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Abstract

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1 Introduction

1.1 Friedkin-Johsen Model and Opinion Formation Games

In the Friedkin-Johsen model (FJ-model) an undirected graph $G(V, E)$ with n nodes, is assumed, where V denotes the agents and E the social relations between them. Each agent i poses an internal opinion $s_i \in [0, 1]$ and a self confidence coefficient $\alpha_i \in [0, 1]$. At each round $t \geq 1$, agent i updates her opinion as follows:

$$x_i(t) = (1 - \alpha_i) \frac{\sum_{j \in N_i} x_j(t-1)}{|N_i|} + \alpha_i s_i, \tag{1}$$

where N_i is the set of her neighbors. The simplicity of the update rule (1) makes the FJ-model plausible, because in real social networks it is very unlikely that agents change their opinions according to complex rules. Based on the FJ-model, in [BKO11], they propose an *opinion formation game* in which the strategy that each agent i plays, is the opinion $x_i \in [0, 1]$ that she publicly expresses incurring her a cost

$$C_i(x_i, x_{-i}) = (1 - a_i) \sum_{j \in N_i} (x_i - x_j)^2 + a_i (x_i - s_i)^2. \tag{2}$$

In [**?**] they prove that it always admits a unique Nash Equilibrium x^* and has Price of Anarchy 9/8 if G is undirected and $O(n)$ in the directed case. We denote an instance of this game as $I = (G, s, a)$. The FJ-model can be viewed as the best-response dynamics of the of the opinion formation game. More precisely, each agent i has an opinion $x_1(0), \ldots, x_n(0)$ and at round $t \geq 1$, suffers the cost (3) and updates her opinion so as to minimize her individual cost

$$x_i(t) = \underset{x \in [0,1]}{\operatorname{argmin}} (1 - a_i) \sum_{j \in N_i} (x - x_j(t-1))^2 + a_i (x - s_i)^2 \tag{3}$$

In [GS14] it is proved that for any instance $I = (G, s, a)$ the opinion vector $x(t) = (x_1(t), \ldots, x_n(t))$ converges to the unique equilibrium point x^*. Moreover, this is achieved by an update rule that is very simple but more importantly it is *rational*, in the sense that each agent i adopts this rule in order to minimize her individual cost.

1.2 Opinion Formation Games with Random Payoffs

In the game defined by (2), agent's i cost $C_i(x_i, x_{-i})$ is a deterministic function of the opinion vector x. Many recent works (see e.g. [ZLZ17]) study games with random payoffs, that is agent's i cost $(C_i(x_i, x_{-i}))$ is a random variable. The random payoff setting can be much more realistic, since randomness may naturally occur because of incomplete information, noise or other stochastic factors. Motivated by this line of research we introduce a random payoff variant of the opinion formation game (2).

Definition 1. *Let $I = (G, s, a)$ an instance of the opinion formation game and x the opinion vector. Each agent i,*

- *picks uniformly at random one of her neighbors $j \in N_i$*
- *suffers cost $C_i(x_i, x_{-i}) = (1 - a_i) \sum_{j \in N_i} (x_i - x_j)^2 + a_i (x_i - s_i)^2$*

This game is more compatible to a realistic settings since in real world networks (e.g. Facebook, Twitter e.t.c.), each agent may have several hundreds of friends. As a result is far more reasonable to assume that every day, each agent meets a random small subset of her acquaintances and suffers a cost based on how much she disagrees with them. Since the expected cost of our random payoff variant equals the cost of game (2), $\mathbf{E}[C(x_i, x_{-i})] = (1 - a_i) \sum_{j \in N_i} (x_i - x_j)^2 + a_i (x_i - s_i)^2$, the results about the existence of a unique equilibrium, and the PoA bounds also hold in our variant. However, the best-response dynamics in the random payoff game does not converge to the equilibrium. In this work, we investigate whether there exists natural dynamics that leads the system to the equilibrium point x^*.

1.3 Our Results and Techniques

In this work we propose Algorithm 1 as a dynamics for the random payoff game of Definition 1. Note that the dynamics described in Algorithm 1 is the *fictitious play* in the game defined by the instance $I = (G, s, a)$. Generally speaking *fictitious play* does not guarantee convergence to the equilibrium. In Section 3 we show

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Algorithm 1 Fictitious Play

Initially, each agent i has an opinion $x_1(0), \ldots, x_n(0)$.
At round $t \geq 1$, each agent i meets uniformly at random one of her friends $j \in N_i$
Suffers cost $C_i^t(x_i(t-1), x_j(t-1)) = (1-a_i)(x_i(t-1) - x_j(t-1))^2 + a_i(x_i(t-1) - s_i)^2$
Updates opinion

$$x_i(t) = \underset{x \in [0,1]}{\operatorname{argmin}} \sum_{\tau=1}^{t} C_i^\tau(x, x_j(\tau - 1)) \tag{4}$$

---

that, for our game with random payoffs, fictitious play converges to equilibrium, with the following rate.

Theorem 1. *Let $I = (G, s, \alpha)$ be any instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector x^t produced by Algorithm 1 after t rounds satisfies*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq C\sqrt{\log n}\,\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

The update rule (4) guarantees convergence while vastly reducing the information exchange between the agents at each round. Namely, each agent learns the opinion of only one of her neighbors whereas in the classical FJ-model (1) each agent learns the opinions of all of her neighbors. In terms of total communication needed to get within distance ε of the equilibrium, the update rule (4) needs $O(|V| \log |V|)$ communication while (1) needs $O(|E|)$. Of course for this difference to be significant we need each agent to have at least $O(\log |V|)$ neighbors. A large social network like Facebook has approximately 2 billion users and each user has usually more that 100 friends which far more than $\log(2\ 10^9)$.

Apart from converging to the equilibrium, our update rule (4) is also a rational behavioral assumption since it ensures *no-regret* for the agents. Having no-regret means that the average cost for each agent i after T rounds is close to the average cost that she would suffer by expressing any fixed opinion. This is a very important feature of our update rule because even players that selfishly will to minimize their incurred cost, could choose to play according to it. In Section 4 we show the following theorem.

