A Motivic Snaith Decomposition

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Contents

1. Introduction

Theorem 1.1. *Over a field k, there is a $\mathbb{P}^1$–stable splitting* $\mathrm{BGL}_{m,+} \simeq \bigvee_{i=1}^{m} \mathrm{BGL}_i/\mathrm{BGL}_{i-1}$.

Notation

In what follows S will be an arbitrary base scheme. For a scheme X we write $\mathcal{H}(X)$ for the ∞–category of presheaves of spaces on Sm_X localized at Nisnevich-local equivalences and projections $Y \times \mathbb{A}^1 \longrightarrow Y$. It is a presentable ∞–category in the sense of [Lur09]. We refer to $\mathcal{H}(X)$ interchangably as the $\mathbb{A}^1$–homotopy category of X or the motivic homotopy category of X. The associated pointed ∞–category will be denoted by $\mathcal{H}_\bullet(X)$. Inverting $(\mathbb{P}^1, \infty) \in \mathcal{H}_\bullet(X)$ with respect to the smash product yields the stable motivic homotopy category $\mathcal{SH}(X)$ of X. It is a symmetric monoidal, presentable, stable ∞–category in the sense of [Lur12]. An account of this definition of $\mathcal{H}(X)$, $\mathcal{H}_\bullet(X)$ and $\mathcal{SH}(X)$ for noetherian schemes and its equivalence to the approach of [MV99] is given in [Rob15], the generalization to arbitrary schemes can be found in [Hoy14, Appendix C].

We follow [Lev18] in writing $X/S \in \mathcal{SH}(S)$ for the $\mathbb{P}^1$–suspension spectrum of a smooth scheme X over S. We will write $X_+ \in \mathcal{H}_\bullet(S)$ for X with a disjoint basepoint added. We sometimes do not distinguish notationally between the pointed motivic space $X_+ \in \mathcal{H}_\bullet(S)$ and its $\mathbb{P}^1$–suspension spectrum $X/S = X_+ \in \mathcal{SH}(S)$.

When dealing with ind-schemes we have elected not to speak of "ind-smooth" schemes and morphisms. Instead, for us a smooth morphisms between ind-schemes will be what is usually called an ind-smooth morphism, namely a formal colimit of smooth morphisms.

2. Becker–Gottlieb Transfers in Motivic Homotopy Theory

Becker and Gottlieb introduced their eponymous transfer maps in [BG75] as a tool for giving a simple proof of the Adams conjecture. They considered a compact Lie group G and a fiber bundle $E \longrightarrow B$ over a finite CW complex with structure group G and whose fiber F is a closed smooth manifold with a smooth action by G. There is a smooth G-equivariant embedding $F \subset V$ of F into a finite dimensional representation V of G. There is an associated Pontryagin–Thom collapse map $S^V \longrightarrow F^\nu$ where ν is the normal bundle of F in V and F^ν is its Thom space. Denoting by τ the tangent bundle of F one obtains a morphism

$$S^V \longrightarrow F^\nu \longrightarrow F^{\tau\oplus\nu} \simeq F_+ \wedge S^V$$

in G-equivariant homotopy theory. Assuming that $E \longrightarrow B$ is associated to a principal G-bundle $\widetilde{E} \longrightarrow B$ one gets a map

$$\widetilde{E} \times S^V \longrightarrow \widetilde{E} \times (F_+ \wedge S^V)$$

and passing to homotopy orbits with respect to the diagonal G-actions yields the transfer map $B_+ \longrightarrow E_+$ in the stable homotopy category.

This construction of the transfer was generalized in [DP80]. The map $S^V \longrightarrow F_+ \wedge S^V$ arises from a *duality datum* in parameterized stable homotopy theory over the base space B.

Definition 2.1. A *duality datum* in a symmetric monoidal category consists of a pair of objects X and $X^\vee$ with morphisms $\mathbf{1} \xrightarrow{\text{coev}} X \otimes X^\vee$ and $X^\vee \otimes X \xrightarrow{\text{ev}} \mathbf{1}$ such that the compositions

$$X \xrightarrow{\text{coev}\otimes\text{id}} X \otimes X^\vee \otimes X \xrightarrow{\text{id}\otimes\text{ev}} X$$

and

$$X^\vee \xrightarrow{\text{id}\otimes\text{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev}\otimes\text{id}} X^\vee$$

are identities. In this situation $X^\vee$ is said to be a *right dual* of X and X is said to be a *left dual* of $X^\vee$. If X is additionally a right dual of $X^\vee$, then X is said to be *strongly dualizable* with dual $X^\vee$.

A duality datum in a symmetric monoidal ∞–category $\mathscr{C}$ is a duality datum in the homotopy category $\mathrm{h}\mathscr{C}$, see [Lur12, section 4.6.1].

Remark 2.2. In [Lev18], Levine defines a dual $X^\vee = \mathrm{Map}(X, \mathbf{1})$ for any object X in a *closed* symmetric monoidal category. Then X is called strongly dualizable whenever the induced morphism $X^\vee \otimes X \longrightarrow \mathrm{Map}(X, X)$ is an equivalence. By [Lur12, Lemma 4.6.1.6] this coincides with our definition.

