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23/9/20 Cont distributions :-

★ **Uniform distribution** $\longrightarrow$ Also called rectangular distribution

A random var X over finite interval (a,b) where $-\infty<a<b<\infty$, if

$$f(x,a,b) = \begin{cases} \frac{1}{b-a} & ; \ a<x<b \\ 0 & ; \ \text{otherwise} \end{cases}$$

Mean, $\mu = \frac{(a+b)}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

Median $= \frac{(a+b)}{2}$

Mode
↳ any pt x in interval (a,b).

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt.$$

If $x \le a$, then, $F(x) = 0$.

If $a < x < b$, then, $F(x) = \int_a^x \frac{1}{b-a}\,dt = \frac{1}{b-a}[t]_a^x = \frac{x-a}{b-a}$

If $x \ge b$,

$$F(x) = \int_a^b \frac{1}{b-a}\,dt = \frac{1}{b-a}|b-a| = 1.$$

Combining,

$$F(x) = \begin{cases} 0 & ; \ x \le a \\ \frac{x-a}{b-a} & ; \ a<x<b \\ 1 & ; \ x \ge b. \end{cases}$$

Q) $f(x) = \begin{cases} 1/4 & , \ |x|<2 \\ 0 & , \ \text{otherwise} \end{cases}$

Find :- i) $P(X<1)$, ii) $P(|X|>1)$, iii) $P(2X+3>5)$.

$\rightarrow$ $f(x) = \frac{1}{4}$ $\quad |x|<2 \quad (-2<x<2)$

$a=-2, \ b=2.$

$F(x) = \frac{(x-a)}{(b-a)} = \frac{x+2}{4}.$

i) $P(X<1) = F(1) = 3/4$

ii) $P(|X|>1) = 1 - P(|X| <= 1)$
$= 1 - P(-1 \le X \le 1)$
$= 1 - (F(1) - F(-1)) = 1 - (3/4 - 1/4)$
$= 1/2$

iii) $P((2X+3)>5) = P(X>1)$
$= 1 - F(1)$
$= \frac{1}{4}$

7/10/20

★ Exponential distribution ⟶

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & \forall\ x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$X \sim Exp(\lambda)$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \left[e^{-\lambda x}\right]_0^{\infty} = 1$$

Let F be distribution func of X, i.e., $F(x) = P(X \leq x)$

If $x < 0$, then $F(x) = 0$

If $x \geq 0$, then,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \lambda e^{-\lambda t}\,dt$$

Integrating, we have,

$$F(x) = \left[-e^{-\lambda x}\right]_0^{x} = 1 - e^{-\lambda x}$$

$$\therefore F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0. \end{cases}$$

i) $\mu_r' = \dfrac{r!}{\lambda^r}$.

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)\,dx = \int_0^{\infty} x^r \lambda e^{-\lambda x}\,dx = \int_{y=0}^{\infty} \frac{y}{\lambda^r} e^{-y}\,dy$$

subs $y = \lambda x$

$$\therefore \mu_r' = \frac{1}{\lambda^r} \int_0^{\infty} y\, e^{-y}\,dy = \frac{1}{\lambda^r}\,\Gamma(r+1) = \frac{r!}{\lambda^r}.$$

ii) Taking $r=1$ in (i),

$$\text{Mean}(X) = \mu_1' = \frac{1}{\lambda}$$

iii) Taking $r=2$ in (i), $\mu_2' = \dfrac{2}{\lambda^2}$

Hence, variance of X, $Var(X) = \mu_2' - \mu_1'^2 = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}$

Q) The time (in hours) required to repair machine exponentially distributed with $\lambda = \frac{1}{3}$ what is the prob that repair time exceeds 3 hours?

→ Let X be time (in hours) required.

