Linear Prediction of Signal

Given a sequence of observation

$Y[n\text{-}1],\ Y[n\text{-}2],\ \ldots.\ Y[n\text{-}M]$, what is the best prediction for $Y[n]$? ?

(one-step ahead prediction)

The minimum mean square error prediction $\hat{Y}[n]$ for $Y[n]$ is given by $\hat{Y}[n] = E\{Y[n] \mid Y[n\text{-}1], Y[n\text{-}2], \ldots, Y[n\text{-}M]\}$

Which is a nonlinear predictor.

A linear prediction is

$$\hat{Y}[n] = \sum_{i=-1}^{M} h[i]\, Y[n-i]$$

where $h[i], i = 1....M$ are the prediction parameters.

Very wide range of applications for this type prediction.

For an exact AR(M) process, linear prediction model of order M and the corresponding AR model have the same parameters. For other signals LP model gives an approximation.

Area of application :

- Speech modeling
- Low-bit rate speech coding
- Speech recognition
- ECG modeling
- DPCM coding
- Internet traffic prediction

LPC (10) is the popular linear prediction model used for speech coding. For a frame of speech samples, the prediction parameters are estimated and coded. In CELP (Code book Excited Linear Prediction) the prediction $e[n] = y[n] - \hat{Y}[n]$ is vector quantized and transmitted.

$\hat{Y}[n] = \sum_{i=1}^{M} h[i]\, Y[n\text{-}i]$ is a FIR Wiener filter shown in the following figure. It is called the linear prediction filter.

Therefore

$$\begin{aligned} e[n] &= Y[n] - \hat{Y}[n] \\ &= Y[n] - \sum_{i=1}^{M} h[i]\, Y[n\text{-}i] \end{aligned}$$

is the prediction error and the corresponding filter is called prediction error filter.

Linear Minimum Mean Square error estimates for the prediction parameters are given by the orthogonality relation

$$E\, e[n]\; Y[n\text{-}j] = 0 \quad for\; j = 1,\ 2,\ \ldots ,\ M$$

$$\therefore E(Y[n] - \sum_{\mathrm{i=1}}^{\mathrm{M}} h[i]\, Y[n\text{-}i])Y[n-j] = 0 \qquad j = 1,\ 2,\ \ldots,\ M$$

$$\Rightarrow R_{YY}\;[j]\text{-}\;\sum_{\mathrm{i=1}}^{\mathrm{M}} h[i]\, R_{YY}[j\text{-}i] \;= 0$$

$$\Rightarrow\; R_{YY}\;[j] = \;\sum_{\mathrm{i=1}}^{\mathrm{M}} h[i]\, R_{YY}[j\text{-}i] \quad j = 1,\ 2,\ \ldots,\ M$$

which is the Wiener Hopf equation for the linear prediction problem and same as the Yule Walker equation for AR(M) Process.

In Matrix notation

$$\begin{bmatrix} \mathrm{R_{YY}[0]\;\; R_{YY}[1]\; .... R_{YY}\,[M-1]} \\ \mathrm{R_{YY}[1]\;\; R_{YY}[o]\; ... R_{YY}[M-2]} \\ . \\ . \\ . \\ \mathrm{R_{YY}[M-1]\;\; R_{YY}[M-2]......\qquad R_{YY}[0]} \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ . \\ . \\ . \\ h[M] \end{bmatrix} = \begin{bmatrix} \mathrm{R_{YY}\,[1]} \\ \mathrm{R_{YY}[2]} \\ . \\ . \\ . \\ \mathrm{R_{YY}\,[M]} \end{bmatrix}$$

$$\mathbf{R_{YY}h} = \mathbf{r_{YY}}$$

$$\therefore \mathbf{h} = (\mathbf{R_{YY}})^{-1}\mathbf{r_{YY}}$$

Mean Square Prediction Error (MSPE).

$$E(e^2[n] = E(Y[n] - \sum_{i=1}^{M} h[i]y[n-i])e[n]$$

$$= EY[n]e[n]$$

$$= Ey[n](Y[n] - \sum_{i=1}^{M} h[i]Y[n-i])$$

$$= R_{YY}[0] - \sum_{i=1}^{M} h[i]R_{YY}[i])$$

Forward Prediction Problem The above linear prediction problem is the forward prediction problem. For notational simplicity let us rewrite the prediction equation as

$$\hat{Y}[n] = \sum_{i=1}^{M} h_M[i]\, Y[n\text{-}i]$$

where the prediction parameters are being denoted by $h_M[i], i = 1...M$.

Backward Prediction Problem :

Given $Y[n],\ Y[n\text{-}1],\ \ldots.\ Y[n\text{-}M+1]$, we want to estimate $Y[n-M]$.

