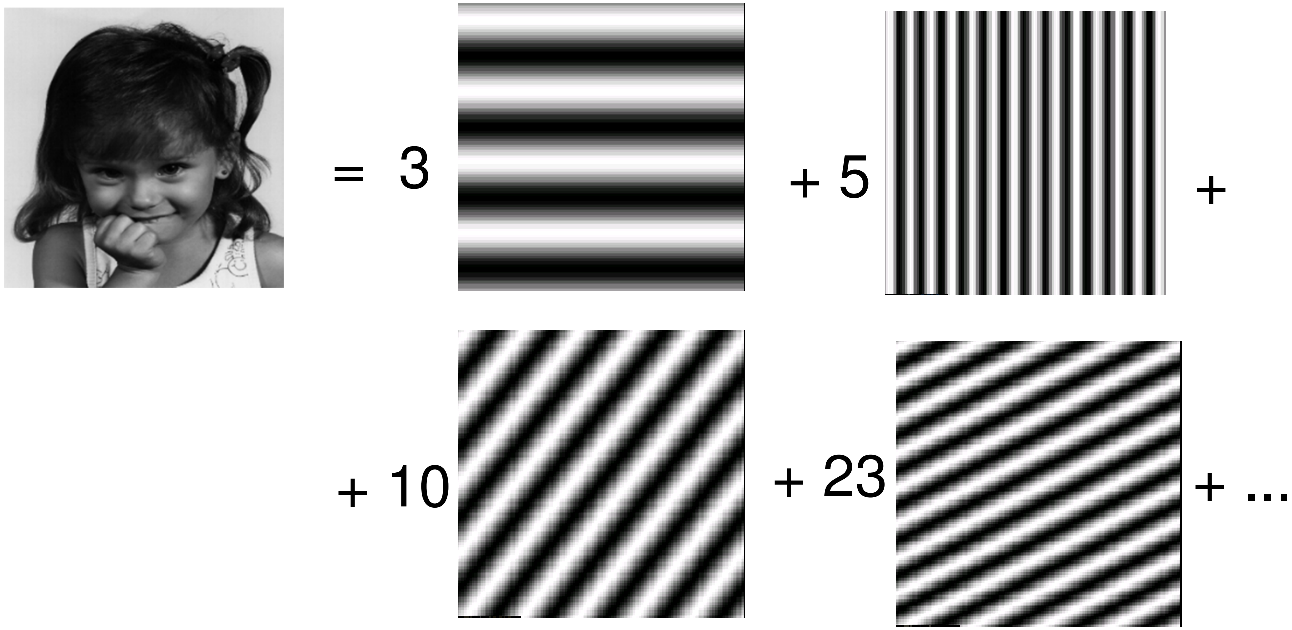
Why Fourier Transform?

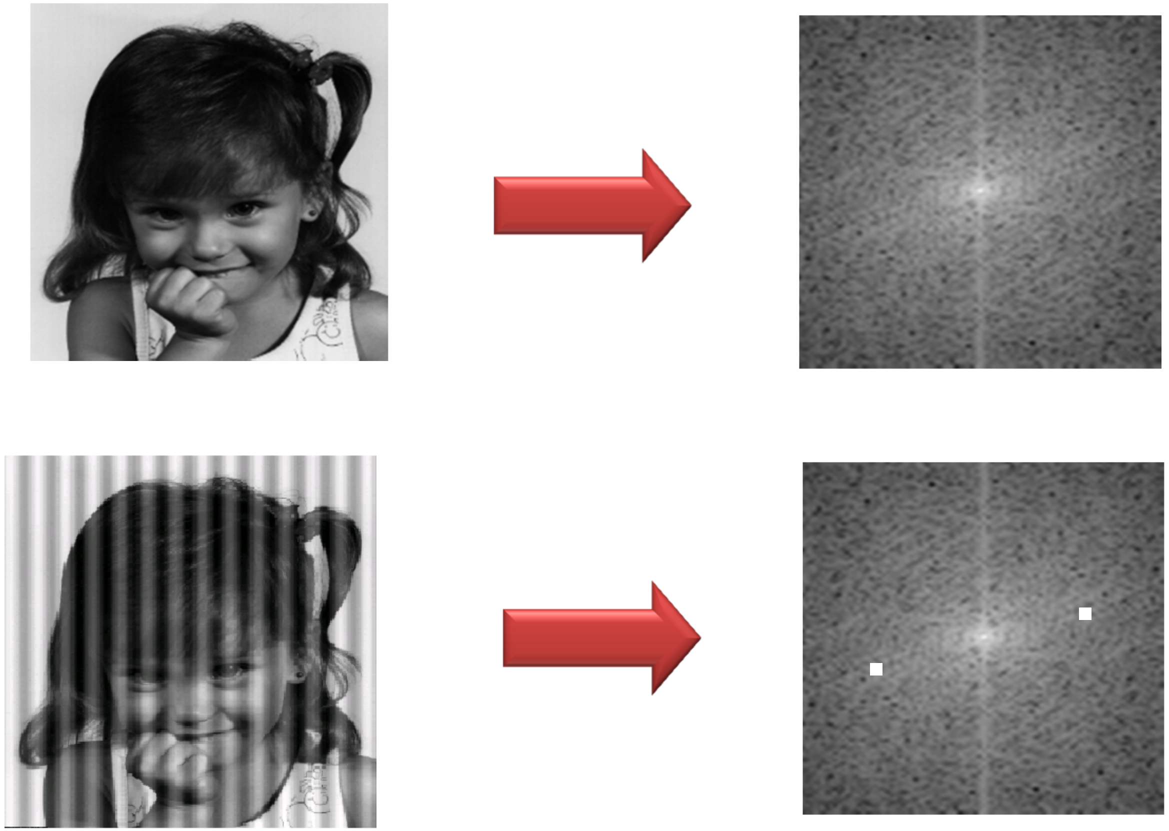
How can we enhance such an image?

Solution: Image Representation

$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & 8 & 7 \\ 0 & 3 & 5 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

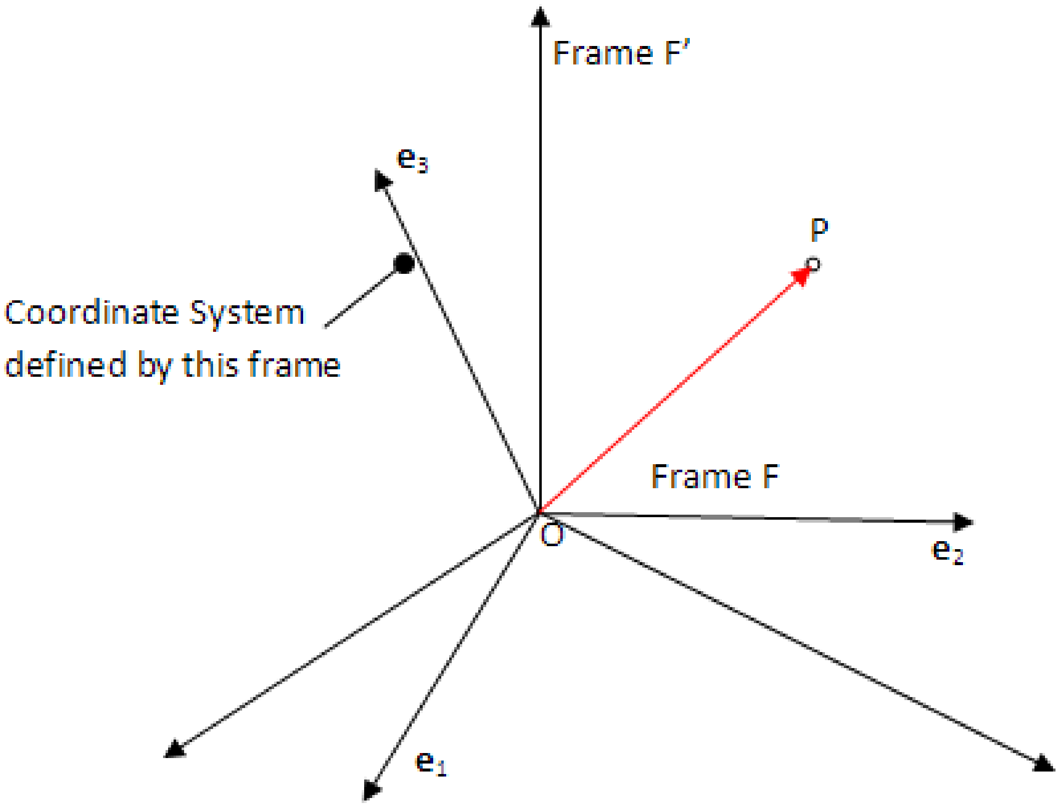
$$+ 3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$


= 3
+ 5
+
+ 10
+ 23
+ ...



