Contribution Title*

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1 Introduction

1.1 Friedkin-Johsen Model

In the Friedkin-Johsen model (FJ-model) an undirected graph $G(V, E)$ with n nodes, is assumed, where V denotes the agents and E the social relations between them. Each agent i poses an internal opinion $s_i \in [0, 1]$ and a self confidence coefficient $\alpha_i \in [0, 1]$. At each round $t \geq 1$, agent i updates her opinion as follows:

$$x_i(t) = (1 - \alpha_i)\frac{\sum_{j \in N_i} x_j(t-1)}{|N_i|} + \alpha_i s_i, \tag{1}$$

where N_i is the set of her neighbors. The simplicity of its update rule makes it plausible, because in real social networks it is very unlikely that agents change their opinions according to complex rules. This model has been studies from several perspectives. Based on the above model, in [?], they propose a game in which the strategy that each agent i plays, is the opinion $x_i \in [0, 1]$ that she publicly expresses incurring her a cost

$$C_i(x) = (1 - a_i) \sum_{j \in N_i} (x_i - x_j)^2 + a_i(x_i - s_i)^2. \tag{2}$$

They prove that this game admits a unique equilibrium point and that the price of anarchy is 9/8 for undirected graphs and $O(n)$ for directed networks. In [?] they study they show that the update rule (1) is the *best response* play for the above game and they also study its convergence properties to the equilibrium point $x^* \in [0, 1]^n$. Therefore, Friedkin-Johsen model admits several nice properties as it very simple to execute by the agents, rapidly converges to its equilibrium, and it is a natural behavior for selfish agents.

Our work is motivated by the fact that the FJ-model requires that agents interact with all their neighbors at each round. This requirement could be under discussion in today's large social networks, where each individual could have several hundreds of friends (neighbors). In such graphs it is quite unnatural to assume that everybody updates her opinion using the opinions of all her friends. In this work we propose simple and natural models, similar to the original FJ-model, that require minimal interaction between agents at each round with similar convergence properties.

We now describe our setting precisely. Similar to the FJ-model, our setting consists of an instance $I = (G, s, a)$, where $G(V, E)$ is a graph and $s, a \in [0, 1]^n$. We depart from the FJ-model by imposing agents to only interact with only one of their neighbors at each round. We assume that the interaction between the agents is possible only through the following oracles.

- The *unlabeled* oracle $O_i^u(t)$: when agent i calls the oracle $O_i^u(t)$ at round t the oracle picks one of her neighbors j uniformly at random and returns $x_j(t-1)$.
- The *labeled* oracle $O_i^l(t)$: when agent i calls the oracle at round t the oracle $O_i^l(t)$ picks one of her neighbors j uniformly at random and returns $(j, x_j(t-1))$, namely the label and the opinion of her neighbor j.

In the *labeled* resp. *unlabeled* setting, at the beginning of each round t, each agent i calls $O^l(i)$ resp. $O^u(i)$. With these settings in mind we study the existence of simple and natural update rules that converge in reasonable to the same equilibrium point x^*.
MOTIVATION FOR THE TWO SETTINGS.
We now describe the update rules that we will study. In the unlabeled setting we consider the following update rule that closely resembles the original FJ-model

$$\begin{aligned} x_i(0) &= s_i \\ x_i(t) &= (1 - \alpha_i)\frac{\sum_{\tau=1}^{t} O_i^u(\tau)}{t} + \alpha_i s_i. \end{aligned} \tag{3}$$

In the labeled setting each agent i keeps a vector $M_i \in [0, 1]^{|N_i|}$ with the opinions of all her neighbors. At each round t she receives the value $(j, x_j(t-1))$ from $O_i^l(t)$ and sets $M_i(j) = x_j(t-1)$. Initially all agents have $M_i = 0$. Her update rule is

$$\begin{aligned} x_i(0) &= s_i \\ x_i(t) &= (1 - a_i)\frac{\sum_{j \in N_i} M_i(j)}{|N_i|} + a_i s_i \end{aligned} \tag{4}$$

1.2 Our Results

We first show that in the unlabeled setting, using the update rule (3) gives a protocol that converges to the same equilibrium point as the FJ-model. In the next theorem we provide the rate of convergence of our model.

Theorem 1. *In the unlabeled setting, for any instance $I = (G, s, a)$ the update rule (3) has convergence rate*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i\in V} a_i$ and C is a universal constant.

For the labeled case we show that we even in our restricted setting we can still achieve an almost linear rate of convergence as stated in the following theorem

Theorem 2. *In the labeled setting, for any instance $I = (G, s, a)$ the update rule (4) has convergence rate*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq 2(1-a)^{\overline{\frac{\sqrt{t}}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)}}}$$

2 Full memory with limited information exchange

In each round, every agent learns a neighbour's opinion and updates the appropriate cell in memory. Because an agent learns only one opinion in each round, he uses outdated information about his other neighbours in order to compute his opinion. We first introduce some convinient notation.

Definition 1. *$\pi_{ij}(t)$ is the last time that i learned j's opinion until time t.*

Obviously, if at time t an agent i learned k's opinion, then $\pi_{ik}(t) = t$. The rule, according to which agent i updates his opinion at time t is the following(for simplicity, assume that $\alpha_i = \frac{1}{d+1}$ and that the graph is d-regular):

$$x_i(t+1) = (1-\alpha_i)\frac{\sum_{j\in N_i} x_j(\pi_{ij}(t))}{d_i} + \alpha_i s_i$$

Notice that in contrast with the traditional model, instead of $x_j(t)$ we have $x_j(\pi_{ij}(t))$. That is because the information about j is probably outdated. Denote with x^* the solution vector of the system. We would like to prove that such a process converges to the solution of the linear system involving the laplacian matrix of the graph. In order to analyse the algorithm, we will divide the time into "epochs". Each epoch has a length $D \geq d$, so the first epoch is from time 1 to time D, the second from $D+1$ to $2D$ e.t.c. Time 0 does not belong at any epoch. The numbering of epochs begins from 1. We will also assume that during each epoch, each agent picks every one of his neighbours for update at least once. Later we are going to find a suitable epoch length D, such that this holds with high probability. In other words, for a specific time in epoch i every agent has information which has "bounded" outdateness, since every coordinate was updated at least once during the $i-1$ epoch. We are going to use this fact to prove the following lemma.

Lemma 1. *Denote with x^* the solution vector of the system. For every time t, which belongs to epoch T, it holds:*

$$||x(t) - x^*||_\infty \leq (1-\alpha_i)^T\, ||x(0) - x^*||_\infty$$

Proof. We are going to use induction in time. The base case is for time $t = 1$. We are going to prove that:

$$||x(1) - x^*||_\infty \leq (1-\alpha_i)||x(0) - x^*||_\infty$$

We assume that everybody has the same initial vector $x(0)$ stored in memory, so for each i holds:

$$x_i(1) = (1-\alpha_i)\frac{\sum_{j\in N_i} x_j(0)}{d} + \alpha_i s_i \tag{5}$$

Since x^* is the solution of the system, we have:

$$x_i^* = (1-\alpha_i)\frac{\sum_{j\in N_i} x_j^*}{d} + \alpha_i s_i \tag{6}$$

Subtracting (6) from (5) we get:

$$\begin{aligned}
|x_i(1) - x_i^*| &= |(1-\alpha_i)\frac{\sum_{j\in N_i}(x_j(0) - x_j^*)}{d}| \\
&\leq (1-\alpha_i)\frac{\sum_{j\in N_i}|x_j(0) - x_j^*|}{d} \\
&\leq (1-\alpha_i)||x(0) - x^*||_\infty
\end{aligned}$$

$$\Rightarrow ||x(1) - x^*||_\infty \leq (1-\alpha_i)||x(0) - x^*||_\infty$$

So the base case is verified. Let the inductive hypothesis hold for all times until t. Suppose that time $t+1$ belongs to epoch T. We fix an agent i. Then, if $\pi_{ij}(t)$ belongs to the previous epoch, by the induction hypothesis it holds:

$$|x_j(\pi_{ij}(t)) - x_j^*| \leq (1-\alpha_i)^{T-1}||x(0) - x^*||_\infty$$

On the other hand, if $\pi_{ij}(t)$ belongs to the current epoch T, by the induction hypothesis we have:

$$|x_j(\pi_{ij}) - x_j^*| \leq (1-\alpha_i)^{T}||x(0) - x^*||_\infty \leq (1-\alpha_i)^{T-1}||x(0) - x^*||_\infty$$