The linear Prediction is given by

$$\hat{Y}[n\text{-}M] = \sum_{i=1}^{M} b_M\ [i]\, Y[n+1-i]]$$

Applying orthogonality principle.

$$\mathrm{E}(Y[n\text{-}M]\text{-}\sum_{i=1}^{M} b_M\ [i] Y[n+1-i])\ \ Y[n+1-j] \quad = 0 \quad j = 1,2...,M.$$

This will give

$$R_{YY}\ [M+1-j] = \ \sum_{i=1}^{M} b_M[i]\, R_{YY}[j\text{-}i] \quad j = 1,\ 2,\ \ldots,\ M$$

Corresponding matrix form

$$\begin{bmatrix} \mathrm{R_{YY}}[0]\ \ \mathrm{R_{YY}}[1]\ \ldots.\ \mathrm{R_{YY}}\ [\mathrm{M}-1] & \\ \mathrm{R_{YY}}[1]\ \ \mathrm{R_{YY}}[\mathrm{o}]\ \ldots\ \mathrm{R_{YY}}[\mathrm{M}-2] & \\ . & \\ . & \\ . & \\ \mathrm{R_{YY}}[\mathrm{M}-1]\ \ \mathrm{R_{YY}}[\mathrm{M}-2]\ldots\ldots & \mathrm{R_{YY}}[0] \end{bmatrix} \begin{bmatrix} b_M[1] \\ b_M[2] \\ . \\ . \\ . \\ b_M[M] \end{bmatrix} = \begin{bmatrix} \mathrm{R_{YY}}\ [M] \\ \mathrm{R_{YY}}[M-1] \\ . \\ . \\ . \\ \mathrm{R_{YY}}\ [1] \end{bmatrix} \qquad (1)$$

Forward Prediction

Rewriting the Mth-order forward prediction problem, we have

$$\begin{bmatrix} R_{YY}[0] & R_{YY}[1] \; .... & R_{YY}[M-1] \\ R_{YY}[1] & R_{YY}[o] \; ... & R_{YY}[M-2] \\ . & & \\ . & & \\ . & & \\ R_{YY}[M-1] & R_{YY}[M-2] ...... & R_{YY}[0] \end{bmatrix} \begin{bmatrix} h_M[1] \\ h_M[2] \\ . \\ . \\ . \\ h_M[M] \end{bmatrix} = \begin{bmatrix} R_{YY}[1] \\ R_{YY}[2] \\ . \\ . \\ . \\ R_{YY}[M] \end{bmatrix} \qquad (2)$$

From (1) and (2) we conclude
$b_M[i] = h_M[M+1-i], i = 1,2..., M$
Thus forward prediction parameters in reverse order will give the backward prediction parameters.
M S prediction error

$$\varepsilon_M = E\left(Y[n-M] \; - \sum_{i=1}^{M} b_M \; [i] \; y\,[n+1-i] \right) \; y\,[n-M]$$

$$= R_{YY}[0] - \sum_{i=1}^{M} b_M \; [i] \; R_{YY} \; [M+1-i]$$

$$= R_{YY}[0] - \sum_{i=1}^{M} h_M \; [M+1-i] \; R_{YY} \; [M+1-i]$$

which is same as the forward prediction error.
Thus
Backward prediction error = Forward Prediction error.

Example Find the second order predictor for $y[n]$ given $Y[n] = X[n] + V[n]$, where $v[n]$ is a 0-mean white noise with variance 1 and uncorrelated with $x[n]$ and $X[n] = 0.6X[n\text{-}1] + W[n]$, $W[n]$ is a 0-mean random variable with variance 0.82

The linear predictor is given by

$$\hat{Y}[n] = h_2\,[1] \; Y[n-1] \; + h_2\,[2] \; Y[n-2]$$

We have to find $h_2\,[1]$ and $h_2\,[2]$.
Corresponding Yule Walker equations are

$$\begin{bmatrix} R_{YY}[0] & R_{YY}[1] \\ R_{YY}[1] & R_{YY}[0] \end{bmatrix} \begin{bmatrix} h_2[1] \\ h_2[2] \end{bmatrix} = \begin{bmatrix} R_{YY}\,\mathbf{[1]} \\ R_{YY}\,\mathbf{[2]} \end{bmatrix}$$

To find out $R_{YY}[0]$, $R_{YY}[1]$ and $R_{YY}[2]$
$Y[n] = X[n] + V[n]$,
$R_{YY}[m] = R_{XX}[m] + \delta[m]$

$$X[n] = 0.6X[n\text{-}1] + W[n]$$
$$\therefore R_{XX}[m] = \frac{0.82}{1-(0.6)^2}(0.6)^{|m|} = 1.28\times(0.6)^{|m|}$$
$$R_{YY}[0] = 2.28, R_{YY}[1] = 0.77 \text{ and } R_{YY}[2] = 0.462$$

Solving $h_2[1] = 0.3$ and $\mathrm{h}_2[2] = 0.1$.