Theorem 2. *For every instance $I = (G, s, a)$, for every agent i*

$$\sum_{\tau=1}^{t} C_i^\tau(x_i(\tau)) \leq min_{x \in [0,1]} \sum_{\tau=1}^{t} C_i^\tau(x) + O(\log t)$$

Even though our update rule (4) has the above desired properties it only achieves convergence rate of $\widetilde{O}(1/\sqrt{t})$ for a fixed instance I with $\rho > 1/2$. For such an instance the original FJ-model outperforms our update rule since it achieves convergence rate $O(1/2^t)$. We investigate whether this gap is due to an inherent characteristic of the random payoff setting i.e. learning the opinion of just one random neighbor vs learning all of them or we could find another natural dynamics that converges exponentially fast. Can the agents select another no-regret algorithm that guarantees an asymptotically faster convergence rate? In Section 4 we investigate the following question

Question. *Is there a no-regret algorithm that the agents can choose such that for any instance $I = (G, s, a)$, $\mathbf{E}\left[\|x_t - x^*\|_\infty\right] = O(1/t)$?*

To answer this question we first show that the existence of a no-regret algorithm (that achieves this convergence rate to the equilibrium) implies the existence of an algorithm that uses i.i.d samples from a Bernoulli random variable $B(p)$ to estimate its success probability p with the same asymptotic error rate. We use sample complexity lower bound techniques from statistics to show that such a no-regret algorithm is unlikely to exist. We show that updating the opinion according to no-regret algorithms leads to update rules that are totally ignorant of the specific graph structure G. Perhaps surprisingly, in Section 7 we present an update rule that takes into account only the maximum degree of the graph and for any fixed instance I achieves convergence rate $O(1/2^{\sqrt{t}})$.

1.4 Related Work

2 Preliminaries

2.1 Notation

Let $G(V,E)$ be a graph. We denote by V the set of agents, and $N_i \subset V$ the set of neighbors of agent i. Let $n = |V|$. We denote by $x(t) \in [0,1]^n$ the vector of opinions of the agents at round t and $x_i(t)$ its i-th coordinate. We denote by $Q([0,1])$ the set of rationals in $[0,1]$.

2.2 Online Convex Optimization and No Regret Algorithms

The *Online Convex Optimization* (OCO) framework can be seen as a game played between a player and an adversary. Let $K \subseteq \mathbf{R}^n$ be a convex set and a set of functions $\mathcal{F}$ defined over K. *At round t*,

1. *the agent chooses* $x_t \in K$.
2. *the adversary observes the* x_t *and selects a function* $f_t(x) \in \mathcal{F}$.
3. *the player receives cost* $f_t(x_t)$.

The goal of the player is to pick x_t based on the history $(f_1, \dots, f_{t-1})$ in a way that minimizes the total cost. We emphasize that the agent has to choose x_t before seeing f_t, otherwise the problem becomes trivial.

Definition 2. *An OCO algorithm A, at round t selects a vector* $x^t \in K$ *according to the history,* $x^t = A_t(f_1, \dots, f_t)$.

Definition 3. *Let the OCO algorithm A and the sequence of functions* $\{f_1, \dots, f_T\} \in F^T$. *The regret of A is defined by*

$$R_A(T) = \sum_{t=1}^{T} f_t(x_t) - min_{x \in K} \sum_{t=1}^{T} f_t(x)$$

if $R_A(T) = o(T)$ *for any sequence* $\{f_1, \dots, f_T\}$ *then A is no-regret algorithm*

According to the feasibility set K and the set of functions F, different no-regret algorithms with differnt regret bounds can be derived. Some seminal examples areZinkevic's algorithm that acheives regret $R_A(T) = O\left(\sqrt{T}\right)$, when K is convex, closed and bounded and F is the set of convex functionswith bounded first derivative. Hazan et al [] proposed an algorithm with $R_A(T) = O(\log T)$, when F is the set of twice diffentiable strongly convex functions.

No regret dynamics and repeated games: Agent i with s_i, a_i is the player. At round t, she adopts an opinion $x_i(t) \in [0,1]$ and then the adversary selects a function $C_i^t(x) = (1-a_i)(x-y_t)^2 + a_i(x-s_i)^2$, where $y^t \in [0,1]$. Let A a no-regret in the above OCO setting. Clearly, each agent i is willinf to adopt as $x_i(t)$ the suggestion of A, ensuring that average cost is similar to the average cost of the best fixed opinion. As a result if all agents adopt the no-regret algorithm A, for a fixed instance I of the game the following a dynamical process is defined.

Algorithm 2 no-regret dynamics

Let an instance $I = (G, s, a)$ of the opinion formation game and $x_1(0), \dots, x_n(0)$ the initial opinions.
At round $t \geq 1$, each agent i:

1: Adopts an opinion $x_i(t) = A_t(C_i^1, \dots, C_i^{t-1})$
2: Meets uniformly at random one of her friends $j \in N_i$
3: Suffers cost $C_i^t(x_i(t))$, where $C_i^t(x) = (1 - a_i)(x - x_j(t))^2 + a_i(x - s_i)^2$

We denote the as $x_A(t)$ the opinion vector when the algorithm A is selected. In case $A_t(C_i^1, \dots, C_i^{t-1}) = \text{argmin}_{x \in [0,1]} \sum_{\tau=1}^{t-1} C_i^t(x)$ we obtain the *fictitious play* defined in 1 and for simplicity we denote it as $x(t)$. As we have already menioned, in Section 2 we bound the convergence rate of the *fictitious play*, $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$ and in Section 3, we prove that it is no-regret algorithm for the specific OCO setting. The latter does not hold in more general OCO settings. In section 4, we investigate whether there exists an algorithm A with significantly better asymptotic rate of convergence.

2.3 Estimating Bernoulli distributions

Assume that we have access to the random variables $Y_1, \dots, Y_t$, that are mutually independent and each Y_i is distributed according to a Bernoulli distribution with $p \in [0, 1]$, $X_i \sim B(p)$. Initially the parameter p is unkown. The task is to observe the realization of $y_1, \dots, y_t$ and output an estimate $\hat{p} = \theta_t(y_1, \dots, y_t)$ according to a function $\hat{\theta}_t : \{0, 1\}^t \mapsto [0, 1]$, that is close to the unkown parameter p.

Definition 4. *An estimator $\hat{\theta}$ is a collections of function, $\{\hat{\theta}_t\}_{t=1}^\infty$, where $\hat{\theta}_t : \{0, 1\}^t \mapsto [0, 1]$.*

A well known example of such an estimator is the *sample mean*, where $\hat{\theta}_t = \frac{\sum_{i=1}^t Y_i}{t}$. Obviously for any efficient estimator, the estimate $\hat{p}$ converges to the unkown parameter p as t grows. The following definition is the standard metric for the efficiency of an estimator $\hat{\theta}$.

