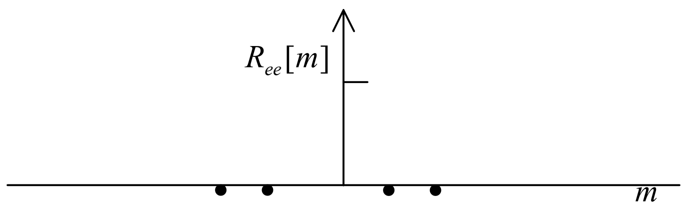
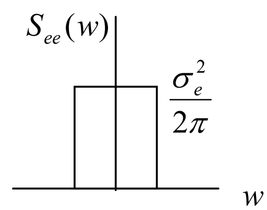
Random Signal Modelling

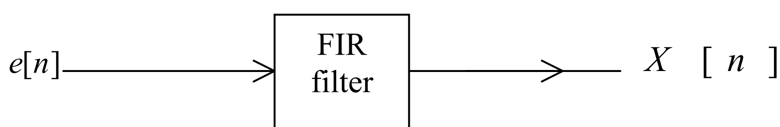
In statistics, random-process modeling using difference equations t is known as time series analysis.

White Noise Sequence: The simplest model is the white noise $e[n]$. We shall assume that $e[n]$ is of 0-mean and variance σ_e^2.





Moving Average model $MA(q)$ model.



The difference equation model is

$$X[n] = \sum_{i=o}^{q} b_i e[n-i]$$

$\mu_e = 0 \Rightarrow \mu_Y = 0$

and $e[n]$ is an uncorrelated sequence means

$$\sigma_Y{}^2 = \sum_{i=0}^{q} b_i{}^2$$

The autocorrelations are given by

$$R_{XX}[m] = E\,X[n]\,X[n\text{-}m]$$
$$= \sum_{i=0}^{q}\sum_{j=0}^{q} b_i b_j E e[n-i]e[n-m-j]$$
$$= \sum_{i=0}^{q}\sum_{j=0}^{q} b_i b_j R_{ee}[m-i+j]$$

Noting that $R_{ee}[m] = \sigma_e^2 \delta[m]$, we get

$R_{ee}[m] = \sigma_e^2$ when

$m\text{-}i + j = 0$

$\Rightarrow i = m + j$

The maximum value for $m + j$ is q so that

$$R_{XX}[m] = \sum_{j=0}^{q-m} b_j b_{j+m} \sigma_e^2 \qquad m \geq 0$$

and

$$R_{XX}[-m] = R_{XX}[m]$$

Notice that $R_{XX}[m]$ is related by a nonlinear relationship with model parameters. Thus finding the model parametes is not simple.

The power spectral density is given by

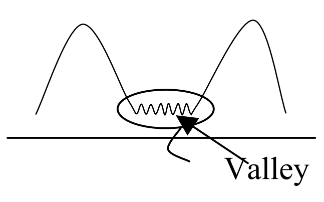
$$S_{XX}(w) = \frac{\sigma_e^2}{2\pi}\left|B(w)\right|^2$$

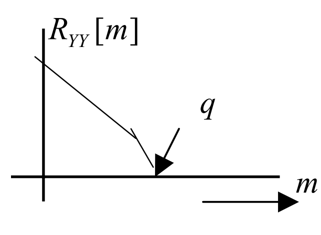
where

$$B(w) == b_o + b_1\, e^{-jw} + \ldots\ldots b_q e^{-jqw}$$

FIR system will give some zeros. So if the spectrum has some valleys then MA will fit well.

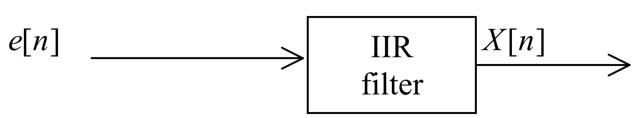
Test for MA process: $R_{YY}\,[m]$ becomes zero suddenly after some value of m.