So, for every neighbour of agent i, the value that i stores for this neighbour is "close" to the optimal. Now, using again equation (1), we have:

$$\begin{aligned}
|x_i(t+1) - x_i^*| &= |(1-\alpha_i)\frac{\sum_{j\in N_i}(x_j(\pi_{ij}(t)) - x_j^*)}{d}| \\
&\leq (1-\alpha_i)\frac{\sum_{j\in N_i}|x_j(\pi_{ij}(t)) - x_j^*|}{d} \\
&\leq (1-\alpha_i)\frac{\sum_{j\in N_i}||x(\pi_{ij}(t)) - x^*||_\infty}{d} \\
&\leq \frac{\sum_{j\in N_i}(1-\alpha_i)^{T-1}||x(0) - x^*||_\infty}{d+1} \\
&= (1-\alpha_i)(1-\alpha_i)^{T-1}||x(0) - x^*||_\infty \\
&= (1-\alpha_i)^{T}||x(0) - x^*||_\infty
\end{aligned}$$

Corollary 1. *If the algorithm runs for $O(D\log\frac{1}{\epsilon})$ iterations, it gets ϵ-close to the solution x^*.*

We now turn our attention to the problem of computing the appropriate length D of epochs. A useful fact concerning the coupons collector problem is the following.

Lemma 2. *Suppose that the collector picks $n\ln n + cn$ coupons, where n is the number of distinct coupons. Then:*

$$\mathbf{P}\left[\textit{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

Theorem 3. *After t rounds, with probability at least $1-p$: $\|x^t - x^*\|_\infty \leq (1-a)^{\frac{t}{2d\ln(\frac{dt}{p})}}$*

Proof. In our setting, coupon i corresponds to the selection of neighbour i. Each node is a collector and wants to gather all d_i coupons during each epoch. Suppose $d = \max_i d_i$ is the maximum degree of the graph. Then, if we set $n = d$ and $c = \ln(\frac{dt}{p})$, using the previous lemma we get that a node hasn't seen at least one neigbour after $cd + d\ln d$ samples with probability at most $\frac{p}{dt}$. This means that if we set $D = cd + d\ln d = d\ln\frac{dt}{p} + d\ln d \geq 2d\ln\frac{dt}{p}$ when p is small enough, then the probability that a specific agent at a specific epoch hasn't collected all neighbouring opinions at least once is at most $\frac{p}{dt}$. By a simple union bound argument, we get that all agents have seen all their neighbours during all epochs with probability at least $1-p$. We observe that at time t approximately $\frac{t}{D}$ epochs have passed. Therefore, using the previous result about the convergence rate, we get that with probability at least $1-p$:

$$\|x^t - x^*\|_\infty \le (1-a)^{\frac{t}{D}} \le (1-a)^{\frac{t}{2d\ln(\frac{dt}{p})}}$$

We are now going to translate this result to one that involves the expected value of the error.

Theorem 4. *If we set $u(t) = \|x^t - x^*\|_\infty$, the error after t rounds, then:*

$$\mathbf{E}\left[u(t)\right] \le 2(1-a)^{\frac{\sqrt{t}}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)}}$$

Proof. Using the result of the previous theorem, we obtain:

$$\mathbf{P}\left[u(t) > (1-a)^{\frac{t}{2d\ln(\frac{dt}{p})}}\right] \le p$$

for every probability $p \in [0,1]$. Also, since all the parameters of the problem lie in $[0,1]$, we have

$$\mathbf{E}\left[u(t)|u(t) > r\right] \le 1$$

Now, by the conditional expectations identity, we get:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &= \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \le r\right]\mathbf{P}\left[u(t) \le r\right] \\ &\le p + r\end{aligned}$$

where $r = (1-a)^{\frac{t}{2d\ln(\frac{dt}{p})}}$. If we set $p = (1-a)^{\frac{\sqrt{t}}{2d\ln dt}}$, then:

$$\mathbf{E}\left[u(t)\right] \le (1-a)^{\frac{\sqrt{t}}{2d\ln dt}} + (1-a)^{\frac{t}{2d\ln(\frac{dt}{p})}}$$

We now evaluate r for our choice of probability p:

$$\begin{aligned} r &= (1-a)^{\frac{t}{2d\ln\left(\frac{dt}{p}\right)}} \\ &= (1-a)^{\frac{t}{2d\ln\left(\frac{dt}{(1-a)^{\frac{\sqrt{t}}{2d\ln dt}}}\right)}} \\ &= (1-a)^{\frac{t}{2d\ln dt + 2d\frac{\sqrt{t}}{2d\ln dt}\ln\left(\frac{1}{(1-a)}\right)}} \\ &\le (1-a)^{\frac{t}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)\sqrt{t}}} \\ &= (1-a)^{\frac{\sqrt{t}}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)}} \end{aligned}$$

Using the previous calculation, we obtain:

$$\mathbf{E}\left[u(t)\right] \le (1-a)^{\frac{\sqrt{t}}{2d\ln dt}} + (1-a)^{\frac{\sqrt{t}}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)}} \le 2(1-a)^{\frac{\sqrt{t}}{4d\ln(dt)\ln\left(\frac{1}{1-a}\right)}}$$

Therefore, this strategy achieves subexponential convergence rate in expectation.

3 Constant Memory with full window

In this work we investigate an variant of the above process. We denote by V the set of agents, and $N_i \subset V$ the set of neighbors of agent i. Let $x(t) \in [0,1]^n$ be the opinion vector at round t. We denote by $x_i(t)$ the opinion of agent i at round t. At round $t+1$, $x(t+1)$ is constructed in the following way. At first, each agent

i gets to see the opinion $x_j(t)$, where $j \in N_i$ is picked uniformly at random from the set of her neighbors. Let W_i^t be the random variable corresponding to the selected neighbor. Now each agent $i \in V$ updates her opinion as follows

$$x_i(t+1) = (1-\alpha_i)\frac{\sum_{\tau=1}^{t+1} x_{W_i^\tau}(\tau-1)}{t+1} + \alpha_i s_i.$$

We remark that each agent i get to see *only* the opinion $x_{W_i^t}$ and not the label W_i^t of her neighbor.

Let $x^* \in [0,1]^n$ be the unique equilibrium point of the given instance I. We prove that the above stochastic process has the following convergence rate to x^*.

Theorem 5.

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] = \begin{cases} O\left(\sqrt{\log n}\frac{(\log t)^2}{t^\alpha}\right) & \text{if } \alpha \le 1/2 \\ O\left(\sqrt{\log n}\frac{(\log t)^2}{t^{1/2}}\right) & \text{if } \alpha > 1/2 \end{cases}$$

Proof. Setting $p = 1/\sqrt{t}$ in Lemma 5 yields the result.