Definition 5. *For an estimator $\hat{\theta} = \{\hat{\theta}_t\}_{t=1}^\infty$ we define its error rate $R_p(t) = E_p[|\hat{\theta}_t(Y_1, \dots, Y_t) - p|]$*

$$\text{where } E_p[|\hat{\theta}_t(Y_1, \dots, Y_t) - p|] = \sum_{(y_1, \dots, y^t) \in \{0,1\}^t} |\hat{\theta}_t(y^1, \dots, y^t) - p| \cdot p^{\sum_{i=1}^t y^i} (1 - p)^{t - \sum_{i=1}^t y^i}$$

The quantity $E_p[|\hat{\theta}_t(Y_1, \dots, Y_t) - p|]$ is the expected distance of the estimated value $\hat{\theta}_t$ from the parameter p, when the distribution of the samples is $B(p)$. To simplify notation we also denote it as $E_p[|\hat{\theta}_t - p|]$. The error rate $R_p(t)$ quantifies the rate of convergence of the estimated value $\hat{p} = \theta_t(Y_1, \dots, Y_t)$ to the real parameter p.
Since p is unkown, any meaningful estimator $\hat{p}$ must guarantee that for all $p \in [0, 1]$, $\lim_{t \to \infty} R_p(t) = 0$. For example the *sample mean* has error rate $R_p(t) \leq \frac{1}{2\sqrt{t}}$ for ant $p \in [0, 1]$ and clearly statisfies the above requirement. In section 4 we investigate the following question,

Question. *Is there an estimator $\hat{\theta}$ such that for all $p \in Q[0, 1]$, $\lim_{t \to \infty} t R_p(t) = 0$?*

3 Fictitious Play Convergence Rate

In this Section we prove that the fictitious play described as Algorithm 1 convergences to the equilibrium. We prove that the dynamics Fictitious Play defined as Algorithm 1 has the following convergence rate to x^*.

Theorem 3. *Let $I = (G, s, \alpha)$ be any instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0, 1]^n$. The opinion vector x^t produced by Algorithm 1 after t rounds satisfies*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^2}{t^{\min(1/2, \rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

Proof. Let W_i^t be the random variable corresponding to the selected neighbor of agent i, at round t of the dynamics 1. Therefore, each agent $i \in V$ updates her opinion as follows

$$x_i(t) = \underset{x\in[0,1]}{\operatorname{argmin}} \sum_{\tau=1}^{t} C_i^\tau(x, x_{W_i^\tau}(\tau-1)) = (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x_{W_i^\tau}(\tau-1)}{t} + \alpha_i s_i$$

We want to find a bound $e(t)$ for the random variable of the error $\|x^t - x^*\|_\infty$ such that with probability $1-p$ it holds $\|x^t - x^*\|_\infty \le e(t)$. We derive a recursion for the error $e(t)$ as follows. First we assume that at round t each agent i observes the opinion of a random neighbor at the equilibrium, namely $x^*_{W_i^t}$. Using Lemma 1 we get a recursion for the error $e(t)$. Setting $p = 1/\sqrt{t}$ in Lemma 2 yields the result. ■

Lemma 1. *With probability at least* $1-p$, $\|x^t - x^*\|_\infty \le e(t)$, *where* $e(t)$ *satisfies the following recursive relation*

$$e(t) = \delta(t) + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \text{ and } e(0) = \|x^0 - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$.

Corollary 1. *The function* $e(t)$ *satisfies the following recursive relation*

$$e(t+1) - e(t) + \alpha\frac{e(t)}{t+1} = \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1}$$

Proof. As we have already mentioned for any instance I there exists a unique equilibrium vector x^*. Since $W_i^\tau \sim \mathrm{U}(N_i)$ we have that $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \frac{\sum_{j\in N_i} x_j^*}{|N_i|}$. Since W_i^τ are independent random variables, we can use Hoeffding's inequality (Lemma **??**) to get

$$\mathbf{P}\left[\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] < \frac{6}{\pi^2}\frac{p}{nt^2},$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Therefore, by the union bound

$$\begin{aligned}
&\mathbf{P}\left[\text{for all } t\ge 1 : \max_{i\in V}\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \le \\
&\sum_{t=1}^{\infty}\mathbf{P}\left[\max_{i\in V}\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \le \\
&\sum_{i=1}^{\infty}\frac{6}{\pi^2}\frac{1}{t^2}\sum_{i=1}^{n}\frac{p}{n} = p
\end{aligned}$$

As a result with probability $1-p$ we have that for all t

$$\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| \le \delta(t) \tag{5}$$

We will prove our claim by induction. We assume that $\|x^{\tau} - x^*\|_{\infty} \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$
\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x_{W_i^{\tau}}(\tau-1)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x^*_{W_i^{\tau}} + \sum_{\tau=1}^{t} e(\tau-1)}{t} + \alpha_i s_i \qquad (6)\\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=1}^{t} x^*_{W_i^{\tau}}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\frac{\sum_{j\in N_i} x^*_j}{|N_i|} + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \qquad (7)\\
&\leq x_i^* + \delta(t) + (1-\alpha)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}
$$

We get (6) from the induction step and (7) from inequality (5). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\alpha)\frac{\sum_{\tau=1}^{t} e(\tau)}{t}$. As a result $||x_i(t) - x^*||_{\infty} \leq e(t)$. ■

In order to bound the convergence time of the system, we just need to bound the convergence rate of the function $e(t)$. The following lemma provides us with a simple upper bound for the convergence rate of our process.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion of Corollary 1. Then with probability at least $1-p$ we have that*

$$
e(t) = \begin{cases} O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^{\alpha}}\right) & \text{if } \alpha \leq 1/2 \\ O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^{1/2}}\right) & \text{if } \alpha > 1/2 \end{cases}
$$

Proof. At first, since $\frac{\delta(t)}{t}$ is a decreasing function for $p \leq 1/4$, we have that $e(t) \leq (1-\frac{\alpha}{t}) + g(t)$, where $g(t) = \frac{\delta(t)}{t}$.