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$P(X > 3) = \int_{x=3}^{\infty} f(x)\,dx = \int_{3}^{\infty} \frac{1}{3} e^{-x/3}\,dx$$

$$= \left[-e^{-x/3}\right]_{3}^{\infty} = e^{-1} = \frac{1}{e} = 0.3679$$

☆ Memoryless property

$$P(X > (s+t) \mid X > s) = P(X > t) \quad \forall s \ (t > 0)$$

For any $k > 0$,

$$P(X > k) = \int_{x=k}^{\infty} f(x)\,dx = \int_{k}^{\infty} \lambda e^{-\lambda x}\,dx = \int_{x=k}^{\infty} d\left[-e^{-\lambda x}\right]$$

$$= \left[-e^{-\lambda x}\right]_{k}^{\infty} = \left[0 + e^{-\lambda k}\right] = e^{-\lambda k}$$

$$\therefore P(X > s+t \mid X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$\therefore P(X > s+t \mid X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

Thus, X satisfies memoryless property.

Converse is also true.
If X is a cont. R.V taking only positive values & with memoryless property, then X can have exponential distribution.
Among all cont. prob. distributions.

9/10/20

Q) A fast food chain finds that avg time customers have to wait for service is 45 seconds. If waiting time is treated as exponential random variable, what is the probability that a cust will have to wait more than 5 min given that already he waited for 2 minutes?

→ $\lambda = \frac{1}{45}$

$$P(X > 5\text{min} \mid X > 2\text{min}) = P(X > 3\text{min})$$
$$= P(X > 180\text{sec})$$
$$= 1 - P(X <= 180)$$
$$= 1 - F(180)$$

$\boxed{F(x) = P(X <= x)}$ $= 1 - e^{(-\lambda x)}$

$$P(X > 5\text{min} \mid X > 2\text{min}) = 1 - \left(1 - e^{(-180/45)}\right)$$
$$= e^{(-4)}$$
$$= 0.0183.$$

Q) The time req. to repair a machine exponentially distributed with parameter 1/2. What is the prob. that repair time exceeds 2 hours? What is the conditional prob that a repair time takes at least 10 hours given that its duration exceeds 9 hours?

→ Since X is exponential R.V with $\lambda = \frac{1}{2}$.

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{x}{2}} \text{ for } x > 0$$

$$\therefore P(X > 2) = \int_{x=2}^{\infty} f(x)\,dx = \int_{2}^{\infty} \frac{1}{2} e^{-1/2}\,dx$$
$$= \left[-e^{-x/2}\right]_{x=2}^{\infty} = e^{-1} = 3.679.$$

$$\therefore P(X \geq 10 \mid X \geq 9) = P(X \geq 1) = \int_{x=1}^{\infty} f(x)\,dx \quad \text{memoryless prop.}$$

$$\therefore P(X \geq 10 \mid X \geq 9) = \left[-e^{-x/2}\right]_{1}^{\infty} = e^{-1/2} = 0.6065$$

★ Gamma Distribution ⟶

A cont. R.V with parameters (α, λ) where $\alpha > 0$ & $\lambda > 0$,

$$f(x;\alpha,\lambda) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\boxed{\begin{aligned} \text{Mean}(X) &= \frac{\alpha}{\lambda} \\ \text{Var}(X) &= \frac{\alpha}{\lambda^2} \end{aligned}}$$

Q) Daily consumption of milk in excess of 20000 gallons is distributed as gamma variable with $\alpha = 2$ & $\lambda = \frac{1}{10000}$

The city has daily stock of 30000 gallons. What is the prob. that stock is insufficient in a day?

→ Let X be daily consumption

& $Y = X - 20000$

Given that Y follows $\alpha = 2, \lambda = 1/10000$

$$f(y) = \frac{1 \times y^{2-1}}{(10000)^2\,\Gamma(2)} \times e^{-y/10000} \text{ for } y > 0.$$

$$\therefore f(y) = \frac{y}{(10000)^2} e^{-y/10000} \text{ for } y > 0.$$

Since the city has a daily stock of 30000 gallons, prob. that stock is insufficient on p'clr day:

$$P(X > 30000) = P(Y > 10000) = \int_{10000}^{\infty} \frac{y}{(10000)^2} e^{\frac{-y}{10000}}\,dy$$

$$\therefore P(X > 30000) = \frac{1}{10000} \int_{10000}^{\infty} y\,d\left[-e^{\frac{-y}{10000}}\right]$$

$$P(X > 30000) = \frac{1}{10{,}000} \left[\left[y\,(-e^{-y} \right.\right.$$

★ Weibull Distribution →

A cont. R.V with parameters (α, β) where $\alpha > 0$, $\beta > 0$.

$$f(x; \alpha, \beta) = \begin{cases} \alpha\beta\, x^{\beta-1} e^{-\alpha x^{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

★ Normal distribution → Gaussian distribution.