Dold and Puppe show that, for a fiber bundle $E \longrightarrow B$ with fiber a compact smooth manifold, there is a duality datum in the homotopy category of B–parameterized spectra. It exhibits the fiberwise Thom spectrum of the fiberwise stable normal bundle to E as a dual of the suspension spectrum of E. They then show that the transfer in [BG75] is an instance of the following general construction.

Definition 2.3. In a symmetric monoidal ∞–category $\mathscr{C}$, suppose that an object X is equipped with a map $\Delta\colon X \longrightarrow X \otimes C$ for some other object C. Furthermore, suppose that X is strongly dualizable. The *transfer of X with respect to Δ* is defined as the composition

$$\mathrm{tr}_{X,\Delta}\colon \mathbf{1} \xrightarrow{\mathrm{coev}} X \otimes X^{\vee} \xrightarrow{\mathrm{switch}} X^{\vee} \otimes X \xrightarrow{\mathrm{id}\otimes\Delta} X^{\vee} \otimes X \otimes C \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} \mathbf{1} \otimes C \simeq C.$$

If there can be no risk of confusion we write $\mathrm{tr}_X = \mathrm{tr}_{X,\Delta}$.

In Appendix A we construct a symmetric monoidal ∞–category $\mathcal{SH}(B)$ for every smooth ind-scheme B over a base scheme S. This enables us to extend the definition of the motivic Becker–Gottlieb transfer in [Lev18].

Definition 2.4. For a smooth map $f\colon E \longrightarrow B$ between smooth ind-schemes over S with $E/B \in \mathcal{SH}(B)$ strongly dualizable we define the *relative transfer* $\mathrm{Tr}(f/B)\colon \mathbf{1}_B \longrightarrow E/B$ as follows: Applying $f_\#$ to the diagonal $E \longrightarrow E \times_B E$ gives a morphism $\Delta\colon E/B \longrightarrow E/B \wedge E/B$ in $\mathcal{SH}(B)$ and we set $\mathrm{Tr}(E/B) = \mathrm{Tr}(f/B) = \mathrm{tr}_{E/B,\Delta}$.

Additionally, since $\pi\colon B \longrightarrow S$ is a smooth ind-scheme, we can define the *absolute transfer* of f as

$$\mathrm{Tr}(f/S) = \pi_\#(\mathrm{Tr}(f/B))\colon E/S \longrightarrow B/S.$$

Proposition 2.5. *The motivic Becker–Gottlieb transfer enjoys the following properties.*

(i) *The transfer is additive in homotopy pushouts: Suppose X, Y, U and V are smooth ind-schemes over a smooth ind-scheme B over S. Further suppose that there is a homotopy cocartesian square*

$$\begin{array}{ccc} X/B & \longrightarrow & U/B \\ \downarrow & & \downarrow \\ V/B & \longrightarrow & Y/B \end{array}$$

in $\mathcal{SH}(B)$. Assume that Y/B, U/B and V/B are strongly dualizable. Then $\mathrm{Tr}(Y/B)$ is a sum of the compositions

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(U/B)} U/B \longrightarrow Y/B$$

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(V/B)} V/B \longrightarrow Y/B$$

and

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(X/B)} X/B \longrightarrow Y/B$$

in $\mathcal{SH}(B)$.

(ii) *The relative transfer is compatible with pullback: If* $p\colon B' \longrightarrow B$ *and* $f\colon E \longrightarrow B$ *are maps of smooth ind-schemes over* S *and* E/B *is strongly dualizable in* $\mathcal{SH}(B)$ *then the pullback* $p^*(E/B) \simeq (E \times_B B')/B'$ *is strongly dualizable in* $\mathcal{SH}(B')$ *and* $\mathrm{Tr}(p^* f/B') \simeq p^* \mathrm{Tr}(f/B)$.

(iii) *The absolute transfer is natural in cartesian squares: If*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f'"'] & E \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

is a cartesian square of smooth ind-schemes over S *and the vertical maps are smooth, then the square*

$$\begin{tikzcd} E'/S \arrow[r] & E/S \\ B'/S \arrow[r] \arrow[u, "\mathrm{Tr}(f'/S)"] & B/S \arrow[u, "\mathrm{Tr}(f/S)"'] \end{tikzcd}$$

commutes in $\mathcal{SH}(S)$.

To prove part (i) of Proposition 2.5 we appeal to a general additivity result of May's. In the context of symmetric monoidal triangulated categories, [May01] proves that the transfer is additive in distinguished triangles. However, since duality in symmetric monoidal ∞-categories is characterised at the level of homotopy categories, May's theorem admits the following reformulation.