$$
\begin{aligned}
e(t) &\leq (1-\frac{\alpha}{t})e(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})(1-\frac{\alpha}{t-1})e(t-2) + (1-\frac{\alpha}{t})g(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})\cdots(1-\alpha)e(0) + \sum_{\tau=1}^{t} g(\tau)\prod_{i=\tau+1}^{t}(1-\frac{\alpha}{i}) \\
&\leq \frac{e(0)}{t^{\alpha}} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^{\alpha}} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha(H_t - H_{\tau})} \\
&\leq \frac{e(0)}{t^{\alpha}} + e^{-\alpha H_t}\sum_{\tau=1}^{t} g(\tau)e^{\alpha H_{\tau}} \\
&\leq \frac{e(0)}{t^{\alpha}} + \frac{O\left(\sqrt{\log(\frac{n}{p})}\right)}{t^{\alpha}}\sum_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}
\end{aligned}
$$

We observe that

$$
\sum_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}} \leq \int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau,
$$

since $\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}$

- If $\alpha \leq 1/2$ then
$$\int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau \leq \int_{\tau=1}^{t}\tau^{1/2}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau = O((\log t)^{3/2}))$$
- If $\alpha > 1/2$ then

$$\begin{aligned}\int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau &= \int_{\tau=1}^{t}\tau^{\alpha-1/2}\frac{\sqrt{\log\tau}}{\tau}d\tau \\ &= \frac{2}{3}\int_{\tau=1}^{t}\tau^{\alpha-1/2}((\log\tau)^{3/2})'d\tau \\ &= \frac{2}{3}(\log t)^{3/2} - (\alpha-1/2)\frac{2}{3}\int_{\tau=1}^{t}\tau^{\alpha-3/2}(\log\tau)^{3/2}d\tau \\ &= O\left((\log t)^{3/2}\right)\end{aligned}$$

∎

4 Fictitious Play is no-regret

In this section we consider the Online Convex Optimization problem that we defined in Subsection 2.2. We remind that our class of functions $\mathcal{F}$ contains all functions of the form
$$f_t(x) = \alpha(x-s_i)^2 + (1-\alpha)(x-y_t)^2,$$
where $s_i, \alpha \in [0,1]$ and are independent of t and $y_t \in [0,1]$. In other words, the function f^t is uniquely determined by the number y_t. We show that for this class of functions *fictitious play* admits no regret.

Theorem 4. *Let f^t be a sequence of functions, where each function has the form $f^t(x) = \alpha(x-s_i)^2 + (1-\alpha)(x-y_t)^2$, where $s_i, \alpha \in [0,1]$ and do not dependent on t and $y_t \in [0,1]$. If we define $x^t = \mathrm{argmin}_{x\in[0,1]}\sum_{\tau=1}^{t-1} f^{\tau}(x)$ then for each time T*
$$\sum_{t=1}^{T} f^t(x^t) \leq \min_{x\in[0,1]}\sum_{t=1}^{T} f^t(x) + O(\log T).$$

In order to prove this claim, we will first define a similar rule y^t that takes into account the function f^t for the "prediction" at time t. Intuitively, this guarantees that the rule admits no regret.

Lemma 3. *Let $y^t = \mathrm{argmin}_{x\in[0,1]}\sum_{\tau=1}^{t} f^{\tau}(x)$ then*
$$\sum_{t=1}^{T} f^t(y^t) \leq \min_{x\in[0,1]}\sum_{t=1}^{T} f^t(x)$$

Proof. By definition of y^t, $\min_{x\in[0,1]}\sum_{t=1}^{T} f^t(x) = \sum_{t=1}^{T} f^t(y^T)$, so
$$\begin{aligned}\sum_{t=1}^{T} f^t(y^t) - \min_{x\in[0,1]}\sum_{t=1}^{T} f^t(x) &= \sum_{t=1}^{T} f^t(y^t) - \sum_{t=1}^{T} f^t(y^T) \\ &= \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^T) \\ &\leq \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^{T-1}) \\ &= \sum_{t=1}^{T-2} f^t(y^t) - \sum_{t=1}^{T-2} f^t(y^{T-1})\end{aligned}$$

Continuing in the same way, we get $\sum_{t=1}^{T} f^t(y^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x)$. ■

Now we can derive some intuition for the reason that *fictitious play* admits no regret. Since the cost incurred by the sequence y^t is at most that of the best fixed strategy, we can compare the cost incurred by x^t with that of y^t. However, for each t the numbers x^t and y^t are quite close and as a result the difference in their cost must be quite small.

Lemma 4. *For all t, $f^t(x^t) \leq f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-a)^2}{t^2}$.*

Proof. We first prove that for all t,

$$\left|x^t - y^t\right| \leq \frac{1-\alpha}{t}. \tag{8}$$

By definition $x^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1}$ and $y^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t} a_\tau}{t}$.

$$\begin{aligned}
\left|x^t - y^t\right| &= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1} - \frac{\sum_{\tau=1}^{t} a_\tau}{t}\right| \\
&= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau - (t-1)y_t}{t(t-1)}\right| \\
&\leq \frac{1-\alpha}{t}
\end{aligned}$$

The last inequality follows from the fact that $a_\tau \in [0,1]$. We now use inequality (8) to bound the difference $f^t(x^t) - f^t(y^t)$.

$$\begin{aligned}
f^t(x^t) &= \alpha(x^t - s_i)^2 + (1-\alpha)(x^t - y_t)^2 \\
&\leq \alpha(y^t - s_i)^2 + 2\alpha\left|y^t - s_i\right|\left|x^t - y^t\right| + \alpha\left|x^t - y^t\right|^2 \\
&\quad + (1-\alpha)(y_t - y_t)^2 + 2(1-\alpha)\left|y^t - y_t\right|\left|x^t - y^t\right| + (1-\alpha)\left|x^t - y^t\right|^2 \\
&\leq f^t(y^t) + 2\left|x^t - y^t\right| + \left|y^t - x^t\right|^2 \\
&\leq f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-\alpha)^2}{t^2}
\end{aligned}$$

■

Theorem 4 easily follows since

$$\begin{aligned}
\sum_{t=1}^{T} f^t(x^t) &\leq \sum_{t=1}^{T} f^t(y^t) + \sum_{t=1}^{T} 2\frac{1-\alpha}{t} + \sum_{t=1}^{T} \frac{(1-\alpha)^2}{t^2} \\
&\leq \min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + 2(1-\alpha)(\log T + 1) + (1-\alpha)\frac{\pi^2}{6} \\
&\leq \min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + O(\log T)
\end{aligned}$$

5 Final Lower Bound

As we have already discussed in **??** a no-regret algorithm A defines no-regret dynamics for any fixed instance $I = (G, s, a)$. Let $\mathbf{E}_I\left[\|x(t) - x^*\|_\infty\right]$ denote the convergence rate of the above dynamics. Since A ensures no-regret for each agent i, this dynamical process can be considered as a natural dynamical process. In this section we investigate whether there exists such a no-regret algorithm that guarantees significantly better asymptotic convergence rate than those that *fictitious play* provides. In this section we present some results that indicate that the following question admits an negative answer.

