Fictitious Play in Opinion Formation Games with Random Payoffs

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Abstract. We study opinion formation games based on the famous model proposed by Friedkin and Johsen. In today's huge social networks the assumption that in each round agents update their opinions by taking into account the opinions of *all* their friends could be unrealistic. Therefore, we assume that in each round each agent gets to meet with only one random friend of hers. Since it is more likely to meet some friends than others we assume that agent i meets agent j with probability p_{ij}. Specifically, we define an opinion formation game, where at round t, agent i with intrinsic opinion $s_i \in [0,1]$ and expressed opinion $x_i(t) \in [0,1]$ meets with probability p_{ij} neighbor j with opinion $x_j(t)$ and suffers a disagreement cost that is a convex combination of $(x_i(t) - s_i)^2$ and $(x_i(t) - x_j(t))^2$.

For a dynamics in the above setting to be considered as natural it must be simple, converge to the equilibrium x^*, and perhaps most importantly, it must be a rational choice for selfish agents. In this work we show that an intuitive game play, *fictitious play*, is a natural dynamics for the above game. We prove that, after $O(1/\varepsilon^2)$ rounds, the opinion vector is within error ε of the equilibrium. Moreover, to show that selfish agents would choose to update their opinions according to fictitious play, we show that it has no-regret.

The classical Friedkin-Johsen dynamics converges to the equilibrium within error ε after only $O(\log(1/\varepsilon))$ rounds whereas, in our imperfect information setting, fictitious play needs $\widetilde{O}(1/\varepsilon^2)$ rounds. We ask whether there exists a natural dynamics for our problem with better rate of convergence. We answer this question in the negative by showing that dynamics based on no-regret algorithms cannot converge with less than $\mathrm{poly}(1/\varepsilon)$ rounds.

1 Introduction

The formation and dynamics of opinions are an important aspect in modern society and have been studied extensively for decades (see e.g., [Jac08]). Opinion formation is based on information exchange, between that socially connected people (e.g., family, friends, colleagues) who interact often and affect each other's opinion. Moreover, opinion formation is often *dynamic* in the sense that discussions and interactions lead to changes in the expressed opinions. With the advent of the internet and social media the dynamic aspects of opinion formation have become ever more dominant. To capture opinion formation on a formal level, several models have been proposed (see e.g., [DeG74,FJ90,HK02,BKO11,GS14,BGM13] for continuous opinions and [FGV16a,YOA+13,BFM16] for discrete ones). A common assumption, that dates back to DeGroot [DeG74], is that opinions evolve through a form of repeated averaging of information collected from the agent social neighborhoods.

Our work builds on the influential model of Friedkin and Johnsen [FJ90]. According to FJ-model, each agent i holds an internal opinion $s_i \in [0,1]$, which is private and invariant over time and a public opinion $x_i \in [0,1]$ Initially, agents start with their internal opinion and at each round $t \geq 1$, update their public opinion $x_i(t)$ to a weighted average of public opinion of their social neighbors and their internal opinion, i.e.

$$x_i(t) = \frac{\sum_{j \neq i} w_{ij} x_j(t-1) + w_{ii} s_i}{\sum_{j \neq i} w_{ij} + w_{ii}} \tag{1}$$

where the weights w_{ij} indicate the influence between the agents and w_{ii} the self confidence towards their internal belief. The FJ-model is one of the most intensively studied models in opinion dynamics. It admits a unique equilibrium point $x^* \in [0,1]^n$ to which the opinion vector $x(t)$ converges exponentially fast [GS14].

Latter Bindel et al. introduced a game theoretic view point in opinion formation processes [BKO11]. They considered the update rule of the FJ-model as the minimizer of disagreement cost function and based on this they defined an one shot opinion formation game. The strategy of each agent i is her public opinion x_i, incuring her a disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_{ii}(x_i - s_i)^2 \tag{2}$$

By definition the FJ-model is the *simultaneous best response dynamics* in the repeated version of their one shot game and the Nash equilibrium (NE) of their game concides with the equilibrium point x^* of the FJ-model. The reason for this definition was to measure the inefficiency of the limiting behavior of the FJ-model with respect to the total disagreement cost. They proved that Price of Anarchy is less than 9/8 in case $w_{ij} = w_{ji}$. Additionally, their work also introduced an alternative framework for opinion dynamics. Instead of defining update rules that model the way opinions evolve, agents are considered to repeatedly play a suitable one shot game. This consideration is much more fruitful since it permits the agents to update their opinions in more abstract ways. For example it would be reasonable to consider that in the above repeated game, agents instead of adopting *best response*, they update their opinions according to a *no-regret* learning algorithm.

