Heute: Beispiele zu Seifert-van Kampen

- S^4 ist einfach zusammenhängend

vorher: Jede Schleife in S^4 ist homotop zu einer Schleife die mind. einen Punkt verfehlt

mit Seifert-van Kampen:

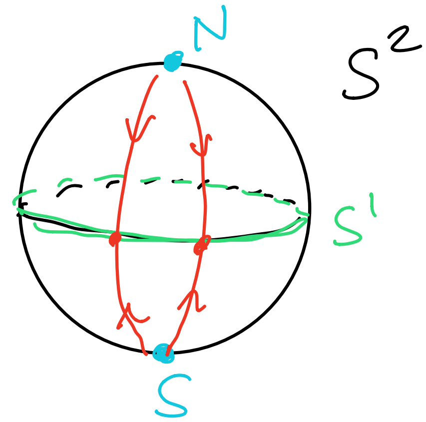
Satz: Sei $X = U \cup V$ mit offenen Teilmengen $U, V \subset X$, so dass $U, V, U \cap V$ wegzshgd. sind, und $x \in U \cap V$. Dann induzieren

$$U \hookrightarrow X \quad , \quad V \hookrightarrow X$$

einen Isomorphismus

$$\pi_1(U,x) \underset{\pi_1(U \cap V, x)}{*} \pi_1(V,x) \longrightarrow \pi_1(X,x)$$

$$
\begin{tikzcd}
\pi_1(U\cap V, x) \arrow[rr] \arrow[dd] & & \pi_1(U,x) \arrow[dl] \arrow[dd] \\
& \cdot * \cdot \arrow[dr, dashed, "\cong"] & \\
\pi_1(V,x) \arrow[ur] \arrow[rr] & & \pi_1(X,x)
\end{tikzcd}
$$



__Zur S^n:__

$U = S^n \setminus \{N\} \cong \mathbb{R}^n$

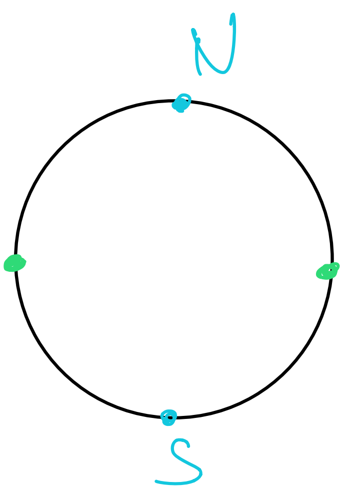
$V = S^n \setminus \{S\} \cong \mathbb{R}^n$

$U \cap V = S^n \setminus \{N, S\} \simeq S^{n-1}$ wegzshgd. für $n \geq 2$

S-vK: $\pi_1(S^n) \cong \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$

$\cong \{e\} \underset{\pi_1(U\cap V)}{*} \{e\} \cong \{e\}$

__Für $n=1$__



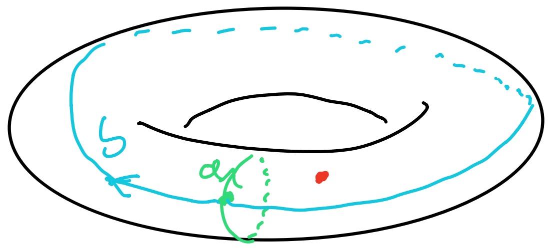
S^0 ist nicht wegzshgd.!

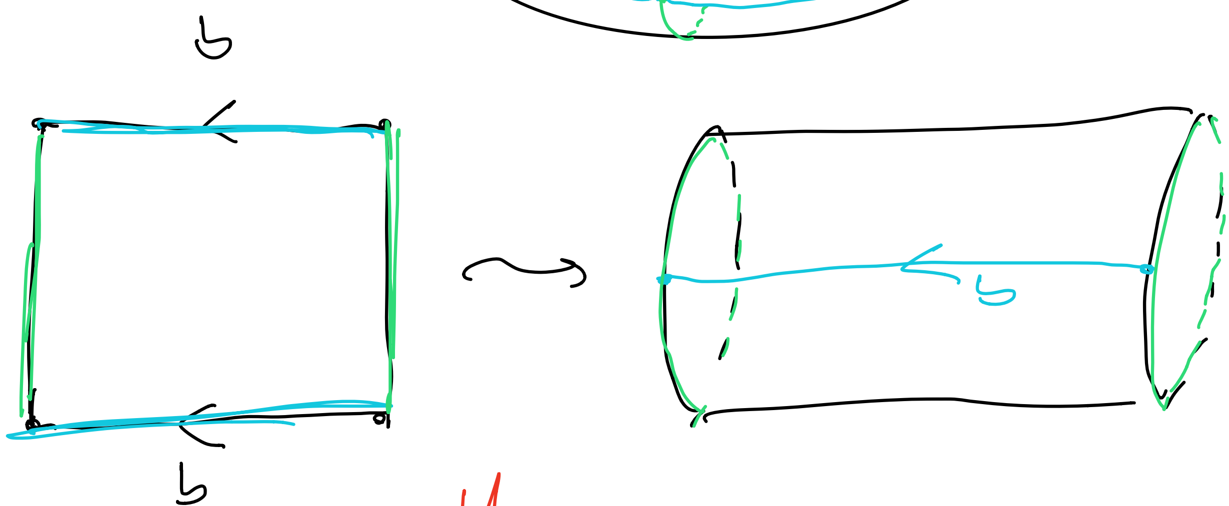
In der Tat $\pi_1(S^1) \cong \mathbb{Z}$