We start by stating the standard Hoeffding bound

Lemma 3 (Hoeffding's Inequality). *Let $X = (X_1 + \cdots + X_t)/t$, $X_i \in [0,1]$. Then,*

$$\mathbf{P}\left[|X - \mathbf{E}[X]| > \lambda\right] < 2e^{-2t\lambda^2}.$$

Lemma 4. *With probability at least $1-p$, $\|x^t - x^*\|_\infty \le e(t)$, where $e(t)$ satisfies the following recursive relation*

$$e(t) = \delta(t) + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \text{ and } e(0) = \|x^0 - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$.

Corollary 2. *The function $e(t)$ satisfies the following recursive relation*

$$e(t+1) - e(t) + \alpha\frac{e(t)}{t+1} = \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1}$$

Proof. As we have already mentioned for any instance I there exists a unique equilibrium vector x^*. Since $W_i^\tau \sim \mathrm{U}(N_i)$ we have that $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \frac{\sum_{j\in N_i} x_j^*}{|N_i|}$. Since W_i^τ are independent random variables, we can use Hoeffding's inequality (Lemma 3) to get

$$\mathbf{P}\left[\left|\frac{\sum_{\tau=1}^t x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] < \frac{6}{\pi^2}\frac{p}{nt^2},$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Therefore, by the union bound

$$\mathbf{P}\left[\text{for all } t \ge 1 : \max_{i\in V}\left|\frac{\sum_{\tau=1}^t x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \le$$
$$\sum_{t=1}^{\infty}\mathbf{P}\left[\max_{i\in V}\left|\frac{\sum_{\tau=1}^t x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \le$$
$$\sum_{i=1}^{\infty}\frac{6}{\pi^2}\frac{1}{t^2}\sum_{i=1}^{n}\frac{p}{n} = p$$

As a result with probability $1-p$ we have that for all t

$$\left|\frac{\sum_{\tau=1}^t x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| \le \delta(t) \tag{7}$$

We will our claim by induction. We assume that $\|x^\tau - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x_{W_i^\tau}(\tau-1)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau} + \sum_{\tau=1}^{t} e(\tau-1)}{t} + \alpha_i s_i \qquad (8)\\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\frac{\sum_{j\in N_i} x^*_j}{|N_i|} + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \qquad (9)\\
&\leq x_i^* + \delta(t) + (1-\alpha)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}$$

We get (8) from the induction step and (9) from inequality (7). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\alpha)\frac{\sum_{\tau=1}^{t} e(\tau)}{t}$. As a result $||x_i(t) - x^*||_\infty \leq e(t)$.

In order to bound the convergence time of the system, we just need to bound the convergence rate of the function $e(t)$. The following lemma provides us with a simple upper bound for the convergence rate of our process.

Lemma 5. *Let $e(t)$ be a function satisfying the recursion of Corollary 2. Then with probability at least $1-p$ we have that*

$$e(t) = \begin{cases} O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^\alpha}\right) & \text{if } \alpha \leq 1/2 \\ O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^{1/2}}\right) & \text{if } \alpha > 1/2 \end{cases}$$

Proof. At first, since $\frac{\delta(t)}{t}$ is a decreasing function for $p \leq 1/4$, we have that $e(t) \leq (1-\frac{\alpha}{t}) + g(t)$, where $g(t) = \frac{\delta(t)}{t}$.

$$\begin{aligned}
e(t) &\leq (1-\frac{\alpha}{t})e(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})(1-\frac{\alpha}{t-1})e(t-2) + (1-\frac{\alpha}{t})g(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})\cdots(1-\alpha)e(0) + \sum_{\tau=1}^{t} g(\tau)\prod_{i=\tau+1}^{t}(1-\frac{\alpha}{i}) \\
&\leq \frac{e(0)}{t^\alpha} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^\alpha} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha(H_t - H_\tau)} \\
&\leq \frac{e(0)}{t^\alpha} + e^{-\alpha H_t}\sum_{\tau=1}^{t} g(\tau)e^{\alpha H_\tau} \\
&\leq \frac{e(0)}{t^\alpha} + \frac{O\left(\sqrt{\log(\frac{n}{p})}\right)}{t^\alpha}\sum_{\tau=1}^{t}\tau^\alpha\frac{\sqrt{\log\tau}}{\tau^{3/2}}
\end{aligned}$$

We observe that

$$\sum_{\tau=1}^{t}\tau^\alpha\frac{\sqrt{\log\tau}}{\tau^{3/2}} \leq \int_{\tau=1}^{t}\tau^\alpha\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau,$$

since $\tau^\alpha\frac{\sqrt{\log\tau}}{\tau^{3/2}}$

– If $\alpha \leq 1/2$ then

$$\int_{\tau=1}^{t} \tau^{\alpha} \frac{\sqrt{\log \tau}}{\tau^{3/2}} d\tau \leq \int_{\tau=1}^{t} \tau^{1/2} \frac{\sqrt{\log \tau}}{\tau^{3/2}} d\tau = O((\log t)^{3/2}))$$

– If $\alpha > 1/2$ then

$$\begin{aligned} \int_{\tau=1}^{t} \tau^{\alpha} \frac{\sqrt{\log \tau}}{\tau^{3/2}} d\tau &= \int_{\tau=1}^{t} \tau^{\alpha-1/2} \frac{\sqrt{\log \tau}}{\tau} d\tau \\ &= \frac{2}{3} \int_{\tau=1}^{t} \tau^{\alpha-1/2} ((\log \tau)^{3/2})' d\tau \\ &= \frac{2}{3} (\log t)^{3/2} - (\alpha - 1/2) \frac{2}{3} \int_{\tau=1}^{t} \tau^{\alpha-3/2} (\log \tau)^{3/2} d\tau \\ &= O\left((\log t)^{3/2}\right) \end{aligned}$$

4 Lower bound for unbiased estimators

We first state a well known result from point estimation:

Lemma 6 (Cramer-Rao bound). *Let $\hat{\theta}$ be an estimator for the parameter θ of a distribution P_θ, where θ is a continuous parameter. Suppose that the estimator is unbiased, that is:$E[\hat{\theta}] = 0$, for all distributions P_θ. Under suitable regularity conditions, which are met by bernoulli distributions, it holds:*

$$Var[\hat{\theta}] \geq \frac{1}{nI_\theta}$$

, where I_θ is the Fischer information of distribution P_θ.

In the case of Bernoulli random variables, we can easily show that for a Bernoulli with probability θ, $I_\theta = \frac{1}{\theta(1-\theta)}$. Applying the above result we obtain

$$var[\hat{\theta}] \geq \frac{\theta(1-\theta)}{n} \geq \frac{1}{4n}$$

for every unbiased estimator $\hat{\theta}$. We can use this fact to show that a specific class of distributed protocols that solve our problem has "large" variance. Let P be a protocol that when restricted to the star topology acts like an unbiased estimator for the weighted mean value of the neighbours, which can be an arbitrary real number in $[0, 1]$. We will show that for all star topologies the variance of the solution is $\geq \frac{(1-a)^2}{4n}$ after n samples. Suppose that for a specific star topology the preceding claim doesn't hold. Then, we notice that if $\hat{\theta}$ is the output of the protocol is this topology, then $\frac{(\hat{\theta}-as)}{1-a}$ is an unbiased estimator of the mean value of the neighbours, and this holds for every possible value of the mean. By our hypothesis, we have:

$$Var[\frac{\hat{\theta} - as}{1-a}] = \frac{1}{(1-a)^2} Var[\hat{\theta}] < \frac{1}{4n}$$

, a contradiction, since our constructed estimator is unbiased.

