Week 10 Lecture Notes: Beam Deflections Using Double Integration

Mechanics of Materials

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Outline

Week 10 Overview: Beam Deflections

- Topic: Analyzing beam deflections with the **double integration method**.
- Goals:
  - Understand *elastic curve*, *slope*, and *displacement*.
  - Solve common beam problems step-by-step.
- Why it matters: Ensures safe design of structures like bridges and buildings.

Introduction

- Double integration: Key method to find how beams bend under loads.
- Outputs: Elastic curve (shape), slope (angle), displacement (movement).
- **Sign Convention**: Positive y is upward; downward deflections are negative.

Theoretical Background

- Euler-Bernoulli beam theory:

$$EI\frac{d^2y}{dx^2} = M(x)$$

- y: deflection, x: position, $M(x)$: bending moment.
- E: modulus of elasticity, I: moment of inertia (second moment of area).
- Double integration:
  1. $\frac{dy}{dx}$ (slope) from first integral.
  2. $y(x)$ (deflection) from second integral.

Steps of Double Integration

1. Define $M(x)$ (watch for piecewise regions!).
2. Integrate: $EI\frac{dy}{dx} = \int M(x)\,dx + C_1$.
3. Integrate again: $EIy = \iint M(x)\,dx + C_1x + C_2$.
4. Apply boundary conditions:
   - Pinned/Roller: $y = 0$, slope $\frac{dy}{dx} \neq 0$.
   - Fixed: $y = 0$, $\frac{dy}{dx} = 0$.
5. Constants C_1, C_2 depend on supports!

Example 1: Simply Supported Beam with Central Load

- Setup: Length L, load P at $x = L/2$.
- Piecewise Moment:

$$M(x) = \begin{cases} \frac{Px}{2}, & 0 \leq x \leq \frac{L}{2} \\ \frac{P(L-x)}{2}, & \frac{L}{2} < x \leq L \end{cases}$$

- Equation: $EI\frac{d^2y}{dx^2} = M(x)$.

Figure: Elastic curve (Source: thestructuralengineer.info)

Example 1: Solution

- For $0 \leq x \leq L/2$:
- First integration: $EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1$.
- Second integration: $EIy = \frac{Px^3}{12} + C_1x + C_2$.
- Boundaries: $y(0) = 0$ (pin), $y(L) = 0$ (roller).
- Solve: $C_2 = 0$, $C_1 = -\frac{PL^2}{16}$ (using symmetry or full span).
- Max deflection: $\delta_{\max} = -\frac{PL^3}{48EI}$ at $x = L/2$.

See video: youtube.com/watch?v=EWhL-mixfaI

Example 2: Cantilever Beam with End Load

- Setup: Length L, load P at free end ($x = L$).
- Moment: $M(x) = P(L - x)$ (single region).
- Equation: $EI\frac{d^2y}{dx^2} = P(L - x)$.

Figure: Cantilever deflection (Source: thestructuralengineer.info)

Example 2: Solution

- First integration: $EI\frac{dy}{dx} = P(Lx - \frac{x^2}{2}) + C_1$.
- Second integration: $EIy = P(\frac{Lx^2}{2} - \frac{x^3}{6}) + C_1x + C_2$.
- Boundaries: $y(0) = 0$, $\frac{dy}{dx}(0) = 0$ (fixed end).
- Solve: $C_1 = 0$, $C_2 = 0$.
- Deflection at end: $y(L) = -\frac{PL^3}{3EI}$ (downward).

See video: youtube.com/watch?v=MJxIjG-32JA

Example 3: Simply Supported Beam with Uniform Load

- Setup: Length L, uniform load w.
- Moment: $M(x) = \frac{wx}{2}(L - x)$ (single region).
- Equation: $EI\frac{d^2y}{dx^2} = \frac{wx}{2}(L - x)$.

Figure: Uniform load deflection (Source: thestructuralengineer.info)

Example 3: Solution

- First integration: $EI\frac{dy}{dx} = \frac{w}{2}(\frac{Lx^2}{2} - \frac{x^3}{3}) + C_1$.
- Second integration: $EIy = \frac{w}{2}(\frac{Lx^3}{6} - \frac{x^4}{12}) + C_1x + C_2$.
- Boundaries: $y(0) = 0$ (pin), $y(L) = 0$ (roller).
- Solve: $C_2 = 0$, $C_1 = -\frac{wL^3}{24}$.
- Max deflection: $\delta_{\max} = -\frac{5wL^4}{384EI}$ at $x = L/2$.

See video: youtube.com/watch?v=6I5pjdIAtIc

Practical Tips for Solving Problems

- Sketch beam, loads, supports—label everything!
- Check $M(x)$ with equilibrium (forces/moments balance).
- **Boundary Conditions**:
  - Pinned/Roller: $y = 0$, slope can vary.
  - Fixed: $y = 0$, $\frac{dy}{dx} = 0$.
- Watch piecewise $M(x)$—define regions clearly.
- Units: E (Pa), I (m^4), M (N·m).

Resources for Week 10

- Textbook: Beam deflection chapter.
- Figures: `https://www.thestructuralengineer.info/education/professional-examinations-preparation/calculation-examples/calculation-example-calculate-the-equation-of-the-elas`
- Videos:
  - Ex 1: youtube.com/watch?v=EWhL-mixfaI.
  - Ex 2: youtube.com/watch?v=MJxIjG-32JA.
  - Ex 3: youtube.com/watch?v=6l5pjdlAtlc.
- Office hours: Bring questions!

Conclusion

- Double integration: Step-by-step tool for beam deflections.
- Classic examples: Simply supported (point/uniform load), cantilever.
- Key for safe structural design.
- Next week: Macaulay's method for complex loads.