- Global phenomena becomes local
- Spatial correction in possible in the new representation

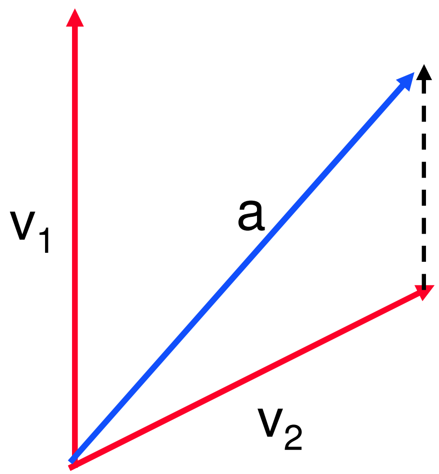
Part I: Vector Spaces and Basis Vectors



Basis Vectors

- A given vector value is represented with respect to a *coordinate system*.
- A coordinate system is defined by a set of linearly independent vectors forming the system *basis*.
- Any vector value is represented as a linear sum of the basis vectors.

$$\boxed{a = 0.5 * v_1 + 1 * v_2 \equiv (0.5\ ,\ 1)_{\mathbf{v}}}$$



- v1, v2 are basis vectors
- The representation of ***a*** with respect to this basis is (0.5,1)

Orthonormal Basis Vectors

- If the basis vectors are mutually orthogonal and are unit vectors, the vectors form an *orthonormal basis*.
- Example:

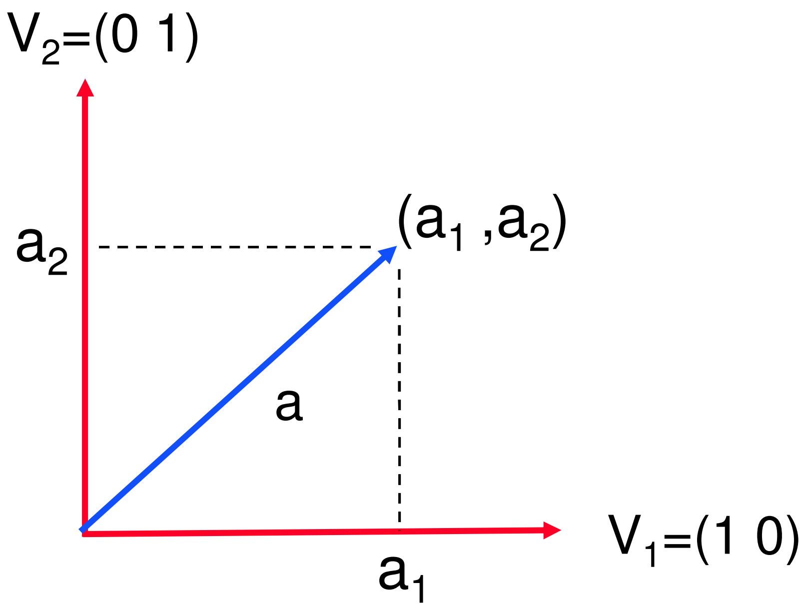
  The *standard basis* is orthonormal:

$V_1=(1\ 0\ 0\ 0\ \ldots)$

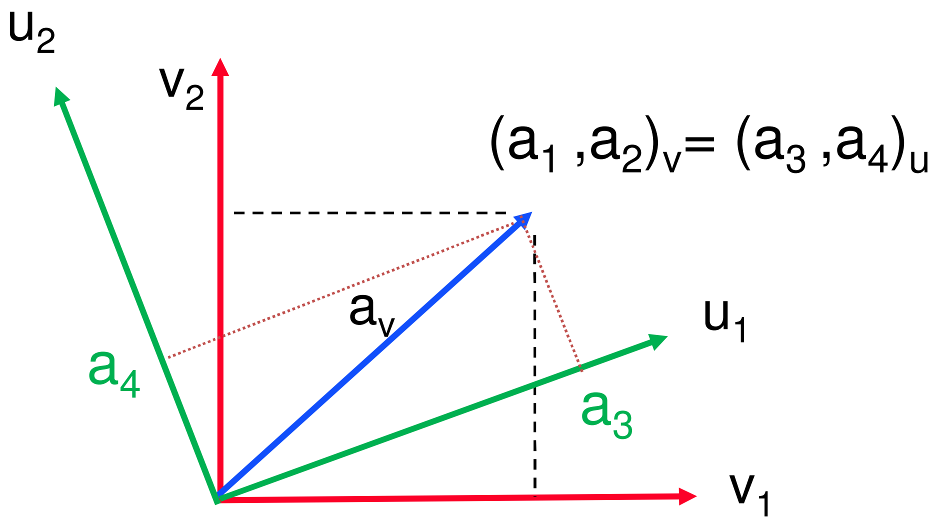
$V_2=(0\ 1\ 0\ 0\ \ldots)$

$V_3=(0\ 0\ 1\ 0\ \ldots)$

$\ldots.$



Change of Basis



Given a vector $\mathbf{a_v}$, represented in orthonormal basis $\{v_i\}$, what is the representation of $\mathbf{a_v}$ in a different orthonormal basis $\{u_i\}$?

$$\mathrm{a_u(i)} = \langle \mathrm{a_v}, \mathrm{u_i} \rangle$$

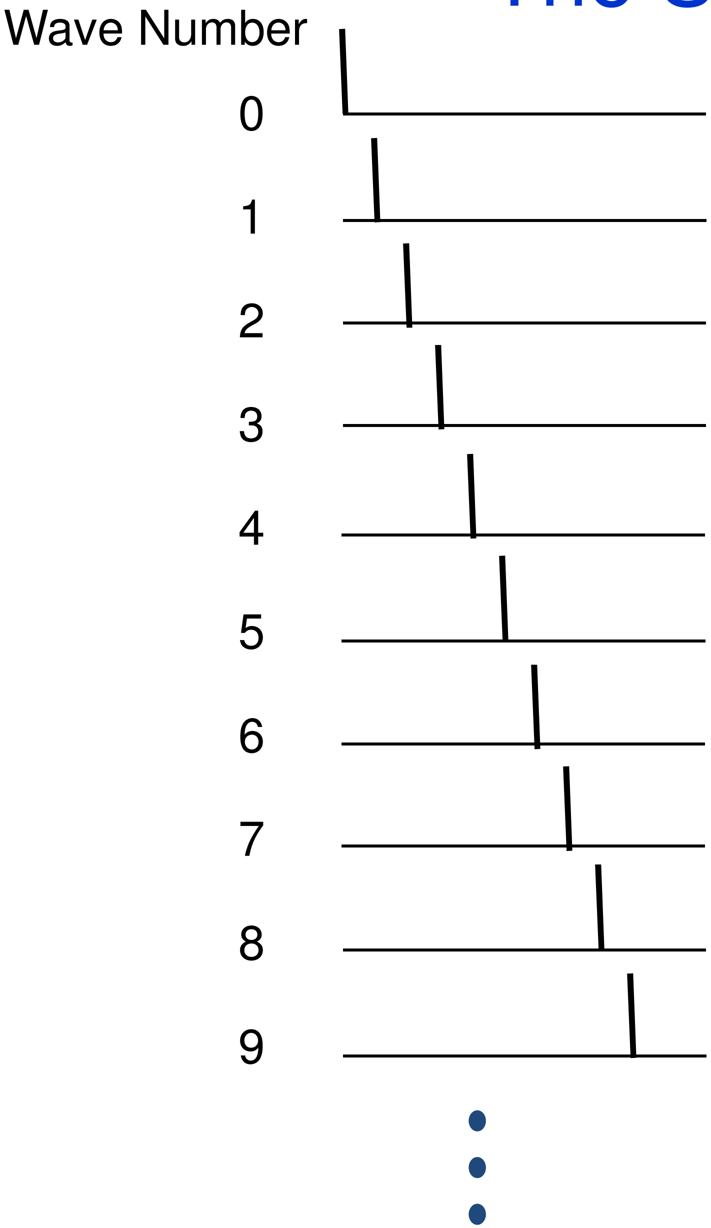
$$a_v = \sum_i a_u(i)\, u_i$$

$$\textit{where} \quad \langle c, b \rangle = c^T b = \sum_i c(i) b(i)$$

Signal (Image) Transform

1. Basis Functions.
2. Method for finding the transform coefficients given a signal (in the standard basis).
3. Method for finding the signal given the transform coefficients.