5 No Regret

We consider the following online convex optimization problem. At each time step t, the player i selects a real number x^t and a function $f^t(x)$ arrives. The player then suffers $f^t(x^t)$ cost. The functions $f^t(x)$ have the following form:

$$f^t(x) = \alpha(x - s_i)^2 + (1 - \alpha)(x - a_t)^2$$

where $s_i, \alpha \in [0, 1]$ and are independent of t and $a_t \in [0, 1]$. In other words, the function f^t is uniquely determined by the number a_t.

We show that for this class of functions *fictitious play* admits no regret.

Theorem 6. *Let f^t be a sequence of functions, where each function has the form: $f^t(x) = \alpha(x - s_i)^2 + (1 - \alpha)(x - a_t)^2$, where $s_i, \alpha \in [0,1]$ and are independent of t and $a_t \in [0,1]$. If we define $x^t = \arg min_{x\in[0,1]} \sum_{\tau=1}^{t-1} f^\tau(x)$ then for each time T*

$$\sum_{t=1}^{T} f^t(x^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + O(\log T)$$

In order to prove this claim, we will first define a similar rule y^t that takes into account the function f^t for the "prediction" at time t. Intuitively, this guarrantees that the rule admits no regret.

Lemma 7. *Let $y^t = argmin_{x\in[0,1]} \sum_{\tau=1}^{t} f^\tau(x)$ then*

$$\sum_{t=1}^{T} f^t(y^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x)$$

Proof. By definition of y^t, $min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) = \sum_{t=1}^{T} f^t(y^T)$, so

$$\begin{aligned}
\sum_{t=1}^{T} f^t(y^t) - min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) &= \sum_{t=1}^{T} f^t(y^t) - \sum_{t=1}^{T} f^t(y^T) \\
&= \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^T) \\
&\leq \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^{T-1}) \\
&= \sum_{t=1}^{T-2} f^t(y^t) - \sum_{t=1}^{T-2} f^t(y^{T-1})
\end{aligned}$$

Continuing in the same way, we get $\sum_{t=1}^{T} f^t(y^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x)$.

Now we can derive some intuition for the reason that *fictitious play* admits no regret. Since the cost incurred by the sequence y^t is at most that of the best fixed strategy, we can compare the cost incurred by x^t with that of y^t. However, for each t the numbers x^t and y^t are quite close and as a result the difference in their cost must be quite small.

Lemma 8. *For all t, $f^t(x^t) \leq f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-a)^2}{t^2}$.*

Proof. We first prove that for all t,

$$\left|x^t - y^t\right| \leq \frac{1-\alpha}{t} \tag{10}$$

.By definition $x^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1}$ and $y^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t} a_\tau}{t}$.

$$\begin{aligned}
\left|x^t - y^t\right| &= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1} - \frac{\sum_{\tau=1}^{t} a_\tau}{t}\right| \\
&= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau - (t-1)a_t}{t(t-1)}\right| \\
&\leq \frac{1-\alpha}{t}
\end{aligned}$$

The last inequality follows from the fact that $a_\tau \in [0,1]$. We now use inequality (10) to bound the difference $f^t(x^t) - f^t(y^t)$.

$$\begin{aligned}
f^t(x^t) &= \alpha(x^t - s_i)^2 + (1-\alpha)(x^t - a_t)^2 \\
&\le \alpha(y^t - s_i)^2 + 2\alpha\left|y^t - s_i\right|\left|x^t - y^t\right| + \alpha\left|x^t - y^t\right|^2 \\
&\quad + (1-\alpha)(y_t - a_t)^2 + 2(1-\alpha)\left|y^t - a_t\right|\left|x^t - y^t\right| + (1-\alpha)\left|x^t - y^t\right|^2 \\
&\le f^t(y^t) + 2\left|x^t - y^t\right| + \left|y^t - x^t\right|^2 \\
&\le f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-\alpha)^2}{t^2}
\end{aligned}$$

Theorem 6 easily follows since:

$$\begin{aligned}
\sum_{t=1}^{T} f^t(x^t) &\le \sum_{t=1}^{T} f^t(y^t) + \sum_{t=1}^{T} 2\frac{1-\alpha}{t} + \sum_{t=1}^{T} \frac{(1-\alpha)^2}{t^2} \\
&\le min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + 2(1-\alpha)(\log T + 1) + (1-\alpha)\frac{\pi^2}{6} \\
&\le min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + O(\log T)
\end{aligned}$$

6 Lower Bound

Theorem 7. *Suppose $\hat{\theta}^t$ is an estimator for the probability p of a Bernoulli random variable that takes t samples. Then, for every interval $[a,b]$ which is contained in $[0,1]$ and for every $c > 0$, it holds that:*

$$\lim_{t\to\infty} t^{1+c} \int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = \infty$$

In order to prove this, we need the following result:

Lemma 9 (Fano's Inequality). *Suppose we have an estimator $\hat{\theta}^t$ for the probability of a Bernoulli distribution. The unknown distribution belongs to a family of distributions $\mathcal{P}$ with the property that the probabilities of every two distributions in this family differ by at least 2δ, where δ is a positive real number. Then, for every subset $P_1, P_2, ..., P_n$ of distributions from this family, where p_i is the probability of distribution P_1, the following inequality holds:*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right] \ge \delta\left(1 - \frac{I(X;V) + \log 2}{\log n}\right)$$

We are now going to use Theroem (9) to prove the following lemma, which provides a lower bound for the mean of the expectation of errors of the estimator.

Lemma 10. *Suppose we choose t Bernoulli distributions $\{P_i\}_{i=0}^{t-1}$ with the following probabilities:*

$$p_i = a + \frac{b-a}{2t} + i\frac{b-a}{t}, i = 0, ..., t-1$$

. Then:

$$\frac{1}{t}\sum_{i=1}^{t} \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right] \ge \frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right)$$

where $c_1, c_2 > 0$ and $c_1 > \frac{1}{2}$.

Proof. Without loss of generality, we can assume that t is a multiple of 5. For simplicity, we set $E_i = \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right]$. By the choice of p_i, we have that

$$|p_i - p_j| \geq \frac{b-a}{t}$$

, for every distinct i, j in the family. This means that for the family of t distributions that we picked, the property of Lemma (9) is satisfied for $\delta = \frac{b-a}{2t}$. Thus, by dividing the t distributions into groups of 5 and applying Lemma (9) to each one, we obtain:

$$\begin{aligned} \frac{\sum k = i^{i+4}}{5} &\geq \frac{b-a}{2t}\left(1 - \frac{I(X;V) + \log 2}{\log n}\right) \\ &= \frac{b-a}{2t}\left(1 - \frac{\log 2}{\log 5} - \frac{I(X;V)}{\log 5}\right) \\ &= \frac{b-a}{2t}\left(c_1 - \frac{I(X;V)}{\log 5}\right) \end{aligned} \tag{11}$$

where $c_1 = 1 - \frac{\log 2}{\log 5} > \frac{1}{2}$. We are now going to upper bound the mutual information $I(X;V)$ for every group of 5 distributions. We denote by P_i^t the distribution on vectors with t coordinates, where each coordinate is sampled independently from P_i. Let $U = \{i, i+1, i+2, i+3, i+4\}$ be a family of 5 distributions. Using a well known inequality REFERENCE NEEDED HERE:

$$\begin{aligned} I(X;V) &\leq \frac{1}{5^2} \sum_{\substack{i,j \in U \\ i \neq j}} D_{kl}\left(P_i^t, P_j^t\right) \\ &\leq D_{kl}\left(P_i^t, P_{i+4}^t\right) \\ &= tD_{kl}\left(P_i, P_{i+4}\right) \\ &\leq t\frac{(p_i - p_{i+4})^2}{p_i(1-p_i)} \\ &\leq t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)(p_i - p_{i+4})^2 \\ &= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\left(\frac{b-a}{2t} + i\frac{b-a}{t} - \frac{b-a}{t} - (i+4)\frac{b-a}{t}\right)^2 \\ &= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\frac{16(b-a)^2}{t^2} \\ &= \frac{16(b-a)^2}{t}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right) \end{aligned}$$

