Opinion Dynamics with Imperfect Information

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Abstract. We study opinion formation games based on the famous model proposed by Friedkin and Johsen. In today's huge social networks the assumption that in each round agents update their opinions by taking into account the opinions of *all* their friends could be unrealistic. Therefore, we assume that in each round each agent gets to meet with only one random friend of hers. Since it is more likely to meet some friends than others we assume that agent i meets agent j with probability p_{ij}. In this imperfect information setting, we are interested in simple and natural variants of the FJ model that converge to the same equilibrium point x^*. Specifically, we define an opinion formation game, where at round t, agent i with intrinsic opinion $s_i \in [0,1]$ and expressed opinion $x_i(t) \in [0,1]$ meets with probability p_{ij} neighbor j with opinion $x_j(t)$ and suffers a disagreement cost that is a convex combination of $(x_i(t)-s_i)^2$ and $(x_i(t)-x_j(t))^2$. We show that the agents can adopt an intuitive and simple update rule that ensures *no regret* to the experienced disagreement cost and at the same time the produced opinion vector converges to the equilibrium x^* within error ε after roughly $O(\text{poly}(\log n/\varepsilon))$ rounds, where n is the number of agents. In our imperfect information setting, we prove that, for dynamics that ensure no regret for the agents, $\Omega(1/\varepsilon)$ rounds are also necessary. Finally, we present an update rule that requires only $\widetilde{O}(\log^2(1/\varepsilon))$ rounds to acheive error ε, resembling the convergence of the original FJ model. Slow convergence is not a generic property of our imperfect information setting but rather a characteristic of no regret dynamics.

1 Introduction

The study of *Opinion Formation* has a long history (see e.g. [Jac08]). Opinion Formation is a *dynamic process* in the sense that socially connected people (e.g. family, friends, colleagues) exchange information and this leads to changes in their expressed opinions over time. Today, the advent of the internet and social media makes the study of opinion formation in large social networks even more important; realistic models of how people form their opinions by interacting with each other, are of great practical interest for prediction, advertisement etc. In an attempt to formalize the process of opinion formation, several models have been proposed over the years (see e.g., [DeG74,FJ90,HK02,DNAW00]. The common assumption underlying all these models, which dates back to DeGroot [DeG74], is that opinions evolve through a form of repeated averaging of information collected from the agents' social neighborhoods.

Our work builds on the model proposed by Friedkin and Johnsen [FJ90]. The FJ model is a variation on the DeGroot model capturing the fact that consensus on the opinions is rarely reached. According to FJ model each person i has a public opinion $x_i \in [0,1]$ and an internal opinion $s_i \in [0,1]$, which is private and invariant over time. There also exists a weighted graph $G(V,E)$ representing a social network where V stands for the persons ($|V|=n$) and E their social relations. Initially, all nodes start with their internal opinion and at each round t, update their public opinion $x_i(t)$ to a weighted average of the public opinions of their neighbors and their internal opinion,

$$x_i(t) = \frac{\sum_{j \in N_i} w_{ij} x_j(t-1) + w_{ii} s_i}{\sum_{j \in N_i} w_{ij} + w_{ii}}, \tag{1}$$

where $N_i = \{j \in V : (i,j) \in E\}$ is the set of i's neighbors, the weight w_{ij} associated with the edge $(i,j) \in E$ measures the extend of the influence that j poses on i and the weight $w_{ii} > 0$ quantifies how susceptible is i in adopting opinions that differ from her internal opinion s_i.

The FJ model is one of most influential models for opinion formation. It has a very simple update rule, making it plausible for modeling natural behavior and its basic assumptions are aligned with empirical findings on the way opinions are formed [AFH+05,Kra09]. At the same time, it admits a unique stable point point $x^* \in [0,1]^n$ to which it converges with a *linear* rate [GS14]. The FJ model has also been studied under a game theoretic viewpoint. Bindel et al. considered its update rule as the minimizer of a quadratic disagreement cost function and based on it they defined the following opinion formation game [BKO11]. Each node i is a selfish agent whose strategy is the public opinion x_i that she expresses incurring her a disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \in N_i} w_{ij}(x_i - x_j)^2 + w_{ii}(x_i - s_i)^2 \tag{2}$$

Note that the FJ model is the *simultaneous best response dynamics* and its stable point x^* is the unique Nash equilibrium of the above game. In [BKO11] they quantified its inefficiency with respect to the total disagreement cost. They proved that the *Price of Anarchy* (PoA) is 9/8 in case G is undirected and $w_{ij} = w_{ji}$. They also provided PoA bounds in the case of unweighted Eulerian directed graphs.

We remark that in [BKO11] an alternative framework, for studying the way opinions evolve, were introduced. Namely, the opinion formation process can be described as the *dynamics* of an opinion formation game. This framework is much more comprehensive for two reasons. At first, different aspects of the opinion formation process can be easily captured by defining suitable games. Secondly, it permits considering different *natural dynamics* for the same opinion formation game (e.g. *best response dynamics, no regret dynamics, fictitious play*). Following this framework subsequent works [BGM13,BFM16,EFHS17] considered variants of the above game and studied the convergence properties of the *best response dynamics*.

1.1 Motivation and our setting

Many recent works study the Nash equilibrium x^* of the opinion formation game defined in [BKO11] under various perspectives. In [CCL16] they extended the bounds for PoA in more general classes of directed graphs, while many recently introduced influence maximization problems [GTT13,AKPT18,MMT17] are defined with respect to x^*. For example, Gionis et al. [GTT13] considered the problem of identifying k nodes in a network to set their internal opinion equal to 1 so as to maximize the sum of the opinions in the equilibrium point x^* ($\|x^*\|_1$). The reason for this scientific interest is evident: the equilibrium x^* is considered as an appropriate way to model the final opinions formed in a social network, since the *well established* FJ model converges to it.

Our work is motivated by the fact that there are notable cases in which the FJ model is not an appropriate model for the dynamic of the opinions, due to the large amount of information exchange that it implies. More precisely, at each round its update rule (1) requires that every agent learns all the opinions of her social neighbors. In today's large social networks where users usually have several hundreds of friends it is highly unlikely that, each day, they learn the opinions of all their social neighbors. In such environments it is far more reasonable to assume that individuals randomly meet a small subset of their acquaintances and these are the only opinions that they learn. Such information exchange constraints render the FJ model unsuitable for modeling the opinion formation process in such large networks and therefore, it is not clear whether x^* captures the limiting behavior of the opinions. In this work we ask:

Question 1. Is the equilibrium x^* an adequate way to model the final formed opinions in a large social network? Namely, are there simple variants of the FJ model that require limited information exchange and converge fast to x^*? Can they be justified as natural behavior for selfish agents under a game-theoretic solution concept?

To address these questions, one could define precise dynamical processes whose update rules require limited information exchange between the agents and study their convergence properties. Instead of doing that, we describe the opinion formation process in such large networks as *dynamics* of an adequate opinion formation game that captures these information exchange constraints. This way we can precisely define which *dynamics* are *natural*, and, more importantly, to study general classes of *dynamics* (e.g. no regret dynamics) without explicitly defining their update rule. The opinion formation game that we consider is a straightforward variant of the game defined in [BKO11] based on interpreting the weight w_{ij} as a measure on how frequently i meets j.