The Standard Basis (Orthogonal)



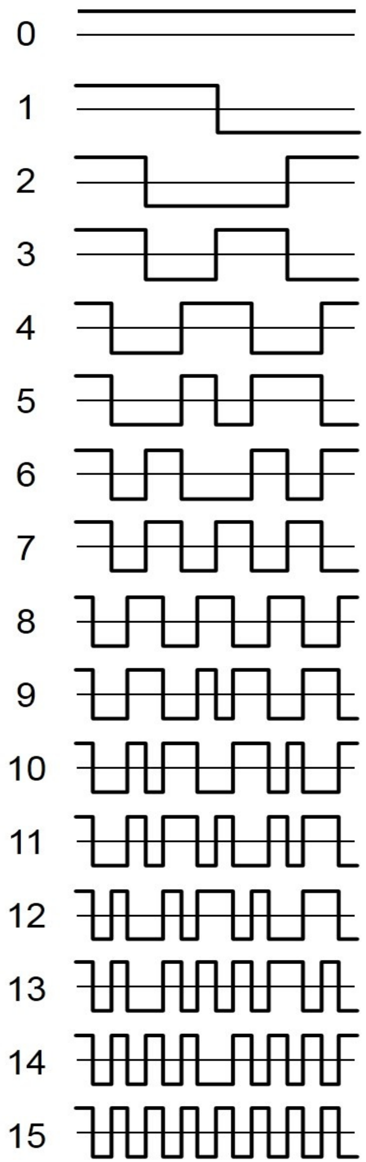
N = 16

Standard Basis Functions

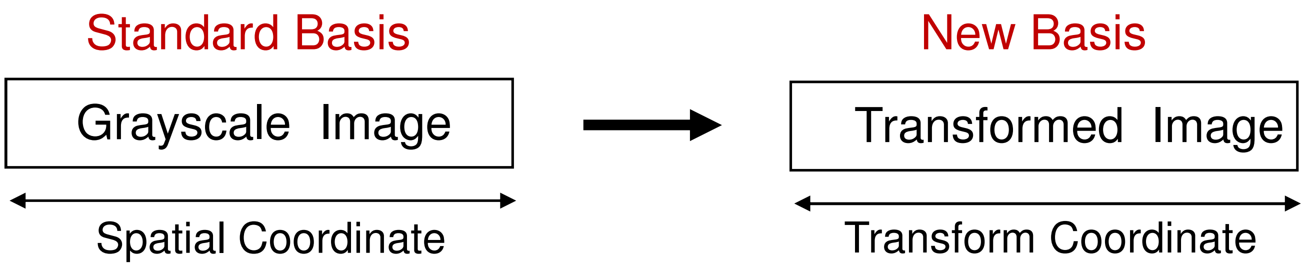
The Hadamard Basis (Orthogonal)

Wave Number

N = 16



Hadamard Transform (1D)



Standard Basis:

$$[\,2\ \ 1\ \ 6\ \ 1\,]_{standard} = \mathbf{2}\,[\,1\ 0\ 0\ 0\,] + \mathbf{1}\,[\,0\ 1\ 0\ 0\,] + \mathbf{6}\,[\,0\ 0\ 1\ 0\,] + \mathbf{1}\,[\,0\ 0\ 0\ 1\,]$$

Hadamard Transform:

$$[\,2\ \ 1\ \ 6\ 1\,]_{standard} = \mathbf{5}\,[\,1\ 1\ 1\ 1\,]/2 + \mathbf{-2}\,[\,1\ 1\ -1\ -1\,]/2 +$$

$$+ \mathbf{-2}\,[\,1\ -1\ -1\ 1\,]/2 + \mathbf{3}\,[\,1\ -1\ 1\ -1\,]/2 \equiv$$

$$\equiv [\,\mathbf{5}\ \ \mathbf{-2}\ \ \mathbf{-2}\ \ \mathbf{3}\,]_{Hadamard}$$

Finding the Transform Coefficients

Signal:

$\mathbf{X} = [\ 2\ 1\ 6\ 1\ ]_{\text{standard}}$

Hadamard Basis:

$\mathbf{T_0} = [\ 1\ \ 1\ \ 1\ \ 1\ ]\ /2$

$\mathbf{T_1} = [\ 1\ \ 1\ \ -1\ \ -1\ ]\ /2$

$\mathbf{T_2} = [\ 1\ -1\ \ -1\ \ 1\ ]\ /2$

$\mathbf{T_3} = [\ 1\ -1\ \ 1\ -1\ ]\ /2$

Hadamard Coefficients:

$$a_0 = \langle \mathbf{X}, \mathbf{T_0} \rangle = \langle\ [\ 2\ 1\ 6\ 1\ ]\ ,\ [\ 1\ \ 1\ \ 1\ \ 1\ ]\ /2\ \rangle = 5$$

$$a_1 = \langle \mathbf{X}, \mathbf{T_1} \rangle = \langle\ [\ 2\ 1\ 6\ 1\ ]\ ,\ [\ 1\ \ 1\ \ -1\ \ -1\ ]\ /2\ \rangle = -2$$

$$a_2 = \langle \mathbf{X}, \mathbf{T_2} \rangle = \langle\ [\ 2\ 1\ 6\ 1\ ]\ ,\ [\ 1\ \ -1\ \ -1\ \ 1\ ]\ /2\ \rangle = -2$$

$$a_3 = \langle \mathbf{X}, \mathbf{T_3} \rangle = \langle\ [\ 2\ 1\ 6\ 1\ ]\ ,\ [\ 1\ \ -1\ \ 1\ \ -1\ ]\ /2\ \rangle = 3$$

$$[\ 2\ 1\ 6\ 1\ ]_{\text{Standard}} \equiv [\ 5\ \ -2\ \ -2\ \ 3\ ]_{\text{Hadamard}}$$

Reconstructing the Image from the transform coefficients

Transform:

$\mathbf{Y} = [\ 5\ -2\ -2\ \ 3\ ]_{\text{Hadamard}}$

Hadamard Basis:

$\mathbf{T_0} = [\ 1\ \ 1\ \ 1\ \ 1\ ]\ /2$

$\mathbf{T_1} = [\ 1\ \ 1\ -1\ -1\ ]\ /2$

$\mathbf{T_2} = [\ 1\ -1\ \ -1\ \ 1\ ]\ /2$

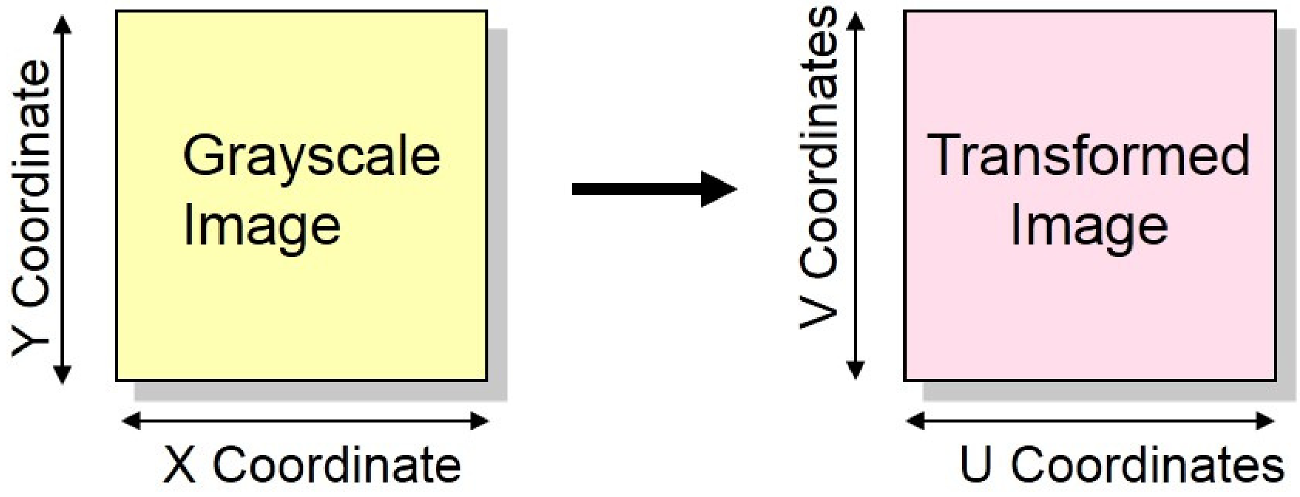
$\mathbf{T_3} = [\ 1\ -1\ \ 1\ \ -1\ ]\ /2$

Reconstruction: $\sum_i Y(i) T_i$

$\mathbf{5}\ [\ 1\ 1\ 1\ 1\ ]/2 \ +\ \mathbf{-2}\ [\ 1\ 1\ -1\ -1\ ]\ /2\ +\ \mathbf{-2}\ [\ 1\ -1\ -1\ 1\ ]\ /2\ +$

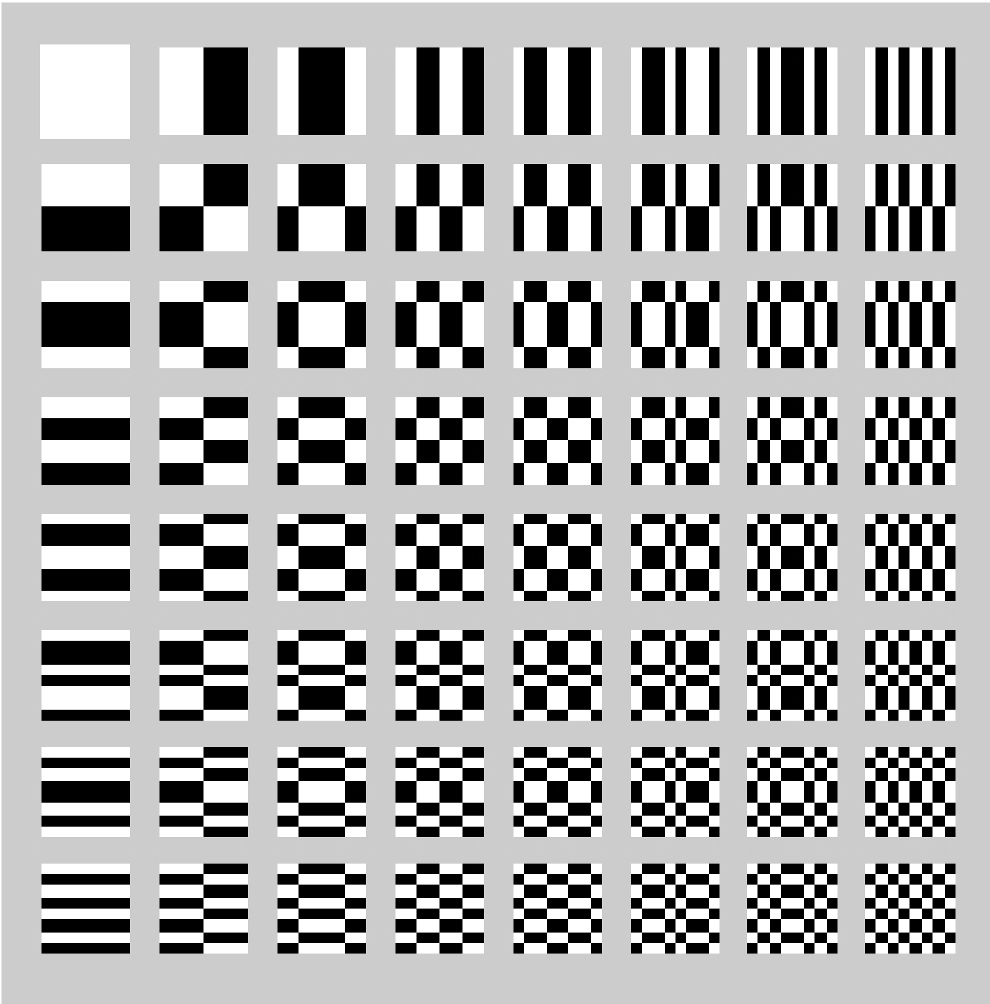
$+\ \mathbf{3}\ [\ 1\ -1\ \ 1\ -1\ ]\ /2\ \ = [\ 2\ \ \ 1\ \ \ 6\ \ \ 1\ ]_{\text{standard}}$

Transform: Change of Basis in 2D Images



- The coefficients are arranged in a 2D array.

2D Hadamard Basis Functions



size = 8x8

White = +1   Black = -1

Transform: Change of Basis

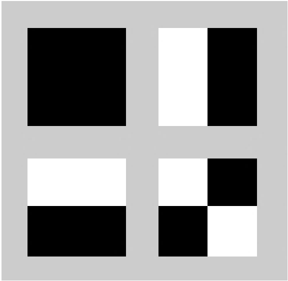
Standard Basis:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hadamard Transform:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} = 5\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 + -2\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 + -2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2 + 3\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2$$

$$\equiv \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{\text{Hadamard}}$$



Finding the Transform Coefficients (2D)

Signal:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}_{\text{standard}}$$

New Basis:

$$\mathbf{T_{11}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 \qquad \mathbf{T_{12}} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2$$

$$\mathbf{T_{21}} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 \qquad \mathbf{T_{22}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2$$

Signal: $\mathbf{X} = a_{11}\mathbf{T_{11}} + a_{12}\mathbf{T_{12}} + a_{21}\mathbf{T_{21}} + a_{22}\mathbf{T_{22}}$

New Coefficients:

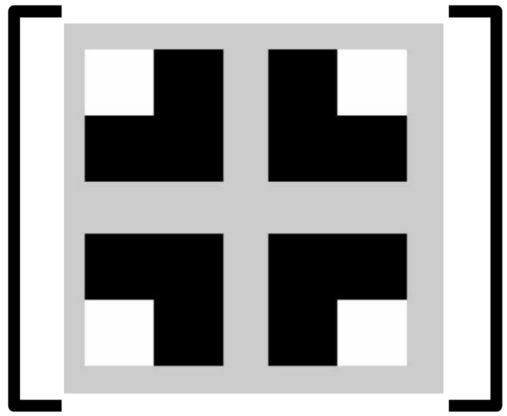
$$a_{11} = \langle \mathbf{X},\mathbf{T_{11}} \rangle = \text{sum}(\text{sum}(\mathbf{X}.{*}\mathbf{T_{11}})) = 5$$
$$a_{12} = \langle \mathbf{X},\mathbf{T_{21}} \rangle = \text{sum}(\text{sum}(\mathbf{X}.{*}\mathbf{T_{21}})) = -2$$
$$a_{21} = \langle \mathbf{X},\mathbf{T_{22}} \rangle = \text{sum}(\text{sum}(\mathbf{X}.{*}\mathbf{T_{22}})) = -2$$
$$a_{22} = \langle \mathbf{X},\mathbf{T_{12}} \rangle = \text{sum}(\text{sum}(\mathbf{X}.{*}\mathbf{T_{12}})) = 3$$

$$\mathbf{X} \equiv \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{\text{new}}$$