1.1 Motivation

Our work is motivated by the fact that the FJ-model, despite its simplicity making it a plausible choice for modeling natural behavior, implies a large amount of information exchage between the agents. At each round its update rule requires that every agent learns all the opinions of her social neighbors. In todays's large social networks each user may have several hundrends of friends and obviously she cannot learn all these opinions each day. This introduces some skeptism on how well the FJ-model resembles the opinion formation in such networks. As a result the following questions arise.

Question 1. Can we find variants of the FJ-model that require less information exchange between the agents and share similar convergence properties? Can these models be justified as natural behavior for selfish agents under a game-theoretic solution concept?

To model the opinion formation process in enviroments with limited information exchange, we consider that the agents repeatedly play the following one shot game.

Definition 1. *For a given opinion vector* $x \in [0,1]^n$*, the disagreement cost of agent* i *is the random variable* $C_i(x_i, x_{-i})$ *defined as follows:*

- i *meets one of her neighbors* j *with probability* $p_{ij} = w_{ij} / \sum_{j \in N_i} w_{ij}$
- *suffers cost* $C_i(x_i, x_{-i}) = (1 - a_i)(x_i - x_j)^2 + a_i(x_i - s_i)^2$*, where* $\alpha_i = w_i / (\sum_{j \in N_i} w_{ij} + w_i)$

This one shot game is a straightforward limited information variant of the opinion formation game defined in [BKO11], has the same Nash equilibrium (w.r.t to the expected cost of the agents) and admits a very natural interpretation that we discuss latter. At each round t, the agents play the above one-shot game i.e. each agent i selects an opinion $x_i(t) \in [0,1]$ and suffers a disagreement cost based on the opinion of the neighbor that she met. At the end of the round she is only informed about the opinion and the index of the neighbor that she met. Our work deals with the following questions:

Question 2. Can the agents update their opinions according to the infomation that they receive such that 1) the disagreement cost that they experience is in a sense minimized, 2) the opinion vector $x(t)$ convergences to the Nash equilibrium?

1.2 Contribution

We introduce a simple and intuitive update rule, similar to that of FJ-model, that the agents can adopt and the resulting opinion vector $x(t)$ converges to x^*. Our update rule is a *Follow the Leader algorithm* meaning that each round t, each agent updates her opinion to the minimizer of total disagreement cost that she experienced until round $t-1$. In section 3, we bound its convergence time and we show that in order to acheive ε distance from x^*, $\text{poly}(1/\varepsilon)$ rounds are needed. In section 4, we show that the agents have *no-regret* in adopting it. Namely, the average disagreement cost (that the agent experiences) per round approaches that of expressing the best opinion in hindsight. The latter makes our algorithm a natural choice for agents that selfishly want to minize their incured disagreement cost. Our results contribute to showing that the FJ-model can be extended with simple variants to explain the opinion formation process in enviroments with limited information exchange.

In section 5, we show that for any update rule that ensures *no-regret* to the agents, the resulting opinion vector $x(t)$ cannot converge to x^* faster than polynomially. We prove the latter for a larger class of update rules, *opinion dependent update rules* using information theoretic arguments. This implies that in our limited information setting natural models cannot converge exponentially fast. Finally in Section 6, we present an update rule that is not opinion dependent and acheives exponential convergence. Our results indicate the fundamental reason that the FJ-model converges exponential fast and this has little to do with the "large" information exchange that it requires, they also serve as an "algorithmic guide" for future variants of the FJ-model.

1.3 Related Work

Apart from the forementioned results there exists a large amount of literature concerning the FJ-model. Many recent works [BGM13,CKO13,BFM16,EFHS17] bound the inefficiency of equilibrium in variants of opinion formation game defined in [BKO11]. In [GS14] they bound that the convergence time of the FJ-model in in special graph topologies. In [BFM16], a variant of the opinion formation game in which social relations depend on the expressed opinions, is studied. They prove that, the discretized version of the above game admits a potential function and thus best-response converges to the Nash equilibrium. Convergence results in other discretized variants of the FJ-model can be found in [YOA+13,FGV16b]. In [FPS16] the convergence properties of limited information variants of the Heglesmann-Krause model [HK02] and the FJ model, are examined.