Theorem 2.6 ([May01, Theorem 1.9]). *Let* $\mathscr{C}$ *be a symmetric monoidal stable* ∞*-category and let* $X \longrightarrow Y \longrightarrow Z$ *be a cofiber sequence in* $\mathscr{C}$*. Assume* $C \in \mathscr{C}$ *is such that* $\_\otimes C$ *preserves cofiber sequences. Suppose that* Y *is equipped with a map* $\Delta_Y\colon Y \longrightarrow Y \otimes C$ *and that* X *and* Y *are strongly dualizable. Then* Z *is strongly dualizable and there are maps* Δ_X *and* Δ_Z *such that*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Delta_X"] & Y \arrow[r] \arrow[d, "\Delta_Y"] & Z \arrow[d, "\Delta_Z"] \\ X \otimes C \arrow[r] & Y \otimes C \arrow[r] & Z \otimes C \end{tikzcd}$$

commutes. Furthermore, we have $\mathrm{tr}_{Y,\Delta_Y} = \mathrm{tr}_{X,\Delta_X} + \mathrm{tr}_{Z,\Delta_Z}$ *in* $\pi_0 \mathrm{Map}_{\mathscr{C}}(\mathbf{1}, C)$.

Proof of Proposition 2.5, (i). The homotopy cocartesian square induces a cofiber sequence

$$U/B \vee V/B \longrightarrow Y/B \longrightarrow S^1 \wedge X/B$$

in $\mathcal{SH}(B)$. Shifting this sequence yields and introducing diagonal maps gives a diagram

$$\begin{tikzcd}
X/B \arrow[r] \arrow[d, "\Delta_X"] & U/B \vee V/B \arrow[r] \arrow[d, "\Delta_U \vee \Delta_V"] & Y/B \arrow[dd, "\Delta_Y"] \\
X/B \wedge X/B \arrow[d, "(1)"] & (U/B \wedge U/B) \vee (V/B \wedge V/B) \arrow[d, "(2)"] & \\
X/B \wedge Y/B \arrow[r] & (U/B \vee V/B) \wedge Y/B \arrow[r] & Y/B \wedge Y/B
\end{tikzcd}$$

in which the outer two rows are cofiber sequences and the maps (1) and (2) are induced from the maps $X/B \longrightarrow Y/B$, $U/B \longrightarrow Y/B$ and $V/B \longrightarrow Y/B$ respectively. Then we can conclude using Theorem 2.6. $\square$

Part (ii) of Proposition 2.5 is proven in [Lev18, Lemma 1.6]. We formulate the proof of part (iii) as a lemma.

Lemma 2.7. *Let S be a scheme and B and B' smooth ind-schemes over S. Suppose that $f: E \longrightarrow B$ is smooth with $E/B \in \mathcal{SH}(B)$ strongly dualizable and*

$$\begin{tikzcd}
E' \arrow[r, "i'"] \arrow[d, "f'"] & E \arrow[d, "f"] \\
B' \arrow[r, "i"'] & B
\end{tikzcd}$$

is cartesian. Then the square

$$\begin{tikzcd}
E'/S \arrow[r, "i'/S"] & E/S \\
B'/S \arrow[r, "i/S"'] \arrow[u, "\mathrm{Tr}(f'/S)"] & B/S \arrow[u, "\mathrm{Tr}(f/S)"']
\end{tikzcd}$$

is homotopy commutative.

Proof. Write $p: B' \longrightarrow S$ and $q: B \longrightarrow S$ for the structure morphisms. There is a natural transformation $p_\# i^* \longrightarrow q_\#$ defined as the composition

$$p_\# i^* \xrightarrow{\text{unit}} p_\# i^* q^* q_\# \simeq p_\# p^* q_\# \xrightarrow{\text{counit}} q_\#.$$

Consequently, we obtain a homotopy commutative diagram

$$\begin{tikzcd}
B'/S = p_\# p^* \mathbf{1}_S \arrow[r, "p_\# \mathrm{Tr}(f'/B')"] \arrow[d, equal] & p_\# f'_\# i'^* f^* q^* \mathbf{1}_S = E'/S \arrow[d, "\mathrm{Ex}^*_\#"] \\
p_\# i^* q^* \mathbf{1}_S \arrow[r, "p_\# i^* \mathrm{Tr}(f/B)"] \arrow[d] & p_\# i^* f_\# f^* q^* \mathbf{1}_S \arrow[d] \\
B/S = q_\# q^* \mathbf{1}_S \arrow[r, "q_\# \mathrm{Tr}(f/B)"'] & q_\# f_\# f^* q^* \mathbf{1}_S = E/S.
\end{tikzcd}$$

Chasing through the definition of $p_\# i^* \longrightarrow q_\#$ shows that the leftmost composite vertical map is i/S and that the rightmost vertical map is i'/S. $\square$

Finally, we will need some tools to understand when a smooth morphism $E \longrightarrow B$ of smooth ind-schemes over S determines a strongly dualizable object E/B in $\mathcal{SH}(B)$. We have the following formulation of motivic Atiyah duality.

Theorem 2.8 (see [Voe01], [Rio05], [Ayo07b], [CD09]). *If $Y \longrightarrow X$ is a smooth and proper morphism of schemes, then $Y/X \in \mathcal{SH}(X)$ is strongly dualizable.*

Because the property of being strongly dualizable is formulated in the homotopy category, it is immediate that any smooth scheme $Y \longrightarrow X$ such that Y/X is $\mathbb{A}^1$-homotopy equivalent to a smooth and proper scheme over X defines a strongly dualizable object in $\mathcal{SH}(X)$. Furthermore, dualizability is local in the following sense.

