Mathematical Analysis

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Art of Problem Solving

2020-06-21

$\sim$ To all Mathematical Analysis and Calculus enthusiasts $\sim$

Table of Contents

Section Contents

Why study mathematical analysis

Why study mathematical analysis?

Calculus I: 1D
Calculus II: 2D
Calculus III: 3D

Calculus describes the trends, rates of change, volumes, etc. of curves, surfaces, scalar fields, and vector fields.

So, how are mathematical analysis and calculus related?

Calculus 0: Mathematical analysis
Calculus IV: Quaternions

Mathematical analysis contains the foundations of calculus.

Section Contents

Limits

Limit:

- A value that a sequence approaches forever, but doesn't necessarily reach
- All limits can be expressed in terms of sequences:

$$\lim_{n\to\infty} a_n \text{ is the limit of } a_1, a_2, a_3, \ldots$$

$$\overbrace{\lim_{n\to\infty} \sum_{k=1}^{n} b_k}^{\text{Series}} \text{ is the limit of } \overbrace{b_1, b_1 + b_2, b_1 + b_2 + b_3, \ldots}^{\text{Sequence of partial sums}}$$

$$\lim_{x\to a} f(x) \text{ is the limit of } f(x_1), f(x_2), f(x_3)\ldots$$

for all sequences of $x_n \neq a$ satisfying $\lim_{n\to\infty} x_n = a$

(as long as the limits exist and are all equal)

Notation

Sequences can also be written as :

$\{a_n\}$ $\{a_n\}_1^\infty$ $\{a_1, a_2, a_3, \ldots\}$ $\{a_n \mid n \in \mathbb{N}\}$

"for which"

Limits can also be written as :

$a_n \to L \, (n \to \infty)$ $\lim_{n \to \infty} a_n = L$

Some examples

Infinite sequence:	$1, 1, 1, 1 \ldots \to 1$	converges to 1
Arithmetic sequence:	$1 + 2 + 3 + \ldots + n$	" $\dfrac{n(n+1)}{2}$
Geometric series:	$\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \ldots \to 1$	" 1
Harmonic series:	$\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \ldots$	diverges, tends toward $+\infty$
	$1, 0, 1, 0, \ldots$	diverges
Function limits:	$\lim_{x \to 0} \lvert x \rvert = 0$	equals 0
	$\lim_{x \to 0} \sin(\dfrac{1}{x})$	does not exist

Definition

Definition (Limit of an infinite sequence)

$\{a_n\}_1^\infty$:

- *Converges to a limit L if and only if (iff)*

$$\forall\, \varepsilon > 0,\, \exists\, N > 0 \ s.t.\ \ n > N \Rightarrow |a_n - L| < \varepsilon.$$

- *Diverges iff the limit does not exist.*
- *Tends toward* $+\infty$ *iff*

$$\forall\, M > 0,\, \exists\, N > 0 \ s.t.\ \ n > N \Rightarrow a_n > M.$$

- $''\ -\infty\ ''\ M < 0\ ''\ a_n < M$

Uniqueness

Theorem (Uniqueness of the limit of a sequence)

If $\lim_{n\to\infty} a_n = L$ *exists, it is unique.*

Proof (by contradiction):
Suppose $\lim_{n\to\infty} a_n = L$ exists. Also suppose $\lim_{n\to\infty} a_n = M$ exists. Then $L = M$, otherwise:
Let $\varepsilon = \left|\frac{M-L}{2}\right|$. Then there exists some $N \in \mathbb{N}$ such that

$$\begin{aligned} n > N \Rightarrow & |a_n - L| < \varepsilon, |a_n - M| < \varepsilon \\ \Rightarrow & |M - a_n| + |a_n - L| < |M - L|. \end{aligned}$$

But by Triangle Inequality,

$$|M - a_n| + |a_n - L| \geq |M - L|. \quad \lightning \text{ (contradiction)}$$

$\therefore$ Limit exists $\Rightarrow$ is unique.

Existence

Axiom (Axiom of Completeness)

(Least Upper Bound Property)
A non-empty sequence that is bounded from above has a least upper bound.

" from below " greatest lower bound

The uniqueness theorem and axiom of completeness tell us:

The real numbers are the decimal numbers.
In other words, the real numbers (number line) are continuous.

Axiom of Completeness

There are 6 versions of the Axiom of Completeness:

1. Least Upper Bound Property
2. Dedekind Completeness
3. Bolzano-Weierstrass (Sequential Compactness) Theorem
4. Cauchy Completeness
5. Nested Intervals Theorem
6. Heine-Borel (Finite Covers) Theorem

They are all equivalent (starting from any one, you can prove any other one).

The least upper bound property, Dedekind completeness, and Bolzano-Weierstrass theorem hold for any ordered field and can be generalized to partially ordered fields.

The Heine-Borel theorem, Cauchy completeness, and nested intervals theorem can be generalized to any metric space and topological group.

Other results

Other properties of sequences include the method of infinite descent, monotone convergence theorem, sandwich theorem, $\limsup$ and $\liminf$, and arithmetic operations on limits.

Section Contents

Continuous functions

Continuous function: Continuous at each point, denoted $f \in C$

What does it mean for $f(x)$ to be continuous at $x = a$?