Definition 1. *For a given opinion vector $x \in [0,1]^n$, the disagreement cost of agent i is the random variable $C_i(x_i, x_{-i})$ defined as follows:*

- *Agent i meets one of her neighbors j with probability $p_{ij} = w_{ij}/\sum_{j \in N_i} w_{ij}$.*
- *Agent i suffers cost $C_i(x_i, x_{-i}) = (1-a_i)(x_i - x_j)^2 + a_i(x_i - s_i)^2$, where $\alpha_i = w_{ii}/(\sum_{j\in N_i} w_{ij} + w_{ii})$.*

Note that the FJ model, the game defined in [BKO11], and the game of Definition 1 are all defined using equivalent instances $G(V, E, w)$. Moreover, it is not hard to see that the equilibrium x^* is also the unique Nash equilibrium of the above game with respect to the expected disagreement cost of the agents. This game provides us with a general template of all the *dynamics* examined in this paper. At round t, each agent i selects an opinion $x_i(t)$ and suffers a disagreement cost based on the opinion of the neighbor that she randomly met. At the end of the round t, she is informed only about the opinion and the index of this neighbor and may use this information to update her opinion in the next round. Obviously different update rules lead to different *dynamics*, however all of these respect the information exchange constraints: at every round each agent learns the opinion of *just one* of her neighbors. Question 1 now takes the following more concrete form.

Question 2. Can the agents update their opinions according to the limited information that they receive such that the produced opinion vector $x(t)$ converges fast to the equilibrium x^* and the total disagreement cost that they experience is minimal?

1.2 Contribution

We discuss here the contribution of this work at a high level; for the formal statements of our theorems see Section 2. Since we are most concerned about the dependence of the rate of convergence on the distance ε from the equilibrium, we shall suppress the dependence on other constants of the problem such as the size of the graph, n. We remark that the dependence of our dynamics on these constants is in fact rather good (see Section 2, but we do this only for clarity of exposition.

Definition 2 (Informal). *We say that a dynamics converges slow resp. fast to its stable point x^* if it requires* $\mathrm{poly}(1/\varepsilon)$ *resp.* $\mathrm{poly}(\log(1/\varepsilon))$ *rounds to be within (expected) error ε of x^*.*

We know that the FJ model converges fast to the equilibrium x^*: the opinion vector $x(t)$ is in distance ε in $O(\log 1/\varepsilon)$ rounds [GS14]. We characterize the *limited information dynamics* that can achieve such fast convergence. We prove that if the update rule, that the agents use, only depends on the opinions of the neighbors that they meet, then there exists an instance such that the produced opinion vector $x(t)$ requires at least $\Omega(1/\varepsilon)$ rounds to be ε-close to x^*. We call these update rules *opinion dependent* and the respective dynamics *opinion dependent dynamics*. In Section 5, we show that the existence of opinion dependent update rules with fast convergence rate implies the existence of Bernoulli estimators with small sample complexity and we present a novel, information theoretic argument, to rule out their existence. The reason that we are interested in *opinion dependent dynamics* is that they subsume the class of *no regret dynamics* which is generally accepted as *natural behavior* [EMN09]. We say that an update rule ensures no regret to any agent that adopts it, when the time averaged disagreement cost (that the agent experiences) per round is close to the cost of expressing the best opinion in hindsight independently of the neighbors that she meets. Therefore, update rules that ensure no regret are a reasonable choice for selfish agents.

We introduce a simple and intuitive *opinion dependent update rule* and we show that if the agents adopt it, the resulting opinion vector $x(t)$ converges to x^*. Our update rule is a *Follow the Leader algorithm*, meaning that at round t, each agent updates her opinion to the minimizer of total disagreement cost that she experienced until round $t-1$. It also has a very simple form: it is roughly the time average of the opinions that the agent observes. In Section 3, we bound its convergence rate and we show that in order to achieve ε distance from x^*, $\text{poly}(1/\varepsilon)$ rounds are needed. In Section 4, we show that this rule also ensures *no regret* to the agents. Its no regret property can be derived by the more general results in [HAK07]. However, we give a short and simple proof that may be of some interest.

In Section 6 we show that "slow convergence" is not a generic property of the *limited information dynamics*. We present an update rule that apart from the observed opinions, also uses the weights w_{ij} and converges fast to x^*. This update rule reveals that the slow convergence of *opinion dependent* update rules is not due to the reduced information exchange (learning the opinion of only one agent), but due to the fact that the agents are "oblivious" to the weighted graph of the instance and they "learn" it during the game play.

Our results reveal that the equilibrium x^* is a robust choice for modeling the limiting behavior of the opinions of agents since even in our limited information setting, there exist natural dynamics that converge to it. Our lower bound indicates that rationality comes at the price of "slow" convergence.

1.3 Further Related Work

There exists a large amount of literature concerning the FJ model. Many recent works [BGM13,CKO13], [BFM16,EFHS17] bound the inefficiency of equilibrium in variants of opinion formation game defined in [BKO11]. In [GS14] they bound the convergence time of the FJ model in special graph topologies. In [BFM16], a variant of the opinion formation game in which social relations depend on the expressed opinions is studied. They prove that the discretized version of the above game admits a potential function and thus best-response converges to the Nash equilibrium. Convergence results in other discretized variants of the FJ model can be found in [YOA+13,FGV16]. In [FPS16] they provide convergence results for limited information variants of the Heglesmann-Krause model [HK02] and the FJ model. Although they considered limited information variant of the FJ model is very similar to ours, their convergence results are much weaker, since they concern the expected value of the opinion vector.

Other works that relate to ours, concern the convergence properties of dynamics based on no regret learning algorithms. In [FV97,FS99,SA00,SALS15] it is proved that in a finite n-person game if each agent updates her mixed strategy according to a no regret algorithm the resulting *time-averaged* strategy vector converges to Coarse Correlated Equilibrium. The convergence properties of no regret dynamics for games with infinite strategy spaces were considered in [EMN09]. They proved that for a large class of games with concave utility functions (socially concave games), the time-averaged strategy vector converges to Pure Nash Equilibrium (PNE). More recent work investigates a stronger notion of convergence of no regret dynamics. In [CHM17] they show that, in n-person finite generic games that admit unique Nash equilibrium, the strategy vector converges *locally* and fast to it. They also provide conditions for *global* convergence. Our results fit in this line of research since we show that for a game with *infinite* strategy space, the strategy vector (and not the time-averaged) converges to the Nash equilibrium x^*.

No regret dynamics in imperfect information settings have recently received substantial attention from the scientific community since they provide realistic models for the practical applications of game theory. Perfect payoff information is rare in practice; agents act based on random or noisy past payoff observations. Kleinberg et al. in [KPT09] treated load-balancing in distributed systems as a repeated game and analyzed the convergence properties of no regret learning algorithms under the *full information assumption* that each agent learns the load of every machine. In a subsequent work [KPT11], the same authors consider the same problem in a *imperfect information setting* ("bulletin board model"), in which each agent learns the load of just the machine that served him. Most relevant to ours, are the works [HCM17,MS17,BM17,CHM17], where they examine the convergence properties of no regret learning algorithms when the agents observe their payoffs with some additive zero-mean random noise. In our imperfect information setting the agents experience random disagreement cost with expected value equal to the actual cost. The main difference is that our *noise* is not additive but due to a sampling process.

2 Our Results and Techniques

As previously mentioned, an instance of the game in [BKO11] is also an instance the game of Definition 1. Following the notation introduced earlier we have that $p_{ij} = w_{ij}/\sum_{j\in N_i} w_{ij}$ if $j \in N_i$ and 0 otherwise. Moreover, $\alpha_i = w_{ii}/(\sum_{j\in N_i} w_{ij} + w_{ii}) > 0$ since $w_{ii} > 0$ by the definition of the game in [BKO11]. If an agent i does not have outgoing edges ($N_i = \emptyset$) then $p_{ij} = 0$ for all j. Therefore $\sum_{j=1}^n p_{ij} = 0$, $\alpha_i = 1$ if $N_i = \emptyset$ and $\sum_{j=1}^n p_{ij} = 1$, $\alpha_i \in (0,1)$ otherwise. For simplicity we adopt the following notation for an instance of the game of Definition 1.