• **Torus** $T^2 \cong \underset{a}{S^1} \times \underset{b}{S^1}$ liefert

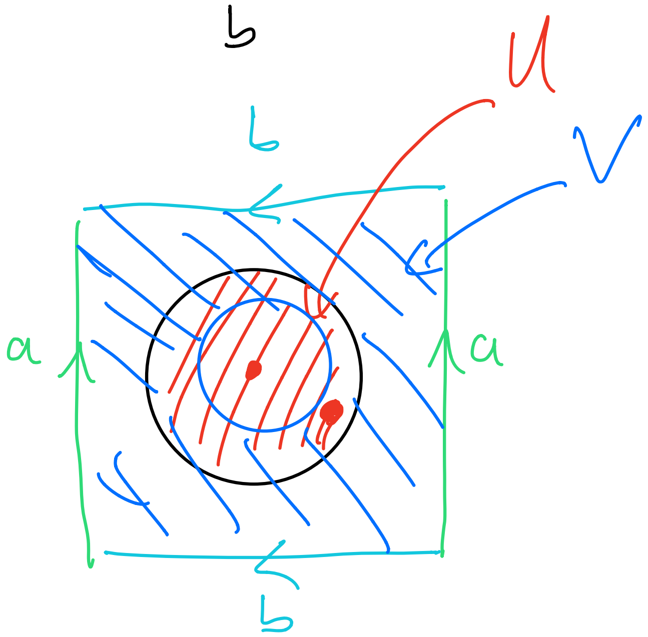
$$\pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

geometrischer:

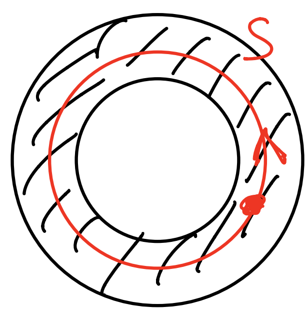




$T^2 \cong$

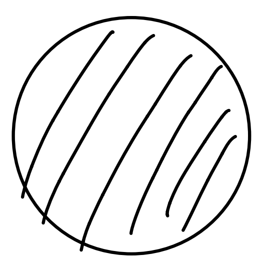


$U \cap V =$

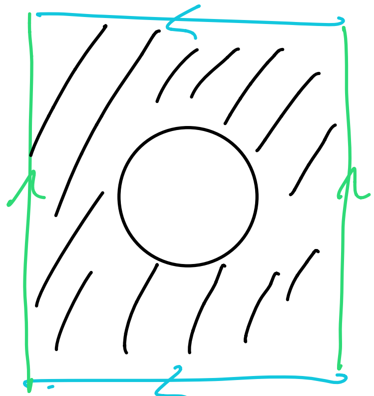


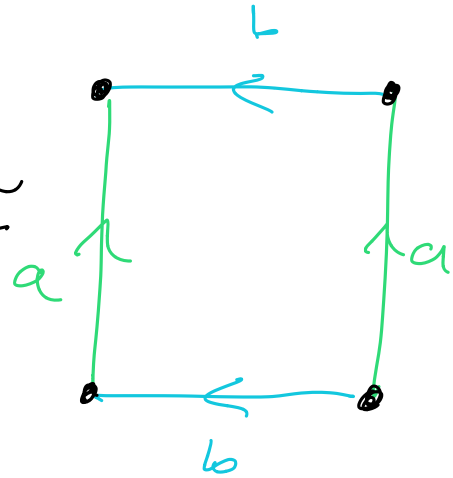
$\simeq S^1$

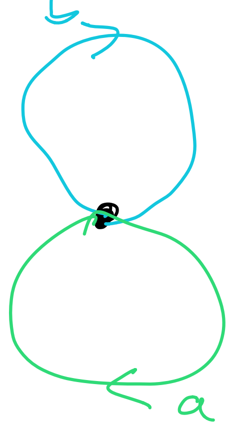
$U =$



$\simeq *$

$V =$ 

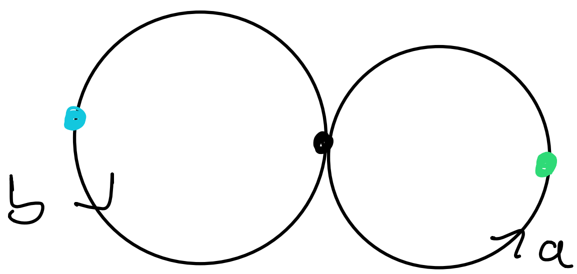
$\simeq$ 

$\simeq$ 

$\cong S^1 \vee S^1$

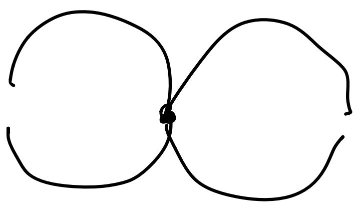
„Bouquet von S^1"

S-vK für $S^1 \vee S^1$:



$U = (S^1 \vee S^1) \setminus \bullet \simeq S^1_a$

$V = (S^1 \vee S^1) \setminus \bullet \simeq S^1_b$

$U \cap V =$  $\simeq \bullet$

$$\Rightarrow \quad \pi_1(S^1 \vee S^1) \cong \pi_1(S^1) \underset{\{e\}}{*} \pi_1(S^1)$$

$$\cong \langle a, b \mid \ \rangle$$

Zurück zu T^2:

$$\pi_1(V) \cong \langle a, b \mid \rangle$$

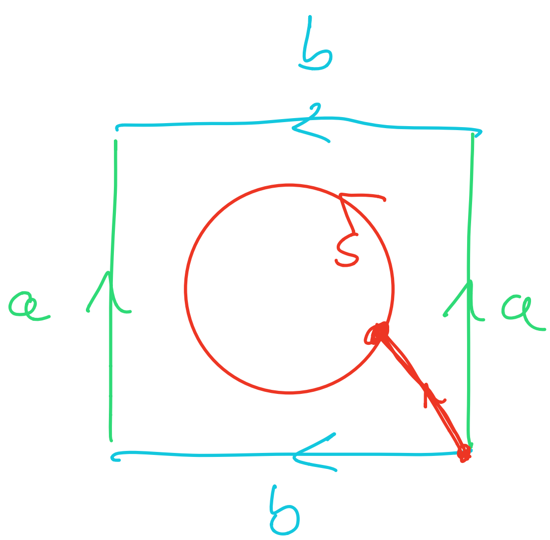
$$\pi_1(U) \cong \{e\}$$

$$\pi_1(U \cap V) \cong \mathbb{Z}$$

$$\Rightarrow \pi_1(T^2) \cong \langle a, b \mid \rangle \underset{\mathbb{Z}}{*} \{e\}$$

Wir brauchen die Abbildung

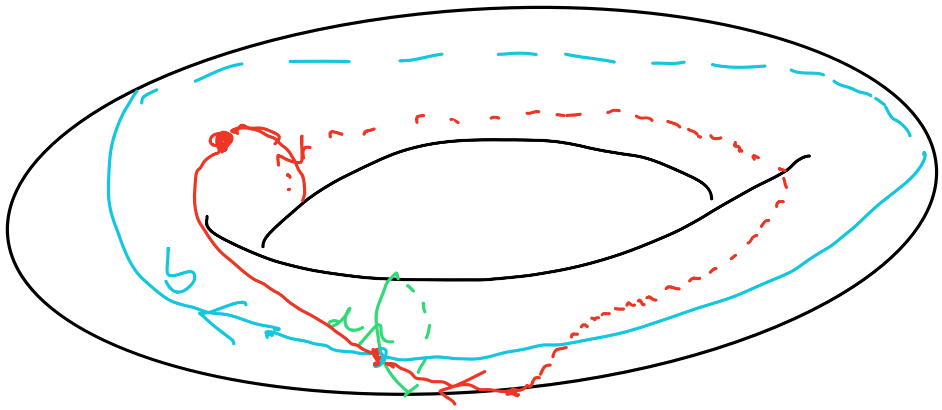
$$\mathbb{Z} \xrightarrow{\varphi} \langle a, b \mid \rangle$$