Definition (Continuous function)

$f(x)$ is continuous at $x = a$ iff

1. $\exists \lim_{x \to a} f(x)$
2. $\exists f(a)$
3. $\lim_{x \to a} f(x) = f(a)$

Here, how is $\lim_{x \to a} f(x)$ defined?

Limit of a function

Definition (Limit of a function)

$\lim_{x \to a} f(x) = L$ *iff*

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ \ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

What the limit of a function says is: One can make the distance between $f(x)$ and L as small as one wants by making the distance between x and a smaller, where $x \neq a$.

Definition in higher dimensions

The definition is similar for higher dimensions:

Definition (Continuous function)

$\vec{f}(\vec{x})$ *is continuous at* $\vec{x} = \vec{a}$ *iff*

1. $\exists \lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x})$
2. $\exists \vec{f}(\vec{a})$
3. $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$

Definition (Limit of a function)

$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ *iff*

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ 0 < |\vec{x} - \vec{a}| < \delta \Rightarrow |\vec{f}(\vec{x}) - \vec{L}| < \varepsilon.$$

What this says is: As $\vec{x} \to \vec{a}$ along any path, then $\vec{f}(\vec{x}) \to \vec{L}$.

One-dimensional definition 2

Thus for single-variable functions, the limit of a function can also be defined as:

Definition (Limit of a function)

$\lim_{x\to a} f(x) = L$ *iff*

1. $\exists \lim_{x\to a^+} f(x)$ *(limit from the right, $x > a$)*
2. $\exists \lim_{x\to a^-} f(x)$ *(limit from the left, $x < a$)*
3. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Uniqueness

Theorem (Uniqueness of limit of a function)

$\lim_{x\to a} f(x)$ *exists* $\Rightarrow \lim_{x\to a} f(x)$ *is unique.*

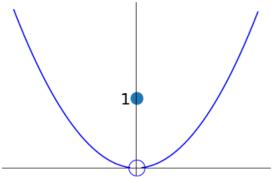
$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ *exists* $\Rightarrow \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ *is unique.*

Proof:
One can prove that a necessary and sufficient condition for $\lim_{x\to a} f(x) = L$ is:

$$\lim_{n\to\infty} f(x_n) = L \, \forall \{x_n\} \text{ s.t. } \lim_{n\to\infty} x_n = a, x_n \neq a.$$

Then use the uniqueness of the limit of a sequence, and the result follows.

Example 1. Suppose $f(x) = \begin{cases} 1 & x = 0 \\ x^2 & x \neq 0 \end{cases}$:



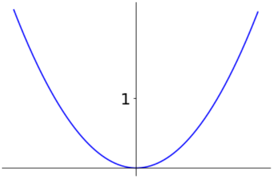
$$\lim_{x \to 0} f(x) = 0$$

$$f(0) = 1$$

$\rightarrow f(x)$ is not continuous at $x = 0$

$\rightarrow f(x)$ is not continuous

Here $f(0)$ is **a removable discontinuity**, because we can redefine $f(0)$ so that:



$$\lim_{x \to a} f(x) = a^2 \, \forall \, a \in \mathbb{R}$$

and $f(a) = a^2$

$$\rightarrow \lim_{x \to a} f(x) = f(a)$$

$\rightarrow f(x)$ is continuous at $x = 0$

Example 2. Use ε-δ to prove $\lim_{x\to 1} \dfrac{1-x}{1-\sqrt{x}} = 2$ for $x \geq 0$:

In fact we know $\frac{1-x}{1-\sqrt{x}} = \frac{(1+\sqrt{x})\cancel{(1-\sqrt{x})}}{\cancel{1-\sqrt{x}}}$ except $x = 1, x = 0$ aren't in the domain of $\frac{1-x}{1-\sqrt{x}}$.

For $\varepsilon > 0$, find $\delta > 0$ s.t. $0 < |x-1| < \delta \Rightarrow |\frac{1-x}{1-\sqrt{x}} - 2| < \varepsilon$:

We must show $|\frac{1-x}{1-\sqrt{x}} - \frac{2(1-\sqrt{x})}{1-\sqrt{x}}| = |\frac{x-2\sqrt{x}+1|}{\sqrt{x}-1}| = |\frac{(\sqrt{x}-1)^2}{\sqrt{x}-1}| < \varepsilon$.
In other words show $|\sqrt{x}-1|^2 < \varepsilon|\sqrt{x}-1|$. Can we divide both sides by $|\sqrt{x}-1|$?

So we want to show $|\sqrt{x}-1| < \varepsilon$, that is $1-\varepsilon < \sqrt{x} < 1+\varepsilon$.
Using $|x-1| < \delta$ we get $-\delta < x-1 < \delta$, that is $1-\delta < x < 1+\delta$.
Using Triangle Inequality we get $1-\sqrt{\delta} \leq \sqrt{x} \leq 1+\sqrt{\delta}$.
Thus $\delta = \varepsilon^2$ suffices. ■

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In other words show $|\sqrt{x}-1|^2 < \varepsilon|\sqrt{x}-1|$. Can we divide both sides by $|\sqrt{x}-1|$? First we have to show $|\sqrt{x}-1| \neq 0$.