Definition 3. *We denote an instance of the opinion formation game of Definition 1 as $I = (P, s, \alpha)$, where P is a $n \times n$ matrix with non-negative elements p_{ij}, with $p_{ii} = 0$ and $\sum_{j=1}^n p_{ij}$ is either 0 or 1, $s \in [0,1]^n$ is the internal opinion vector, $\alpha \in (0,1]^n$ the self confidence vector.*

An instance $I = (P, s, \alpha)$ is also an instance of the FJ model, since by the update rule (1) $x_i(t+1) = (1-\alpha_i)\sum_{j\in N_i} p_{ij}x_j(t-1) + a_i s_i$. It also defines the opinion vector $x^* \in [0,1]^n$ which is the stable point of the FJ model and the Nash equilibrium of the game in [BKO11].

Definition 4. *For a given instance $I = (P, s, \alpha)$ the equilibrium $x^* \in [0,1]^n$ is the unique solution of the following linear system, for every $i \in V$, $x_i^* = (1-\alpha_i)\sum_{j\in N_i} p_{ij}x_j^* + a_i s_i$.*

The fact that the above linear systems always admits a solution follows by matrix norm properties. Throughout the paper we study *dynamics* in the game of Definition 1. We denote as W_i^t the neighbor that agent i met at round t, which is a random variable whose probability distribution is determined by the instance $I = (P, s, \alpha)$ of the game, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$. Another parameter of an instance I that we often use is $\rho = \min_{i\in V} \alpha_i$.

In Section 3, we study the convergence properties of the opinion vector $x(t)$ when all agents update their opinions according to the "Follow the Leader" principle. Since each agent i must select $x_i(t)$, before knowing which of her neighbors she will meet and what opinion her neighbor will express, their update rule says "*play the best according to what you have observed*". For a given instance (P, s, a) of the game the Follow the Leader dynamics $x(t)$ is defined in Dynamics 1 and Theorem 1 shows its convergence rate to x^*.

Dynamics 1 Follow the Leader dynamics

Initially $x_i(0) = s_i$ for all agents i.
At round $t \geq 0$ each agent i:
  Meets neighbor with index W_i^t, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$.
  Suffers cost $(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$ and learns the opinion $x_{W_i^t}(t)$.
  Updates her opinion $x_i(t+1) = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t}(1-\alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i(x-s_i)^2 \quad (3)$

Theorem 1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\,\frac{(\log t)^{3/2}}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i\in V} a_i$ and C is a universal constant.

In Section 4 we argue that, apart from its simplicity, update rule (3) ensures no regret for the selfish agents, and therefore is a *rational game play*. Since each agent i selfishly wants to minimizing the disagreement cost that she experiences, it is natural to assume that she selects $x_i(t)$ according to an *no regret algorithm* for the *online convex optimization problem* where the adversary chooses a function $f_t(x) = (1-\alpha_i)(x-b_t)^2 + \alpha_i(x-s_i)^2$ at each round t. In Theorem 2 we prove that "Follow the Leader" is a no regret algorithm for the above OCO problem. We remark that this does not hold, if the adversary can pick functions from a different class (see e.g. chapter 5 in [Haz16]).

Theorem 2. *Consider the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Let $\{b_t\}_{t=1}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. If $x_t = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x,b_\tau)$ then for all t, $\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$.*

On the positive side, the FTL dynamics is a simple and natural stochastic process that ensures convergence to x^* while vastly reducing the information exchange between the agents per round; it only requires each agent to learn the opinion of just one neighbor. In terms of total communication exchange needed to get within distance ε of the equilibrium x^*, the FTL dynamics requires $\widetilde{O}(n)$ communication exchange while the FJ model needs $O(|E|)$. On the negative side, its convergence rate is outperformed by the rate of FJ model. For a fixed instance $I = (P, s, \alpha)$, the FTL dynamics converges with rate $\widetilde{O}(1/t^{\min(\rho,1/2)})$ while FJ model converges with rate $O(e^{-\rho t})$ [GS14]. As a result the following question arises.

Question 3. Can the agents adopt other no regret algorithms such that the resulting dynamics $x(t)$ converges fast to x^*?

In Section 5 we answer this question in the negative. The reason that FTL dynamics converges slowly is that update rule (3) only depends on the opinions of the neighbors that agent i meets, α_i, and s_i. This is also true for any update rule that ensures no regret to the agents (see Section 5). As already mentioned, we call such update rules "*opinion dependent*" and the produced dynamics *opinion dependent dynamics*.

Definition 5 (opinion dependent update rule). *An opinion dependent update rule A is a sequence of functions $(A_t)_{t=0}^{\infty}$ where $A_t : [0,1]^{t+2} \mapsto [0,1]$.*

Definition 6 (opinion dependent dynamics). *Let an opinion dependent update rule A. For a given instance $I = (P, s, \alpha)$ the rule A produces an opinion depedent dynamics $x_A(t)$ defined as follows:*

– *Initially each agent i selects her opinion $x_i^A(0) = A_0(s_i, \alpha_i)$*
– *At round $t \geq 1$, each agent i selects her opinion $x_i^A(t) = A_t(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), \alpha_i, s_i)$, where W_i^t is the neighbors that i meets at round t.*

Note that FTL dynamics is an opinion dependent dynamics since update rule (3) can be written equivalently, $x_i(0) = s_i$ and $x_i(t) = (1-\alpha_i)\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i$. In Theorem 3 we show that for any opinion dependent dynamics there exists an instance $I = (P, s, \alpha)$ where $\Omega(1/\varepsilon)$ rounds are required to achieve convergence within error ε.

Theorem 3. *Let A be an* opinion dependent *update rule, which all agents use to update their opinions. For any $c > 0$ there exists an instance $I = (P, s, a)$ such that $\mathbf{E}[\|x_A(t) - x^*\|_\infty] = \Omega(1/t^{1+c})$, where $x_A(t)$ denotes the opinion vector produced by A for the instance $I = (P, s, \alpha)$.*

To prove Theorem 3, we show that opinion dependent rules that need a small number of rounds to converge, imply the existence of estimators for Bernoulli distributions with small sample complexity. Then with a simple argument presented in Lemma 6, we show that such estimators cannot exist. In Section 5 we also briefly discuss two well-known sample complexity lower bounds from the statistics literature and explain why they do not work in our case.

In Section 6, we present a simple update rule that is not opinion dependent and achieves error rate $e^{-O(\sqrt{t})}$. This update rule is a function of the opinions and the indices of the agents that i met, α_i, s_i and the i-th row the matrix P. We mention that the lower bound presented in Theorem 3 applies for "opinion dependent rules" that also depend on the agents' indices that i met. Therefore, the dependency on the row P_i is inevitable in order to obtain fast convergence.

3 The Convergence Rate of FTL Dynamics

In this section we prove Theorem 1 which bounds the convergence time of FTL dynamics to the unique equilibrium point x^*. Notice that for an instance $I = (P, s, \alpha)$, the opinion vector $x(t) \in [0,1]^n$ of the FTL dynamics (see Dynamics 1) can be written equivalently as follows:

– Initially all agents adopt their internal opinion, $x_i(0) = s_i$.
– At round $t \geq 1$, each agent i updates her opinion $x_i(t) = (1-\alpha_i)\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i$, where W_i^τ is the neighbor that i met at round t.