Other works, that relate to ours, concern the convergence properties of dynamics based on no-regret learning algorithms. In [FV97,FS99,SA00,SALS15] it is proved that in a finite n-person game if each agent updates her mixed strategy according to a no-regret algorithm the resulting *time-averaged* strategy vector converges to Coarse Correlated Equilibrium. The convergence properties of no-regret dynamics for games with infinite strategy spaces were considered in [EMN09]. They proved that for a large class of games with concave utility function (socially concave games), the time-averaged strategy vector converges to the PNE.

More recent work investigate a stronger notion of convergence of no-regret dynamics. In [CHM17] they show that, in n-person finite generic games that admit unique Nash equilibrium, the strategy vector converges *locally* and exponentially fast to it. They also provide conditions for *global* convergence. Our results fit in this line of research since we show that for a game with *infinite* strategy space, the strategy vector (and not the time-averaged) converges to the unique Nash equilibrium.

No-regret dynamics under limited information are also examined in other settings. In Kleinberg et al. in [KPT09] treated load-balancing in distributed systems as a repeated game and analyzed the convergence properties no-regret online algorithms under the *full information assumption* that each agent learns the load of every machine. In a subsequent work [KPT11], the same authors consider the same problem in a *limited information setting* ("bulletin board model") in which each agent learns the load of just the machine that served him. In [HCM17,MS17] they examine the convergence properties of online learnig algorithms in case the payoffs oberved by the agents are contaminated with some random noise.

2 Our Results and Techniques

We now introduce the necessary notions to present our results in detail. We start with some remarks on our limited information opinion formation game defined in Definition 1. In the original opinion formation game an opinion vector $x \in [0,1]^n$ defines *deterministically* the cost $C_i(x_i, x_{-i})$ of each agent i, whereas in our variant it defines a probability distribution on the cost $C_i(x_i, x_{-i})$ that i suffers. The cost $C_i(x_i, x_{-i})$ in (2) can be written equivalently

$$C_i(x_i, x_{-i}) = W_i \left((1-\alpha_i) \sum_{j \in N_i} p_{ij}(x_i - x_j)^2 + \alpha_i (x_i - s_i)^2 \right), \tag{3}$$

where $W_i = \sum_{j \in N_i} w_{ij} + w_i$ and $\alpha_i = w_i / (\sum_j w_{ij} + w_i)$ are positive constants independent of the opinion vector x. The random cost in Definition 1 has a natural interpretation: the coefficient α_i measures the reluctance of agent i to adopt an opinion other than s_i, while p_{ij} can be seen as the *real* influence that j poses on i. In our repeated version of the limited information game of Definition 1, p_{ij} is the frequency that i meets j, meaning that the influence that j poses on i can also be interpreted as a measure of how often they meet. The latter aligns with the common belief that we are influenced more by those we interact more often. Equation (3) also helps to establish the existence of NE for the game of Definition 1. In our case, the notion of NE extends with respect to the expected cost of each agent. Namely, $x^* \in [0,1]$ is a NE if and only if $\mathbf{E}\left[C(x_i^*, x_{-i}^*)\right] \le \mathbf{E}\left[C(x_i, x_{-i}^*)\right]$ for each agent i. Since $\mathbf{E}\left[C_i(x_i, x_{-i})\right] = (1-\alpha_i)\sum_{j \in N_i} p_{ij}(x_i - x_j)^2 + \alpha_i(x_i - s_i)^2$, it follows from (3) that the opinion formation game with random payoffs has the same equilibrium x^* as the original opinion formation game.

In the following, we shall adopt the following notation for an instance of the game of Definition 1.

Definition 2. *We denote an instance of the opinion formation game of Definition 1 as* (P, s, α).

- P *is a* $n \times n$ *matrix with non-negative elements* p_{ij}, *with* $p_{ii} = 0$ *and* $\sum_{j=1}^n p_{ij}$ *is either* 0 *or* 1.
- $s \in [0,1]^n$ *is the internal opinion vector.*
- $\alpha \in (0,1]^n$ *the self confidence coefficient vector.*

We use the matrix P to simplify notation, $p_{ij} = w_{ij}/(\sum_{j \in N_i} w_{ij} + w_i)$ if $j \in N_i$ and 0 otherwise. Abusing notation we will sometimes refer to the graph G. Another parameter of an instance $I = (P, s, \alpha)$ that we often use is $\rho = \min_{i \in V} \alpha_i$.