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In other words show $|\sqrt{x}-1|^2 < \varepsilon|\sqrt{x}-1|$. Can we divide both sides by $|\sqrt{x}-1|$? First we have to show $|\sqrt{x}-1| \neq 0$.
We know:
$0 < |x-1| = |\sqrt{x}+1||\sqrt{x}-1| < \delta \Rightarrow |\sqrt{x} \pm 1| \neq 0$ ✓

So we want to show $|\sqrt{x}-1| < \varepsilon$, that is $1-\varepsilon < \sqrt{x} < 1+\varepsilon$.
Using $|x-1| < \delta$ we get $-\delta < x-1 < \delta$, that is $1-\delta < x < 1+\delta$.
Using Triangle Inequality we get $1-\sqrt{\delta} \leq \sqrt{x} \leq 1+\sqrt{\delta}$.
Thus $\delta = \varepsilon^2$ suffices. ■

When the limit does not exist

When does $\lim_{x \to a} f(x)$ not exist?

Example 3 (Infinity) Prove: $\displaystyle\lim_{x \to 0} \frac{1}{x}$ does not exist

Example 4 (Jump) Prove: $\displaystyle\lim_{x \to 0} f(x)$ does not exist

for $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Example 5 (Alternating) Prove: $\displaystyle\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist

See Appendix for answers.

Calculating limits of functions

How to quickly calculate $\lim_{x\to a} f(x) = L$?

Theorem (Arithmetic operations on limits of functions)

- $f \in C \Rightarrow \lim_{x\to a} f(x) = f(a)$
- $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M \Rightarrow$

$$\lim_{x\to a}(f \pm g) = L \pm M$$

$$\text{"} \quad fg \text{ "} \quad LM$$

$$M \neq 0 \Rightarrow \lim_{x\to a} \frac{f}{g} = \frac{L}{M}$$

- $\lim_{x\to M} f(x) = L, \lim_{x\to a} g(x) = M \Rightarrow \lim_{x\to a} f \circ g = L$

Proving continuity

How to quickly prove continuity?

Theorem (Arithmetic operations on continuity)

$f, g \in C \Rightarrow$

$$\bullet f \pm g \in C$$

$$\bullet fg \in C$$

$$\bullet g(x) \neq 0 \, \forall \, x \Rightarrow \frac{f}{g} \in C$$

$$\bullet f \circ g \in C \qquad \textit{(composition of functions)}$$

* If you can draw the graph of $f(x)$ without lifting your pen, then $f \in C$.

Higher dimensions

In higher dimensions:

Theorem (Arithmetic operations)

- $\vec{f}, \vec{g} \in C \Rightarrow$
$$\vec{f} \pm \vec{g} \in C$$
$$\vec{f} \cdot \vec{g} \in C \ ?$$
$$\exists\, \vec{f} \circ \vec{g} \Rightarrow \vec{f} \circ \vec{g} \in C$$
- $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = L, \lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) = M \Rightarrow \lim_{\vec{x} \to \vec{a}} (\vec{f} \pm \vec{g}) = L \pm M$
$$\text{"} \ \vec{f} \cdot \vec{g} \ \text{"} \ LM \ ?$$
- $\lim_{\vec{x} \to M} \vec{f}(\vec{x}) = L, \lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) = M \Rightarrow \lim_{\vec{x} \to \vec{a}} \vec{f} \circ \vec{g} = L$

Sandwich theorem

Single-variable functions also satisfy:

Theorem (Sandwich theorem)

On an open interval containing c:

$$
\begin{aligned}
f(x) \leq g(x) \leq h(x),\\
\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L\\
\Rightarrow \lim_{x\to c} g(x) = L
\end{aligned}
$$

Example 6. Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Hint:

Sandwich theorem

Single-variable functions also satisfy:

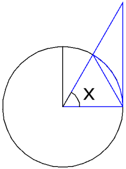
Theorem (Sandwich theorem)

On an open interval containing c:

$$\begin{aligned} & f(x) \leq g(x) \leq h(x), \\ & \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L \\ \Rightarrow & \lim_{x \to c} g(x) = L \end{aligned}$$

Example 6. Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Hint:



Sandwich theorem

Example 6. Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Proof:

For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:
For $x > 0$ ($x < 0$ is similar):
The various areas satisfy:

$$\frac{\sin x}{2} < \frac{x}{2\pi}\pi < \frac{\tan x}{2}$$

$$\sin x < x < \tan x$$

$$\frac{1}{\sin x} > \frac{1}{x} > \frac{1}{\tan x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

$$\lim_{x\to 0} 1 = \lim_{x\to 0} \cos x = 1$$

$$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1 \blacksquare$$

Sandwich theorem

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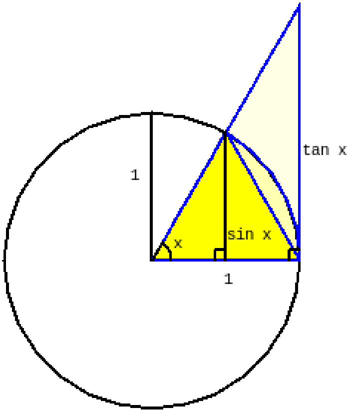
Proof:



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$$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1 \blacksquare$$

Other results

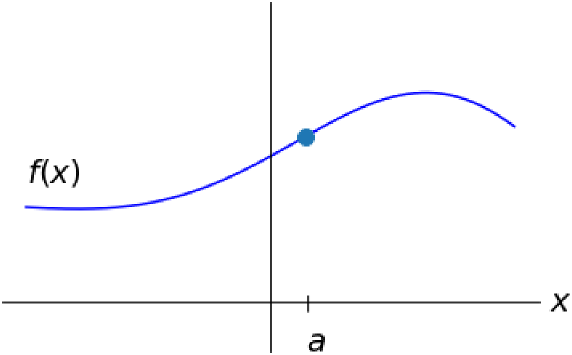
Other properties of continuous functions include boundedness, extreme value theorem, and intermediate value theorem.