Since the opinion vector $x(t)$ is a random vector, the convergence metric used in Theorem 1 is $\mathbf{E}[\|x(t) - x^*\|_\infty]$ where the expectation is taken over the random meeting of the agents. At first we present a high level idea of the proof. We remind that the unique equilibrium $x^* \in [0,1]^n$ of the instance $I = (P, s, \alpha)$ satisfies the following equations for each agent $i \in V$,

$$x_i^* = (1-\alpha_i)\sum_{j \in N_i} p_{ij} x_j^* + \alpha_i s_i$$

Since our metric is $\mathbf{E}[\|x(t) - x^*\|_\infty]$, we can use the above equations to bound $|x_i(t) - x_i^*|$.

$$\begin{aligned}
|x_i(t) - x_i^*| &= (1-\alpha_i)\left|\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*\right| \\
&= (1-\alpha_i)\left|\sum_{j \in N_i} \frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j] x_j(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*\right| \\
&\leq (1-\alpha_i)\sum_{j \in N_i}\left|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j] x_j(\tau)}{t} - p_{ij} x_j^*\right|
\end{aligned}$$

Now assume that $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}| = 0$ for all $t \geq 1$, then with simple algebraic manipulations one can prove that $\|x(t) - x^*\|_\infty \leq e(t)$ where $e(t)$ satisfies the recursive equation $e(t) = (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$, where $\rho = \min a_i$. It follows that $\|x(t) - x^*\|_\infty \leq 1/t^\rho$ meaning that $x(t)$ converges to x^*. Obviously the latter assumption does not hold, however since W_i^τ are independent random variables with $\mathbf{P}[W_i^\tau = j] = p_{ij}$, $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}|$ tends to 0 with probability 1. In Lemma 1 we use this fact to obtain a similar recursive equation for $e(t)$ and then in Lemma 2 we upper bound its solution.

Lemma 1. *Let $e(t)$ the solution of the recursion $e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$ where $e(0) = \|x(0) - x^*\|_\infty$, $\delta(t) = \sqrt{\ln(\pi^2 n t^2/6p)/t}$ and $\rho = \min_{i \in V} \alpha_i$. Then,*

$$\mathbf{P}\left[\text{for all } t \geq 1,\ \|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - p$$

Proof. At first we prove that with probability at least $1-p$, for all $t \geq 1$ and all agents i:

$$\left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| \leq \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}} := \delta(t). \tag{4}$$

Since W_i^τ are independent random variables with $\mathbf{P}[W_i^\tau = j] = p_{ij}$ and $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \sum_{j \in N_i} p_{ij} x_j^*$. By the Hoeffding's inequality we get

$$\mathbf{P}\left[\left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| > \delta(t)\right] < 6p/(\pi^2 n t^2).$$

To bound the probability of error for all rounds $t \geq 1$ and all agents i, we apply the union bound

$$\sum_{t=1}^{\infty} \mathbf{P}\left[\max_i \left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| > \delta(t)\right] \leq \sum_{t=1}^{\infty} \frac{6}{\pi^2} \frac{1}{t^2} \sum_{i=1}^{n} \frac{p}{n} = p$$

As a result with probability at least $1-p$ we have that inequality (4) holds for all $t \geq 1$ and all agents i. We now prove our claim by induction. Let $\|x(\tau) - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau} + \sum_{\tau=0}^{t-1} e(\tau)}{t} + \alpha_i s_i \qquad (5) \\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\sum_{j \in N_i} p_{ij} x_j^* + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \qquad (6) \\
&\leq x_i^* + \delta(t) + (1-\rho)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}$$

We get (5) from the induction step and (6) from inequality (4). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. As a result $\|x(t) - x^*\|_\infty \leq e(t)$ and the induction is complete. Therefore, we have that with probability at least $1-p$, $\|x(t) - x^*\|_\infty \leq e(t)$ for all $t \geq 1$.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion $e(t) = \delta(t) + (1-\rho)\sum_{\tau=0}^{t-1} e(\tau)/t$ and $e(0) = \|x(0) - x^*\|_\infty$, where $\delta(t) = \sqrt{\ln(Dt^{2.5})/t}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\,1/2)}}$.*

Theorem 1 follows by direct application of Lemma 2 and both proofs can be found in the Appendix A.

4 *Follow the Leader* ensures no regret

In this section we explain why update rule (3) ("Follow the Leader") ensures no regret to any agent that repeqtedly play the game of Definition 1. Based on the cost that the agents experience, we consider an appropriate *Online Convex Optimization* problem. This problem can be viewed as a "game" played between an adversary and a player. *At round $t \geq 0$,*

1. *the player selects a value $x_t \in [0, 1]$.*
2. *the adversary observes the x_t and selects a $b_t \in [0, 1]$*
3. *the player receives cost $f(x_t, b_t) = (1-\alpha)(x_t - b_t)^2 + \alpha(x_t - s)^2$.*

where α, s are constants in $[0, 1]$. The goal of the player is to pick x_t based on the history $(b_0, \ldots, b_{t-1})$ in a way that minimizes her total cost. Generally, different OCO problems can be defined by a set of functions $\mathcal{F}$ that the adversary chooses from and a feasibility set $\mathcal{K}$ from which the player picks her value (see [Haz16] for an introduction to the OCO framework). In our case the feasibility set is $\mathcal{K} = [0, 1]$ and the set of functions is $\mathcal{F}_{\alpha,s} = \{(1-\alpha)(x-b)^2 + \alpha(x-s)^2$, for all $b \in [0, 1]\}$. As a result, each selection of the constants s, α lead to a different OCO problem.

Definition 7. *An algorithm A for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0, 1]$ is a sequence of functions $(A_t)_{t=1}^{\infty}$ where $A_t : [0, 1]^t \mapsto [0, 1]$.*

Definition 8. *An algorithm A is no regret for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0, 1]$ if and only if for all sequences $(b_t)_{t=0}^{\infty}$ that the adversary may choose, if $x_t = A_t(b_0, \ldots, b_{t-1})$ then for all t $\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + o(t)$.*

Informally speaking if the player selects the value x_t according to a *no regret algorithm* then she does not regret for not playing any fixed value no matter what the choices of the adversary are. Theorem 2 states that "Follow the Leader" i.e. $x_t = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ is a no regret algorithm for all the OCO problems with $\mathcal{F}_{\alpha,s}$.

Returning to the dynamics of the game in Definition 1, it is reasonalbe to consider that each agent i selects $x_i(t)$ according to no regret algorithm A_i for the OCO problem with $\mathcal{F}_{s_i,\alpha_i}$, since by Definition 8,

$$\frac{1}{t}\sum_{\tau=0}^{t} f_i(x_i(\tau), x_{W_i^\tau}(\tau)) \leq \frac{1}{t}\min_{x\in[0,1]} \sum_{\tau=0}^{t} f_i(x, x_{W_i^\tau}(\tau)) + \frac{o(t)}{t}$$

The latter means that the time averaged total disagreement cost that she suffers is similar to the time averaged cost by expressing the best fixed opinion and this holds no matter the opinions of the neighbors that i meets. Under this perspective FTL dynamics is a *natural limited information dynamics*, since update rule (3) ensures no regret to the agents.

We now present the key steps for proving Theorem 2. We first prove that a similar strategy that also takes into account the value b_t admits no regret (Lemma 3). Obviously knowing the value b_t before selecting x_t is in direct contrast with the OCO framework, however proving the no regret property for this algorithm easily extends to establishing the no regret property of *Follow the Leader*. The proofs of the subsequent lemmas and Theorem 2 can be found in the Appendix B.

Lemma 3. *Let* $\{b_t\}_{t=0}^{\infty}$ *be an arbitrary sequence with* $b_t \in [0,1]$. *Let* $y_t = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t} f(x,b_\tau)$ *then for all* t, $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.

Now we can understand reason why "Follow the Leader" admits no regret. Since the cost incurred by the sequence y_t is at most that of the best fixed value, we can compare the cost incurred by x_t with that of y_t. Since the functions in $\mathcal{F}_{\alpha,s}$ are quadratic, the extra term $f(x, b_t)$ that y_t takes into account doesn't change dramatically the minimum of the total sum. Meaning that x_t, y_t are relatively close.

Lemma 4. *For all* $t \ge 0$, $f(x_t, b_t) \le f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.