Standard Basis:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}_{\text{Standard}}$$

coefficients

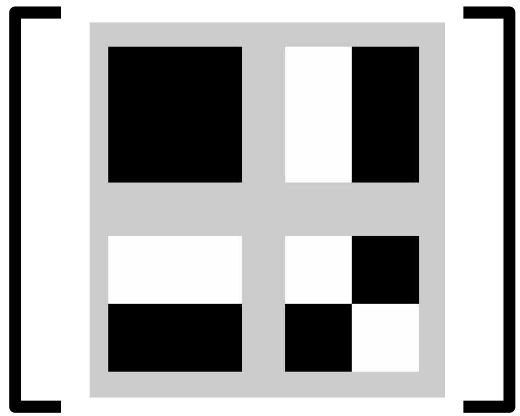


Standard

Hadamard Transform:

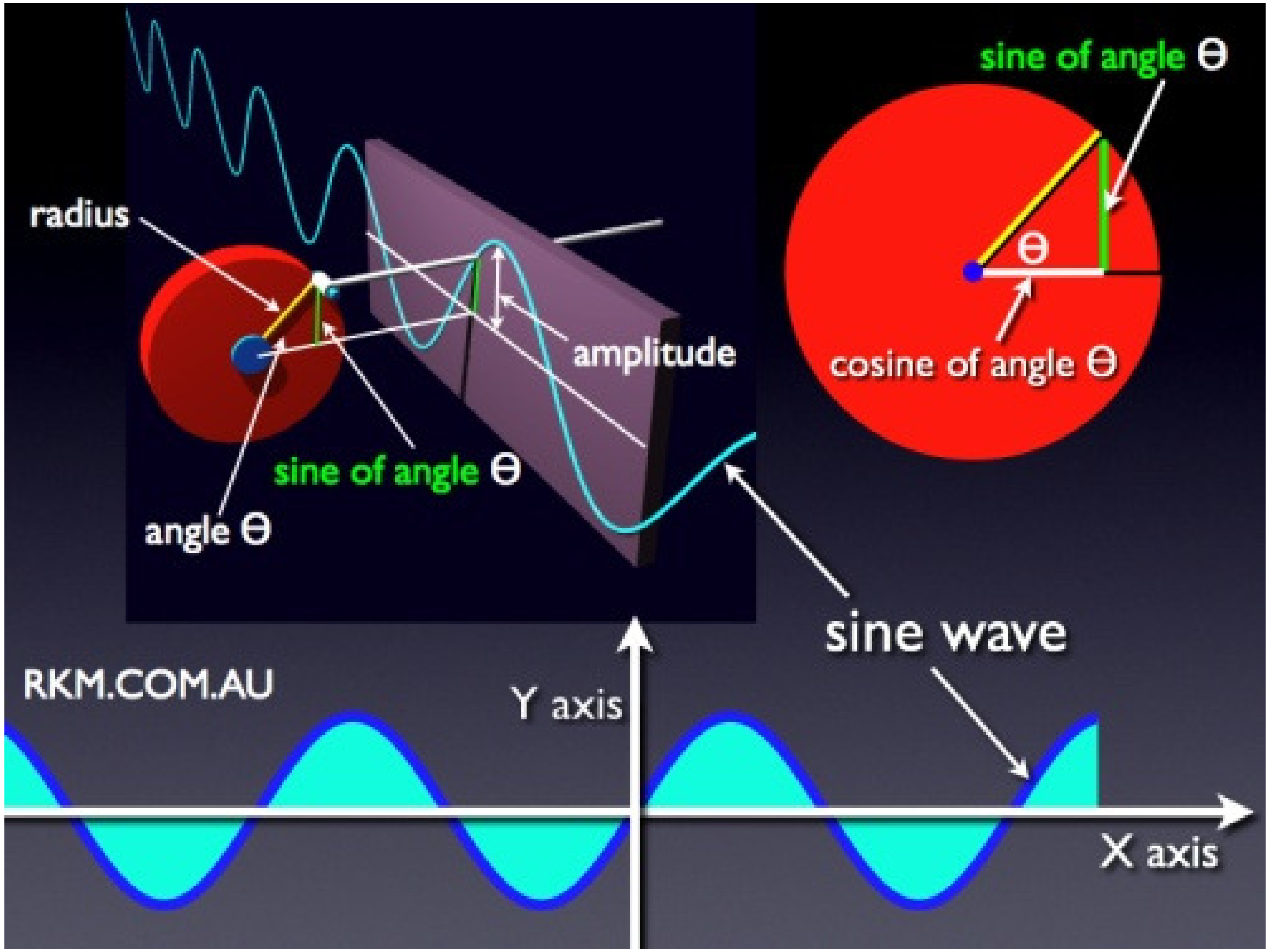
$$\begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{\text{Hadamard}}$$