In Section 3, we study the convergence properties of the dynamics $x(t)$ when all agents update their opinions according to the "follow the leader" principle. Each agent i must select $x_i(t)$, before knowing which of her neighbors she will meet and what opinion her neighbor will express. The update rule (4) says "*play the best according to what you have observed*". For a given instance (P, s, a) of the game of Definition 1 the FTL dynamics $x(t)$ is defined in 1 and Theorem 1 shows the convergence rate of $x(t)$ to x^*.

Theorem 1. *Let* $I = (P, s, \alpha)$ *be an instance of the opinion formation game of Definition 1 with equilibrium* $x^* \in [0,1]^n$. *The opinion vector* $x(t) \in [0,1]^n$ *produced by update rule 4 after* t *rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \le C\sqrt{\log n}\,\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

Algorithm 1 FTL Dynamics

1: Initially $x_i(0) = s_i$ for all agents i.
2: At round $t \geq 0$ each agent i:
  3: Meets neighbor with index W_i^t.
  4: Suffers cost $(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$.
  4: Updates her opinion

$$x_i(t+1) = \operatorname*{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t} (1-\alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i(x - s_i)^2 \tag{4}$$

where $\rho = \min_{i \in V} a_i$ *and* C *is a universal constant.*

In Section 4 we argue that, apart from its simplicity, update rule 4 ensures no-regret for the agents, and therefore is a *rational game play* for selfish agents. Since each agent i selfishly wants to minimizing her individual cost, it is natural to assume that she selects $x_i(t)$ according to an *no-regret algorithm* for the *online convex optimization problem* where the adversary chooses a function $f_t(x) = (1-\alpha_i)(x - b_t)^2 + \alpha_i(x - s_i)^2$ at each round t. In Theorem 2 we prove that fictitious play is a no-regret algorithm for the above OCO problem. We remark that, in general, fictitious play does not guarantee no-regret if the adversary can pick functions from a larger class (see e.g. chapter 5 in [Haz16]).

Theorem 2. *Consider the function* $f : [0,1]^2 \mapsto [0,1]$ *with* $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ *for some constants* $s, \alpha \in [0,1]$. *Let* $\{b_t\}_{t=1}^{\infty}$ *be an arbitrary sequence with* $b_t \in [0,1]$. *If* $x_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ *then for all* t,

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$$

The FTL-dynamics can efficiently simulate the opinion formation process in limited information exchange environments. It ensures convergence to the same equilibrium as the original FJ-model while vastly reducing the information exchange between the agents per round; it only requires each agent to learn the opinion of just one neighbor. In terms of total communication needed to get within distance ε of the equilibrium x^*, the update rule (4) needs $O(n \log n)$ communication while (1) needs $O(|E|)$. Of course for this difference to be significant we need each agent to have at least $O(\log n)$ friend which not unusual in today's large social networks.

Even though the update rule (4) has the above desired properties, the convergence rate of the produced dynamics is outperformed by the convergence rate of the classical FJ-model. For a fixed instance $I = (P, s, \alpha)$, fictitious play converges with rate $\widetilde{O}(1/t^{\min(\rho, 1/2)})$ while FJ-model converges with rate $O(e^{-\rho t})$ [GS14]. As a result the following question arises

Question 3. Can the agents adopt other no-regret algorithms such that the resulting dynamics $x(t)$ converges exponential fast to x^*?

In Section 5 we answer this question in the negative. The reason that fictitious play converges slowly is that update rule (4) only depends on the opinions of the agents that agent i meets, α_i, and s_i. This is also true for any no-regret algorithm that i uses to select $x_i(t)$ (see Section 5). We call such update rules "*opinion dependent*". In Theorem 3 we show that for any opinion dependent update rule there exists an instance $I = (P, s, \alpha)$ where $\text{poly}(1/\varepsilon)$ rounds are required to achieve convergence within error ε.

Theorem 3. *Let* A *be an* opinion dependent *update rule, which all agents use to update their opinions. For any* $c > 0$ *there exists an instance* $I = (P, s, a)$ *such that*

$$\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = \Omega(1/t^{1+c}),$$

where $x_A(t)$ *denotes the opinion vector produced by* A *for the instance* $I = (P, s, \alpha)$.