Theorem 2.9 ([Lev18, Proposition 1.2, Theorem 1.10]). *Let B be a scheme over S. Suppose that $E \in \mathcal{SH}(B)$ and there is a finite Nisnevich covering family $\{j_i \colon U_i \longrightarrow B\}$ and that $j_i^* E \in \mathcal{SH}(U_i)$ is strongly dualizable. Then E is strongly dualizable as well.*

If $E \longrightarrow B$ is a Nisnevich–locally trivial fiber bundle with smooth fiber F and B is smooth over S, then E/B is strongly dualizable in $\mathcal{SH}(B)$ if F/S is strongly dualizable in $\mathcal{SH}(S)$

3. Transfers of Grassmannians

Definition 3.1. The ind-scheme Gr_r is the sequential colimit of the Grassmannians $\mathrm{Gr}_r(n)$ of r–planes in n–space along the canonical closed immersions $\mathrm{Gr}_r(n) \hookrightarrow \mathrm{Gr}_r(n+1)$.

It is well known that Gr_r is a model for BGL_r in the $\mathbb{A}^1$–homotopy category. In fact, let $U_r(N)$ be the scheme of monomorphisms $\mathcal{O}^r \longrightarrow \mathcal{O}^N$. Along $\mathcal{O}^N \oplus 0 \subset \mathcal{O}^{N+1}$, there are closed embeddings $U_r(N) \hookrightarrow U_r(N+1)$ and [MV99, Proposition 4.3.7] shows that the colimit $U_r(\infty) = \operatorname{colim}_N U_r(N)$ along these embeddings is contractible in $\mathcal{H}(S)$. Also, the quotient $U_r(N)/\mathrm{GL}_r$ is isomorphic to $\mathrm{Gr}_r(N)$ and consequently $U_r(\infty)/\mathrm{GL}_r \cong \mathrm{Gr}_r$ is a model for BGL_r.

Direct sum defines a morphism $U_r(N) \times U_{n-r}(N) \longrightarrow U_n(2N)$ which is equivariant with respect to the block diagonal inclusion $\mathrm{GL}_r \times \mathrm{GL}_{n-r} \longrightarrow \mathrm{GL}_n$. Passing to the colimit $N \to \infty$ and taking quotients yields a morphism

$$i_{r,n} \colon \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow \mathrm{Gr}_n.$$

This morphism is equivalent in $\mathcal{H}(S)$ to the map $\mathrm{BGL}_r \times \mathrm{BGL}_{n-r} \longrightarrow \mathrm{BGL}_n$ induced by the block diagonal inclusion $\mathrm{GL}_r \times \mathrm{GL}_{n-r} \subset \mathrm{GL}_n$. The goal of this section is to develop a partial inductive description of the absolute transfer $\mathrm{tr}_{n,r} \colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ of $i_{r,n}$ in $\mathcal{SH}(S)$. For this purpose a different version of $i_{r,n}$ in $\mathcal{H}(S)$ will be more convenient.

Lemma 3.2. *In $\mathcal{H}(S)$ there is an equivalence* $\mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$*. Along this equivalence, $i_{r,n}$ corresponds to the quotient*

$$\overline{i_{r,n}}\colon U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow U_n(\infty)/\mathrm{GL}_n \cong \mathrm{Gr}_n$$

by GL_n.

Proof. Writing $\varphi\colon U_r(N) \times U_{n-r}(N) \longrightarrow U_n(2N)$ for the map induced by taking direct sums, we obtain a commutative diagram

$$\begin{tikzcd}
U_r(N) \times U_{n-r}(N) \arrow[r, "\varphi"] \arrow[d] & U_n(2N) \arrow[d] \\
\mathrm{Gr}_r(N) \times \mathrm{Gr}_{n-r}(N) \arrow[r] \arrow[dr, "i_{r,n}"'] & U_n(2N)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \arrow[d] \\
& \mathrm{Gr}_n(2N).
\end{tikzcd}$$

Passing to the colimit $N \to \infty$ the horizontal maps become equivalences. □

Lemma 3.3. *The morphism $\overline{i_{r,n}}$ is a Zariski–locally trivial bundle over Gr_n. Its fiber is the quotient* $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$.

Proof. By construction, the morphism $\overline{i_{r,n}}$ is isomorphic to the colimit of the quotient maps $U_n(N)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow U_n(N)/\mathrm{GL}_n \cong \mathrm{Gr}_n(N)$. But these are all Zariski–locally trivial with fiber $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$. □

We note that $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$ is equivalent to $\mathrm{Gr}_r(n)$ in $\mathcal{H}(S)$ and this equivalence is compatible with the respective GL_n actions. This is shown in [AHW18, Lemma 3.1.5] and implies in particular that the image in $\mathcal{SH}(\mathrm{Gr}_n)$ of the associated bundle $U_n(\infty) \times^{\mathrm{GL}_n} \mathrm{Gr}_r(n) \longrightarrow \mathrm{Gr}_n$ is equivalent to that of the quotient $U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow \mathrm{Gr}_n$.

