Mechanics of Materials - Week 12: Thin-Walled Pressure Vessels

Cylindrical & Spherical Vessels

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Outline

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What Are Pressure Vessels?

- Containers designed to hold gases or liquids under pressure.
- Examples include boilers, gas tanks, water pipes, and even aerosol cans.
- A **thin-walled vessel** assumption applies when the wall thickness t is relatively small compared to the inner radius r (rule of thumb: $r/t \geq 10$).

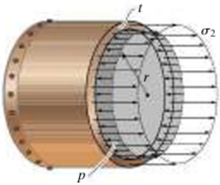

Figure: Schematic of a thin-walled cylindrical vessel

Why Thin-Walled Assumption?

- Simplifies stress analysis significantly.
- Enables closed-form solutions for hoop and longitudinal stresses.
- Common in practice for pipes, pressurized hoses, and large storage tanks.

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Stresses in Cylindrical Vessels

- Internal gauge pressure: p.
- Two primary normal stresses:
  - **Hoop (circumferential) stress**: σ_θ or σ_{hoop}.
  - **Longitudinal (axial) stress**: σ_{long}.

Key Equations (Thin-Walled)

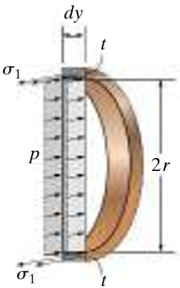
$$\sigma_{\text{hoop}} = \sigma_1 = \frac{pr}{t}, \qquad \sigma_{\text{long}} = \sigma_2 = \frac{pr}{2t}.$$

Derivation of Hoop Stress (Brief)

- Consider a half-cylinder cut by a plane parallel to the axis.
- Force due to internal pressure acts on the cross-sectional area (length $\approx 2r$).
- Hoop stress acts along the circumference on the "cut" edges.
- Equilibrium of horizontal forces leads to:

$$\sigma_{\text{hoop}} \cdot (t \times \text{length}) \;=\; p \cdot (\text{internal area}).$$

$$\sigma_{\text{hoop}} = \frac{pr}{t}.$$



Engineering Considerations

- **Factor of Safety (F.S.)**: Typically, use $\sigma_{\text{allowed}} = \frac{\sigma_{\text{yield}}}{\text{F.S.}}$.
- **End Caps**: Axial loads on heads can introduce additional stresses.
- **Material Selection**: Must withstand stresses and possible corrosion, temperature effects.
- **Welding/Joints**: Often the weak link in pressure vessels.

Full 2D Stress Tensor in Cylindrical Vessels

- Combines hoop and longitudinal stresses into a 2D stress state at a point.
- Assume: longitudinal direction (x), hoop direction (y), no shear ($\tau_{xy} = 0$).

Stress Tensor

$$\sigma = \begin{bmatrix} \sigma_{\text{long}} & 0 \\ 0 & \sigma_{\text{hoop}} \end{bmatrix} = \begin{bmatrix} \frac{pr}{2t} & 0 \\ 0 & \frac{pr}{t} \end{bmatrix}$$

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Stresses in Spherical Vessels

- Due to symmetry, stress is uniform in all directions in a sphere.
- Magnitude of spherical stress is identical to the *longitudinal* stress in a cylindrical vessel.

Key Equation (Thin-Walled Sphere)

$$\sigma_{\text{sphere}} = \frac{pr}{2t}.$$

Free-Body Diagram (Spherical)

- If you “cut” a sphere by a plane, the internal pressure acts over a circular cross section.
- The internal tensile stress in the spherical wall resists the force due to pressure.

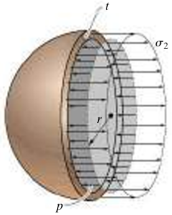

Figure: Spherical vessel free-body diagram

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Key Takeaways

- **Thin-Walled Cylinders:**

$$\sigma_{\text{hoop}} = \frac{pr}{t}, \quad \sigma_{\text{long}} = \frac{pr}{2t}.$$

- **Thin-Walled Spheres:**

$$\sigma_{\text{sphere}} = \frac{pr}{2t}.$$

- **When to apply**:
  - $r/t \gtrsim 10$ for the thin-walled assumption.
  - Uniform internal pressure.
- **Engineering Concern**: Always ensure proper safety factors for material strength, welds, and design codes (ASME, etc.).

Further Reading

- Mechanics of Materials texts (e.g., Gere, Beer & Johnston).
- ASME Boiler and Pressure Vessel Code (BPVC) for practical design standards.
- API standards for piping and petroleum-related vessels.