coefficients

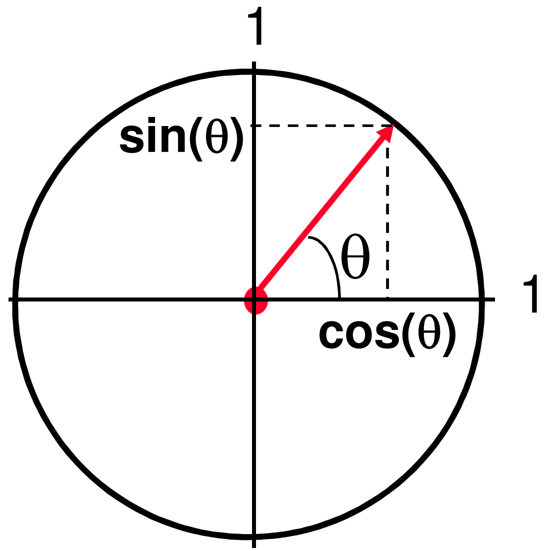


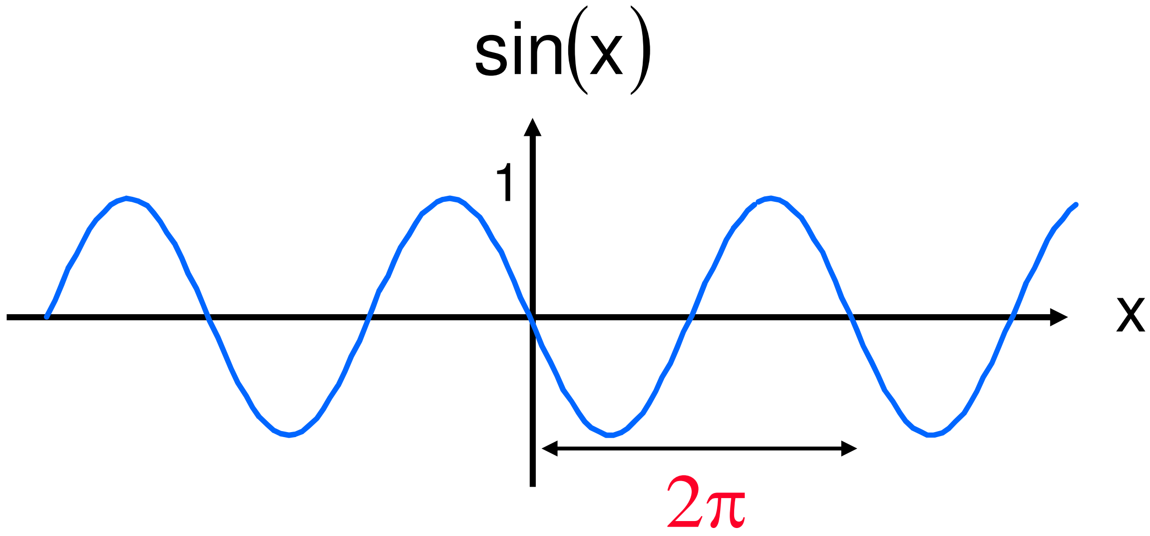
Hadamard

Part II: Sine and Cosine



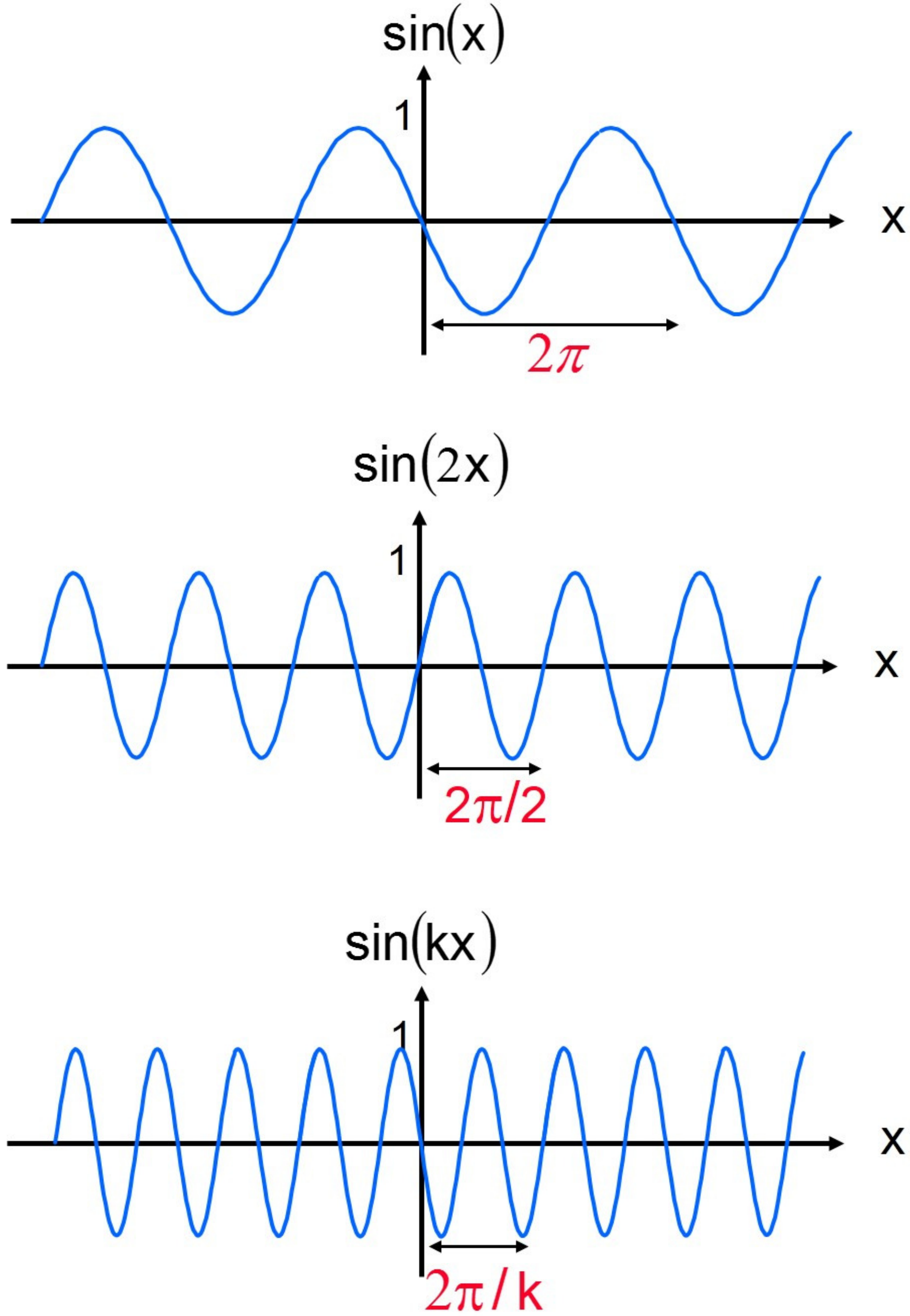
Wavelength and Frequency of Sine/Cosin





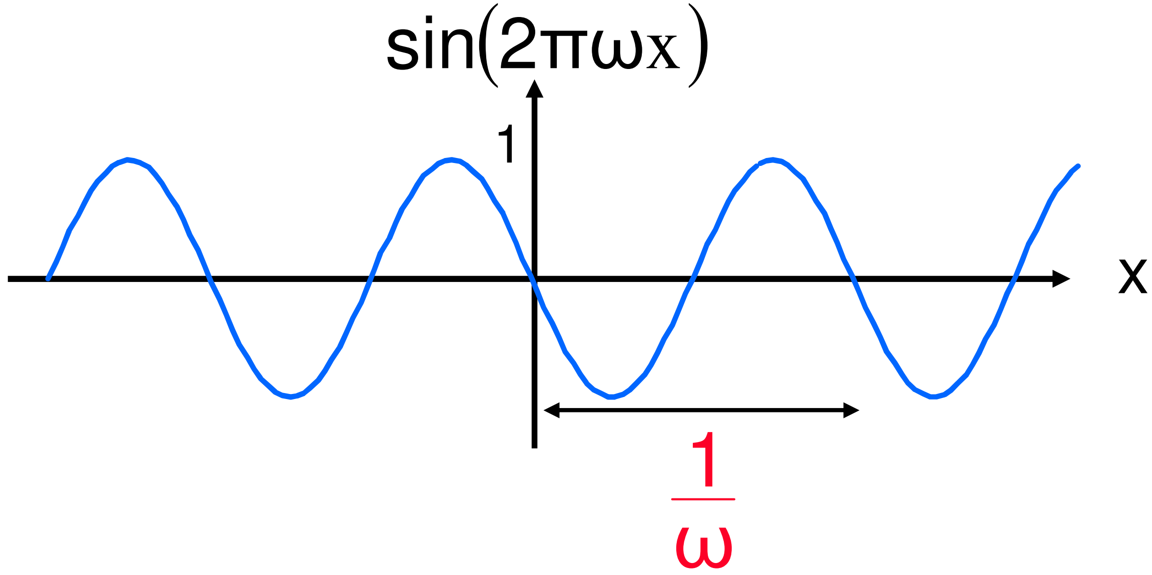
– The wavelength of sin(x) is 2π .

– The frequency is $1/(2\pi)$.



