Random Process (Continued)

Consider two real random processes $X(t)$ and $Y(t)$.

Joint stationarity of $X(t)$ and $Y(t)$ => Joint densities are invariant with shift of time.

Cross-correlation function for a jointly wss processes $X(t)$ and $Y(t)$ is defined as

$$R_{X,Y}(\tau) = E\,X(t+\tau)Y(t)$$

so that $R_{YX}(\tau) = E\,Y(t+\tau)X(t)$

$$= E\,X(t)Y(t+\tau)$$

$$= R_{X,Y}(-\tau)$$

$$\therefore \quad R_{YX}(\tau) = R_{X,Y}(-\tau)$$

Cross power spectral density

$$S_{X,Y}(w) = \int_{-\alpha}^{\alpha} R_{X,Y}(\tau)e^{-jw\tau}d\tau$$

For real processes $X(t)$ and $Y(t)$

$$S_{X,Y}(w) = S^*_{Y,X}(w)$$

If we sample a stationary random process uniformly we get a stationary random sequence.
Sampling theorem is valid in terms of PSD.
If we define

$$R_X[m] = E\,X[n+m]\,X[n]$$

Then corresponding PSD is given by

$$S_X(w) = \sum_{m=-\infty}^{\infty} R_x[m]e^{-j\omega m}$$

$$\therefore R_X[m] = \frac{1}{2\pi}\int_{2\pi} S_x(\omega)e^{j\omega m}d\omega \qquad -\pi \le \omega \le \pi$$

For a discrete sequence the generalized PSD is defined in the $z-$domain as follows

$$S_X(z) = \sum_{m=-\alpha}^{\alpha} R_x[m]z^{-m}$$

<u>Gaussian Random Process</u>

Let $X(t)$ be a random process.

Given any n we can have random vector given by

$$\mathbf{X} = \begin{bmatrix} \mathbf{X(t_1)} \\ \mathbf{X(t_2)} \\ . \\ . \\ \mathbf{X(t_n)} \end{bmatrix}$$

with $$\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} \mu_X(\mathbf{t_1}) \\ \mu_{\mathbf{X}}(\mathbf{t_2}) \\ . \\ . \\ \mu_{\mathbf{X}}(\mathbf{t_n}) \end{bmatrix}$$

and a positive definite $n \times n$ matrix $\mathbf{C_X}$ given by

$$\mathbf{C_X} = \mathbf{E(X - \boldsymbol{\mu}_X)(X - \boldsymbol{\mu}_X)'}$$

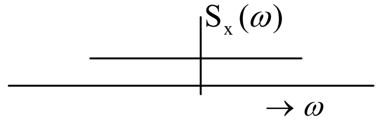
$$f_{X(1),X(2).....X(n)}(x_1, x_2.....x_n) = \frac{1}{\sqrt{(2\pi)^n}\sqrt{\det(\mathbf{C_x})}} e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_\mathbf{X})\mathbf{C}_\mathbf{X}^{-1}(\mathbf{X}-\boldsymbol{\mu}_\mathbf{X})'}$$

If $X(t)$ is an uncorrelated Gaussian process, then $\mathbf{C_X}$ is a digonal matrix and

$$f_{X(1),X(2).....X(n)}(x, x_2.....x_n) = \prod_{i=1}^{n} f_{X(i)}(x_i)$$

So in case Gaussian uncorrelatedness implies independence.

White noise process



A white noise process $X(t)$ is defined by

$$S_X(w) = \alpha^2 \qquad\qquad -\infty < w < \infty$$

The corresponding autocorrelation function is given by

$$R_X(\tau) = \alpha^2 \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta.