Lemma 3.4. *The morphism $\overline{i_{r,n}}\colon U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow \mathrm{Gr}_n$ defines a strongly dualizable object $G_{r,n} \in \mathcal{SH}(\mathrm{Gr}_n)$.*

Proof. By Lemma A.3 it will be enough to show that the pullback $i\colon E \longrightarrow \mathrm{Gr}_n(N)$ of $\overline{i_{r,n}}$ along the inclusion $\mathrm{Gr}_n(N) \longrightarrow \mathrm{Gr}_n$ defines a dualizable object in $\mathcal{SH}(\mathrm{Gr}_n(N))$ for all N. But, by Lemma 3.3 the morphism i is a Zariski-locally trivial fiber bundle over $\mathrm{Gr}_n(N)$ with fiber $X = \mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$. Hence, to show that i defines a strongly dualizable object in $\mathcal{SH}(\mathrm{Gr}_n(N))$, by Theorem 2.9 it is enough to show that $X/S \in \mathcal{SH}(S)$ is strongly dualizable.

But we have seen that $X \simeq \mathrm{Gr}_r(n)$ in $\mathcal{H}(S)$ and therefore also in $\mathcal{SH}(S)$. The scheme $\mathrm{Gr}_r(n)$ is smooth and proper over S, so motivic Atiyah duality, Theorem 2.8, implies that $\mathrm{Gr}_r(n)/S$ and therefore also X/S is strongly dualizable in $\mathcal{SH}(S)$, see for example [Lev18, Proposition 1.2]. □

Lemma 3.5. *Suppose $r < n$. The open complement of the closed immersion* $\mathrm{Gr}_r(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ *is the total space of an affine space bundle of rank $n - r$ over* $\mathrm{Gr}_{r-1}(n-1)$.

Dually, the complement of the closed immersion $\mathrm{Gr}_{r-1}(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ *is the total space of an affine space bundle of rank r over* $\mathrm{Gr}_r(n-1)$.

Proof. Suppose $\mathrm{Spec}(A)$ is an affine scheme mapping to S. On $\mathrm{Spec}(A)$–valued points, the inclusion $\mathrm{Gr}_r(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ is given by considering a projective submodule P of A^{n-1} as a submodule of $A^n = A^{n-1} \oplus A$. It follows that the complement U of $\mathrm{Gr}_r(n-1)$ has $\mathrm{Spec}(A)$–valued points

$$U(\mathrm{Spec}\, A) = \{P \subset A^n : P \text{ is projective of rank } r \text{ and } P \not\subset A^{n-1} \oplus 0\}.$$

Given $P \in U(\mathrm{Spec}\, A)$, the module $P \cap (A^{n-1} \oplus 0)$ will be locally free of rank $r-1$. This gives a map $\varphi\colon U \longrightarrow \mathrm{Gr}_{r-1}(n-1)$ which is trivial over the standard Zariski–open cover of $\mathrm{Gr}_{r-1}(n-1)$ with fiber $\mathbb{A}^{n-r}$.

The dual statement is proved similarly. In fact, the bundle $V \longrightarrow \mathrm{Gr}_r(n-1)$ in question is the tautological r-plane bundle on $\mathrm{Gr}_r(n-1)$. □

The decomposition $\mathrm{Gr}_r(n) = U \cup V$ of the last lemma yields a homotopy cocartesian square

$$\begin{tikzcd}
U \smallsetminus \mathrm{Gr}_{r-1}(n-1) = U \cap V \arrow[r] \arrow[d] & V \simeq \mathrm{Gr}_r(n-1) \arrow[d] \\
\mathrm{Gr}_{r-1}(n-1) \simeq U \arrow[r] & \mathrm{Gr}_r(n)
\end{tikzcd}$$

in the $\mathbb{A}^1$–homotopy category $\mathcal{H}(S)$. It is immediate that this decomposition of $\mathrm{Gr}_r(n)$ is stable under the action of $\mathrm{GL}_{n-1} \times 1 \subset \mathrm{GL}_n$. We can therefore pass to the bundles over Gr_{n-1} associated to the universal GL_{n-1}–torsor $U_{n-1}(\infty)$ over Gr_{n-1} and obtain a homotopy cocartesian square

$$\begin{tikzcd}
(U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} (U \cap V))/\mathrm{Gr}_{n-1} \arrow[r] \arrow[d] & G_{r,n-1} \arrow[d] \\
G_{r-1,n-1} \arrow[r] & (U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} \mathrm{Gr}_r(n))/\mathrm{Gr}_{n-1}
\end{tikzcd}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$.

Proposition 3.6. *Suppose $r < n$ and consider the composition*

$$\varphi\colon \mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{incl}} \mathrm{Gr}_{n+} \xrightarrow{\mathrm{tr}_{n,r}} \mathrm{Gr}_{r+} \wedge \mathrm{Gr}_{n-r,+}$$

where incl *is given by the assignment $P \longmapsto P \oplus A$ on* $\mathrm{Spec}(A)$*–valued points. Then there is a map $\psi\colon \mathrm{Gr}_{n-1,+} \longrightarrow \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+}$ in $\mathcal{SH}(S)$ such that φ is the sum of the compositions*

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id} \wedge \mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r-1}} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

and

$$\mathrm{Gr}_{r-1,+} \xrightarrow{\psi} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}.$$