To prove Theorem 3, we show that opinion dependent rules with "small round complexity", imply the existence of estimators for Bernoulli distributions with "small" sample complexity. Then with a simple argument presented in Lemma 6, we show that such estimators cannot exist. In Section 5 we also briefly discuss two well-known sample complexity lower bounds from the statistics literature and explain why they do not work in our case.

In Section 6, we present a simple update rule that is not opinion dependent and achieves error rate $e^{-O(\sqrt{t})}$. This update rule is a function of the opinions and the indices of the agents that i met, α_i, s_i and the i-th row the matrix P. We mention that the lower bound presented in Theorem 3 applies for "opinion dependent rules" that also depend on the agents' indices that i met. Therefore, the dependency on the row P_i is inevitable in order to obtain exponential convergence. Although, the assumption that the agents are aware of the influence matrix P is up to discussion, this update rule reveals that the slow convergence of *opinion dependent* update rules is not due to the reduced information exchange (learning the opinion of only one agent), but due to the fact that the agents are "oblivious" to the influence matrix P of the game and they learn it during the game play.

3 Fictitious Play Convergence Rate

In this section that we prove if all agents select their opinion according to update rule (4) then the produced dynamics $x(t)$ converges to the unique equilibrium point x^*. For an instance (P, s, α) the opinion vector $x(t) \in [0,1]^n$ produced by (4) is defined as follows:

- Initially all agents adopt their internal opinion, $x_i(0) = s_i$
- At round $t \geq 1$, each agent i updates her opinion as follows:

$$x_i(t) = \operatorname*{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} (1-\alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i (x - s_i)^2 \tag{4}$$

where W_i^τ is the neighbor that i met at round t. Since the opinion vector $x(t)$ is a random vector, our convergence metric is $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$ where the expectation is taken over the random meeting of the agents. Our convergence result is stated in Theorem 1 and it is the main result of the section.

Theorem 1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule 4 after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\,\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

At first we present a high level idea of the proof. Update rule (4) can be written equivalently as:

$$x_i(t) = (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i$$

Notice that the unique equilibrium point $x^* \in [0,1]^n$ of the instance $I = (P, s, \alpha)$ satisfies the following equations for each agent $i \in V$,

$$x_i^* = (1-\alpha_i)\sum_{j \in N_i} p_{ij} x_j^* + \alpha_i s_i$$

Since we are interested in bounding the $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$, we can use the above equations to bound $|x_i(t) - x_i^*|$.

$$\begin{aligned}
|x_i(t) - x_i^*| &= (1-\alpha_i)\left|\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*\right| \\
&= (1-\alpha_i)\left|\sum_{j \in N_i} \frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j] x_j(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*\right| \\
&\leq (1-\alpha_i)\sum_{j \in N_i}\left|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j] x_j(\tau)}{t} - p_{ij} x_j^*\right|
\end{aligned}$$

Now assume that $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}| = 0$ for all $t \geq 1$, then with simple algebraic manipulations one can prove that $\|x(t) - x^*\|_\infty \leq e(t)$ where $e(t)$ satisfies the recursive equation $e(t) = (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. It follows that $\|x(t) - x^*\|_\infty \leq 1/t^\rho$ meaning that $x(t)$ converges to x^*. Obviously the latter assumption does not hold, however since W_i^τ are independent random variables with $\mathbf{P}[W_i^\tau] = p_{ij}$, $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}|$ tends to 0 with probability 1. In Lemma 1 we use this fact to obtain a similar recursive relation for $e(t)$ and then in Lemma 2 we upper bound the solution of this recursive equation.

Lemma 1. *Let $e(t)$ the solution of the following recursion,*

$$e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$$

where $e(0) = \|x(0) - x^*\|_\infty$,$\delta(t) = \sqrt{\frac{\ln(\pi^2 n t^2/6p)}{t}}$ and $\rho = \min_{i\in V} \alpha_i$. Then,

$$\mathbf{P}\left[\text{for all } t \geq 1,\ \|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - p$$

Proof. At first we prove that with probability at least $1-p$, for all $t \geq 1$ and all agents i:

$$\left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\in N_i} p_{ij} x_j^* \right| \leq \delta(t) \tag{5}$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$.