So, if we set

$$c_2 = 16\frac{(b-a)^2}{\log 5}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)$$

then by equation (11) we obtain:

$$\sum_{k=i}^{i+4} \geq \frac{5(b-a)}{2t}\left(c_1 - \frac{c_2}{t}\right) \tag{12}$$

Inequality (12) holds for every group of 5 distributions. By summing up for all groups:

$$\begin{aligned} \sum_{i=0}^{t-1} E_i &\geq t\frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right) \\ &= \frac{b-a}{2}\left(c_1 - \frac{c_2}{t}\right) \end{aligned}$$

which is what we wanted to prove.

Next, we state a lemma regarding an upper bound on the expectation of the error in each p_i.

Lemma 11. *For every* $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$ *:*

$$\mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right]_{p_i} \le c_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p + |p - p_i|$$

where c_a^b *is a constant that depends on* a, b.

Now, we find an upper bound for the quantity $\sum_{i=0}^{t-1} E_i$. Our goal is to find an upper bound where the coefficient of $(b-a)/t$ is less than $1/4$.

Lemma 12.

$$\frac{1}{t}\sum_{i=0}^{t-1} E_i \le \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right] dp + \frac{b-a}{4t}$$

Proof. By Lemma (11) we get that for every $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$:

$$E_i \le c_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p + |p_i - p|$$

We integrate both sides of the equation with respect to p and get:

$$\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} E_i dp \le c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} |p_i - p|\, dp$$

Hence:

$$\begin{aligned}\frac{b-a}{t} E_i &\le c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + 2\int_{p_i - \frac{b-a}{2t}}^{p_i} |p_i - p|\, dp \\ &= c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + \frac{1}{4}\frac{(b-a)^2}{t^2}\end{aligned}$$

Therefore:

$$\frac{E_i}{t} \le \frac{c_a^b}{b-a}\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + \frac{b-a}{4t^2} \tag{13}$$

By summing up Equation (13) for all t intervals, we get:

$$\begin{aligned}\frac{\sum_{i=0}^{t-1} E_i}{t} &\le \frac{c_a^b}{b-a}\sum_{i=0}^{t-1}\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + \frac{b-a}{4t} \\ &= \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp + \frac{b-a}{4t}\end{aligned}$$

Now, by combining Lemmas (10) and (12) we get:

$$\frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp \ge \frac{b-a}{2t}\left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right)$$

By multiplying by t^{1+c} we get:

$$\frac{c_a^b}{b-a} t^{1+c}\int_a^b \mathbf{E}\left[\left|\hat{\theta}^t - p\right|\right]_p dp \ge \frac{b-a}{2} t^c \left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right) \tag{14}$$

The coefficient of t^c in the right hand side of (14) is positive, so by sendig $t \to \infty$ we get:

$$\lim_{t\to\infty} t^{1+c}\int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = \infty$$

7 Lower bound

Theorem 8. *Let P be a protocol run by each agent that produces vector output x^t after t iterations. Suppose that for every graph G and for every choice of parameters $s_i, \alpha_i \in [0,1]$ the error of the output is:*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] = f(n,t)$$

where n is the number of nodes in the graph. Then,

$$\lim_{t\to\infty} \sqrt{t} f(t,t) > 0$$

Proof. Let G be a graph that consists of 2 star topologies. That is, we have 2 center nodes v_1, v_2. Each of the two center nodes has exactly t neighbours, which have degree 1. We will call these nodes the leaves of the graph. This graph has $2t+2$ nodes. We set $\alpha_i = 1$ for all the leaves and $\alpha_i = 1/2$ for the two centers. For exactly $t/2 + \sqrt{t}/2$ leaves of v_1 we set $s_i = 1$, for the rest $s_i = 0$. Similarly, for exactly $t/2 - \sqrt{t}/2$ leaves of v_2 we set $s_i = 1$, for the rest $s_i = 0$. By direct calculation, we get that the equilibrium point for this graph is the following:

$$x_i^* = \begin{cases} s_i & \text{if i is a leaf node} \\ \frac{1}{4} + \frac{1}{4\sqrt{t}} & \text{if } i = v_1 \\ \frac{1}{4} - \frac{1}{4\sqrt{t}} & \text{if } i = v_2 \end{cases}$$

Since the graph has $2t+2$ nodes:

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] = f(2t+2,t) \tag{15}$$

On the other hand, since $\|x^t - x^*\|_\infty \geq \max\{\left|x_{v_1} - x^*_{v_1}\right|, \left|x_{v_2} - x^*_{v_2}\right|\}$ we get:

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \geq \frac{1}{2}\mathbf{E}\left[\left|x^t_{v_1} - x^*_{v_1}\right|\right] + \frac{1}{2}\mathbf{E}\left[\left|x^t_{v_2} - x^*_{v_2}\right|\right] \tag{16}$$

Suppose we have a star graph with center node c. At each time t, the oracle returns to the center node either the value of a leaf with $s_i = 1$ (e.g.$f(1,1,t)$) with probability :

$$p = \frac{\#\text{ leaves with } s_i = 1}{\#\text{ leaves}}$$

or the value of a leaf with $s_i = 0$ (e.g.$f(1,0,t)$) with probability $1-p$. As a result, knowing the 0-1 choice of the oracle at time step t, one can compute x_c^t, meaning that $\hat{\theta}^t = 2x_c^t$ is an estimator for a Bernoulli random variable with probability p. This obviously holds for every choice of $s_i \in \{0,1\}$. This means that in our graph, $\hat{\theta}_1 = 2x^t_{v_1}$ is an estimator for the Bernoulli with probability $p_1 = \frac{1}{2} + \frac{1}{2\sqrt{t}}$ and $\hat{\theta}_2 = 2x^t_{v_2}$ is an estimator for the Bernoulli with probability $p_2 = \frac{1}{2} - \frac{1}{2\sqrt{t}}$. To be more precise, since the protocol is executed in the same way in all nodes, $\hat{\theta}_1$ and $\hat{\theta}_2$ are the ouputs of one estimator $\hat{\theta}$, depending on the distribution from which this estimator gets its input. By combining inequalities (15) and (16) we get:

$$f(2t+2,t) \geq \frac{1}{4}E_{p_1}\{\left|\hat{\theta} - p_1\right|\} + \frac{1}{4}E_{p_2}\{\left|\hat{\theta} - p_2\right|\} \tag{17}$$

We will now try to bound the right hand side of equation (17). We first notice that:

$$\begin{aligned} E_{p_1}\left|\hat{\theta} - p_1\right| &\geq \delta\mathbf{E}\left[\mathbb{1}\{\left|\hat{\theta} - p_1\right| > \delta\}\right] \\ &= \delta\mathbf{P}\left[\left|\hat{\theta} - p_1\right| > \delta\right] \end{aligned}$$

The same holds for p_2. It suffices now to provide a lower bound for this probability. In order to do this, we use a standard reduction from estimation problems to testing problems(see lecam). In a testing problem, there exist two Bernoulli distributions P_1 and P_2. First, nature selects with equal probability one of them and let $V \in \{1,2\}$ denote this random choice. Then, we draw a random vector $(Y_1, \ldots, Y_t)$ from the t-fold product distribution P_V^t. The goal is to decide whether $V = 1$ or $V = 2$. More precisely, we have to select a suitable function $\psi : \{0,1\}^t \mapsto \{1,2\}$ that minimizes $\mathbf{P}\left[\psi(Y_1, \ldots, Y_t) \neq V\right]$ First, we state a Lemma that provides a lower bound for this probability.