Section Contents

Derivatives

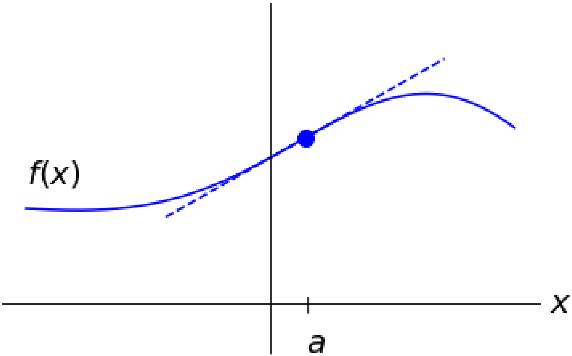
Derivative : The slope (rate of change) of a curve at any point

What is the slope of $f(x)$ at $x = a$?



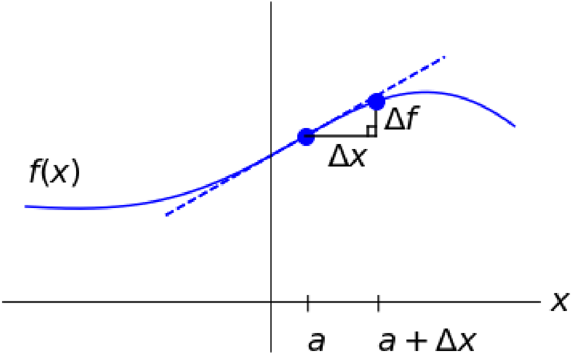
Derivatives

It is the slope of the tangent line:



Derivatives

It is approximately $\frac{\Delta f}{\Delta x}$:



But it needs $\Delta x \to 0$:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

This is the definition of the derivative of $f(x)$ at $x = a$.

Definition

Definition (Derivative)

The derivative of $f(x)$ is

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It is also written as $f'(x)$. It is sometimes called the differential quotient.

The study of derivatives is the study of rates of change.

Definition

Definition (Differentiable)

$f(x)$ *is differentiable* $\iff$ $f(x)$ *is differentiable at each point*

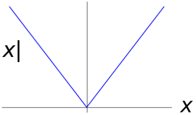
$f(x)$ *is differentiable at* $x = a$ $\iff$ *the derivative of* $f(x)$ *at* $x = a$ *exists*

Differentiability

Not all $f(x)$ are differentiable :

Example 7

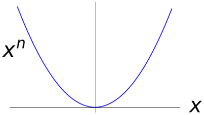
$f(x) = |x|$



Differentiable at $x = 0$?

Example 8
Powers

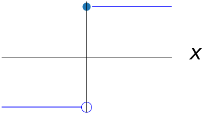
$f(x) = x^n$



Differentiable?

Example 9

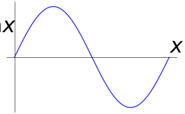
$f(x)$



Differentiable at $x = 0$?

Example 10

$f(x) = \sin x$



Differentiable?

Differentiability

Example 7

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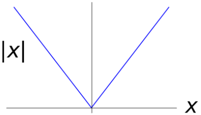

Not differentiable

Example 8
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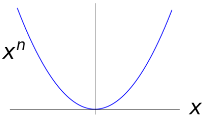


Differentiable

Calculate the limit:

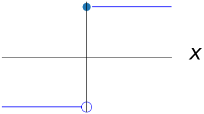
$$f'(x) = \begin{cases} nx^{n-1} & n \in \mathbb{N} \\ 0 & n = 0 \end{cases}$$

For the proof for $n \in \mathbb{R}$,

see Example 14

Example 9

$f(x)$



Not differentiable

Example 10

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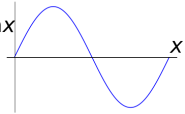

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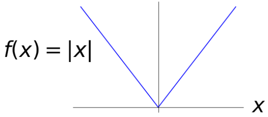
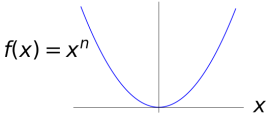
Differentiability

Example 7



Not differentiable

Example 8
Powers



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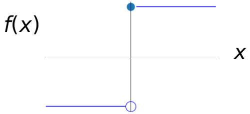

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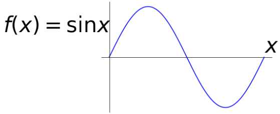
Example 10



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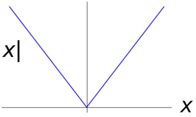
Differentiability

