MLE

$$\boxed{f(\rho, \overbrace{n, R, q}^{\text{const.}}) = \frac{\Gamma(R-\rho+1)}{\Gamma(n-\rho+1)} q^{-\rho}}$$

$$\max_{\boxed{\rho = Rp}} f \iff \max_{p} \binom{R(1-p)}{n-Rp} q^{n-Rp} (1-q)^{R-n}$$

$$\iff p_{MLE}$$

"p estimate"

Discrete

$$\boxed{\rho \in \{0, 1, \ldots, n\}} \le R$$

$$n, R \in \mathbb{N}$$

$$q \in [0,1]$$

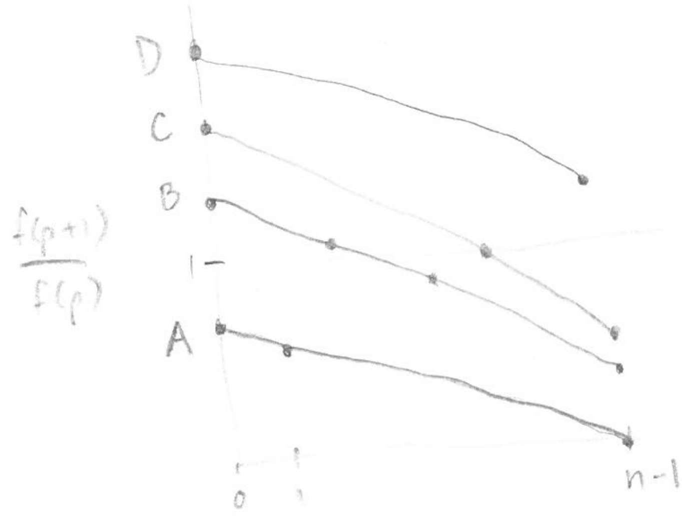
$$\boxed{\frac{f(\rho+1)}{f(\rho)}} = \frac{\Gamma(R-\rho)\,\Gamma(n-\rho+1)}{\Gamma(R-\rho+1)\,\Gamma(n-\rho)} \frac{q^{-\rho-1}}{q^{-\rho}} = \frac{(R-\rho-1)!\,(n-\rho)!}{q(R-\rho)!\,(n-\rho-1)!} = \frac{n-\rho}{q(R-\rho)}$$

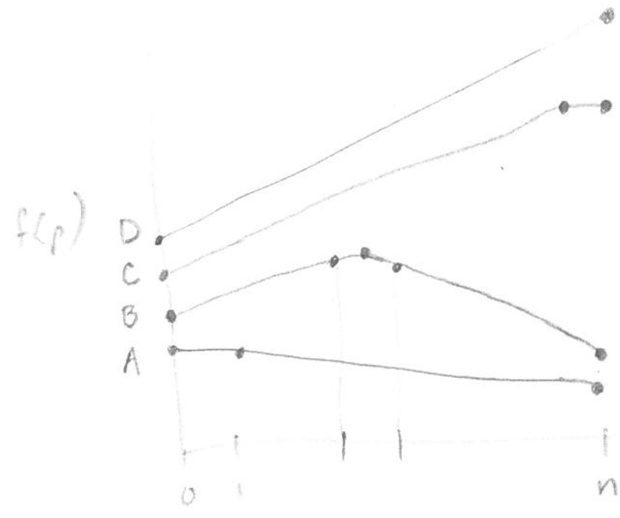
$$\boxed{= \left(\frac{1}{q}\right)\frac{n-\rho}{R-\rho}} = \frac{1}{q}\left(1 - \frac{R-\rho-(n-\rho)}{R-\rho}\right) = \frac{1}{q}\left(1 - \frac{R-n}{R-\rho}\right)$$

monotonic w/ ρ

strictly mono dec w/ ρ from $\frac{1}{q}\frac{n}{R}$ ↓

Cases:





MLE

$n=10,\ R=12$ $\qquad \boxed{X \in \{0,1,\dots,n-1\}}$

$q=.9$ A. $\frac{1}{q}\frac{n}{R} < 1 \rightarrow \boxed{q > \frac{n}{R}}$

$.7$ B. $\frac{1}{q}\frac{n}{R} > 1$, $\exists X$ s.t. $\frac{f(X+1)}{f(X)} < 1$, $\nexists X$ s.t. $\frac{f(X+1)}{f(X)} = 1$

$\rightarrow \boxed{q < \frac{n}{R},\ \exists X \text{ s.t. } \frac{f(X+1)}{f(X)} < 1,\ \nexists X \text{ s.t. } \frac{f(X+1)}{f(X)} = 1}$

$.6$ C. $\boxed{q \le \frac{n}{R},\ \exists X \text{ s.t. } \frac{f(X+1)}{f(X)} = 1}$

$n=10,\ R=10,\ q=.6$ D. $\boxed{q < \frac{n}{R},\ \frac{f(X+1)}{f(X)} > 1 \text{ always}}$

$$\boxed{\frac{f(X+1)}{f(X)} = 1} \iff \frac{1}{q}\left(1 - \frac{R-n}{R-X}\right) = 1 \rightarrow q = 1 - \frac{R-n}{R-X},\ 1-q = \frac{R-n}{R-X},$$

$$R-X = \frac{R-n}{1-q},\quad X = R - \frac{R-n}{1-q} = \frac{\cancel{R} - Rq \cancel{-R} + n}{1-q} = \iff \boxed{X = \frac{n-Rq}{1-q}}$$

A. max at $p=0 \rightarrow \boxed{P_{MLE} = 0}$

B. max at $p = \left\lceil \frac{n-Rq}{1-q} \right\rceil$ or $\left\lfloor \frac{n-Rq}{1-q} \right\rfloor$ s.t. $f(p)$ bigger

$\rightarrow \boxed{P_{MLE} = \frac{1}{R}\left\lceil \frac{n-Rq}{1-q} \right\rceil \text{ or } \frac{1}{R}\left\lfloor \frac{n-Rq}{1-q} \right\rfloor \text{ s.t. } f(p = Rp) \text{ bigger}}$

Python: Hsu data too large, use Stirling's Approx

C. max at $p = \frac{n-Rq}{1-q}$ and $\frac{n-Rq}{1-q} + 1$

$\rightarrow \boxed{P_{MLE} = \frac{n-Rq}{R(1-q)} \text{ and } \frac{n-Rq}{R(1-q)} + \frac{1}{R}}$

D. max at $p = n \rightarrow \boxed{P_{MLE} = \frac{n}{R}}$

MLE

$$\boxed{\begin{array}{l}\text{Stirling's Approx} \\ n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right)\end{array}}$$

there may be asymptotic expansions for $\ln(n!)$

$$\rightarrow f(p,n,R,q) = \frac{(R-p)!}{(n-p)!}q^{-p} \approx \frac{\sqrt{2\pi(R-p)}\,(R-p)^{(R-p)}}{(n-p)!\,e^{(R-p)}}q^{-p}$$

$$\rightarrow \boxed{\ln f(p,n,R,q) \approx \frac{1}{2}\ln(2\pi)+(R-p+\tfrac{1}{2})\ln(R-p)-\ln((n-p)!)-(R-p)-p\ln q}$$

$$\ln n! = \ln\sqrt{2\pi} + \frac{\ln n}{2} + n\ln\left(\frac{n}{e}\right) + \cancel{\ln\left(1+O\left(\frac{1}{n}\right)\right)}$$

$$O\left(\frac{1}{n}\right).$$

Comment

$$\ln(n!) = \sum_{i=1}^{n}\ln(i) = \sum_{i=1}^{n} f(i) \sim \int_1^n f(x)\,dx$$

error!!