Question. *Is there a no-regret algorithm A such that for all instance $I = (G, s, a)$, $\mathbf{E}_I [\|x(t) - x^*\|_\infty] \in O(1/t)$?*

Altough we were not able to answer provide a negative answer to the above question, we present to main results that indicate that the existence of such an algorithm is higly unlikely. More precisely, we were able to prove two following theorems.

Theorem 5. *Let a no-regret alogrithm A such that for all instances I, $\lim_{t\to\infty} t\mathbf{E}_I [\|x(t) - x^*\|_\infty] = 0$. Then there exists an estimator $\hat{\theta}$ such that for all $p \in Q[0,1]$, $\lim_{t\to\infty} tR_p(t) = 0$*

Theorem 6. *Let a Bernoulli estimator $\hat{\theta}$ with error rate $R_p(t)$. Then, for all $[a, b] \subseteq [0, 1]$,*

$$\lim_{t\to\infty} t \int_a^b R_p(t)dp = +\infty$$

As result the existence of such an algorithm A implies the existence of estimator $\hat{\theta}$ with error rate $R_p(t)$ that satisfies the following two properties:

- for all $p \in Q([0, 1])$, $\lim_{t\to\infty} tR_p(t) = 0$
- for all $[a, b] \subseteq [0, 1]$, $\lim_{t\to\infty} t \int_a^b R_p(t)dp = +\infty$

We conjecture that the does not exist such a function $R_p(t)$. Even if it is not the case, such a fuction admits an very degenerate form making very likely to be the error rate of any reasonable estimator $\hat{\theta}$.

5.1 Proof of Theorem 4

Lemma 5. *For any graph oblivious update rule $F = \{ f\}_{t=1}^\infty$, there exists a Bernulli estimator $\hat{\theta_F}$ such that for all $q \in Q([0, 1])$, there exists an instance I_q such that*

$$E_q[|\hat{\theta}_t - q|] \leq 2E_{I_q}[||x^t - x^*||_\infty]$$

Proof. At round t, the graph oblivious update rule $F = \{ f_t\}_{t=1}^\infty$ has selected the functions $f_t(O_1, \ldots, O_t, s_i, a_i)$ and $g_t(s_i, a_i)$. An estimator $\hat{\theta}$ receives the samples $Y_1, \ldots, Y_t$ and outputs the value $\hat{\theta}_t(Y_1, \ldots, Y_t) \in [0, 1]$. Now consider the estimator $\hat{\theta_t} = \frac{1}{2} f(g_1(Y_1, 1), \ldots, g_t(Y_t, 1))$ and observe that $\hat{\theta_t}$ is a function $\{0, 1\}^t \mapsto [0, 1]$. Now for any $q \in Q([0, 1])$, construct an appropriate instance I_q such that $E_q[|\hat{\theta_t} - q|] \leq 2E_{I_q}[||x^t - x^*||_\infty]$. Since $q = \frac{k}{n}$ some $k, n \in \mathbb{N}$, the following instance I_q with $n + 1$ agents:

- A central agent with $s_c = 0$ and $a_c = 1/2$.
- Directed edges from the central agent to all the other agents.
- k agents with $s_i = 0$ and $a_i = 1$
- $n - k$ agents with $s_i = 1$ and $a_i = 1$

Notice that in this instance for all $i \neq c$, $x_i^* = s_i$ and $x_c^* = \frac{q}{2}$. We show that $E_q[|\hat{\theta_t} - q|] \leq 2E_{I_q}[||x^t - x^*||_\infty]$. At round t, if the oracle returns to the center agent the value $g_t(1, 1)$ of a 1-agent, then $Y_t = 1$ otherwise $Y_t = 0$. As a result, $\mathbf{P}[Y_t = 1] = q$ and

$$\begin{aligned} E_{I_q}[||x^t - x^*||_\infty] &\geq E_{I_q}[|x_c^t - x_c^*|] \\ &= E_q[|\frac{\hat{\theta_t}}{2} - \frac{q}{2}|] \end{aligned}$$

■

Corollary 2. *Let the update rule $F = \{f_t\}_{t=1}^\infty$ s.t. for all instances I, $\lim_{t\to\infty} tE_I[||x^t - x^*||] = 0$. Then for any $q \in Q[0, 1]$,*

$$\lim_{t\to\infty} tE_q[|\hat{\theta_F} - q|] = 0$$

5.2 Proof of Theorem 5

6 Lower Bound

Lemma 6 (Fano's Inequality)**.** *Suppose we have an estimator $\hat{\theta}^t$ for the probability of a Bernoulli distribution. The unknown distribution belongs to a family of distributions $\mathcal{P}$ with the property that the probabilities of every two distributions in this family differ by at least 2δ, where δ is a positive real number. Then, for every subset $P_1, P_2, ..., P_n$ of distributions from this family, where p_i is the probability of distribution P_1, the following inequality holds:*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\left[|\hat{\theta}^t - p_i|\right] \geq \delta\left(1 - \frac{I(X;V) + \log 2}{\log n}\right)$$

We are now going to use Theroem (6) to prove the following lemma, which provides a lower bound for the mean of the expectation of errors of the estimator.

Lemma 7. *Suppose we choose t Bernoulli distributions $\{P_i\}_{i=0}^{t-1}$ with the following probabilities:*

$$p_i = a + \frac{b-a}{2t} + i\frac{b-a}{t}, i = 0, ..., t-1$$

. Then:

$$\frac{1}{t}\sum_{i=1}^{t} \mathbf{E}\left[|\hat{\theta}^t - p_i|\right] \geq \frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right)$$

where $c_1, c_2 > 0$ and $c_1 > \frac{1}{2}$.

Next, we state a lemma regarding an upper bound on the expectation of the error in each p_i.

Lemma 8. *For every $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$:*

$$\mathbf{E}\left[|\hat{\theta}^t - p_i|\right]_{p_i} \leq c_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p + |p - p_i|$$

where c_a^b is a constant that depends on a, b.