Proof. Consider the homotopy pullback

$$\begin{tikzcd} E = U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} \mathrm{Gr}_r(n) \arrow[r] \arrow[d] & \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \arrow[d] \\ \mathrm{Gr}_{n-1} \arrow[r, "\mathrm{incl}"'] & \mathrm{Gr}_n \end{tikzcd}$$

in $\mathcal{H}(S)$. By the discussion following Lemma 3.5 we obtain a cofiber sequence

$$X/\mathrm{Gr}_{n-1} \longrightarrow G_{r,n-1} \vee G_{r-1,n-1} \longrightarrow E/\mathrm{Gr}_{n-1}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$ where $X = U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} (U \cap V)$. Theorem 2.6 then shows that

$$\mathrm{tr}_{E/\mathrm{Gr}_{n-1}} = \mathrm{tr}_{G_{r,n-1}} + \mathrm{tr}_{G_{r-1,n-1}} - \mathrm{tr}_{X/\mathrm{Gr}_{n-1}}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$. Passing to the absolute transfer and using Lemma 2.7 yields that φ is the sum of the compositions

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id} \wedge \mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r-1}} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

and

$$\mathrm{Gr}_{n-1,+} \longrightarrow X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

in $\mathcal{SH}(S)$. Here, the map $X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ is obtained from the inclusion $U \cap V \subset \mathrm{Gr}_r(n)$ by passing to associated bundles. Now, this inclusion factors through the inclusion of U into $\mathrm{Gr}_r(n)$. By Lemma 3.5 the inclusion $\mathrm{Gr}_{r-1}(n-1) \subset U$ is an $\mathbb{A}^1$–equivalence, being the zero section of an affine space bundle. Therefore $X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ factors through the map $\mathrm{incl} \wedge \mathrm{id} \colon \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$. This way we obtain the map ψ and the required decomposition of $\mathrm{tr}_{n,r} \circ \mathrm{incl}$. □

4. Proof of the Theorem

We have the filtration

$$\mathrm{Gr}_{0,+} \xrightarrow{i_1} \mathrm{Gr}_{1,+} \xrightarrow{i_2} \dots \xrightarrow{i_n} \mathrm{Gr}_{n,+} \longrightarrow \dots \xrightarrow{i_m} \mathrm{Gr}_{m,+}$$

and we have seen that for $r \leq n$ the map $i_{r,n} \colon \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow \mathrm{Gr}_n$ admits an absolute transfer $\mathrm{tr}_{n,r} \colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ in the motivic stable homotopy category $\mathcal{SH}(S)$. Write $f_{n,r} \colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+}$ for the composition

$$\mathrm{Gr}_{n,+} \xrightarrow{\mathrm{tr}_{n,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

and $\phi_{n,r}$ for the composition

$$\mathrm{Gr}_{n,+} \xrightarrow{f_{n,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}.$$

Lemma 4.1. *With notation as above, for $r < n$ the compositions*

$$\mathrm{Gr}_{n-1,+} \xrightarrow{i_n} \mathrm{Gr}_{n,+} \xrightarrow{f_{n,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

and

$$\mathrm{Gr}_{n-1,+} \xrightarrow{f_{n-1,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

coincide.

Proof. By Proposition 3.6 the composition $f_{n,r} \circ i_n$ is a sum of two compositions

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id}\wedge\mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

and

$$\mathrm{Gr}_{n-1,+} \longrightarrow \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl}\wedge\mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

in $\mathcal{SH}(S)$. But the composition

$$\mathrm{Gr}_{r-1,+} \xrightarrow{\mathrm{incl}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

vanishes. Therefore, $f_{n,r} \circ i_n$ coincides with the composition

$$\mathrm{Gr}_{n-1,+} \xrightarrow{f_{n-1,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

in $\mathcal{SH}(S)$. □

Proof of Theorem 1.1. Proceeding by induction on n, assume that

$$\Phi = \bigvee_{r=0}^{n-1} \phi_{n-1,r} \colon \mathrm{Gr}_{n-1,+} \longrightarrow \bigvee_{r=0}^{n-1} \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

is an equivalence in $\mathcal{SH}(S)$. Because of Lemma 4.1 we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Gr}_{n,+} \arrow[dr, "\Phi'"] & \\
\mathrm{Gr}_{n-1,+} \arrow[ur, "i_n"] \arrow[rr, "\Phi"'] & & \bigvee_{r=0}^{n-1} \mathrm{Gr}_r/\mathrm{Gr}_{r-1}
\end{tikzcd}$$

where $\Phi' = \bigvee_{r=0}^{n-1} \phi_{n,r}$. It follows that $\Phi^{-1} \circ \Phi' \circ i_n \simeq \mathrm{id}$, i. e. i_n admits a left inverse. That is to say, the cofiber sequence

$$\mathrm{Gr}_{n-1,+} \xrightarrow{i_n} \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_n/\mathrm{Gr}_{n-1}$$

splits and yields an equivalence

$$\mathrm{Gr}_{n,+} \xrightarrow{(\Phi^{-1}\Phi')\vee\phi_{n,n}} \mathrm{Gr}_{n-1,+} \vee \mathrm{Gr}_n/\mathrm{Gr}_{n-1}$$

since $\phi_{n,n}$ is by definition the canonical projection. Post-composing with $\Phi \vee \mathrm{id}$ then shows that the stable map $\Phi' \vee \phi_{n,n} \colon \mathrm{Gr}_{n,+} \longrightarrow \bigvee_{r=0}^{n} \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$ is an equivalence in $\mathcal{SH}(S)$ as well. □