5 Lower Bound for Opinion Dependent Dynamics

In this section we prove that any no regret dynamics cannot converge much faster than FTL dynamics (Dynamics 1). For example, assume that for every instance $I = (P, s, \alpha)$, each agent i update her opinion according to Online Gradient Descent which is a no regret algorithm proposed by Zinkevic in [Zin03], i.e. $x_i(t+1) = x_i(t) - 1/\sqrt{t}(x_i(t) - (1-\alpha_i)x_{W_i^t}(t) - \alpha_i s_i)$ and $x(t)$ is the produced opinion vector. Our results imply that there exists an instance I such that $x(t)$ converge to x^* slowly. The reason is fairly simple: let us select for each $(s,\alpha) \in [0,1]^2$ a no regret algorithm $A_{s,\alpha}$[3]for the OCO problem with $\mathcal{F}_{s,\alpha}$. Now for every instance $I = (P, s, \alpha)$ each agent i updates her opinion according to A_{s_i,α_i}, resulting in an opinion vector $x(t)$. Theorem 3 applies since such a selection can be encoded as an opinion dependent update rule. Specifically, the function $A_t : \{0,1\}^{t+2} \mapsto [0,1]$ is defined as $A_t(b_0, \ldots, b_{t-1}, a, s) = A^t_{a,s}(b_0, \ldots, b_{t-1})$.

At first we show that any opinion dependent A, achieving the previous convergence rate, can be used as an estimator of the parameter $p \in [0,1]$ of Bernoulli random variable with the same asymptotic error rate. This reduction is formally stated in Lemma 5. Since we prove Theorem 3 using a reduction to an estimation problem we shall first briefly introduce some definitions and notation. For simplicity we will restrict the following definitions of estimators and risk to the case of estimating the parameter p of Bernoulli random variables. Given t independent samples from a Bernoulli random variable $B(p)$ an estimator is an algorithm that takes these samples as input and outputs an answer in $[0,1]$.

Definition 9. *An estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *is a sequence of functions,* $\theta_t : \{0,1\}^t \mapsto [0,1]$.

Perhaps the first estimator that comes to one's mind is the *sample mean*, that is $\theta_t = \sum_{i=1}^{t} X_i/t$. To measure the efficiency of an estimator we define the *risk*, which corresponds to the expected error of an estimator.

Definition 10. *Let* P *be a Bernoulli distribution with mean* p *and* P^t *be the corresponding* t*-fold product distribution. The risk of an estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *is* $\mathbf{E}_{(X_1,\ldots,X_t)\sim P^t}[|\theta_t(X_1,\ldots,X_t) - p|]$, *which we will denote by* $\mathbf{E}_p[|\theta_t(X_1,\ldots,X_t) - p|]$ *or* $\mathbf{E}_p[|\theta_t - p|]$ *for brevity.*

The risk $\mathbf{E}_p[|\theta_t - p|]$ quantifies the error rate of the estimated value $\hat{p} = \theta_t(Y_1, \ldots, Y_t)$ to the real parameter p as the number of samples t grows. Since p is unknown, any meaningful estimator $\theta = (\theta_t)_{t=1}^{\infty}$ must guarantee that $\lim_{t\to\infty} \mathbf{E}_p[|\theta_t - p|] = 0$ for all p. For example, *sample mean* has error rate $\mathbf{E}_p[|\theta_t - p|] \le \frac{1}{2\sqrt{t}}$.

We show now that any opinion dependent update rule A, can be used as a Bernoulli estimator with the same error rate.

[3] These s, α are scalars in $[0,1]$ and should not be confused with the internal opinion vector s and the self confidence vector α of an instance $I = (P, s, \alpha)$.

Lemma 5. *Let A an opinion dependent update rule such that for all instances I,*

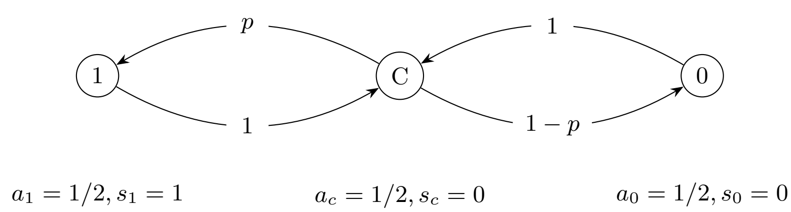
$$\lim_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t)-x^*\|_\infty\right]=0.$$

Then there exists an estimator $\theta_A=(\theta_t^A)_{t=1}^\infty$ such that for all $p\in[0,1]$, $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t^A-p|\right]=0$.

Proof. We construct an estimator $\theta_A=(\theta_t^A)_{t=1}^\infty$ using the update rule A. Consider the instance I_p described in Figure ??. By straightforward computation, we get that the equilibrium point of the graph is $x_c^*=p/3, x_1^*=p/6+1/2, x_0^*=p/6$. Now consider the opinion vector $x_A(t)$ produced by the update rule A for the instance I_p. Note that for $t\geq 1$,

- $x_1^A(t)=A_t(x_c(0),\dots,x_c(t-1),1/2,1)$
- $x_0^A(t)=A_t(x_c(0),\dots,x_c(t-1),1/2,0)$
- $x_c^A(t)=A_t(x_{W_c^0}(0),\dots,x_{W_c^{t-1}}(t-1),1/2,1)$

The key observation is that the opinion vector $x_A(t)$ is a deterministic function of the index sequence $W_c^0,\dots,W_c^{t-1}$ and does not depend on p. Thus, we can construct the estimator θ_A with $\theta_t^A(W_c^0,\dots,W_c^{t-1})=3x_c^A(t)$. For a given instance I_p the choice of neighbor W_c^t is given by the value of the Bernoulli random variable with parameter p ($\mathbf{P}\left[W_c^t=1\right]=p$). As a result, $\mathbf{E}_p\left[|\theta_t^A-p|\right]=3\mathbf{E}\left[|x_c^A(t)-p/3|\right]\leq 3\mathbf{E}\left[\|x_A(t)-x^*\|_\infty\right]$. Since for any instance I_p, we have that $\lim_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t)-x^*\|_\infty\right]=0$. It follows that $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t^A-p|\right]=0$ for all $p\in[0,1]$.



It follows by Lemma 5 that in order to prove Theorem 3 we just need to prove the following claim.

Claim. For any estimator $\theta=(\theta_t)_{t=1}^\infty$ there exists a $p\in[0,1]$ such that $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t-p\right]>0$.

The above claim states that for any estimator $\theta=(\theta_t)_{t=1}^\infty$, we can inspect the functions $\theta_t:\{0,1\}^t\mapsto[0,1]$ and then choose a $p\in[0,1]$ such that the function $\mathbf{E}_p\left[|\theta_t-p|\right]=\Omega(1/t^{1+c})$. As a result, we have reduced the construction of a lower bound concerning the round complexity of a dynamical process to a lower bound concerning the sample complexity of estimating the parameter p of a Bernoulli distribution. The claim follows by Lemma 6, which we present at the end of the section.

At this point we should mention that it is known that $\Omega(1/\varepsilon^2)$ samples are needed to estimate the parameter p of a Bernoulli random variable within additive error ε. Another well-known result is that taking the average of the samples is the *best* way to estimate the mean of a Bernoulli random variable. These results would indicate that the best possible rate of convergence for an *opinion dependent dynamics* would be $O(1/\sqrt{t})$. However, there is some fine print in these results which does not allow us to use them. In order to explain the various limitations of these methods and results we will briefly discuss some of them. We remark that this discussion is not needed to understand the proof of Lemma 6.

The oldest sample complexity lower bound for estimation problems is the well-known Cramer-Rao inequality. Let the function $\theta_t:\{0,1\}^t\mapsto[0,1]$ such that $\mathbf{E}_p\left[\theta_t\right]=p$ for all $p\in[0,1]$, then

$$\mathbf{E}_p\left[(\theta_t-p)^2\right]\geq\frac{p(1-p)}{t}. \tag{7}$$

Since $\mathbf{E}_p\left[|\theta_t - p|\right]$ can be lower bounded by $\mathbf{E}_p\left[(\theta_t - p)^2\right]$ we can apply the Cramer-Rao inequality and prove our claim in the case of *unbiased* estimators, $\mathbf{E}_p\left[\theta_t\right] = p$ for all t. Obviously, we need to prove it for any estimator θ, however this is a first indication that our claim holds.