Now, we find an upper bound for the quantity $\sum_{i=0}^{t-1} E_i$. Our goal is to find an upper bound where the coefficient of $(b-a)/t$ is less than $1/4$.

Lemma 9.

$$\frac{1}{t}\sum_{i=0}^{t-1} E_i \leq \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right] dp + \frac{b-a}{4t}$$

Theorem 7. *Suppose $\hat{\theta}^t$ is an estimator for the probability p of a Bernoulli random variable that takes t samples. Then, for every interval $[a, b]$ which is contained in $[0, 1]$ and for every $c > 0$, it holds that:*

$$\lim_{t\to\infty} t^{1+c}\int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = \infty$$

Proof. By combining Lemmas (12) and (14) we get:

$$\frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp \geq \frac{b-a}{2t}\left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right)$$

By multiplying by t^{1+c} we get:

$$\frac{c_a^b}{b-a}t^{1+c}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp \geq \frac{b-a}{2}t^c\left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right) \tag{9}$$

The coefficient of t^c in the right hand side of (9) is positive, so by sendig $t \to \infty$ we get:

$$\lim_{t\to\infty} t^{1+c}\int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = +\infty$$

∎

7 A Graph Aware Protocol

Let P be a discrete distribution over $[d]$, and let $S_1, \dots, S_t$ be t i.i.d samples drawn from P, i.e. $(S_1, \dots, S_t) \sim P^t$. The empirical distribution $\hat{P}_t$ is the following estimator of the density of P.

$$\hat{P}_N(A) = \frac{\sum_{i=1}^{N} \mathbb{1}_{[S_i \in A]}}{N}, \tag{10}$$

where $A \subseteq [d]$. In words, $\hat{P}_N$ simply counts how many times the value i appeared in the samples $S_1, \dots S_N$. We will use the following version of the classical Vapnik and Chervonenkis inequality.

Lemma 10. *Let $\mathcal{A}$ be a collection of subsets of $\{0, \dots, d\}$ and Let $S_{\mathcal{A}}(N)$ be the Vapnik-Chervonenkis shatter coefficient, defined by*

$$S_{\mathcal{A}(N)} = \max_{x_1, \dots, x_t \in [d]} |\{\{x_1, \dots, x_t\} \cap A : A \in \mathcal{A}\}| .$$

Then

$$\mathbf{E}_{P^N} \left[\max_{A \in \mathcal{A}} \left| \hat{P}_t(A) - P(A) \right| \right] \leq 2 \sqrt{\frac{\log 2 S_{\mathcal{A}}(N)}{N}}$$

An interesting question is whether $\mathrm{poly}(1/\varepsilon)$ rounds are necessary when protocols have access to the unlabeled oracle. We show that this is not true by showing a protocol that needs $\ln^2(1/\varepsilon)$ rounds to be within error ε from the solution x^*. Our protocol is graph-aware in the sense that it is allowed to depend on the graph G. Therefore, we show that without any constraints on the protocols it is hard to prove strong lower bounds for our problem. The problem in this case is that all agents could agree to stop updating their public opinions for enough rounds so that everybody learns, with high probability, exactly the average of the opinions of their neighbors. Having the exact average they can perform a step of the original update of the FJ-model which results in a similar and fast convergence rate. Of course if the opinions of the neighbors are all different we can still use an argument similar to that of the protocol of section COUPONS COLLECTOR. Since some neighbors may share opinions we have to come up with a different solution for this problem. Let $L \leq d$ be the number of different opinions and k_i be the number of neighbors who have opinion ξ_i. Then the average of the opinions of the neighbors is $(\sum_{j=1}^{L} k_i \xi_i)/d_i$. Instead of using the average like we did in section SAMPLE MEAN, we will instead find *exactly* the frequencies k_i and then compute their average exactly with high probability. We next describe our protocol. Since it is the same for all agents, For simplicity we describe it for agent i.

Theorem 8. *The update rule 3 for $a > 1/2$ achieves convergence rate*

$$\mathbf{E} \left[\|x_t - x^*\|_\infty \right] \leq C e^{-\sqrt{t}/(d\sqrt{\log d})},$$

where C is a universal constant.

Proof. According to the update rule 3 all agents fix their opinions $x_i(t)$ for $M_1 \times M_2$ rounds. To estimate the sum of the opinions each agent estimates the frequencies k_j/d_i. Since the neighbors have at most d_i different opinions we can think of the opinions as natural numbers in $[d_i]$. The oracle returns the opinion of a random neighbor and therefore the samples X_i returned by the oracle O_i^u are drawn from a discrete distribution P supported on $[d_i]$. The probability $P(j)$ of the j-th opinion is the number of neighbors having this opinion k_j/d_i. To learn the probabilities $P(j)$ using samples from P. Letting $\mathcal{A} = \{\{1\}, \{2\}, \dots, \{d_i\}\}$ we have from Lemma 10 that

$$\mathbf{E}_{P^m} \left[\max_{j \in [d_i]} \left| \hat{P}_m(j) - P(j) \right| \right] \leq 2 \sqrt{\frac{\log 2 d_i}{m}},$$

since $S_{\mathcal{A}} \leq d_i$. Therefore, an agent can draw $m = 100 d^2 \log(2d)$ to learn the frequencies k_j/d_i within expected error $1/(5d)$. Notice now that the array A after line 15 corresponds to the empirical distribution

Algorithm 3 Graph Aware Update Rule

$x_i(0) \leftarrow s_i$, $t \leftarrow 0$.
for $l = 1, \ldots, \infty$ **do**
  $\varepsilon \leftarrow 1/2^l$.
  for $t = 1, \ldots, \ln(1/\varepsilon))$ **do**
    Keep a map h from $[0,1]$ to $[d_i]$ and array of counters A of length d_i and an array B of length M_1.
    $M_1 = O(\ln(1/\varepsilon))$, $M_2 = O(d^2 \ln(d))$
    for $j = 1, \ldots, M_1$ **do**
      for $k = 1, \ldots, M_2$ **do**
        Call the unlabeled oracle $O_i^u(t)$ to get an opinion X_i.
        if X_i is not in h **then**
          Insert X_i to h.
        else
          $A(h(X_i)) \leftarrow A(h(X_i)) + 1$.
        $t \leftarrow t + 1$
      Divide all entries of A by M_2.
      Round all entries of h to the closest multiple of $1/d$.
      $B(j) = \sum_{i=1}^{d_i} A(i)$.
    $x_i(t) \leftarrow \mathrm{maj}_j\, B(j)$.