Limiting case of many distributions.

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty.$$

$-\infty < \mu < \infty$ & $\sigma > 0$.

OR

$$f(x; a, b) = \frac{1}{b\sqrt{2\pi}} \times e^{(x-a)^2/2b^2}.$$

Mean $= a$

Variance $= b^2$.

i) The curve $y = f(x)$ is bell shaped & symmetrical about line $x = \mu$.

ii) The mean, median & mode of X coincide at $x = \mu$.

iii) Since $x = \mu$, mode of X, max. value of $f(x)$ occurs at $x = \mu$

$$f_{max} = [f(x)]_{x=\mu} = \frac{1}{\sigma\sqrt{2\pi}}.$$

Area property :-

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

If X is normal R.V, $X \sim N(\mu, \sigma^2)$, then std. normal variable,

$$Z = \frac{X - \mu}{\sigma}.$$

$$E(Z) = E\left(\frac{(X-\mu)}{\sigma}\right) = \left(\frac{1}{\sigma}\right)(E(X) - \mu) = 0$$

$$Var(Z) = E\left[\left(\frac{(X-\mu)}{\sigma}\right)^2\right] - 0$$
$$= \left(\frac{1}{\sigma^2}\right) E\left((X-\mu)^2\right)$$
$$= \frac{\sigma^2}{\sigma^2} = 1.$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ where } -\infty < Z < \infty.$$

Q) Light bulbs normally distributed with mean = 800 hrs & S.D of 40 hrs.
Find prob. that bulb burns :-
i) more than 834 hrs
ii) b/w 778 and 834 hrs.

→ $Z = \frac{X - \mu}{\sigma} = \frac{X - 800}{40}$

check graph (ss)

i) $P(X > 834)$

$X = 834$, $Z = \frac{834 - 800}{40} = \frac{34}{40} = 0.85$

$P(X>834) = P(Z>0.85)$
$= 0.5 - P(0<Z<0.85)$
$= 0.5 - 0.3023$
$= 0.1977$

ii) $P(778<X<834)$.

$X = 778$

$Z = \frac{778-800}{40} = \frac{-22}{40} = -0.55$.

$P(778<X<834) = P(-0.55<Z<0) + P(0<Z<0.85)$

$\therefore P(778<X<834) = P(0<Z<0.55) + P(0<Z<0.85)$
$= 0.2088 + 0.3023$
$= 0.5111$

12/10/20

Q) savings bank acc showed avg balance of Rs150 & S.D of Rs50. Assuming acc. balance are normally distributed, find what % of account is :

i) over Rs200

ii) b/w Rs120 & Rs170

iii) less than Rs75.

0.84124

→ X: balance of savings bank acc

$\mu = 150$ & SD $\sigma = 50$.

$Z = \frac{(X-\mu)}{\sigma} = \frac{X-150}{50}$

i) $P(X>200) = P(Z>1)$

$X = 200$, $Z = \frac{200-150}{50} = 1$

$P(X>200) = P(Z>1) = 0.5 - P(0<Z<1)$
$= 0.5 - 0.3413$
$= 0.1587$.

$\therefore$ In % $= 15.87\%$

ii) when $X = 120$, $Z = \frac{120-150}{50} = \frac{-30}{50} = -0.6$.

$X = 170$, $Z = \frac{170-150}{50} = \frac{20}{50} = 0.4$

Hence, $P(120 < X < 170) = P(-0.6 < Z < 0.4)$
$= P(-0.6 < Z < 0) + P(0 < Z < 0.4)$
$= P(0 < Z < 0.6) + P(0 < Z < 0.4)$
$= 0.2257 + 0.1554$
$= 0.3811$.

$\therefore$ In % = 38.11%.

iii) $X = 75$, $Z = \frac{75-150}{50} = \frac{-75}{50} = -1.5$.