<u>Levinson Durbin Algorithm</u>

Yule Walker equation for m th order predictor.

$$\begin{bmatrix} R_{YY}[0] \;\; R_{YY}[1] \;\; \ldots \; R_{YY}[m-1] \\ R_{YY}[1] \;\; R_{YY}[o] \;\; \ldots \; R_{yy}[m\text{-}2] \\ . \\ . \\ . \\ R_{YY}[m\text{-}1] \;\; \ldots\ldots \;\; R_{YY}[0] \end{bmatrix} \begin{bmatrix} h_{\mathrm{m}}[1] \\ h_{\mathrm{m}}[2] \\ . \\ . \\ . \\ h_{\mathrm{m}}[m] \end{bmatrix} = \begin{bmatrix} R_{YY}\;[1] \\ . \\ . \\ . \\ . \\ R_{YY}\;[\mathrm{m}] \end{bmatrix} \qquad (1)$$

Writing in the reverse order

$$\begin{bmatrix} R_{YY}[0] \;\; R_{YY}[1] \;\; \ldots \; R_{YY}[m-1] \\ R_{YY}[1] \;\; R_{YY}[o] \;\; \ldots \; R_{yy}[m\text{-}2] \\ . \\ . \\ . \\ R_{YY}[m\text{-}1] \;\; \ldots\ldots \;\; R_{YY}[0] \end{bmatrix} \begin{bmatrix} h_{\mathrm{m}}[m] \\ h_{\mathrm{m}}[m-1] \\ . \\ . \\ . \\ h_{\mathrm{m}}[1] \end{bmatrix} = \begin{bmatrix} R_{YY}\;[\mathrm{m}] \\ . \\ . \\ . \\ . \\ R_{YY}\;[1] \end{bmatrix} \qquad (2)$$

Then (m+1) the order predictor is given by.

$$\left[\begin{array}{cccc:c} R_{YY}[0] & R_{YY}[1] & \cdots & R_{YY}[m-1] & R_{YY}[m] \\ R_{YY}[1] & R_{YY}[0] & \cdots & R_{YY}[m-2] & R_{YY}[m-1] \\ . & & & & \\ . & & & & \\ . & & & & \\ R_{YY}[m-1] & R_{YY}[m-2] & \cdots & R_{YY}[0] & R_{YY}[1] \\ \hdashline R_{YY}[m] & R_{YY}[m-1] & \cdots & R_{YY}[1] & R_{YY}[0] \end{array}\right] \left[\begin{array}{c} h_{\mathrm{m+1}}[1] \\ h_{\mathrm{m+1}}[2] \\ . \\ . \\ . \\ h_{\mathrm{m+1}}[m] \\ \hdashline h_{\mathrm{m+1}}[m+1] \end{array}\right] = \left[\begin{array}{c} R_{YY}[1] \\ R_{YY}[2] \\ . \\ . \\ . \\ R_{\mathrm{YY}}[\mathrm{m}] \\ \hdashline R_{\mathrm{YY}}[\mathrm{m}+11] \end{array}\right]$$

Let us partition equation 3 as shown. Then

$$\begin{bmatrix} R_{YY}[0] & R_{YY}[1] & \cdots & R_{YY}[m-1] \\ R_{YY}[1] & R_{YY}[0] & \cdots & R_{YY}[m-2] \\ . & & & \\ . & & & \\ R_{YY}[m-1] & R_{YY}[m-2] & \cdots & R_{YY}[0] \end{bmatrix} \begin{bmatrix} h_{\mathrm{m+1}}[1] \\ h_{\mathrm{m+1}}[2] \\ . \\ . \\ . \\ h_{m+1}[m] \end{bmatrix} + \mathrm{h}_{\mathrm{m+1}}[\mathrm{m}+1] \begin{bmatrix} R_{YY}[\mathrm{m}] \\ R_{YY}[\mathrm{m-1}] \\ . \\ . \\ . \\ R_{YY}[1] \end{bmatrix} = \begin{bmatrix} R_{YY}[1] \\ R_{YY}[2] \\ . \\ . \\ . \\ R_{YY}[m] \end{bmatrix} \quad (3)$$

and

$$\sum_{i=1}^{m} h_{m+1}[i]\, R_{YY}[m+1-i] + h_{m+1}[m1] R_{YY}[0] = R_{YY}[m+1] \quad (4)$$