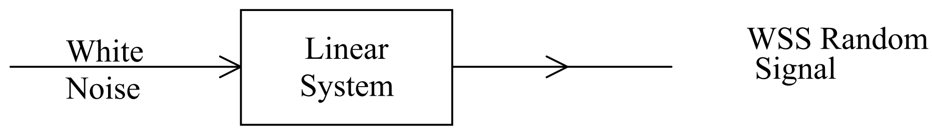
The average power of white noise

$$P_{avg} = \int_{-\infty}^{\infty} \alpha^2 d\omega \rightarrow \infty$$

- White noise is an mathematical abstraction, it cannot be realised.

If the system band-width(BW) is sufficiently narrower than the noise BW and noise PSD is flat , we can model it as a white noise process.

- For a zero-mean white noise process, the correlation of the process at any lag $\tau \neq 0$ is zero.
- White noise plays a key role in random signal modelling.



White Noise Sequence

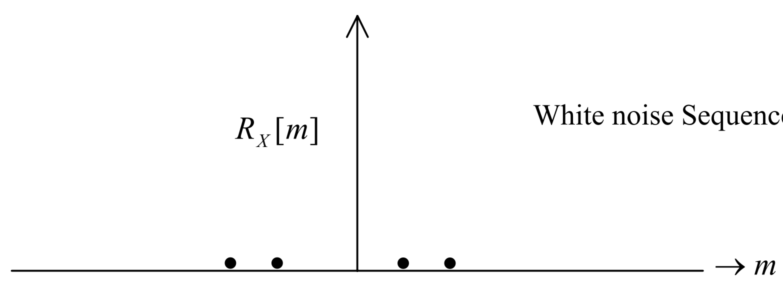
For a white noise sequence $x[n]$,

$$S_X(w) = \alpha^2 \qquad\qquad -\pi < w < \pi$$

Therefore

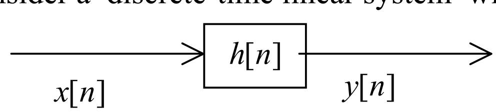
$$R_X(m) = \alpha^2 \delta(m)$$

where $\delta(m)$ is the unit impulse sequence.



Linear Shift Invariant System with Random Inputs

Consider a discrete-time linear system with impulse response $h[n]$.



$$y[n] = x[n] * h[n]$$

$$E\, y[n] = E\, x[n] * h[n]$$

For stationary input $x[n]$

$$\mu_Y = E\, y[n] = \mu_X * h[n] = \mu_x * h(n)$$

$$
\begin{aligned}
R_Y[m] &= E\, y[n]y[n-m] \\
&= E(x[n] * h[n]) * (x[n-m] * h[n-m]) \\
&= R_X[m] * h[m] * h[-m]
\end{aligned}
$$

$R_Y[m]$ is a function of lag m only.

From above

$$S_Y(w) = |H(w)|^2 \, S_X(w)$$

$$S_{XX}(w) \longrightarrow \boxed{|H(w)|^2} \xrightarrow{S_{YY}(w)}$$

Example:

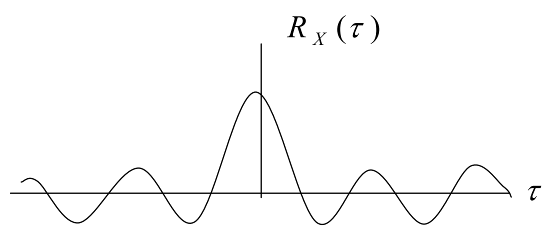
Suppose

$$
\begin{aligned}
H(w) &= 1 \qquad -w_c \le w \le w_c \\
&= 0 \quad \text{otherwise}
\end{aligned}
$$

$$\mathrm{S_X}(w) = \eta^2 \qquad -\infty \le w \le \infty$$

Then $\quad \mathrm{S_Y}(w) = \eta^2 \quad -w_c \le w \le w_c$

and $\quad \mathrm{R_Y}(\tau) = \eta^2 \ \mathrm{sinc}(\mathrm{w_c}\tau)$



Note that though the input is an uncorrelated process, the output is a correlated process.