$P(X < 75) = P(Z < -1.5) = 0.5 - P(-1.5 < Z < 0)$

$\because$ std. normal distribution is symmetrical abt line $Z=0$,

$P(X < 75) = 0.5 - P(0 < Z < 1.5)$
$= 0.5 - 0.4332 = 0.0668$

$\therefore$ In % = 6.68%

2/10/20. MOMENTS OF RANDOM VARIABLE.

Let X be RV, $k \to$ +ve int & $c \to$ const

$\boxed{\text{Moment} = E[(X-c)^k]}$ k^{th} moment of X abt point c.

- $c = 0$

~~abt~~ Moments abt 0 / origin
↳ also called raw moments.

- $\mu_k' = E[X^k]$, $k = 1, 2, \ldots$

$\mu_1' = \mu$

$\mu_0' = 1$

If X is cont. R.V,

$$\mu_k' = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

If X is discrete RV,

$$\mu_k' = \sum_i x_i^k f(x_i)$$

* Moments about mean $\longrightarrow$ Central moments.

$C = \mu$

$\mu_k = E[(X-\mu)^k]$, $k = 1, 2, \ldots$

$\mu_0 = E[(X-\mu)^0] = 1$

$\mu_1 = E[(X-\mu)] = 0$

$\mu_2 = E[(X-\mu)^2] = \sigma^2$ = variance.

If X is cont.

$$\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f(x)\,dx$$

If X is discrete

$$\mu_k = \sum_i (x_i-\mu)^k f(x)$$

Significance of second CM, $\mu_2 = \sigma^2$: describes spread / dispersion of distribution.

Third CM, μ_3 : gives a measure of skewness of the distribution.

For symmetric distribution, $\mu_3 = 0$

If $\mu_3 < 0$, distribution skewed to left (tail of dist is heavier on left)

If $\mu_3 > 0$, skewed to right (tail of distribution is heavier on right).

Fourth CM, μ_4 : measure of peakedness of distribution compared to normal distribution of same variance.

★ Relation b/w Raw & Central moments ⟶

$$\mu_k = \mu_k' - {}^kC_1 \mu_{k-1}' \mu_1' + {}^kC_2 \mu_{k-2}' \mu_1'^2 - \cdots + (-1)^k \mu_1'^k.$$

By binomial expansion,

$$(X-\mu)^k = X^k - {}^kC_1 X^{k-1}\mu + {}^kC_2 X^{k-2}\mu^2 - \cdots + (-1)^k \mu^k.$$

by taking expectation both sides,

$\mu_0 = 1$, $\mu_1 = 0$ (always).

Putting $k=2$,

$$\mu_2 = \mu_2' - {}^2C_1 \mu_1'\mu_1' + \mu_1'^2.$$

$$= \mu_2' - 2\mu_1'^2 + \mu_1'^2.$$

$$= \mu_2' - \mu_1'^2.$$

14/10/20

Q) Find first 4 raw moments & using them the first four central moments for random var. X with PDF :-

$$f(x) = \begin{cases} kx^2 e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

⟶ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} kx^2 e^{-x}\,dx = 1$$

$$k\Gamma(3) = k(2!) = 1$$

$$\therefore k = \frac{1}{2}.$$

Now, calculate μ_r'.

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} \frac{1}{2} x^{r+2} e^{-x}\,dx$$

$$= \frac{1}{2}\Gamma(r+3)$$

$$= \frac{(r+2)!}{2}.$$

$\therefore \mu_1' = \frac{3!}{2} = 3, \ \mu_2' = \frac{4!}{2} = 12, \ \mu_3' = \frac{5!}{2} = 60, \ \mu_4' = \frac{6!}{2} = 360$

$\therefore \mu_2 = \mu_2' - \mu_1'^2 = 12 - 9 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 60 - 108 + 54 = 6$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 360 - 720 + 648 - 243$
$= 45$

Q) $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

i) Show that $E[X^n] = \frac{2}{(n+1)(n+2)}$

ii) Use this to evaluate $E[(2x+1)^2]$

$\rightarrow$ i) $E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx = \int_0^1 x^n \cdot 2(1-x)\,dx = 2\int_0^1 (x^n - x^{n+1})\,dx$

Integrating,

$E[X^n] = 2\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

$= 2\left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{2}{(n+1)(n+2)}$

ii) $E[(2x+1)^2] = E[4x^2+4x+1] = 4\times E[x^2] + 4E[x] + 1$

$= 4\left[\frac{2}{3\times 4}\right] + 4\left[\frac{2}{2\times 3}\right] + 1$

$= \frac{2}{3} + \frac{4}{3} + 1 = 3$.