Sample complexity lower bounds without assumptions about the estimator are usually given as lower bounds for the *minimax risk*, which was defined [4] by Wald in [Wal39] as

$$\min_{\theta_t} \max_{p \in [0,1]} \mathbf{E}_p\left[|\theta_t - p|\right].$$

Minimax risk captures the idea that after we pick the best possible algorithm, an adversary inspects it and picks the worst possible $p \in [0,1]$ to generate the samples that our algorithm will get as input. The methods of Le'Cam, Fano, and Assouad are well-known information-theoretic methods to establish lower bounds for the minimax risk. For more on these methods see [Yu97,Tsy08]. As we stated before, it is well known that the minimax risk for the case of estimating the mean of a Bernoulli is lower bounded by $\Omega(1/\sqrt{t})$ and this lower bound can be established by Le Cam's method. In order to show why such results do no work for our purposes we shall sketch how one would apply Le Cam's method to get this lower bound. To apply Le Cam's method, one typically chooses two Bernoulli distributions whose means are far but their total variation distance is small. Le Cam showed that when two distributions are close in total variation then given a sequence of samples $X_1, \ldots, X_t$ it is hard to tell whether these samples were produced by P_1 or P_2. The hardness of this *testing* problem implies the hardness of *estimating* the parameters of a family of distribution. For our problem the two distributions would be $B(1/2 - 1/\sqrt{t})$ and $B(1/2 + 1/\sqrt{t})$. It is not hard to see that their total variation distance is at most $O(1/t)$, which implies a lower bound $\Omega(1/\sqrt{t})$ for the minimax risk. The problem here is that the parameters of the two distributions depend on the number of samples t. The more samples the algorithm gets to see, the closer the adversary takes the 2 distributions to be. For our problem we would like to *fix* an instance and then argue about the rate of convergence of any algorithm on this instance. Namely, having an instance that depends on t does not work for us.

Trying to get a lower bound without assumptions about the estimators while respecting our need for a fixed (independent of t) p we prove Lemma 6. In fact, we show something stronger: for *almost all* $p \in [0,1]$, any estimator θ cannot achieve rate $o(1/t^{1+c})$. More precisely, suppose we select p uniformly at random in $[0,1]$ and run the estimator θ with samples from the distribution $B(p)$, then with probability 1 the error rate $\mathbf{E}_p\left[|\theta_t - p|\right] = \Omega(1/t^{1+c})$. Although we do not show the sharp lower bound $\Omega(1/\sqrt{t})$ we prove that no fast convergence rate (see Definition 2 is possible and we remark that our proof is fairly simple, intuitive, and could be of independent interest.

Lemma 6. *Let $\theta = (\theta_t)_{t=1}^{\infty}$ be a Bernoulli estimator with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$. For any $c > 0$, if we select p uniformly at random in $[0,1]$ then $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0$ with probability 1.*

Proof. Since θ_t is a function from $\{0,1\}^t$ to $[0,1]$, θ_t can have at most 2^t different values. Without loss of generality we assume that θ_t takes the same value $\theta_t(x)$ for all $x \in \{0,1\}^t$ with the same number of 1's. For example, $\theta_3(\{1,0,0\}) = \theta_3(\{0,1,0\}) = \theta_3(\{0,0,1\})$. This is due to the fact that for any $p \in [0,1]$,

$$\sum_{0\le i\le t} \sum_{\|x\|_1 = i} |\theta_t(x) - p|\, p^i (1-p)^{t-i} \ge \sum_{0 \le i \le t} \binom{t}{i} \left| \frac{\sum_{\|x\|_1 = i} \theta_t(x)}{\binom{t}{i}} - p \right| p^i (1-p)^{t-i}.$$

[4] Although the minimax risk is defined for any estimation problem and loss function, for simplicity, we write the minimax risk for estimating the mean of a Bernoulli random variable.

For any estimator θ with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$ there exists another estimator θ' that satisfies the above property and $\mathbf{E}_p\left[|\theta'_t - p|\right] \le \mathbf{E}_p\left[|\theta_t - p|\right]$ for all $p \in [0,1]$. Thus, we can assume that θ_t takes at most $t+1$ different values. Let A denote the set of p for which the estimator has error rate $o(1/t^{1+c})$, that is

$$A = \{p \in [0,1] : \lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] = 0\}.$$

We show that if we select p uniformly at random in $[0,1]$ then $\mathbf{P}\left[p \in A\right] = 0$. We also define the set

$$A_k = \{p \in [0,1] : \text{for all } t \ge k,\ t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \le 1\}.$$

Observe that if $p \in A$ then there exists t_p such that $p \in A_{t_p}$, meaning that $A \subseteq \bigcup_{k=1}^{\infty} A_k$. As a result,

$$\mathbf{P}\left[p \in A\right] \le \mathbf{P}\left[p \in \bigcup_{k=1}^{\infty} A_k\right] \le \sum_{k=1}^{\infty} \mathbf{P}\left[p \in A_k\right].$$

To complete the proof we show that $\mathbf{P}\left[p \in A_k\right] = 0$ for all k. Notice that $p \in A_k$ implies that for $t \ge k$, the estimator θ must always have a value $\theta_t(i)$ close to p. Using this intuition we define the set

$$B_k = \{p \in [0,1] : \text{for all } t \ge k,\ t^{1+c} \min_{0\le i\le t} |\theta_t(i) - p| \le 1\}.$$

We now show that $A_k \subseteq B_k$. Since $p \in A_k$ we have that for all $t \ge k$

$$t^{1+c} \min_{0\le i\le t} |\theta_t(i) - p| \sum_{i=0}^{t} \binom{t}{i} p^i (1-p)^{t-i} \le t^{1+c} \sum_{i=0}^{t} \binom{t}{i} |\theta_t(i) - p|\, p^i (1-p)^{t-i} = t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \le 1.$$

Thus, $\mathbf{P}\left[p \in A_k\right] \le \mathbf{P}\left[p \in B_k\right]$. We write the set B_k as

$$B_k = \bigcap_{t=k}^{\infty} \{p \in [0,1] :\ \min_{0\le i\le t} |\theta_t(i) - p| \le 1/t^{1+c}\}.$$

As a result, $\mathbf{P}\left[p \in B_k\right] \le \mathbf{P}\left[\min_{0\le i\le t} |\theta_t(i) - p| \le 1/t^{1+c}\right]$, for all $t \ge k$. Each value $\theta_t(i)$ "covers"

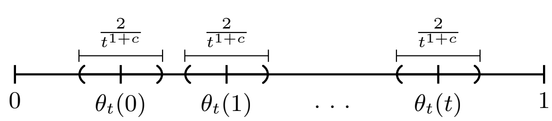


Fig. 1: Estimator output at time t

length $1/t^{1+c}$ from its left and right, as shown in Figure 1, and since there are at most $t+1$ such values, by the union bound we get $\mathbf{P}\left[p \in B_k\right] \le 2(t+1)/t^{1+c}$, for all $t \ge k$. We conclude that $\mathbf{P}\left[p \in B_k\right] = 0$.

6 An Update Rule with Fast Convergence Rate

We already discussed that the reason that opinion dependent dynamics suffer slow convergence is that the update rule depends only on the expressed opinions. Based on works for asynchronous distributed minimization algorithms [BT97,CC16], we provide an update rule showing that information about the graph G combined with agents that do not act selfishly can restore the fast convergence rate.