Example 7

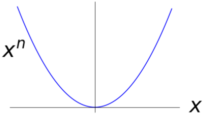
$f(x) = |x|$



Not differentiable

Example 8 Powers

$f(x) = x^n$



Differentiable

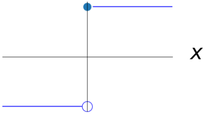
Calculate the limit:

$$f'(x) = \begin{cases} nx^{n-1} & n \in \mathbb{N} \\ 0 & n = 0 \end{cases}$$

For the proof for $n \in \mathbb{R}$, see Example 14

Example 9

$f(x)$



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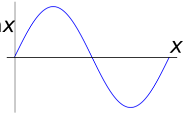
Example 10

$f(x) = \sin x$



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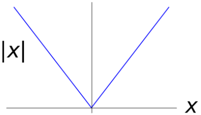

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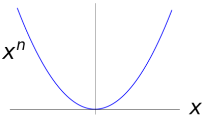

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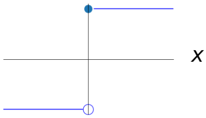

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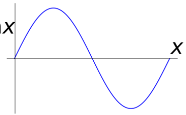

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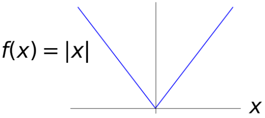

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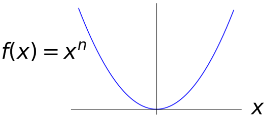

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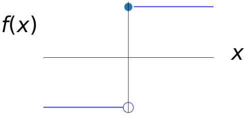

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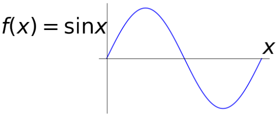

Differentiable

Calculate the limit:

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Differentiability

When is $f(x)$ differentiable?

Clearly, $f(x)$ must be continuous. (Necessary)
If $f(x)$ is continuous then is $f(x)$ differentiable? (Sufficient)

Differentiability

When is $f(x)$ differentiable?

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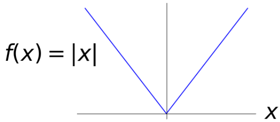
Counterexample:



$\therefore$ Continuity is a necessary but not sufficient condition for differentiability.

Differentiability

It is also easy to prove differentiable $\Longrightarrow$ continuous using ε-δ:

For some a, given $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, let

1. $|x-a| = |h-0| < \delta \Longrightarrow \left|\frac{f(a+h)-f(a)}{h} - f'(a)\right| < \varepsilon_1$
2. $\varepsilon = h(\varepsilon_1 + |f'(a)|)$

then using Triangle Inequality the result follows.

Differentiability

Theorem

$$\textit{Differentiable} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \textit{Continuous}$$

Other properties of differentiable functions include l'Hôpital's rule and mean value theorem.

Calculation

Besides ϵ-δ one can also use:

Theorem (Arithmetic operations on derivatives)

$$\begin{aligned}(f+g)' &= f'+g'\\ (fg)' &= f'g+fg'\\ g\neq 0 \Rightarrow \left(\frac{f}{g}\right)' &= \frac{f'g-g'f}{g^2}{}^*\end{aligned}$$

Theorem (Chain Rule)

$$(f\circ g)'(x) = f'(g)g'(x)$$

to calculate derivatives.

*Just remember: "low d high over high d low, all over low squared"

Examples

Example 11. Derivative of inverse functions

$$y = f(x), \ (f^{-1})'(y) = ?$$

$$f^{-1}(y) = x$$

take derivative, Chain Rule $\rightarrow (f^{-1})'(y) \cdot y' = 1$

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Example 12. Logarithms

$$\frac{d}{dx} \ln x = \frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \to 0} \ldots$$

$$= \lim_{h \to 0} \frac{h \ln\left(\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}\right)}{h} = \ln(e^{\frac{1}{x}})^{\dagger} = \frac{1}{x}$$

† e is defined as $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$, can prove $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

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Examples

Example 13. Exponentials

$$\frac{d}{dx}e^x$$

$$y = e^x, \text{ inverse } \rightarrow (f^{-1})'(y) = \frac{d}{dy}\ln y = \frac{1}{f'(x)} =$$

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Example 14. Power Exponentials

$$\frac{d}{dx}u(x)^{v(x)} = \frac{d}{dx}u(x)^{v(x)} = \frac{d}{dx}e^{\ln(u^v)} = \frac{d}{dx}e^{v\ln u} =$$

$$\text{Chain Rule } \rightarrow = e^{v\ln u}(v\ln u)' = u^v\left(v'\ln u + \frac{v}{u}u'\right)$$

$$\text{Thus } u = x, v = n \rightarrow \frac{d}{dx}x^n = nx^{n-1},\ n \in \mathbb{R}$$

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Higher order derivatives

Repeated differentiation results in:

Definition (Higher order derivative)

The nth derivative of $f(x)$ means to repeatedly take the derivative of $f(x)$ n times, and is written as:

$$\frac{d^n}{dx^n}f(x) = \frac{d^n f}{dx^n} = f^{(n)}(x)$$

This is also a function, and its value at $x = a$ is written as:

$$\frac{d^n}{dx^n}f(x)\bigg|_a = \frac{d^n f}{dx^n}\bigg|_a = f^{(n)}(a)$$

Taylor's theorem

Functions with $n+1$ derivatives also satisfy Taylor's theorem:

Theorem (Taylor's theorem with Lagrange remainder)

If the 0th through $n+1$th derivatives of $f(x)$ exist on $[a, x]$:

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k}_{\textit{nth order Taylor polynomial}} + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}}_{\textit{error}}$$

for some $c \in [a, x]$.