sin(x)
1
x
2π
sin(2x)
1
x
2π/2
sin(kx)
1
x
2π / k

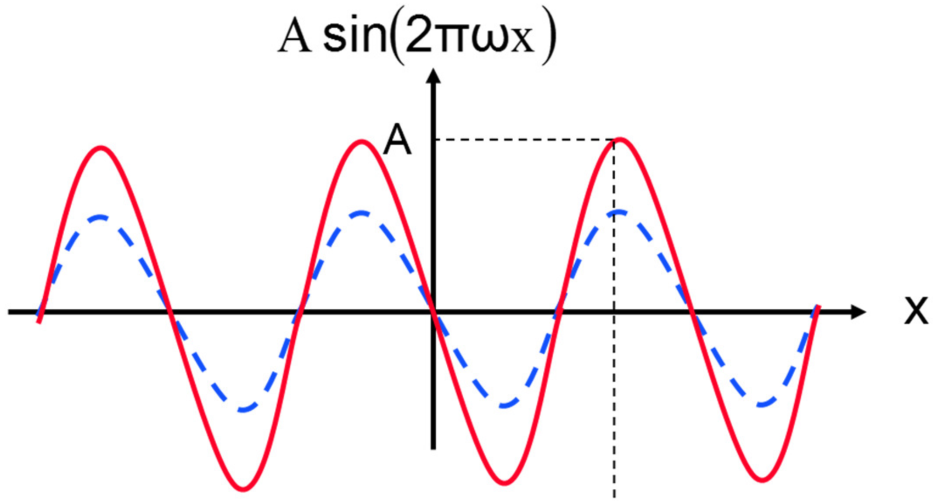
– Define $K=2\pi\omega$



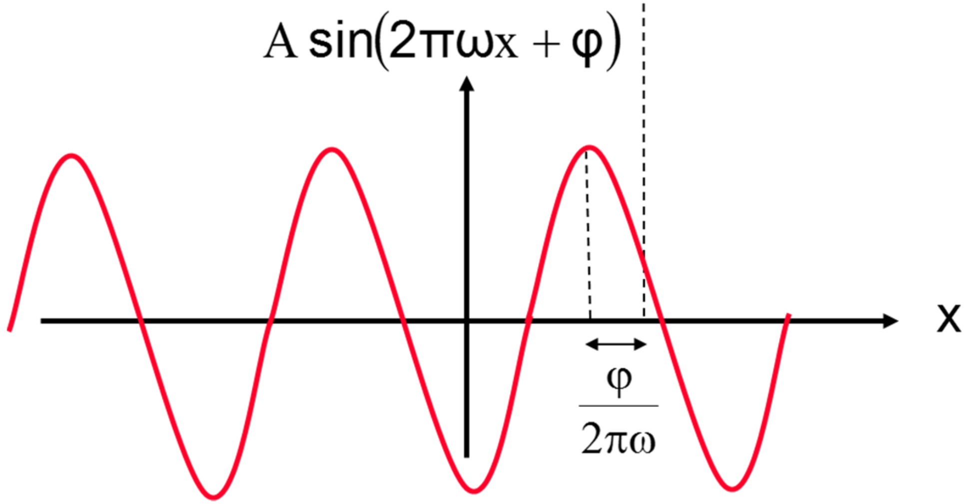
– The *wavelength* of $\sin(2\pi\omega x)$ is $1/\omega$.

– The *frequency* is ω .

– Changing Amplitude:

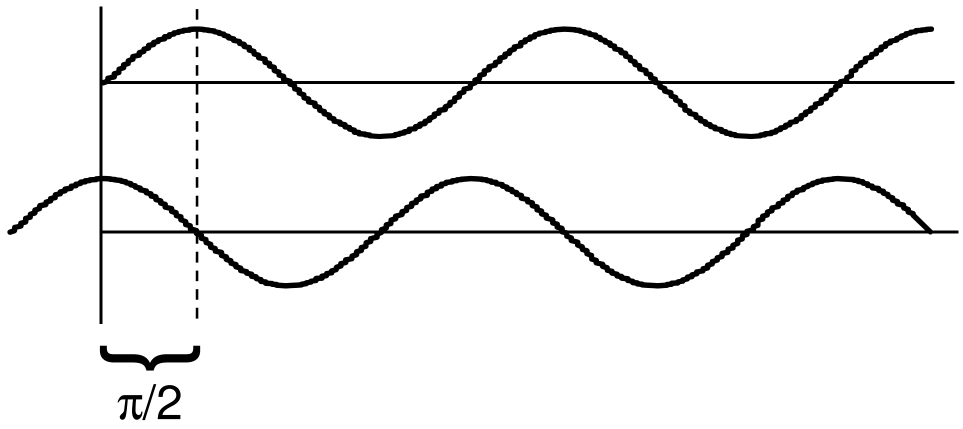


– Changing Phase:



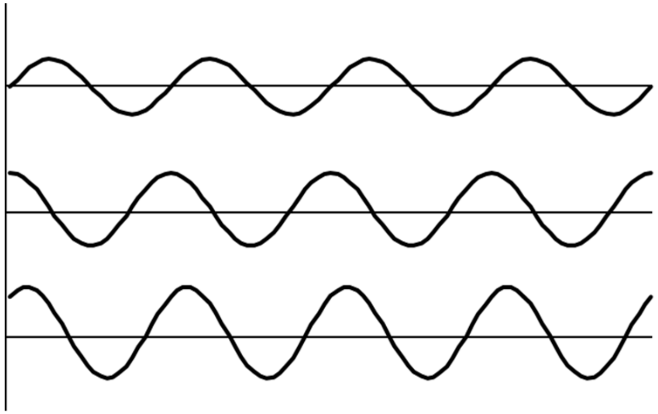
Sine vs. Cosine

- sin(x) is a cos(x) with a phase shift of $\pi/2$.



Combining Sine and Cosine

- If we add a Sine wave to a Cosine wave with the same frequency we get a scaled and shifted (Co-) Sine wave with the same frequency:



3 sin(kx)

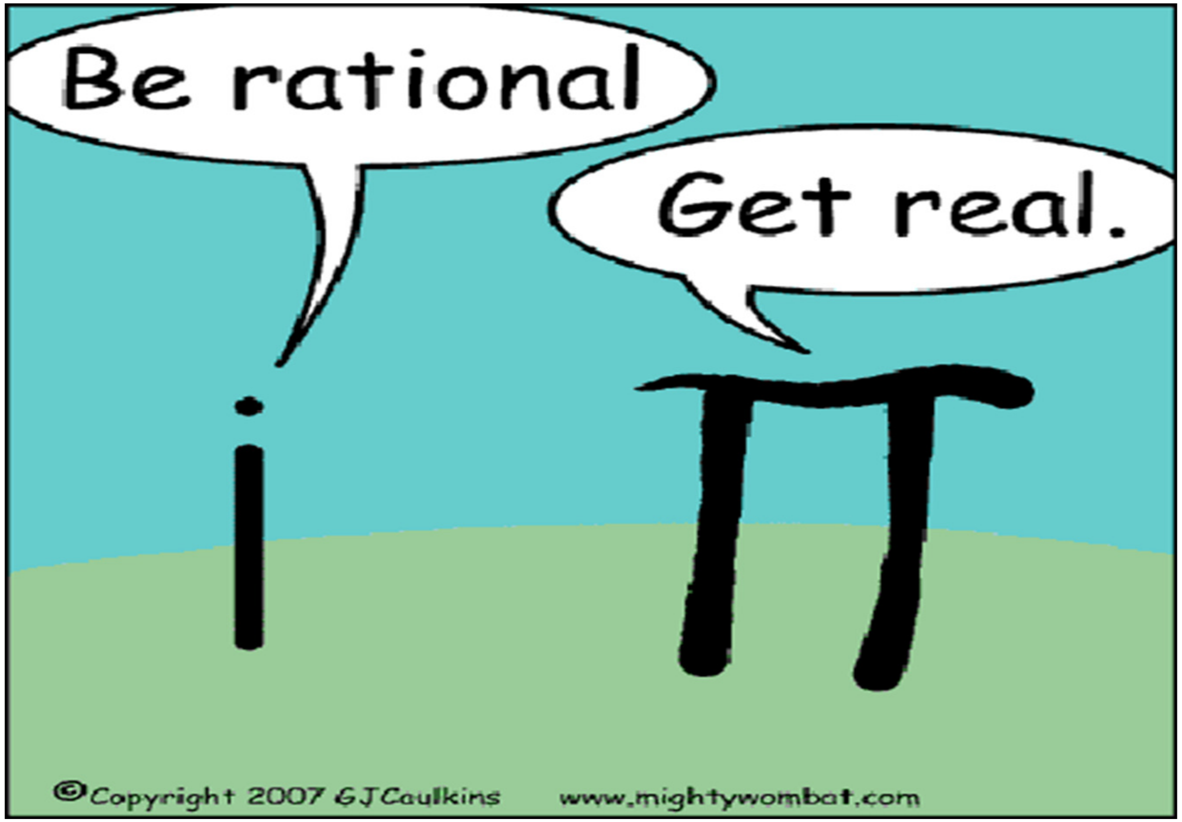
4 cos(kx)

5 sin(kx + 0.3π)

$$\boxed{\begin{gathered} a\sin(kx) + b\cos(kx) = R\sin(kx + \phi) \\ \text{where } \; R = \sqrt{a^2 + b^2} \; \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right) \end{gathered}}$$

(prove it!)