Consider the case of the discrete-time system with a random sequence $x[n]$ as an input.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$R_Y[m] = R_X[m] * h[m] * h[-m]$$

Taking the $z-$transform, we get

$$S_Y(z) = S_X(z)H(z)H(z^{-1})$$

Notice that if $H(z)$ is causal, then $H(z^{-1})$ is anti causal.

Similarly if $H(z)$ is minimum-phase then $H(z^{-1})$ is maximum-phase.

$$R_{XX}[m] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{H(z^{-1})} \xrightarrow[S_{YY}[z]]{R_{YY}[m]}$$

Example: If $H(z) = \dfrac{1}{1-\alpha z^{-1}}$ and $x[n]$ is a unity-variance white-noise sequence, then

$$S_{YY}(z) = H(z)H(z^{-1})$$

$$= \left(\frac{1}{1-\alpha z^{-1}}\right)\left(\frac{1}{1-\alpha z}\right)\frac{1}{2\pi}$$

By partial fraction expansion and inverse $z-$transform, we get

$$R_Y[m] = \frac{1}{1-\alpha^2} a^{|m|}$$

<u>Spectral factorization theorem</u>

A stationary random signal $X[n]$ that satisfies the Paley Wiener condition can be considered as an output of a linear filter fed by a white noise sequence.

$S_X(w)$ is an analytic function of w.

and $\int_{-\pi}^{\pi} |\ln S_X(\omega)|\, d\omega < \infty$, then

$$S_X(z) = \sigma_e^2\, H_c(z)H_a(z)$$

where

$H_c(z)$ is the causal minimum phase transfer function

$H_a(z)$ is the anti-causal maximum phase transfer function

and σ_e^2 a constant and interpreted as the variance of a white-noise sequence.

Innovation sequence

$$e[n] \longrightarrow \boxed{H_c(z)} \longrightarrow X[n]$$

Figure Innovation Filter

Minimum phase filter => its inverse exist.

$$X[n] \longrightarrow \boxed{\dfrac{\quad}{H_c(z)}} \longrightarrow e[n]$$

Figure whitening filter

Since $\ln S_{XX}(z)$ is analytic,

$$\text{In } S_{zz}(z) = \ln S_{XX}(z) = \sum_{k=-\infty}^{\infty} c(k) z^{-k}$$

where $c(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{XX}(w) e^{iwn} dw$

For a real signal $c(k) = c(-k)$

and $c(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{XX}(w) dw$

$$S_{XX}(z) = e^{\sum_{k=-\infty}^{\infty} c(k) z^{-k}}$$

$$= e^{c(0)} e^{\sum_{k=1}^{\infty} c(k) z^{-k}} e^{\sum_{k=-\infty}^{-1} c(k) z^{-k}}$$

Let $Q(z) = e^{\sum_{k=1}^{\alpha} c(k) z^{-k}}$ $\qquad |z| > \rho$

$$= 1 + \mathrm{q}(1)\mathrm{z}^{-1} + q(2) z^{-2} + \ldots\ldots$$

$(\because q(0) = Lim_{z\to\infty} Q(z) = 1$

$Q(z)$ and $\ln Q(z)$ are both analytic

=> $Q(z)$ is a minimum phase filter.

Similarly let

$$Q_1(z) = e^{\sum_{k=-\infty}^{-1} c(k) z^{-k}}$$

$$= e^{\sum_{k=1}^{\infty} c(k) z^{k}} = Q(z^{-1}) \qquad |z| < \frac{1}{\rho}$$

Therefore

$S_{XX}(z) = kQ(z)Q(z^{-1})$

where $k = e^{c(0)}$

<u>Sailent points</u>

- $S_{XX}(z)$ can be factorized into a minimum phase and a maximum phase factors i.e. and $Q(z^{-1})$.
- Since is a minimum phase filter, $\frac{1}{Q(z)}$ exists (=> stable), therefore we can have a filter $\frac{1}{Q(z)}$ to filter the given signal to get the inovation sequence.

- $X[n]$ and $e[n]$ are related through an invertible transform; so they contain the same information.