Lemma 13. *For every function $\psi : \{0,1\}^t \mapsto \{1,2\}$:*

$$\mathbf{P}\left[\psi(Y_1,\ldots,Y_t) \neq V\right] \geq \frac{1}{2}\left(1 - D_{TV}(P_1^t, P_2^t)\right)$$

where t is the number of samples.

We have:

$$\begin{aligned}\frac{1}{4}E_{p_1}\{\left|\hat{\theta}-p_1\right|\} + \frac{1}{4}E_{p_2}\{\left|\hat{\theta}-p_2\right|\} &= \frac{1}{4}E\{\left|\hat{\theta}-p_1\right| |V=1\} + \frac{1}{4}E\{\left|\hat{\theta}-p_2\right| |V=2\} \\ &\geq \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} > \frac{1}{2}|V=1\right] + \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} < \frac{1}{2}|V=2\right]\end{aligned}$$

We define :

$$\psi = \begin{cases} 1 & \hat{\theta} < \frac{1}{2} \\ 2 & \hat{\theta} > \frac{1}{2} \end{cases}$$

As a result,

$$\begin{aligned}\frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} > \frac{1}{2}|V=1\right] + \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} < \frac{1}{2}|V=2\right] &\geq \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\hat{\theta} > \frac{1}{2}, V=1\right] + \mathbf{P}\left[\hat{\theta} > \frac{1}{2}, V=2\right]\right) \\ &= \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\psi=2, V=1\right] + \mathbf{P}\left[\psi=1, V=2\right]\right) \\ &= \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\psi \neq V\right]\right) \\ &\geq \frac{1}{4\sqrt{t}}(1 - D_{TV}(P_1^t, P_2^t)) \\ &\geq \frac{1}{24\sqrt{t}}\end{aligned}$$

So

$$\lim_{t\to\infty} \sqrt{t}f(2t+2,t) > 0$$

7.1 Estimating Bernoulli Distributions

The problem definition is the following:

- There exists an unkown parameter $p \in [0,1]$
- We receive i.i.d. observations X_i , where $X_i \sim B(p)$
- We output an estimate value $\hat{p} = \hat{\theta}_t(X_1,\ldots,X_t)$ according to some function $\hat{\theta}_t :\ \{0,1\}^t \mapsto [0,1]$.

The goal is to appropriately select $\hat{\theta}_t$ such that $\hat{p}$ and p are relatively close. For example a very known selection is

$$\hat{\theta}_t(X1,\ldots,X_t) = \frac{\sum_{i=1}^t X_i}{t}$$

which is the sample mean estimator

Definition 2. *An estimator $\hat{\theta}$ is a colection of functions $\{\hat{\theta}_t\}_{t=1}^{\infty}$, where $\hat{\theta}_t : \{0,1\}^t \mapsto [0,1]$.*

As the number of samples grows we would expect that an effienencient estimator $\hat{\theta}$ produces more and more accurate values for p. The following definition provides us the metric to evaluate the quality of an estimator $\hat{\theta}$.

Definition 3. *For any estimator $\hat{\theta}_t$ we define*

$$E_p[|\hat{\theta}_t(X_1,\ldots,X_t)| - p|] = \sum_{(x_1,\ldots,x^t)\in\{0,1\}^t} |\hat{\theta}_t(x^1,\ldots,x^t) - p| \cdot p^{\sum_{i=1}^t x^i}(1-p)^{t-\sum_{i=1}^t x^i}$$

The quantity $E_p[|\hat{\theta}_t(X_1,\dots,X_t)| - p|]$ is the expected distance of the estimated value $\hat{\theta}_t$ from the parameter p, when the distribution of the samples is $B(p)$. To simplify notation we also denote it as $E_p[|\hat{\theta}_t - p|]$.

Notice that $E_p[|\hat{\theta}_t - p|]$ does not serve as a good metric since an effiencient estimator $\hat{\theta}$ must garantee good estimates for any parameter p. Consider an estimator $\hat{\theta}$ that outputs $1/2$ for all t. In this case, $E_p[|\hat{\theta}_t - p|] = |p - 1/2|$ meaning that this estimator performs optimally when $p = 1/2$, but extremely bad in any other case. As a result the standard metric in the statistics literature is the following.

Definition 4. *For any estimator $\hat{\theta}_t$ we define its risk with t samples as,*

$$R^t(\hat{\theta}) = max_{p\in[0,1]} E_p[|\hat{\theta}_t - p|]$$

Now our goal is clear, we need to design estimators $\hat{\theta}_t$ such that the risk $R^t(\hat{\theta}_t)$ is as small as possible. For example one can easily prove that the risk rate of the *sample mean estimator* ($\hat{\theta}_t = \frac{\sum_{i=1}^t X_i}{t}$) $R^t(\hat{\theta}_t) \leq \frac{1}{\sqrt{2t}}$. Obviously the next question is whether we can find another (probably more complex estimator) that acheives risk smaller than that of $\frac{1}{\sqrt{2t}}$.

Theorem 9. *Any estimator $\hat{\theta}$ has risk $R^t(\hat{\theta}) \geq \frac{1}{16\sqrt{t}}$*

The above theorem can be easily derived by standard techniques (e.g. Le Cam or Fano method) that have been develloped in the statistics literature to provide lower bounds on the risk of estimators. Especially, proving this lower bound for the Bernoulli case is requires a simple applications of these methods. The above theorem also tells us that the risk of the *sample mean estimator* is up to constant factors optimal. Are we done?

Theorem 10. *There exists an estimator $\hat{\theta}$ such that for all $p \in [0,1]$,*

$$E_p[|\hat{\theta}_t - p|] = O\left(\frac{1}{\sqrt{t}}\right)$$

Can this rate be imporoved? For example, is there a an estimator $\hat{\theta}$ such that for all $p \in [0,1]$, $E_p[|\hat{\theta}_t - p|] = O\left(\frac{1}{t}\right)$?

It seems that the latter question can be answered negatively using Theorem 9. Unfortunately, this is not the case. The difference is subtle but very important. The definition of the risk $R^t = \max_{p\in[0,1]} E_p[|\hat{\theta}_t - p|]$, allows the maximum to be attained in a p that depends on t (e.g. $p = 1/2 + 1/t$). As a result, Theorem 9 does not provide us with information about the above question.

Theorem 11. *For any estimator $\hat{\theta}$ there exists a fixed $p \in [0,1]$ such that*

$$E_p[|\hat{\theta}_t - p|] = \Omega(\frac{1}{t})$$

To the best of our knowledge this question is not answered in the statistics literature. However, the same question has been asked for more the estimator of more general and complex distribution. These results ensure the above statement for estimators that satisfy certain propetries (unbaised estimators, locally regular, e.t.c.). In this section we provide a theorem that holds for any estimator and indicates that estimators that don't satisfy the above statement are very unlikely to exists.