Part III: Complex Numbers



Complex Numbers

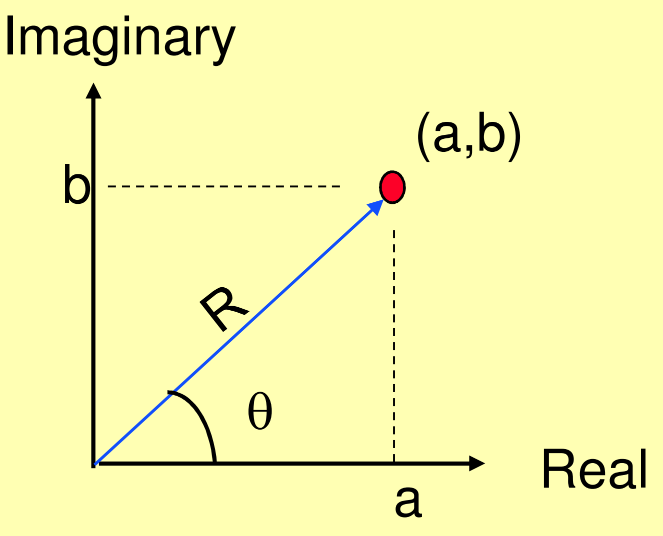
- Two kind of representations in the complex plane:

  - The *Cartesian representation*:

    $$Z = a + ib \qquad where \quad i^2 = -1$$

  - The *Polar representation*:

    $$Z = \mathrm{R}e^{i\theta}$$



The Complex Plane

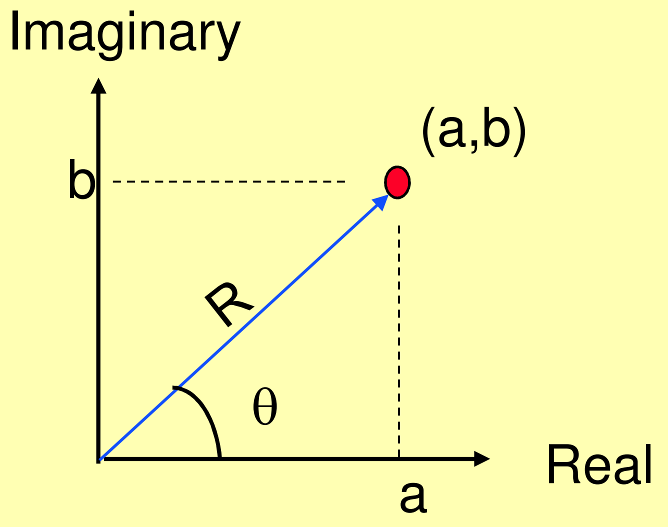
Representation Conversions

– Polar to Cartesian:

$$\mathrm{R}e^{i\theta} = \mathrm{R}\cos(\theta) + i\mathrm{R}\sin(\theta)$$

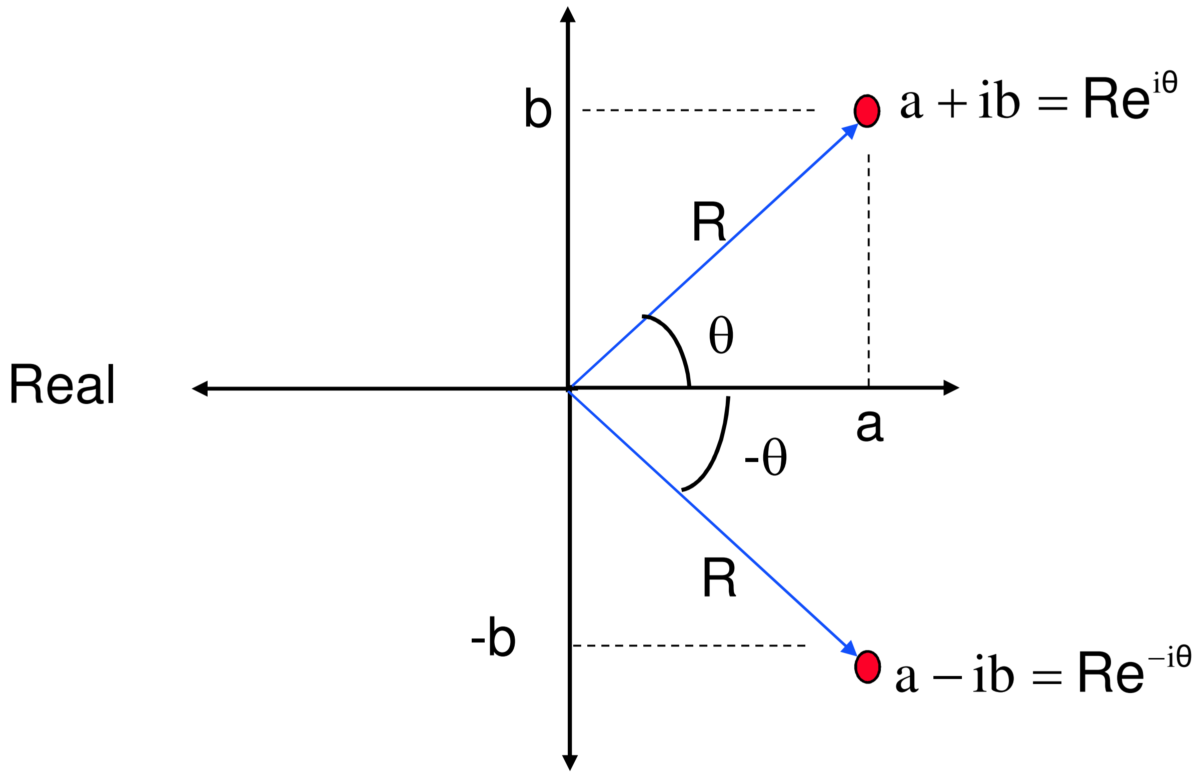
– Cartesian to Polar

$$a + ib = \left(\sqrt{a^2 + b^2}\right) e^{i\,\tan^{-1}(b/a)}$$



The Complex Plane

- Conjugate of Z is Z^*:
  - Cartesian rep: $(a + ib)^* = a - ib$
  - Polar rep: $\left(R\, e^{i\,\theta}\right)^* = \mathrm{R}\mathrm{e}^{-\mathrm{i}\,\theta}$



Algebraic Operations above the Complex

- addition/subtraction: $(a + ib) + (c + id) = (a + c) + i(b + d)$
- multiplication: $(a + ib)(c + id) = (ac - bd) + i(bc + ad)$

$$Ae^{i\alpha}Be^{i\,\beta} = ABe^{i(\alpha+\beta)}$$

- inner Product: $\langle\, (a + ib), (c + id)\, \rangle = \ (a + ib)^*(c + id)$

$$\langle Ae^{i\alpha}, Be^{i\beta} \rangle = \mathrm{Ae}^{-\mathrm{i}\alpha}\mathrm{Be}^{\mathrm{i}\beta} = \mathrm{ABe}^{\mathrm{i}(\beta-\alpha)}$$

- norm: $\|a + ib\|^2 = (\mathrm{a} + \mathrm{ib})^*(\mathrm{a} + \mathrm{ib}) = \mathrm{a}^2 + \mathrm{b}^2$

$$\left\|Re^{i\theta}\right\| = \left(Re^{i\theta}\right)^*\left(Re^{i\theta}\right) = Re^{-i\theta}Re^{i\theta} = R^2$$

The (Co-) Sinusoid as complex exponentoial

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- $\cos(x) = Real\left(e^{ix}\right)$
- $\sin(x) = Imag(e^{ix})$

Or

- $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$
- $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$

- We already saw that

$$S\sin(kx) + C\cos(kx) = R\sin(kx+\theta)$$

where $\sqrt{R = S^2 + C^2}$ and $\theta = \tan^{-1}(\frac{C}{S})$

- Scaling and phase shifting can be represented as a multiplication with $Z = Re^{i\theta}$

$$R\sin(kx+\theta) = imag\left(Re^{i\theta}\, e^{ikx}\right) = imag(Z\, e^{ikx})$$

Or equivalently

$$R\sin(kx+\theta) = \frac{1}{2i}\left(Re^{i\theta}e^{ikx} - Re^{-i\theta}e^{-ikx}\right) =$$
$$= \frac{1}{2i}\left(Ze^{ikx} - Z^{*}e^{-ikx}\right)$$

THE END