A. Stable Motivic Homotopy Theory of Smooth Ind-Schemes

We freely use the theory of presentable ∞–categories as developed in [Lur09, section 5.5.3]. The ∞–category of presentable ∞–categories with left adjoints as morphisms is denoted $\mathcal{P}r^{\mathrm{L}}$ while the ∞–category of presentable ∞–categories with right adjoints as morphisms is denoted $\mathcal{P}r^{\mathrm{R}}$. There is an equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$ of ∞–categories which is the identity on objects and sends a left adjoint functor to its right adjoint. Both $\mathcal{P}r^{\mathrm{L}}$ and $\mathcal{P}r^{\mathrm{R}}$ are complete and cocomplete and the homotopy limits in both $\mathcal{P}r^{\mathrm{L}}$ and $\mathcal{P}r^{\mathrm{R}}$ coincide with homotopy limits in the ∞–category of ∞–categories.

Definition A.1. A *smooth ind-scheme* over S is an object of $\mathrm{Ind}(\mathrm{Sm}_S)$, the ∞–category of ind-objects in the category of smooth schemes over S with arbitrary morphisms between them. A morphism of ind-schemes is smooth if it can be presented as a colimit of smooth morphisms in Sm_S.

The goal of this section will be to generalize the definition of the stable motivic homotopy category $\mathcal{SH}$ to smooth ind-schemes over S. Our approach is to use part of the six functor formalism for $\mathcal{SH}$, as established in [Ayo07b; Ayo07a] for noetherian schemes and extended to arbitrary schemes in [Hoy14, Appendix C]. An overview of the standard functorialities, at least at the level of triangulated categories, can be found in [CD09].

The first functoriality of $\mathcal{SH}$ can be summarized as follows. For every morphism $f \colon X \longrightarrow Y$ between smooth schemes over S we have an adjunction

$$f^* : \mathcal{SH}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(Y) : f_*$$

between the stable presentable ∞–categories $\mathcal{SH}(X)$ and $\mathcal{SH}(Y)$. These adjunctions assemble into functors $\mathcal{SH}^* \colon \mathrm{Sm}_S^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ and $\mathcal{SH}_* \colon \mathrm{Sm}_S \longrightarrow \mathcal{P}r^{\mathrm{R}}$ which are naturally equivalent after composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$. If $f \colon X \longrightarrow Y$ is smooth, then there is an additional adjunction

$$f_\# : \mathcal{SH}(Y) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(X) : f^*.$$

These assemble into a functor $\mathcal{SH}_\#\colon \mathrm{Sm}_{S,\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ from the wide subcategory of Sm_S consisting of smooth morphisms between smooth schemes over S. There are various exchange transformations associated with a cartesian square

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "q"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

in Sm_S, of which we only mention the transformation

$$\mathrm{Ex}^*_\#\colon g_\# q^* \longrightarrow p^* f_\#$$

when f and hence g is smooth. More details on these exchange transformations may be found in [CD09].

Because $\mathcal{P}r^{\mathrm{R}}$ is cocomplete, the functor $\mathcal{SH}_*$ naturally extends to a functor

$$\mathcal{SH}_*\colon\ \mathrm{Ind}(\mathrm{Sm}_S) \longrightarrow \mathcal{P}r^{\mathrm{R}}$$

and we obtain a functor

$$\mathcal{SH}^*\colon\ \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$$

by again composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$.

More explicitly, if $(X_i)_{i\in I}$ is a filtered diagram of smooth schemes over S and $X = \mathrm{colim}_i X_i$ as an ind-scheme over S, then

$$\mathcal{SH}^*(X) = \operatorname*{holim}_i \mathcal{SH}^*(X_i) \quad \text{and} \quad \mathcal{SH}_*(X) = \operatorname*{hocolim}_i \mathcal{SH}_*(X_i).$$

Note that $\mathcal{SH}^*(X)$ and $\mathcal{SH}_*(X)$ are equivalent ∞–categories since homotopy limits along left adjoints in $\mathcal{P}r^{\mathrm{L}}$ correspond to homotopy colimits along their right adjoints in $\mathcal{P}r^{\mathrm{R}}$, see [Lur09, section 5.5.3]. This description of $\mathcal{SH}(X)$ also shows that it inherits the structure of a closed symmetric monoidal, stable, presentable ∞–category, see [Lur12, section 3.4.3, Proposition 4.8.2.18].

The adjunction $f^* \dashv f_*$ for a morphism $f\colon X \longrightarrow Y$ of ind-schemes is obtained by presenting f as a colimit of maps $f_i\colon X_i \longrightarrow Y_i$ between schemes over S and then taking f^* to be the functor induced on the homotopy limits in $\mathcal{P}r^{\mathrm{L}}$ and f_* the functor induced on the homotopy colimits in $\mathcal{P}r^{\mathrm{R}}$.