Theorem 12. *For any Bernoulli estimator $\hat{\theta}$, for all $[a,b] \subseteq [0,1]$,*

$$\lim_{t\to\infty} t \int_a^b E_p[|\hat{\theta}_t - p|]dp = +\infty$$

In case that statement .. is not true than there exists an estimator $\hat{\theta}$ that simultaneous satisfies the following two properties:

1. for all $p \in [0,1]$, $\lim_{t\to\infty} tE_p[|\hat{\theta}_t - p|] = 0$
2. for all $[a,b] \in [0,1]$, $\lim_{t\to\infty} t \int_a^b E_p[|\hat{\theta}_t - p|] = +\infty$

We conjecture that there does not exists a function that satisfies both of these properties. Even in case that this is not true, such a function has a very degenerate form making it very difficult to be the risk function of an estimator.

7.2 Final Lower Bound

Theorem 13. *Let a graph oblivious protocol P such that for any instance I, $E[||x^t - x^*||_\infty] \leq f(I)g(t)$, where $f : I \mapsto R_+$ and $g : N \mapsto R_+$. Then $lim_{t\to\infty} tf(t) \neq 0$.*

Proof. We use the protocol P to construct an estimator $\hat{\theta}$ and then we use the theorem ... For any $k, n \in N$ with $k \leq n$ we construct the following instance.

- A star graph with $n+1$ nodes
- k leaves have $a_i = 1$ and $s_i = 1$
- $n-k$ leaves have $a_i = 1$ and $s_i = 0$
- the central node has $a_{n+1} = 1/2$ and $s_{n+1} = 0$
- directed edges from the central node to all the leaves.

Observe that for all $i \neq n+1$, $x_i^* = s_i$ and $x_{n+1}^* = \frac{k}{2n}$. For all $i \neq n+1$, $x_i^t = P(1,1,t)$ if $s_i = 1$ and $x_i^t = P(0,1,t)$ if $s_i = 0$. Notice that these are deterministic functions with respect to t and are independent of the values of k, n. Now, $x_{n+1}^t = P(P(y_1,1,1), P(y_2,1,2), \ldots, P(y_t,1,t), 1/2, 0, t)$, where $x_i \in 0,1$. As a result, $x_{n+1}^t = g(y_1, \ldots, y_t)$, where $g : \{0,1\}^t \mapsto [0,1]$. As a result, the estimator $\hat{\theta}$ with $\hat{\theta}_t = \frac{x_c^t}{2}$ is valid estimator. Now let $p = k/n$ we have that

$$
\begin{aligned}
E_p[||x^t - x^*||] &\geq E_p[|x_{n+1}^t - \frac{p}{2}|] \\
&= E_p[|\frac{\hat{\theta}_t}{2} - \frac{p}{2}|] \\
&= \frac{1}{2} E_p[|\hat{\theta}_t - p|]
\end{aligned}
$$

Finally, for all $p \in [0,1]$ $\ E_p[|\hat{\theta}_t - p|] \leq 2f(I_{k,n})g(t) = f'(p)g(t)$. By theorem .., $\lim_{t\to\infty} tf(t) \neq 0$.

8 LeCam's lower bound

Suppose $P = B(p), Q = B(q)$ are two Bernoulli distributions. We are first going to bound $D_{TV}(P^t, Q^t)$ by the hellinger distance of the two distributions.

Lemma 14. *Suppose P, Q are two arbitrary distributions. Then:*

$$D_{TV}(P,Q) \leq D_{hel}(P,Q)\sqrt{1 - \frac{D_{hel}(P,Q)^2}{4}}$$

Thus, in order to find a lower bound for $D_{TV}(P,Q)$, it suffices to prove upper and lower bound for $D_{hel}(P,Q)$. This is done in the following lemmas. In the following we denote by $D = D_{hel}(P,Q)^2$.

Lemma 15.

$$D_{hel}(P^t, Q^t)^2 = 2 - 2\left(1 - \frac{1}{2}D\right)^t$$

Lemma 16.

$$2 - 2e^{-\frac{Dt}{2}} \leq D_{hel}(P^t, Q^t)^2 \leq 2 - 2e^{-Dt}$$

Proof. Let's first prove the lower bound. We have:

$$\left(1 - \frac{D}{2}\right)^t \leq e^{-\frac{Dt}{2}} \tag{18}$$

So, by Lemma (15) we get:

$$\begin{aligned} D_{hel}(P^t, Q^t)^2 &= 2 - 2\left(1 - \frac{D}{2}\right)^t \\ &\geq 2 - 2e^{-\frac{Dt}{2}} \end{aligned}$$

For the upper bound, notice that for any $x \in [0, 0.7]$:

$$1 - x \geq e^{-2x} \tag{19}$$

As we shall see later, if p, q are close enough, D will be small, so we can assum that $D/2 \in [0, 0.7]$. So, by using inequality (19) and Lemma (15) we have:

$$\begin{aligned} D_{hel}(P^t, Q^t)^2 &= 2 - 2\left(1 - \frac{D}{2}\right)^t \\ &\leq 2 - 2e^{-Dt} \end{aligned}$$

Lemma 17.

$$D_{TV}(P^t, Q^t) \leq 1 - \frac{e^{-Dt}}{2}$$

Proof. By Lemmata (14) and (16) we get:

$$\begin{aligned} D_{TV}(P^t, Q^t) &\leq D_{hel}(P^t, Q^t)\sqrt{1 - \frac{D_{hel}(P^t, Q^t)^2}{4}} \\ &\leq \sqrt{2 - 2e^{-Dt}}\sqrt{1 - \frac{2 - 2e^{\frac{Dt}{t}}}{4}} \\ &= \sqrt{2}\sqrt{1 - e^{-Dt}}\sqrt{\frac{1}{2} - \frac{e^{-\frac{Dt}{2}}}{2}} \\ &= \sqrt{1 - e^{-Dt}}\sqrt{1 - e^{-\frac{Dt}{2}}} \\ &\leq 1 - \frac{e^{-Dt}}{2} \end{aligned}$$

since $e^{-\frac{Dt}{2}} \leq e^{-Dt}$.

The only thing that remains is to compute $D = D_{hel}(P, Q)$. We proceed to do this in the following lemma:

Lemma 18. *If $p < q$ then*

$$D_{hel}(P, Q)^2 \leq \frac{1}{4}\left(\frac{1}{p} + \frac{1}{1-q}\right)(p - q)^2$$

Proof. Since P and Q are Bernoulli, we have:

$$D_{hel}(P, Q)^2 = (\sqrt{p} - \sqrt{q})^2 + \left(\sqrt{1-p} - \sqrt{1-q}\right)^2$$

We will bound the first term. We define the function $f : [p, q] \to \mathbb{R}$:

$$f(x) = \sqrt{x} - \sqrt{q}$$

where q is assumed to be constant. We have $f(p) = \sqrt{p} - \sqrt{q}$ and $f(q) = 0$. f is clearly continuously differentiable in $[p, q]$ with $f'(x) = \frac{1}{2\sqrt{x}}$, so by the mean value theorem:

$$f(q) - f(p) = f'(\xi)(q - p)$$

for some $\xi \in [p, q]$. We notice that $|f'(\xi)| \leq \frac{1}{2\sqrt{p}}$. So

$$|f(q) - f(p)| \leq \frac{1}{\sqrt{p}} |q - p|$$

Hence,

$$(\sqrt{p} - \sqrt{q})^2 \leq \frac{1}{4p}(p - q)^2 \tag{20}$$

Using the exact same technique, we obtain:

$$\left(\sqrt{1-p} - \sqrt{1-q}\right)^2 \leq \frac{1}{4(1-q)}(p - q)^2 \tag{21}$$