of equation (10). Notice that if the agents have estimations of the frequencies k_j/d_i with error smaller than $1/d$ then by rounding them to the closest multiple of $1/d_i$ they learn the frequencies exactly. By Markov's inequality we have that with probability at least $4/5$ the rounded frequencies are exactly correct. By standard Chernoff bounds we have that if the agents repeat the above procedure $\ln(1/\delta)$ times and keep the most frequent of the answers $B(j)$ then they will obtain the correct answer with probability at least $1-\delta$. From Lemma **??** we have that to achieve error ε we need $\log(1/\varepsilon)/(1/(1-\alpha))$ rounds. Since we need all nodes to succeed at computing the exact averages for $\log(1/\varepsilon)/\log(1/(1-\alpha))$ rounds we have from the union bound that for $\delta < \frac{\varepsilon \ln(1/(1-\alpha))}{n \ln(1/\varepsilon)}$, with probability at least $1-\varepsilon$ the error is at most ε. For the expected error after $T = O(d^2 \log d(\log(n/\varepsilon) + \log\log(1/\varepsilon) + \log\log(1-a))$ rounds we have that $\mathbf{E}\left[\|x_T - x^*\|_\infty\right] = O(\varepsilon)$. ■

A Lower Bound

Lemma 11. *Let $P = B(p)$ and $Q = B(q)$ be two Bernoulli distributions then*

$$D_{\text{kl}}(P\|Q) \leq \left(\frac{1}{p(1-p)} + \frac{1}{q(1-q)}\right)\frac{(p-q)^2}{2}$$

Proof. We have that $D_{\text{kl}}(P\|Q) = p\log(p/q) + (1-p)\log((1-p)/(1-q))$. Let

$$f(x,y) = x\log\left(\frac{x}{y}\right) + (1-x)\log\left(\frac{1-x}{1-y}\right)$$

To simplify notation fix y and let $g(x) = f(x,y)$. We have

$$g'(x) = -\log\left(\frac{1-x}{1-y}\right) + \log\left(\frac{x}{y}\right), \qquad g''(x) = \frac{1}{(1-x)x}$$

If $x < y$ using Taylor in the interval $[x,y]$ we have for a $\xi \in [x,y]$

$$g(x) = g(y) + g'(y)(y-x) + g''(\xi)(y-x)^2/2 = g''(\xi)(x-y)^2/2.$$

If $x > y$ the using Taylor in the interval $[y,x]$ we get the same expression as above. Since $g''(x)$ is convex and is minimized at $x_0 = 1/2$ we have that if $|x-1/2| < |y-1/2|$ then $g''(\xi) \leq g''(y)$ else $g''(\xi) \leq g''(x)$. Therefore,

$$g(x) = f(x,y) \leq \max(g''(x), g''(y))(x-y)^2/2 \leq (g''(x) + g''(y))(x-y)^2/2$$

■

We are now going to use Lemma (6) to prove the following lemma, which provides a lower bound for the mean of the expectation of errors of the estimator.

Lemma 12. *Suppose we choose t Bernoulli distributions $\{P_i\}_{i=0}^{t-1}$ with the following probabilities:*

$$p_i = a + \frac{b-a}{2t} + i\frac{b-a}{t}, i = 0, ..., t-1$$

. Then:

$$\frac{1}{t}\sum_{i=1}^{t}\mathbf{E}\left[\left|\hat{\theta}_t - p_i\right|\right] \geq \frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right)$$

where $c_1, c_2 > 0$ and $c_1 > \frac{1}{2}$.

Proof. Without loss of generality, we can assume that t is a multiple of 5. For simplicity, we set $E_i = \mathbf{E}\left[\left|\hat{\theta}_t - p_i\right|\right]$. By the choice of p_i, we have that

$$\left|p_i - p_j\right| \geq \frac{b-a}{t}$$

, for every distinct i, j in the family. This means that for the family of t distributions that we picked, the property of Lemma (6) is satisfied for $\delta = \frac{b-a}{2t}$. Thus, by dividing the t distributions into groups of 5 and applying Lemma (6) to each one, we obtain:

$$\begin{aligned}
\frac{\sum k = i^{i+4}}{5} &\geq \frac{b-a}{2t}\left(1 - \frac{I(X;V) + \log 2}{\log n}\right) \\
&= \frac{b-a}{2t}\left(1 - \frac{\log 2}{\log 5} - \frac{I(X;V)}{\log 5}\right) \\
&= \frac{b-a}{2t}\left(c_1 - \frac{I(X;V)}{\log 5}\right) \qquad (11)
\end{aligned}$$

where $c_1 = 1 - \frac{\log 2}{\log 5} > \frac{1}{2}$. We are now going to upper bound the mutual information $I(X;V)$ for every group of 5 distributions. We denote by P_i^t the distribution on vectors with t coordinates, where each coordinate is sampled independently from P_i. Let $U = \{i, i+1, i+2, i+3, i+4\}$ be a family of 5 distributions. Using a well known inequality REFERENCE NEEDED HERE:

$$\begin{aligned}
I(X;V) &\leq \frac{1}{5^2} \sum_{\substack{i,j\in U\\ i\neq j}} D_{kl}\left(P_i^t, P_j^t\right)\\
&\leq D_{kl}\left(P_i^t, P_{i+4}^t\right)\\
&= tD_{kl}\left(P_i, P_{i+4}\right)\\
&\leq t\frac{(p_i - p_{i+4})^2}{p_i(1-p_i)}\\
&\leq t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)(p_i - p_{i+4})^2\\
&= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\left(\frac{b-a}{2t} + i\frac{b-a}{t} - \frac{b-a}{t} - (i+4)\frac{b-a}{t}\right)^2\\
&= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\frac{16(b-a)^2}{t^2}\\
&= \frac{16(b-a)^2}{t}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)
\end{aligned}$$

So, if we set

$$c_2 = 16\frac{(b-a)^2}{\log 5}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)$$

then by equation (11) we obtain:

$$\sum_{k=i}^{i+4} \geq \frac{5(b-a)}{2t}\left(c_1 - \frac{c_2}{t}\right) \tag{12}$$

Inequality (12) holds for every group of 5 distributions. By summing up for all groups:

$$\begin{aligned}
\sum_{i=0}^{t-1} E_i &\geq t\frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right)\\
&= \frac{b-a}{2}\left(c_1 - \frac{c_2}{t}\right)
\end{aligned}$$

which is what we wanted to prove. ∎

Next, we state a lemma regarding an upper bound on the expectation of the error in each p_i.