Taylor's theorem

- One can see that for $k = 1, 2, ..., n$:

$$T_n^{(k)}(x) = f^{(k)}(a)$$

where $T_n(x)$ is the nth order Taylor polynomial of f.

- For the proof of the error term（Lagrange remainder） see Appendix.

- Taylor's theorem is highly useful for approximating functions with smaller and smaller errors.

Partial derivatives

Multivariable functions only have partial derivatives:

Definition (Partial derivative)

The partial derivative of $f(\vec{x}) = f(x_1, \ldots, x_n)$ with respect to x_i is:

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and is also written as $\partial_i f$.

Directional derivatives

These can be generalized to directional derivatives:

Definition (Directional derivative)

The directional derivative of $f(\vec{x})$ along $\hat{u}$ is:

$$\frac{\partial f}{\partial \hat{u}} = \partial_{\hat{u}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h}$$

Gradients

Gradients are the nD equivalents of derivatives:

Definition (Gradient)

- *The gradient of* $f(\vec{x}) = f(x_1, ..., x_n)$ *is*

$$\nabla f = < \partial_1 f, \partial_2 f, ..., \partial_n f > .$$

- $f(\vec{x})$ *is differentiable iff*

$$f(\vec{x} + \vec{h}) - f(\vec{x}) = \nabla f \cdot \vec{h} + o(\|\vec{h}\|) \qquad (\|\vec{h}\| \to 0).^{\ddagger}$$

- $\nabla f \cdot \vec{h} \quad (\|\vec{h}\| \to 0)$ *is called the differential of* f.

---

‡ $a = o(b) \quad (h \to 0)$ 的意思是 $\lim_{h \to 0} \frac{a}{b} = 0$。
" $O(b)$ " $= M \in \mathbb{R}$。

Jacobians

These can be generalized to derivatives of multivariable mappings:

Definition (Jacobian)

- *The Jacobian of* $\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix}$ *is*

$$\nabla \vec{f} = J = \begin{bmatrix} & & \\ & J_{ij} = \frac{\partial f_i}{\partial x_j} & \\ & & \end{bmatrix}.$$

- $f(\vec{x})$ *is differentiable iff*

$$\vec{f}(\vec{x} + \vec{h}) - \vec{f}(\vec{x}) = J\vec{h} + o(\|\vec{h}\|) \qquad (\|\vec{h}\| \to 0).$$

- $J\vec{h} \quad (\|\vec{h}\| \to 0)$ *is called the differential of* $\vec{f}$.

Directional derivatives

Theorem (Directional derivative)

$$\partial_{\hat{u}} f = \nabla f \cdot \hat{u}$$

$$\partial_{\hat{u}} \vec{f} = J\hat{u}$$

Note: $\hat{u}$ *must be a* $\boxed{\textit{unit}}$ *vector.*

The proof of this result is left as an exercise for the reader.

Differentiability

Theorem

$$\begin{tikzcd} & & \textit{Continuous} \\ \textit{All directional derivatives continuous (Continuously differentiable, written } f \in C^1\textit{)} \arrow[r, Rightarrow] & \textit{Differentiable} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & & \textit{All directional derivatives exist} \arrow[u, Rightarrow, "/" marking] \end{tikzcd}$$

Example 15. $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Differentiable at $(0,0)$? Continuously differentiable?

Example 16. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

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All directional derivatives exist ~~Continuous~~

Section Contents

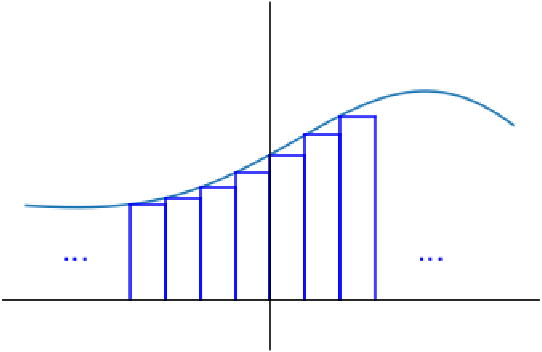
Integrals

Integral: Antiderivative $=$ Area

Why is this true?