Autoregressive Model

In time series analysis it is called AR(p) model.



The model is given by the difference equation

$$X[n] = -\sum_{i=1}^{p} a_i X[n-i] + e[n]$$

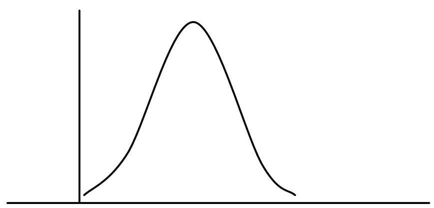
The transfer function $A(w)$ is given by

$$A(w) = \frac{1}{\sum_{i=0}^{n} a_i e^{-j\omega i}}$$

with $a_0 = 1$ (all poles model)
and

$$S_{yy}(\omega) = S_{XX}(w) = \frac{\sigma_e^{\ 2}}{2\pi\,|A(\omega)|^2}$$

If there are sharp peaks in the spectrum, the AR(p) model may be suitable.



The autocorrelation function $R_{XX}[m]$ is given by

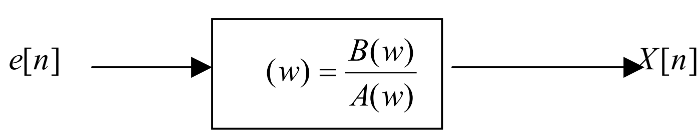
$$\begin{aligned} R_{XX}[m] &= E\,X[n]\,X[n\text{-}m] \\ &= -\sum_{i=1}^{p} a_i E X[n-i] X[n-m] + E e[n] X[n-m] \\ &= -\sum_{i=1}^{p} a_i R_{XX}[m-i] + \sigma_e^2 \delta[m] \end{aligned}$$

$$\therefore R_{XX}[m] = -\sum_{i=1}^{p} a_i R_{XX}[m-i] + \sigma_e^2 \delta[m] \qquad \forall\, m \in I$$

The above relation gives a set of linear equations which can be solved to find a_is.
This set of equations are known as Yule-Walker Equation.

<u>ARMA</u> – Authoregressive Moving Average Model.

Under the most practical situation, the process may be considered as an output of a filter that has both zeros and poles.



The model is given by

$$x[n] = -\sum_{i=1}^{p} a_i X[n-i] + \sum_{i=0}^{q} b_i v[n-i]$$

and is called the $ARMA(p,q)$ model.

The transfer function of the filter is given by

$$H(w) = \frac{B(\omega)}{A(\omega)}$$

$$S_{XX}(w) = \frac{|B(\omega)|^2 \sigma_e^2}{|A(\omega)|^2 2\pi}$$

How do get the model parameters?

For $m \geq \max(p,\ q+1)$, there will be no contributions from b_i terms to $R_{XX}[m]$.

$$R_{yy}[m] = R_{XX}[m] = \sum_{i=1}^{p} a_i R_{XX}[m-i] \qquad m \geq \max(p, q+1)$$

From a set of p Yule Walker equations, a_i parameters can be found out.

Then we can rewrite the equation

$$\widetilde{X}[n] = X[n] + \sum_{i=1}^{p} a_i X[n-i]$$

$$\therefore \widetilde{X}[n] = \sum_{i=0}^{q} b_i v[n-i]$$

From the above equation $b_i s$ can be found out.

The $ARMA(p,q)$ is an economical model. Only $AR(p)$ only $MA(q)$ model may require a large number of model parameters to represent the process adequately. This concept in model building is known as the parsimony of parameters.

<u>General Model building Steps</u>

- Identification of p and q.
- Estimation of model parameters.
- Check the modeling error.
- If it is white noise then stop.
  Else select new values for p and q
  and repeat the process.

<u>Other model</u>:

ARIMA model: Here after differncing the data can be fed to an ARMA model.

SARMA model: Seasonal ARMA model etc.