Lemma 13. *Let $p, q \in [0,1]$ and Let $P = B(p)$, $Q = B(q)$ be two Bernoulli distributions with corresponding product distributions P^t, Q^t. Moreover, let $\hat{\theta}_t : \{0,1\}^t \to [0,1]$ be an estimator. Then*

$$\left|\mathbf{E}_{P^t}\left[|\hat{\theta}_t - p|\right] - \mathbf{E}_{Q^t}\left[|\hat{\theta}_t - q|\right]\right| \leq \sqrt{\mathbf{E}_{P^t}\left[(\hat{\theta}_t - p)^2\right]}\sqrt{\left(\frac{q^2}{p} + \frac{(1-q)^2}{1-p}\right)^t - 1} + |p-q|,$$

Proof. By the triangle inequality we have

$$\left|\mathbf{E}_{P^t}\left[|\hat{\theta}_t - p|\right] - \mathbf{E}_{Q^t}\left[|\hat{\theta}_t - q|\right]\right| \leq \left|\mathbf{E}_{P^t}\left[|\hat{\theta}_t - p|\right] - \mathbf{E}_{Q^t}\left[|\hat{\theta}_t - p|\right]\right| + |p-q| \tag{13}$$

To simplify notation let $X = (X_1, \ldots, X_t)$ be the sample and let $r(X) = |\hat{\theta}_t(X) - p|$. Moreover, denote by $p(x)$ resp. $q(x)$ the density functions of $B(p)$ resp. $B(q)$. $p_t : \{0,1\}^t \to [0,1]$ resp. q_t be the density functions of the product distribution P^t, i.e. namely $p_t(x) = \prod_{i=1}^t p(x_i)$. We have

$$\begin{aligned}
\mathbf{E}_{P_t}\left[r(X)\right] - \mathbf{E}_{Q^t}\left[r(X)\right] &= \sum_{x\in\{0,1\}^t} r(x)p_t(x) - \sum_{x\in\{0,1\}^t} r(x)q_t(x) \\
&= \sum_{x\in\{0,1\}^t} r(x)(p_t(x) - q_t(x)) \\
&= \sum_{x\in\{0,1\}^t} r(x)\sqrt{p_t(x)}\ \frac{p_t(x) - q_t(x)}{\sqrt{p_t(x)}}
\end{aligned}$$

From Cauchy-Schwarz inequality we have

$$\left(\mathbf{E}_{P^t}\left[r(X)\right] - \mathbf{E}_{Q^t}\left[r(X)\right]\right)^2 \leq \left(\sum_{x\in\{0,1\}^t} r^2(x)p_t(x)\right)\left(\sum_{x\in\{0,1\}^t} \frac{(p_t(x)-q_t(x))^2}{p_t(x)}\right) \tag{14}$$

Notice that

$$\begin{aligned}
\sum_{x\in\{0,1\}^t} \frac{(p_t(x)-q_t(x))^2}{p_t(x)} &= \sum_{x\in\{0,1\}^t} p_t(x) - 2q_t(x) + \frac{q_t^2(x)}{p_t(x)} \\
&= -1 + \sum_{x\in\{0,1\}^t} \frac{q_t^2(x)}{p_t^2(x)} p_t(x) \\
&= -1 + \mathbf{E}_{P^t}\left[\frac{q_t^2(X)}{p_t^2(X)}\right]
\end{aligned}$$

Since X_i are independent we have

$$\mathbf{E}_{P^t}\left[\prod_{i=1}^t \frac{q^2(X_i)}{p^2(X_i)}\right] = \left(\mathbf{E}_P\left[\frac{q^2(X_1)}{p^2(X_1)}\right]\right)^t = \left(\frac{q^2}{p} + \frac{(1-q)^2}{1-p}\right)^t \tag{15}$$

Combining equations (13), (14), (15) yields the result. ∎

Now, we find an upper bound for the quantity $\sum_{i=0}^{t-1} E_i$. Our goal is to find an upper bound where the coefficient of $(b-a)/t$ is less than $1/4$.

Lemma 14.

$$\frac{1}{t}\sum_{i=0}^{t-1} E_i \leq \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}_t - p|\right] dp + \frac{b-a}{4t}$$

Proof. By Lemma (13) we get that for every $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$:

$$E_i \leq c_a^b \mathbf{E}\left[|\hat{\theta}_t - p|\right]_p + |p_i - p|$$

We integrate both sides of the equation with respect to p and get:

$$\int_{p_i-\frac{b-a}{2t}}^{p_i+\frac{b-a}{2t}} E_i dp \leq c_a^b \int_{p_i-\frac{b-a}{2t}}^{p_i+\frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}_t - p|\right]_p dp + \int_{p_i-\frac{b-a}{2t}}^{p_i+\frac{b-a}{2t}} |p_i - p|\, dp$$

Hence:

$$\begin{aligned}
\frac{b-a}{t} E_i &\leq c_a^b \int_{p_i-\frac{b-a}{2t}}^{p_i+\frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}_t - p|\right]_p dp + 2\int_{p_i-\frac{b-a}{2t}}^{p_i} |p_i - p|\, dp \\
&= c_a^b \int_{p_i-\frac{b-a}{2t}}^{p_i+\frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}_t - p|\right]_p dp + \frac{1}{4}\frac{(b-a)^2}{t^2}
\end{aligned}$$

Therefore:

$$\frac{E_i}{t} \leq \frac{c_a^b}{b-a} \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}_t - p\right|\right]_p dp + \frac{b-a}{4t^2} \tag{16}$$

By summing up Equation (16) for all t intervals, we get:

$$\begin{aligned} \frac{\sum_{i=0}^{t-1} E_i}{t} &\leq \frac{c_a^b}{b-a} \sum_{i=0}^{t-1} \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}_t - p\right|\right]_p dp + \frac{b-a}{4t} \\ &= \frac{c_a^b}{b-a} \int_a^b \mathbf{E}\left[\left|\hat{\theta}_t - p\right|\right]_p dp + \frac{b-a}{4t} \end{aligned}$$

■

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