Riemann Integral

Because Riemann Integral ("Integral"): Area $=$ Sum



Riemann Integral

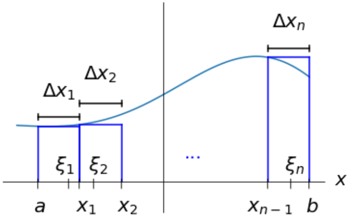
Definition (Riemann Integral)

The (Riemann) integral of $f(x)$ on $[a, b]$ is

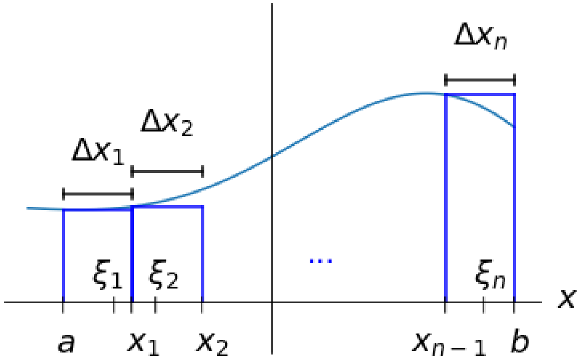
$$\lim_{\|\pi\|\to 0} \sum_{1}^{n} f(\xi_i)\Delta x_i$$

iff it exists and is equal for all $\{\xi_i \in [x_{i-1}, x_i] \mid i = 1, ..., n\}$.

Here π is a partition of $[a, b]$: $a = x_0 < x_1 < ... < x_n = b$.



Riemann Integral



$\{x_i\}$ is the sequence of partition points of π .

$\|\pi\| = \max_{1\le i\le n}\{\Delta x_i = x_i - x_{i-1}\}$ is the norm of π .

$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ is the Riemann sum of f .

$\{\xi_i\}$ are the sample points of the Riemann sum.

Upper sum and lower sum

Definition (Upper sum)

The upper sum of $f(x)$ is just the Riemann sum of $f(x)$ with

$$\xi_i = M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Definition (Lower sum)

" lower sum "

" m_i " inf "

Sup and inf

Here sup is the supremum: least upper bound.

Definition (Supremum)

$\sup X = M$ *iff*

1. $M \geq x \; \forall \; x \in X$
2. $\forall \; \varepsilon > 0 \; \exists \; x_\varepsilon \in X$ *s.t.* $x_\varepsilon > M - \varepsilon$

" inf "infimum " greatest lower bound "

Definition (Infimum)

inf " m "

1. $m \leq$ "
2. " $x_\varepsilon < m + \varepsilon$

Integrability

Definition (Integrable)

$f(x)$ is (Riemann) integrable $\iff$ the integral of $f(x)$ exists

Theorem

$f(x)$ is continuous on $[a, b] \implies f(x)$ is integrable on $[a, b]$

Proof sketch:

Define $x_i, M_i = f(s_i), m_i = f(t_i)$ as previously.

$$\forall\, s_i, t_i \in [a, b] : |s_i - t_i| \leq \Delta x_i \leq \|\pi\| < \delta \Rightarrow$$

$$\Rightarrow |f(s_i) - f(t_i)| < \frac{\varepsilon}{b-a} \Rightarrow$$

$$\Rightarrow \sum M_i \Delta x_i - \sum m_i \Delta x_i < \varepsilon$$

Notation

The integral of f from a to b is written as:

$$\int_a^b f(x)dx$$

("Definite integral")

(a, b are upper, lower limits of integration)

An antiderivative of $f(x)$ is written as:

$$F(x)$$

$$(F'(x) = f(x))$$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

$$\int_a^b f(x)dx = F(b) - F(a)$$

(Part 2: Newton-Leibniz Formula)

$$\frac{d}{dx}\int^x f(x)dx = f(x)$$

(Part 1)

Proof sketch:

$$\int_a^b F'(x)dx = \lim_{\|\pi\|\to 0} \sum_i F'(\xi_i)\Delta x_i$$

$$\lim_{\|\pi\|\to 0} F'(\xi_i) = \lim_{\|\pi\|\to 0} \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

Notation

The indefinite integral of $f(x)$ is written as:

$$\int^x f(x)dx = F(x) \quad \underbrace{+C}_{\text{any constant}}$$

Thus integration is a matter of finding antiderivatives. Other methods for calculating integrals include partial fraction decomposition, integration by parts, u-substitution, etc.

Calculating integrals

Example 17.

Use

1. The geometric meaning of the integral
2. Fundamental theorem of calculus

to find the integral of $f(x) = x$ from 0 to t.

Proof:

1. It is just the area of this triangle: $\frac{t^2}{2}$.

2. The antiderivative of $f(x)$ is $F(x) = \frac{x^2}{2}$.
   Thus $\int_0^t f(x)dx = F(t) - F(0) = \frac{t^2}{2}$.

Calculating integrals

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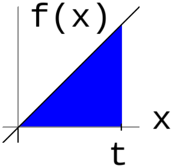


2. The antiderivative of $f(x)$ is $F(x) = \frac{x^2}{2}$.
Thus $\int_0^t f(x)dx = F(t) - F(0) = \frac{t^2}{2}$.

Improper integrals

Definition (Improper integral)

The following are all improper integrals:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx = \lim_{b\to+\infty} F(b) - F(a)$$

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx = F(b) - \lim_{a\to-\infty} F(a)$$

$$\lim_{x\to a^+} f(x) = \infty \text{ or } \lim_{x\to b^-} f(x) = \infty \Rightarrow$$

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^c f(x)dx + \lim_{\varepsilon\to 0^+} \int_c^{b-\varepsilon} f(x)dx$$

Multiple integrals

Definition (Multiple integrals)