If we combine inequalities (20) and (21) we get:

$$(\sqrt{p} - \sqrt{q})^2 + \left(\sqrt{1-p} - \sqrt{1-q}\right)^2 \leq \frac{1}{4}\left(\frac{1}{p} + \frac{1}{1-q}\right)(p - q)^2$$

So, if we set $p = d_1/n$ and $q = d_2/n$, then:

$$\begin{aligned}\mathbf{P}[\psi \neq V] &\geq \frac{|d_1 - d_2|}{4n} e^{-\frac{Dt}{2}} \\ &\geq \frac{|d_1 - d_2|}{4n} e^{-\frac{1}{8}\left(\frac{1}{p} + \frac{1}{1-q}\right)\frac{(d_1-d_2)^2}{n^2}}\end{aligned}$$

9 Graph Aware Protocol - Frequencies

Let P be a discrete distribution over $[d]$, and let $S_1, \ldots, S_t$ be t i.i.d samples drawn from P, i.e. $(S_1, \ldots, S_t) \sim P^t$. The empirical distribution $\hat{P}_t$ is the following estimator of the density of P.

$$\hat{P}_N(A) = \frac{\sum_{i=1}^{N} \mathbb{1}_{[S_i \in A]}}{N}, \tag{22}$$

where $A \subseteq [d]$. In words, $\hat{P}_N$ simply counts how many times the value i appeared in the samples $S_1, \ldots S_N$. We will use the following version of the classical Vapnik and Chervonenkis inequality.

Lemma 19. *Let $\mathcal{A}$ be a collection of subsets of $\{0, \ldots, d\}$ and Let $S_{\mathcal{A}}(N)$ be the Vapnik-Chervonenkis shatter coefficient, defined by*

$$S_{\mathcal{A}(N)} = \max_{x_1,\ldots,x_t \in [d]} |\{\{x_1, \ldots, x_t\} \cap A : A \in \mathcal{A}\}| .$$

Then

$$\mathbf{E}_{P^N}\left[\max_{A \in \mathcal{A}} \left|\hat{P}_t(A) - P(A)\right|\right] \leq 2\sqrt{\frac{\log 2S_{\mathcal{A}}(N)}{N}}$$

An interesting question is whether $\mathrm{poly}(1/\varepsilon)$ rounds are necessary when protocols have access to the unlabeled oracle. We show that this is not true by showing a protocol that needs $\ln^2(1/\varepsilon)$ rounds to be within error ε from the solution x^*. Our protocol is graph-aware in the sense that it is allowed to depend on the graph G. Therefore, we show that without any constraints on the protocols it is hard to prove strong lower bounds for our problem. The problem in this case is that all agents could agree to stop updating their public opinions for enough rounds so that everybody learns, with high probability, exactly the average of the opinions of their neighbors. Having the exact average they can perform a step of the original update of the FJ-model which results in a similar and fast convergence rate. Of course if the opinions of the neighbors are all different we can still use an argument similar to that of the protocol of section COUPONS COLLECTOR. Since some neighbors may share opinions we have to come up with a different solution for this problem. Let $L \leq d$ be the number of different opinions and k_i be the number of neighbors who have opinion ξ_i. Then the average of the opinions of the neighbors is $(\sum_{j=1}^{L} k_i \xi_i)/d_i$. Instead of using the average like we did in section SAMPLE MEAN, we will instead find *exactly* the frequencies k_i and then compute their average exactly with high probability. We next describe our protocol. Since it is the same for all agents, For simplicity we describe it for agent i.

Algorithm 1 Graph Aware Update Rule

1: $x_i(0) \leftarrow s_i$, $t \leftarrow 0$.
2: **for** $l = 1, \ldots, \infty$ **do**
3:     $\varepsilon \leftarrow 1/2^l$.
4:     **for** $t = 1, \ldots, \ln(1/\varepsilon))$ **do**
5:         Keep a map h from $[0, 1]$ to $[d_i]$ and array of counters A of length d_i and an array B of length M_1.
6:         $M_1 = O(\ln(1/\varepsilon))$, $M_2 = O(d^2 \ln(d))$
7:         **for** $j = 1, \ldots, M_1$ **do**
8:             **for** $k = 1, \ldots, M_2$ **do**
9:                 Call the unlabeled oracle $O_i^u(t)$ to get an opinion X_i.
10:                 **if** X_i is not in h **then**
11:                     Insert X_i to h.
12:                 **else**
13:                     $A(h(X_i)) \leftarrow A(h(X_i)) + 1$.
14:                 $t \leftarrow t + 1$
15:             Divide all entries of A by M_2.
16:             Round all entries of h to the closest multiple of $1/d$.
17:             $B(j) = \sum_{i=1}^{d_i} A(i)$.
18:         $x_i(t) \leftarrow \mathrm{maj}_j\, B(j)$.

Theorem 14. *The update rule 1 for $a > 1/2$ achieves convergence rate*

$$\mathbf{E}\left[\|x_t - x^*\|_\infty\right] \leq C \mathrm{e}^{-\sqrt{t}/(d\sqrt{\log d})},$$

where C is a universal constant.

Proof. According to the update rule 1 all agents fix their opinions $x_i(t)$ for $M_1 \times M_2$ rounds. To estimate the sum of the opinions each agent estimates the frequencies k_j/d_i. Since the neighbors have at most d_i different opinions we can think of the opinions as natural numbers in $[d_i]$. The oracle returns the opinion of a random neighbor and therefore the samples X_i returned by the oracle O_i^u are drawn from a discrete distribution P supported on $[d_i]$. The probability $P(j)$ of the j-th opinion is the number of neighbors having this opinion k_j/d_i. To learn the probabilities $P(j)$ using samples from P. Letting $\mathcal{A} = \{\{1\}, \{2\}, \ldots, \{d_i\}\}$ we have from Lemma 19 that

$$\mathbf{E}_{P^m}\left[\max_{j \in [d_i]} \left|\hat{P}_m(j) - P(j)\right|\right] \leq 2\sqrt{\frac{\log 2d_i}{m}},$$

since $S_{\mathcal{A}} \leq d_i$. Therefore, an agent can draw $m = 100d^2 \log(2d)$ to learn the frequencies k_j/d_i within expected error $1/(5d)$. Notice now that the array A after line 15 corresponds to the empirical distribution of equation (22). Notice that if the agents have estimations of the frequencies k_j/d_i with error smaller than $1/d$ then by rounding them to the closest multiple of $1/d_i$ they learn the frequencies exactly. By Markov's inequality we have that with probability at least $4/5$ the rounded frequencies are exactly correct. By standard Chernoff bounds we have that if the agents repeat the above procedure $\ln(1/\delta)$ times and keep the most frequent of the answers $B(j)$ then they will obtain the correct answer with probability at least $1 - \delta$. From Lemma **??** we have that to achieve error ε we need $\log(1/\varepsilon)/(1/(1-\alpha))$ rounds. Since we need all nodes to succeed at computing the exact averages for $\log(1/\varepsilon)/\log(1/(1-\alpha))$ rounds we have from the union bound that for $\delta < \frac{\varepsilon \ln(1/(1-\alpha))}{n \ln(1/\varepsilon)}$, with probability at least $1-\varepsilon$ the error is at most ε. For the expected error after $T = O(d^2 \log d(\log(n/\varepsilon) + \log\log(1/\varepsilon) + \log\log(1-a))$ rounds we have that $\mathbf{E}\left[\|x_T - x^*\|_\infty\right] = O(\varepsilon)$.