From equation (3) premultiplying by $\mathbf{R}_{\mathbf{YY}}^{-1}$, we get

$$\begin{bmatrix} h_{\mathrm{m+1}}[1] \\ h_{\mathrm{m+1}}[2] \\ . \\ . \\ . \\ h_{m+1}[m] \end{bmatrix} + \mathrm{h}_{\mathrm{m+1}}[\mathrm{m}+1]\mathbf{R}_{\mathbf{YY}}^{-1} \begin{bmatrix} R_{YY}[\mathrm{m}] \\ R_{YY}[\mathrm{m-1}] \\ . \\ . \\ . \\ R_{YY}[1] \end{bmatrix} = \mathbf{R}_{\mathbf{YY}}^{-1} \begin{bmatrix} R_{YY}[1] \\ R_{YY}[2] \\ . \\ . \\ . \\ R_{YY}[m] \end{bmatrix}$$

$$
\begin{bmatrix} h_{m+1}[1] \\ h_{m+1}[2] \\ . \\ . \\ . \\ h_{m+1}[m] \end{bmatrix} + h_{m+1}[m+1] \begin{bmatrix} h_m[m] \\ h_m[m-1] \\ . \\ . \\ . \\ h_m[1] \end{bmatrix}^{=} = \begin{bmatrix} h_m[1]) \\ h_m[2] \\ . \\ . \\ . \\ h_m[m] \end{bmatrix}
$$

The equations can be rewritten as

$$h_{m+1}[i] = h_m[i] + k_{m+1} h_m[m+1-i] \quad i = 1,2,...m \tag{5}$$

where $k_m = -h_m[m]$ is called the reflection coefficient or the PARCOR (partial correlation) coefficient.

From equation (4) we get

$$\sum_{i=1}^{m} h_{m+1}[i]\, R_{YY}[m+1-i] + h_{m+1}[m] R_{YY}[0] = R_{YY}[m+1]$$

using equation (5)

$$\sum_{i=1}^{m} \{h_m[i] + k_{m+1} h_m[m+1-i]\} R_{YY}[m+1-i] - k_{m+1} R_{YY}[0] = R_{YY}[m+1]$$

$$\sum_{i=1}^{m} h_m[i] R_{YY}[m+1-i] - k_{m+1} R_{YY}[0] + k_{m+1} \sum_{i=1}^{m} h_m[m+1-i]\, R_{YY}[m+1-i] = R_{YY}[m+1]$$

$$k_{m+1}\{R_{YY}[0] - \sum_{i=1}^{m} h_m[m+1-i]\, R_{YY}[m+1-i]\} = -R_{YY}[m+1] + \sum_{i=1}^{m} h_m[i] R_{YY}[m+1-i]$$

$$k_{m+1} = \frac{-R_{YY}[m+1] + \sum_{i=1}^{m} h_m[i] R_{YY}[m+1-i]}{\varepsilon[m]}$$

$$= \frac{\sum_{i=0}^{m} h_m[i] R_{YY}[m+1-i]}{\varepsilon[m]}$$

where

$\varepsilon[m] = R_{YY}[0] - \sum_{i=1}^{m} h_m[m+1-i]\, R_{YY}[m+1-i]$ is the mean - square prediction error.

$$\therefore \varepsilon[m+1] = R_{YY}[0] - \sum_{i=1}^{m+1} h_{m+1}[m+2-i]\, R_{YY}[m+2-i]$$

Using the recursion for $h_{m+1}[i]$

We get

$$\varepsilon[m+1] = \varepsilon[m]\left[1 - k_{m+1}{}^2\right]$$

will give MSE recursively. Since MSE is non negative

$$k_m^2 \le 1$$

$$\therefore |k_m| \le 1$$

Hardware representation is easier in terms of k_m.

Steps of the algorithm (Levinson, Durbin)
Given $R_{YY}[m], m = 0, 1, 2, \ldots.$

Initialization
Take $h_m[0] = -1$ for all m
For $m = 0$,
$\varepsilon[0] = R_{YY}[0]$

For $m = 1,2,3...$

$$k_m = \frac{\sum_{i=0}^{m-1} h_m[i] R_{YY}[m-i]}{\varepsilon[m-1]}$$

$$h_m[i] = h_{m-1}[i] + k_m h_{m-1}[m-i], \mathrm{i = 1,2..., m - 1}$$

$$h_m[m] = -k_m$$

$$\varepsilon_m = \varepsilon_{m-1}(1 - k_m^2)$$

Go on computing upto given final value of m.