Double, triple, and other multiple integrals are just repeated integration from inside to outside:

$$\int_c^d \int_a^b f(x,y)dxdy \xrightarrow[\text{Treat } y \text{ as constant}]{=} \int_c^d F(b,y) - F(a,y)dy$$

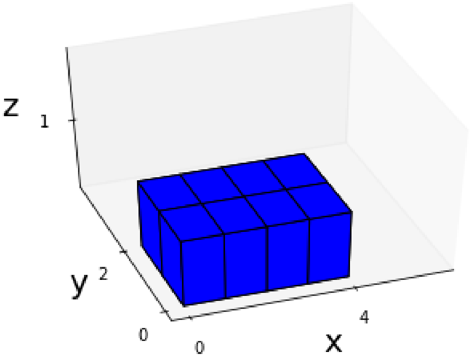
$$\frac{\partial}{\partial x}F(x,y) = f(x,y)$$

Calculating multiple integrals

Example 18.

Find

1. $\int_0^2 \int_0^4 1dxdy$
2. The volume:



Calculating multiple integrals

Proof:

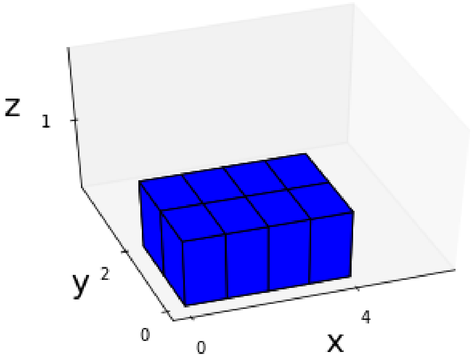
1. $\int_0^2 \int_0^4 1dxdy = \int_0^2 x\Big|_0^4 dy = \int_0^2 4dy = 4y\Big|_0^2 = 8$
2. $4 \cdot 2 \cdot 1 = 8$

   We see that multiple integration is just calculating volume.

Calculating multiple integrals

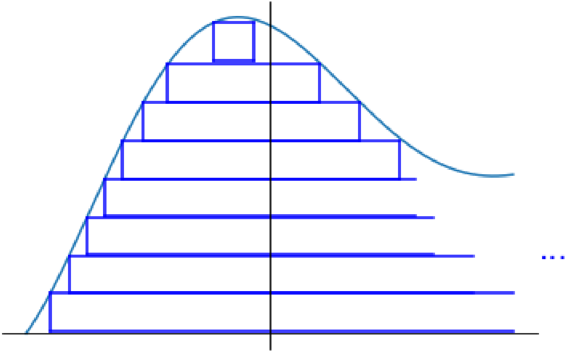
Proof:

1. $\int_0^2 \int_0^4 1 dxdy = \int_0^2 x\Big|_0^4 dy = \int_0^2 4dy = 4y\Big|_0^2 = 8$
2. $4 \cdot 2 \cdot 1 = 8$
   We see that multiple integration is just calculating volume.



Lebesgue integral

Besides Riemann integration there is also Lebesgue integration:



Its main use is that any measurable function on $[a, b]$ is Lebesgue integrable, for example the Dirichlet function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \backslash \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

Section Contents

Appendix A

Notation:

$\forall$	for all
$\in$	element of
$\exists$	there exists
s.t.	such that
$\therefore$	therefore
$\blacksquare$	QED, Quod Erat Demonstrandum
∞	$\pm\infty$, infinity
$\hat{u}$	unit vector, direction vector

Appendix B

Example 3 Answer 1: $\lim_{x\to 0^+} \frac{1}{x} = +\infty \neq \lim_{x\to 0^-} \frac{1}{x} = -\infty$

Answer 2: $\lim_{x\to 0} \frac{1}{x}$ tends toward infinity

Thus $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Example 4 Answer: $\lim_{x\to 0^+} f(x) = +1 \neq \lim_{x\to 0^-} f(x) = -1$

Thus $\lim_{x\to 0} f(x)$ does not exist.

Example 5 Answer: $\lim_{x\to 0} \frac{1}{x}$ alternates between -1 and 1

Thus $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Appendix C

Prove the Lagrange remainder:

$$f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some $c \in [a, x]$.

Proof:

Let $g(t) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(x-t)^k$ and $\lambda(t) = \left(\frac{x-t}{x-a}\right)^{n+1}$.

Using

Theorem (Lagrange mean value theorem)

g and λ are continuous on $[a, b]$, differentiable on (a, b),

$$\lambda(a) = 1,\ \lambda(b) = 0 \Rightarrow$$

$$\exists\ c \in (a, b) \text{ s.t. } g'(c) = \lambda'(c)(g(a) - g(b))$$

Appendix C

Then $\exists\ c \in (a, x)$ s.t. $g'(c) = \lambda'(c)(g(a) - g(x))$.
By direct calculation

$$g'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$
$$\lambda'(t) = -(n+1)\frac{(x-t)^n}{(x-a)^{n+1}}$$
$$g(a) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$
$$g(x) = f(x).$$

Thus $\frac{f^{(n+1)}(c)}{n!}(x-c)^n = -(n+1)\frac{(x-c)^n}{(x-a)^{n+1}}(g(a) - g(x))$.

$$\therefore \text{ Lagrange remainder} = g(x) - g(a)$$
$$= \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1} \blacksquare$$

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