$\Rightarrow$ s ist homotop zu $a * b * a^{-1} * b^{-1}$

$\Rightarrow$ φ ist gegeben durch

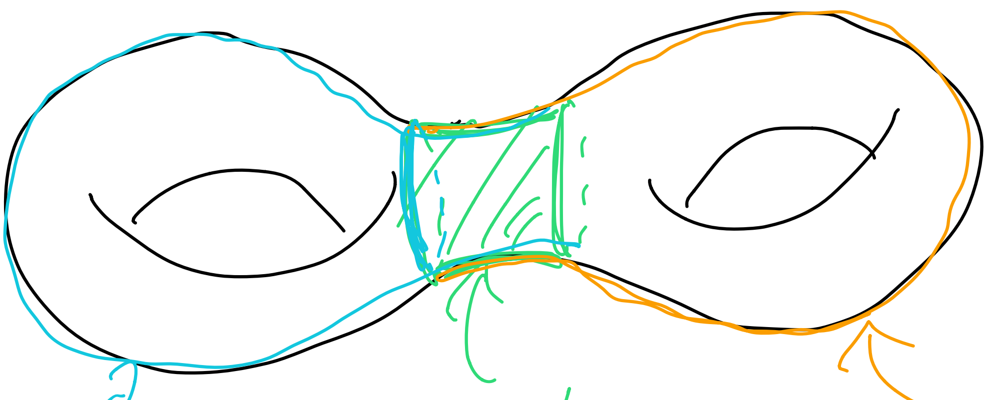
$$1 \longmapsto a b a^{-1} b^{-1}$$



$$\Rightarrow \quad \pi_1(T^2) \cong \langle a,b \mid \rangle \ast_{\mathbb{Z}} \{e\}$$

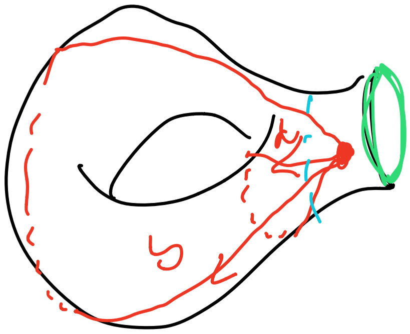
$$\cong \langle a,b \mid aba^{-1}b^{-1} \rangle \cong \underset{a}{\mathbb{Z}} \times \underset{b}{\mathbb{Z}}$$

Fläche mit Genus 2 $\qquad \Sigma_2$

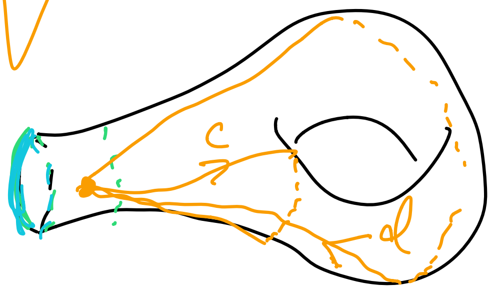


$U \cong S^1 \vee S^1$ $\qquad$ $U \cap V \cong S^1$ $\qquad$ $V \cong S^1 \vee S^1$

U:



V



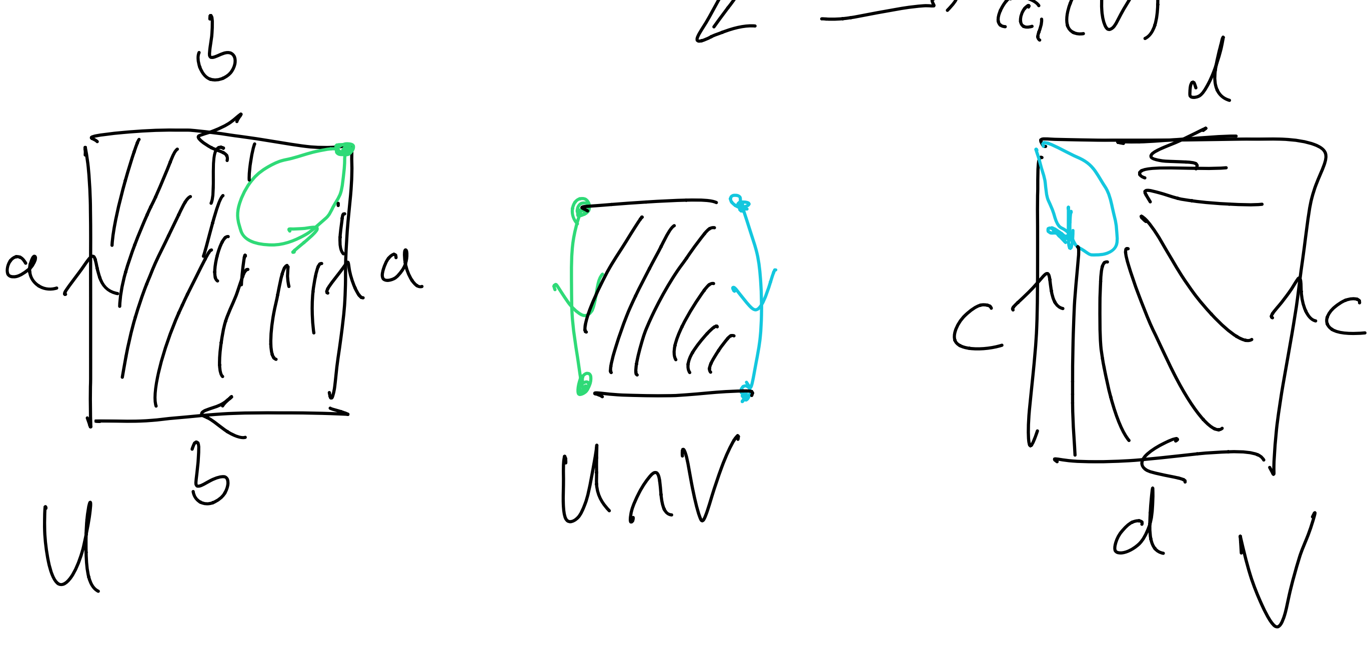
$\pi_1(U) \cong \langle a, b \mid \rangle$

$\pi_1(V) \cong \langle c, d \mid \rangle$

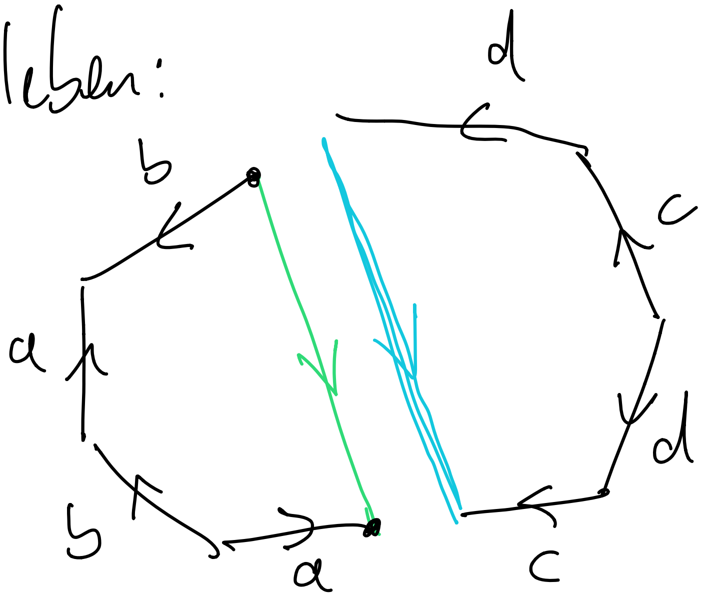
$\pi_1(U \cap V) \cong \mathbb{Z}$

Wieder müssen wir uns überlegen, wie die Abbildungen $\mathbb{Z} \longrightarrow \pi_1(U)$

$\mathbb{Z} \longrightarrow \pi_1(V)$



Verkleben:



$$= \text{Bild von } \mathbb{Z} \longrightarrow \pi_1(U)$$
$$1 \longmapsto b a^{-1} b^{-1} a$$

$$= \text{Bild von } \mathbb{Z} \longrightarrow \pi_1(V)$$
$$1 \longmapsto d^{-1} c^{-1} d c$$

Resultat:

$$\pi_1(\Sigma_2) \cong \pi_1(U) \underset{\mathbb{Z}}{*} \pi_1(V)$$

$$\cong \langle a, b, c, d \mid a^{-1} b a b^{-1} d^{-1} c^{-1} d c \rangle$$

$$\cong \langle a, b, c, d \mid \underbrace{aba^{-1}b^{-1}cdc^{-1}d^{-1}}_{[a,b][c,d]} \rangle$$