★ Moment generating function (MGF) $\longrightarrow$ generates all its (raw) moments.

$$M_X(t) = E(e^{tX}), \quad t \in \mathbb{R}.$$

If X is cont R.V, then,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

If X is discrete with mass pts $\{x_i\}$,

$$M_X(t) = \sum_i e^{tx_i} f(x_i).$$

THEOREM:- Let X be R.V with $M_X(t)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

the rth moment,

$$\mu_r' = M_X^{(r)}(0) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}$$

PROOF:- $M_X(t) = E([e^{tX}]) = E\left[1 + tX + \frac{t^2}{2!}X^2 + \cdots + \frac{t^r}{r!}X^r + \cdots\right]$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$\therefore \mu_r' = M_X^{(r)}(0) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}.$$

Q) $f(x) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ (θ instead of λ)

Find $M_X(t)$. Does $M_X(t)$ exist for all values of t?

$\rightarrow$ $M_X(t) = E(e^{tX}) = \int_{x=0}^{\infty} e^{tx}\cdot\theta e^{-\theta x}\,dx$

$$= \theta \int_0^{\infty} e^{-(\theta - t)x}\,dx.$$

$\hookrightarrow$ converges only when $|t| < \theta$.

$\therefore M_X(t)$ exists only for values of t such that $|t| < \theta$.

$$M_X(t) = \theta\left[-\frac{e^{-(\theta-t)x}}{\theta-t}\right]_0^\infty = \frac{\theta}{\theta-t}.$$

$5-(-6)$

if $\theta = 5, t = -6$.

$$M_X(t) = 5\int_0^\infty e^{-(11)x}\,dx.$$

$$= 5\int_0^\infty e^{-11x}\,dx = 5\,(-11)\left[e^{-11x}\right]_0^\infty$$

$$= \frac{5}{-11}\times\left[e^{\infty} - e^{0}\right]$$

$$= \frac{-5}{11}[0-1] = \frac{5}{11}.$$

Q) Perfect coin is tossed twice. Find MGF of no. of heads. Hence, find mean & variance.

→

X	0	1	2
P(X=x)	1/4	1/2	1/4.

$$M_X(t) = E[e^{tX}] = \sum_x e^{tX}\cdot P(X=x) = \left(e^{t0}\,\frac{1}{4}\right) + \left(e^{t\times1}\times\frac{1}{2}\right) + \left(e^{t\times2}\times\frac{1}{4}\right)$$

$$M_X(t) = \frac{1}{4}\left[1+2e^{t} + e^{2t}\right] = \frac{1}{4}\left[1+e^{t}\right]^2$$

$$M_X'(t) = \frac{1}{2}(1+e^t)e^t = \frac{1}{2}(e^t + e^{2t}) \;\&\; M_X''(t) = \frac{1}{2}(e^t + 2e^{2t}).$$

$\therefore$ Mean$(X) = \mu_1' = M_X'(0) = 1$ & $\mu_2' = M_X''(0) = \frac{3}{2}$.

& Variance$(X) = \mu_2' - \mu_1'^2 = \frac{3}{2} - 1 = \frac{1}{2}$.

Q) Find MGF of discrete RV X with PMF,

$$f(x) = \frac{1}{n} \text{ for } x = 1, 2, \ldots, n.$$

$\rightarrow$ $$M_X(t) = E[e^{tX}] = \sum_{x=1}^{n} e^{tx} \frac{1}{n} = = \frac{1}{n}[e^t + e^{2t} + \cdots + e^{nt}]$$

$$= \frac{e^t}{n}[1 + e^t + (e^t)^2 + \cdots + (e^t)^{n-1}]$$

$$= \frac{e^t}{n} \frac{1 - e^{nt}}{1 - e^t}$$