Our update rule, depends not only on the expressed opinions of the neighbors that an agent i meets but also on the i-th row of matrix P. In update rule (8), each agent stores the *most recent* opinions of the random neighbors that she meets in an array and then updates her opinion according to their weighted sum (each agent knows row i of P). For a given instance $I = (P, s, \alpha)$ we call the produced dynamics *Row Dependent dynamics* and it is defined in Dynamics 2.

The problem with this approach is that the opinions of the neighbors that she keeps in her array are *outdated*, i.e. the opinion of a neighbor of agent i has changed since their last meeting. The good news are that as long as this outdatedness is bounded we can still achieve fast convergence to the equilibrium. By bounded outdatedness we mean that there exists a number of rounds B such that all agents have met all their neighbors at least once from $t - B$ to t. The latter is formally stated in Lemma 7 and its proof can be found Appendix C.

Remark 1. Update rule (8), apart from the opinions and the indices of the neighbors that an agent meets, also depends on the the exact values of the weights p_{ij} and that is why *Row Dependent dynamics* converge fast. We mention that the lower bound of Section 5 still holds even if the agents also use the indices of the neighbors that they meet to update their opinion, since Lemma 5 can be easily modified to cover this case. The latter implies that any update rule that ensures fast convergence must require that each agent i is *aware* of the i-th row of matrix P.

Dynamics 2 Row Dependent dynamics

1: Initially $x_i(0) = s_i$ for all agent i.
2: Each agent i keeps an array M_i of length $|N_i|$, randomly initialized.
3: At round $t \geq 0$ each agent i:
  4: Meets neighbor with index W_i^t, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$.
  5: Suffers cost $(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$ and learns $(x_{W_i^t}(t), W_i^t)$.
  6: Updates her array M_i and opinion: $M_i[W_i^t] \leftarrow x_{W_i^t}(t), \quad x_i(t+1) = (1-\alpha_i)\sum_{j\in N_i} p_{ij} M_i[j] + \alpha_i s_i$ (8)

Lemma 7. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t)$ be the most recent round before round t, that agent i met her neighbor j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t)$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1-\rho)^k$.*

In our randomized setting there does not exist a fixed length window B that satisfies the requirements of Lemma 7. However we can select a length value such that the requirements hold with high probability. To do this observe that agent i simply needs to wait to meet the neighbor j with the smallest weight p_{ij}. Therefore, after $\log(1/\delta)/\min_j p_{ij}$ rounds we have that with probability at least $1 - \delta$ agent i met all her neighbors at least once. Since we want this to be true for all agents we shall roughly take $B = 1/\min_{p_{ij}>0} p_{ij}$. In Section C of the Appendix we give the detailed argument that leads to the Theorem 4, showing that the convergence rate of update rule (8) is fast.

Theorem 4. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$ and let $\rho = \min_{i\in V} a_i$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (8) after t rounds satisfies $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2\exp(-\rho \min_{ij} p_{ij}\sqrt{t}/(4\ln(nt)))$.*

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A The Convergence Rate of FTL Dynamics

We present the proofs of Lemma 2 and Theorem 1 of Section 3

Lemma 2. *Let $e(t)$ be a function satisfying the recursion $e(t) = \delta(t)+(1-\rho)\sum_{\tau=0}^{t-1} e(\tau)/t$ and $e(0) = \|x(0) - x^*\|_\infty$, where $\delta(t) = \sqrt{\ln(Dt^{2.5})/t}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \le \sqrt{2\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\,1/2)}}$.*

Proof. Observe that for all $t \ge 0$ the function $e(t)$ the following recursive relation

$$e(t+1) = e(t)\left(1 - \frac{\rho}{t+1}\right) + \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \tag{9}$$

For $t = 0$ we have that

$$e(1) = (1-\rho)e(0) + \delta(1) = (1-\rho)e(0) + \sqrt{\ln D} \tag{10}$$

Observe that for $D > \mathrm{e}^{2.5}$, $\delta(t)$ is decreasing for all $t \ge 1$. Therefore, $\delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \le \frac{\delta(t)}{t+1}$ and from equations (9) and (10) we get that for all $t \ge 0$

$$e(t+1) \le e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{\ln(D(t+1)^2)}}{(t+1)^{3/2}} \le e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{2\ln(D(t+1))}}{(t+1)^{3/2}}$$

Now let $g(t) = \frac{\sqrt{2\ln(Dt)}}{t^{3/2}}$ to obtain for all $t \ge 1$

$$\begin{aligned}
e(t) &\le (1-\frac{\rho}{t})e(t-1) + g(t) \\
&\le (1-\frac{\rho}{t})(1-\frac{\rho}{t-1})e(t-2) + (1-\frac{\rho}{t})g(t-1) + g(t) \\
&\le (1-\frac{\rho}{t})\cdots(1-\rho)e(0) + \sum_{\tau=1}^{t} g(\tau) \prod_{i=\tau+1}^{t}(1-\frac{\rho}{i}) \\
&\le \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\le \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho(H_t - H_\tau)} \\
&\le \frac{e(0)}{t^\rho} + e^{-\rho H_t}\sum_{\tau=1}^{t} g(\tau)e^{\rho H_\tau} \\
&\le \frac{e(0)}{t^\rho} + \frac{\sqrt{2}}{t^\rho}\sum_{\tau=1}^{t} \tau^\rho \frac{\sqrt{\ln(D\tau)}}{\tau^{3/2}} \\
&\le \frac{e(0)}{t^\rho} + \frac{\sqrt{2\ln D}}{t^\rho}\sum_{\tau=1}^{t} \frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}
\end{aligned}$$

We observe that

$$\sum_{\tau=1}^{t} \frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}} \le \int_{\tau=1}^{t} \frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}\mathrm{d}\tau \tag{11}$$

since, $\tau \mapsto \frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}$ is a decreasing function of τ for all $\rho \in [0,1]$.

– If $\rho \leq 1/2$ then

$$\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} \mathrm{d}\tau \leq \sqrt{\ln t} \int_{\tau=1}^{t} \frac{1}{\tau} \mathrm{d}\tau = (\ln t)^{3/2}$$

– If $\rho > 1/2$ then

$$\begin{aligned}
\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} \mathrm{d}\tau &= \int_{\tau=1}^{t} \tau^{\rho-1/2} \frac{\sqrt{\ln \tau}}{\tau} d\tau \\
&= \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-1/2} ((\ln \tau)^{3/2})' \mathrm{d}\tau \\
&= \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2} - (\rho - 1/2) \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-3/2} (\ln \tau)^{3/2} \mathrm{d}\tau \\
&\leq \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2}
\end{aligned}$$

Theorem 1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

Proof. By Lemma 1 we have that for all $t \geq 1$ and $p \in [0,1]$,

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e_p(t)\right] \geq 1 - p$$

where $e_p(t)$ is the solution of the recursion, $e_p(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Setting $p = \frac{1}{12\sqrt{t}}$ we have that

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - \frac{1}{12\sqrt{t}}$$

where $e(t)$ is the solution of the recursion $e(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(2\pi^2 n t^{2.5})}{t}}$. Since $2\pi^2 \geq e^{2.5}$, Lemma 2 applies and $e(t) \leq C \sqrt{\log n} \frac{\log t^{3/2}}{t^{\min(\rho,1/2)}}$ for some universal constant C. Finally,

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq \frac{1}{12\sqrt{t}} + \left(1 - \frac{1}{12\sqrt{t}}\right) C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho,1/2)}} \leq \left(C + \frac{1}{12}\right) \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho,1/2)}}$$

B FTL dynamics has no-regret

We present the detailed proofs of Lemma 3, Lemma 4 and Theorem 2 of Section 4.