It remains to construct the extra left-adjoint $f_\#$ for a smooth map f between ind-schemes over S. First, a morphism $f\colon X \longrightarrow Y$ between ind-schemes is smooth if and only if it is a filtered colimit of smooth maps $f_i\colon X_i \longrightarrow Y_i$. Each f_i^* admits a left adjoint $f_{i\#}$ and since $\mathcal{P}r^{\mathrm{R}}$ is stable under limits, the functor $f^*\colon\ \mathcal{SH}^*(Y) \longrightarrow \mathcal{SH}^*(X)$ admits a left adjoint as well. That is to say, $\mathcal{SH}^*\colon\ \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ restricts to a functor $\mathcal{SH}^*\colon\ \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}}_{\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{R}}$ from the wide subcategory of $\mathrm{Ind}(\mathrm{Sm}_S)$ consisting of smooth maps between smooth ind-schemes over S. Composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$ then yields the functor

$$\mathcal{SH}_{\#} \colon \mathrm{Ind}(\mathrm{Sm}_S)_{\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{L}}.$$

In summary, we have the following proposition.

Proposition A.2. *For every ind-scheme X over S, there is a closed symmetric monoidal, stable, presentable ∞–category $\mathcal{SH}(X)$. For every morphism $f \colon X \longrightarrow Y$ between ind-schemes there is an associated adjunction*

$$f^* : \mathcal{SH}(Y) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(X) : f_*$$

with f^* a monoidal functor. If f is smooth then there is an additional adjunction

$$f_\# : \mathcal{SH}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(Y) : f^*.$$

These data are functorial in f and admit various natural exchange transformations. If X happens to be a smooth scheme over S then this version of $\mathcal{SH}(X)$ is naturally equivalent to the usual construction.

Following [Lev18], for a smooth morphism $f \colon X \longrightarrow Y$ of ind-schemes over S we define $X/Y = f_\#(\mathbf{1}_X) \in \mathcal{SH}(Y)$ where $\mathbf{1}_X$ denotes the monoidal unit in $\mathcal{SH}(X)$. In particular, if $Y = S$, we see that any smooth ind-scheme X over S determines an object $X/S \in \mathcal{SH}(S)$. If X is a smooth scheme over S, then X/S is canonically equivalent to the $\mathbb{P}^1$–suspension spectrum of X in $\mathcal{SH}(S)$; see [Ayo14, Lemma C.2].

Lemma A.3. *Suppose B is a smooth ind-scheme over S and $E \in \mathcal{SH}(B)$. If B is presented as a filtered colimit $B = \mathrm{colim}_i\, B_i$ of smooth schemes in $\mathrm{Ind}(\mathrm{Sm}_S)$, let $f_i \colon B_i \longrightarrow B$ be the canonical map for each i. Then $E \in \mathcal{SH}(B)$ is strongly dualizable if and only if $f_i^*E \in \mathcal{SH}(B_i)$ is strongly dualizable for every i.*

Proof. This follows from [Lur12, Proposition 4.6.1.11] since we have $\mathcal{SH}(B) \simeq \lim_i \mathcal{SH}(B_i)$. □

Proposition A.4. *Suppose an ind-scheme X is presented as a colimit $X = \mathrm{colim}_i\, X_i$ in $\mathrm{Ind}(\mathrm{Sm}_S)$. Then there is a natural equivalence $X/S \simeq \mathrm{hocolim}_i\, X_i/S$ in $\mathcal{SH}(S)$.*

Proof. Write $\pi \colon X \longrightarrow S$ and $\pi_i \colon X_i \longrightarrow S$ for the structure morphisms. Suppose $Y \in \mathcal{SH}(S)$ is arbitrary. Then we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathcal{SH}(S)}(\pi_\# \mathbf{1}_X, Y) &\simeq \mathrm{Map}_{\mathcal{SH}(X)}(\mathbf{1}_X, \pi^* Y) \\
&\simeq \operatorname*{holim}_i \mathrm{Map}_{\mathcal{SH}(X_i)}(\mathbf{1}_{X_i}, \pi_i^* Y) \\
&\simeq \operatorname*{holim}_i \mathrm{Map}_{\mathcal{SH}(S)}(\pi_{i\#} \mathbf{1}_{X_i}, Y) \\
&\simeq \mathrm{Map}_{\mathcal{SH}(S)}(\operatorname*{hocolim}_i X_i/S, Y)
\end{aligned}$$

of mapping spaces. The Yoneda lemma implies that $X/S = \pi_\# \mathbf{1}_X \simeq \mathrm{hocolim}_i\, X_i/S$ in $\mathcal{SH}(S)$. □

This proposition allows us to extend the definition of the functor $\_/S\colon \mathrm{Sm}_S \longrightarrow \mathcal{SH}(S)$ in [Lev18] to ind-schemes. The functor $\_/S\colon \mathrm{Sm}_S \longrightarrow \mathcal{SH}(S)$ extends uniquely up to natural equivalence to a functor $\_/S\colon \mathrm{Ind}(\mathrm{Sm}_S) \longrightarrow \mathcal{SH}(S)$ because $\mathcal{SH}(S)$ is cocomplete. By Proposition A.4 this coincides on objects with the previous construction $\pi_\#(\mathbf{1}_X)$ for a smooth ind-scheme $\pi\colon X \longrightarrow S$.

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