Lemma 3. *Let $\{b_t\}_{t=0}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. Let $y_t = \mathrm{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$ then for all t, $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.*

Proof. By definition of y_t, $\sum_{\tau=0}^{t} f(y_t, b_\tau) = \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$, so

$$\begin{aligned}
\sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) &= \sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t} f(y_t, b_\tau) \\
&= \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_t, b_\tau) \\
&\le \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_{t-1}, b_\tau)
\end{aligned}$$

The last inequality follows by the fact that $y_{t-1} = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x,b_\tau)$ Inductively, we prove that $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.

Lemma 4. *For all $t \ge 0$, $f(x_t, b_t) \le f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.*

Proof. We first prove that for all t,

$$|x_t - y_t| \le \frac{1-\alpha}{t+1}. \tag{12}$$

By definition $x_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} b_\tau}{t}$ and $y_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t} b_\tau}{t+1}$.

$$\begin{aligned}
|x_t - y_t| &= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau}{t} - \frac{\sum_{\tau=0}^{t} b_\tau}{t+1}\right| \\
&= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau - t b_t}{t(t+1)}\right| \\
&\le \frac{1-\alpha}{t+1}
\end{aligned}$$

The last inequality follows from the fact that $b_\tau \in [0,1]$. We now use inequality (12) to bound the difference $f(x_t, b_t) - f(y_t, b_t)$.

$$\begin{aligned}
f(x_t, b_t) &= \alpha(x_t - s)^2 + (1-\alpha)(x_t - y_t)^2 \\
&\le \alpha(y_t - s)^2 + 2\alpha\,|y_t - s|\,|x_t - y_t| + \alpha\,|x_t - y_t|^2 \\
&\quad + (1-\alpha)(y_t - y_t)^2 + 2(1-\alpha)\,|y_t - y_t|\,|x_t - y_t| + (1-\alpha)\,|x_t - y_t|^2 \\
&\le f(y_t, b_t) + 2\,|x_t - y_t| + |y_t - x_t|^2 \\
&\le f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}
\end{aligned}$$

Theorem 2. *Consider the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Let $\{b_t\}_{t=1}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. If $x_t = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x,b_\tau)$ then for all t, $\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$.*

Proof. Theorem 2 easily follows by Lemma 3

$$\begin{aligned}\sum_{\tau=0}^{t} f(x_\tau, b_\tau) &\leq \sum_{\tau=0}^{t} f(y_\tau, b_\tau) + \sum_{\tau=0}^{T} 2\frac{1-\alpha}{\tau+1} + \sum_{\tau=0}^{t} \frac{(1-\alpha)^2}{(\tau+1)^2} \\ &\leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + 2(1-\alpha)(\log t + 1) + (1-\alpha)\frac{\pi^2}{6} \\ &\leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + O(\log t)\end{aligned}$$

C An Update Rule with Fast Convergence Rate

We are now going to state and prove a series of lemmas that culminate in the proof of Theorem 4. We first prove Lemma 7

Lemma 7. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t)$ be the most recent round before round t, that agent i met her neighbor j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t)$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1-\rho)^k$.*

Proof. To prove our claim we use induction on k. For the induction base $k = 1$,

$$|x_i(t) - x_i^*| = |(1-\alpha_i) \sum_{j\in N_i} p_{ij}(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1-\alpha_i) \sum_{j\in N_i} p_{ij}|x_j(\pi_{ij}(t)) - x_j^*| \leq (1-\rho)$$

Assume that for all $t \geq (k-1)B$ we have that $\|x(t) - x^*\|_\infty \leq (1-\rho)^{k-1}$. For $k \geq 2$, we again have that

$$|x_i(t) - x_i^*| \leq (1-\rho) \sum_{j\in N_i} p_{ij}|x_j(\pi_{ij}(t)) - x_j^*|$$

Since $t - B \leq \pi_{ij}(t)$ and $t \geq kB$ we obtain that $\pi_{ij}(t) \geq (k-1)B$. As a result, the inductive hypothesis applies, $|x_j(\pi_{ij}(t)) - x_j^*| \leq (1-\rho)^{k-1}$ and $|x_i(t) - x_i^*| \leq (1-\rho)^k$.

We now turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdateness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 12. *Suppose that the collector picks coupons with different probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\textit{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

Lemma 13. *Let $\pi_{ij}(t)$ be the most recent round before round t that agent i met agent j and $B = 2\ln(\frac{nt}{\delta})/\min_{ij} p_{ij}$. Then with probability at least $1-\delta$, for all $\tau \geq B$ and for all $i, j \in N_i$*

$$\tau - B \leq \pi_{ij}(\tau) \leq \tau - 1.$$

Proof. For simplicity we denote $w = \min_{ij} p_{ij}$. Consider an agent i at round $\tau \geq B$ where $B = 2\ln(\frac{nt}{\delta})/w$ and assume that there exists an agent $j \in N_i$ such that $\pi_{ij}(\tau) < \tau - B$. Agent i can be viewed as a coupon collector that has buyed B coupons but has not found the coupon corresponding to agent j. Since $N_i < n$ and $\min_{j\in N_i} p_{ij} \geq w$ by Lemma 12 we have that

$$\mathbf{P}\left[\text{there exists } j \in N_i \text{ s.t. } \pi_{ij}(\tau) < \tau - B\right] \leq \frac{\delta}{nt}$$

The proof follows by a union bound for all agent i and all round $B \leq \tau \leq t$.

By direct application of Lemma 7 and Lemma 13, we obtain the following corollary that will be useful in proving Theorem 4.

Corollary 1. *Let $x(t)$ the opinion vector produced by update rule (8) for the instance $I = (P, s, \alpha)$, then with probability at least $1 - \delta$*

$$\|x(t) - x^*\|_\infty \leq \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{\delta})}\right)$$

where $\rho = \min_{i\in V} \alpha_i$.

Proof. Let $B = 2\ln(\frac{nt}{\delta})/\min_{ij} p_{ij}$. By Lemma 13 we have that with probability at least $1 - \delta$, for all $i, j \in N_i$ and for all $\tau \geq B$,

$$\tau - B \leq \pi_{ij}(\tau)$$

As a result, with probability at least $1 - \delta$ the requirements of Lemma 7 are satisfied, meaning that

$$\|x(t) - x^*\|_\infty \leq (1 - \rho)^{\frac{t}{B}} \leq \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{\delta})}\right)$$

We can now prove Theorem 4 using the previous results.

Theorem 4. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0, 1]^n$ and let $\rho = \min_{i\in V} a_i$. The opinion vector $x(t) \in [0, 1]^n$ produced by update rule (8) after t rounds satisfies $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2\exp(-\rho \min_{ij} p_{ij}\sqrt{t}/(4\ln(nt)))$.*

Proof. Let $u(t) = \|x(t) - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 1 we obtain:

$$\mathbf{P}\left[u(t) > \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)\right] \leq \delta$$

for every probability $\delta \in [0, 1]$. Also, since all the parameters of the problem lie in $[0, 1]$, we have

$$\mathbf{E}\left[u(t)|u(t) > r\right] \leq 1$$

Now, by the conditional expectations identity, we get:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &= \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \leq r\right]\mathbf{P}\left[u(t) \leq r\right] \\ &\leq \delta + r\end{aligned}$$

where $r = \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)$. If we set $\delta = \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \leq \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right) + \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)$$

We now evaluate r for our choice of probability δ:

$$\begin{aligned}
r &= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{p}\right)}\right) \\
&= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{\exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)}\right)}\right) \\
&= \exp\left(-\frac{\rho w t}{2\ln nt + 2\frac{\rho w\sqrt{t}}{2\ln nt}}\right) \\
&\leq \exp\left(-\frac{\rho w t}{4\ln(nt)\sqrt{t}}\right) \\
&= \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}$$

Using the previous calculation, we obtain:

$$\begin{aligned}
\mathbf{E}\left[u(t)\right] &\leq \exp\left(-\frac{\rho w\sqrt{t}}{2\ln(nt)}\right) + \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&\leq 2\exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&= 2\exp\left(-\rho\min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}$$