Since W_i^τ are independent random variables with $\mathbf{P}[W_i^\tau = j] = p_{ij}$ and $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \sum_{j\in N_i} p_{ij} x_j^*$. By the Hoeffding's inequality we get

$$\mathbf{P}\left[\left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\in N_i} p_{ij} x_j^* \right| > \delta(t)\right] < 6p/(\pi^2 n t^2).$$

To bound the probability of error for all rounds $t = 1$ to ∞ and all agents i, we apply the union bound

$$\sum_{t=1}^{\infty} \mathbf{P}\left[\max_i \left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\in N_i} p_{ij} x_j^* \right| > \delta(t)\right] \leq \sum_{t=1}^{\infty} \frac{6}{\pi^2}\frac{1}{t^2}\sum_{i=1}^{n}\frac{p}{n} = p$$

As a result with probability at least $1-p$ we have that for all $t \geq 1$ and all agents i,

$$\left| \frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\in N_i} p_{ij} x_j^* \right| \leq \delta(t) \tag{6}$$

Now we can prove our claim by induction. Assume that $\|x(\tau) - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau} + \sum_{\tau=0}^{t-1} e(\tau)}{t} + \alpha_i s_i \qquad (7)\\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\sum_{j\in N_i} p_{ij} x_j^* + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \qquad (8)\\
&\leq x_i^* + \delta(t) + (1-\rho)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}$$

We get (7) from the induction step and (8) from inequality (6). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. As a result $\|x(t) - x^*\|_\infty \leq e(t)$ and the induction is complete. Therefore, we have that with probability at least $1-p$, $\|x(t) - x^*\|_\infty \leq e(t)$ for all $t \geq 1$.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion*

$$e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \text{ and } e(0) = \|x(0) - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\ln(Dt^{2.5})}{t}}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2.5\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\,1/2)}}$.

The proof of Lemma 2 and Theorem 1 can be found in the appendix. Theorem 1 is proved by a direct application of Lemma 1 and 2.

4 Fictitious Play is no-regret

In this section we explain why *fictitious play* is a rational behavioral assumption in the repeated version of the opinion formation game (Definition **??**). Based on this game we consider an appropriate *Online Convex Optimization* problem. This problem can be viewed as the following game played between an adversary and a player. *At round* $t \geq 0$,

1. *the player selects a value* $x_t \in [0,1]$.
2. *the adversary observes the* x_t *and selects a* $b_t \in [0,1]$
3. *the player receives cost* $f(x_t, b_t) = (1-\alpha)(x_t - b_t)^2 + \alpha(x_t - s)^2$.

where α, s are constants in $[0,1]$. The goal of the player is to pick x_t based on the history $(b_0, \dots, b_{t-1})$ in a way that minimizes her total cost. Generally, different OCO problems can be defined by the set of functions $\mathcal{F}$ that the adversary chooses from and the feasibility set $\mathcal{K}$ from which the player picks her value (see [Haz16] for an introduction to the OCO framework). In our case $\mathcal{K} = [0,1]$ and $\mathcal{F}_{\alpha,s} = \{(1-\alpha)(x-b)^2 + \alpha(x-s)^2,$ for all $b \in [0,1]\}$. As a result, each selection of the constants s, α lead to a different OCO problem.

Definition 3. *An algorithm A for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0,1]$ is a sequence of functions $(A_t)_{t=1}^\infty$ where $A_t : [0,1]^t \mapsto [0,1]$.*

Definition 4. *An algorithm A is no-regret for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0,1]$ if and only if for all sequences $(b_t)_{t=0}^\infty$ that the adversary may choose, if $x_t = A_t(b_0, \dots, b_{t-1})$ then for all t*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + o(t)$$

Informally speaking if the player selects the value x_t according to a *no-regret algorithm* then she does not regret for not playing any fixed value no matter what the choices of the adversary are. We prove that *fictitious play* i.e. $x_t = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ is a no-regret algorithm for all OCO problems $\mathcal{F}_{\alpha,s}$. This is formally stated in Theorem 2 and is the main result of this section.

Theorem 2. *Consider the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Let $\{b_t\}_{t=1}^\infty$ be an arbitrary sequence with $b_t \in [0,1]$. If $x_t = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$$

Returning in our repeated game, it is reasonabe to assume that each agent i selects $x_i(t)$ according to no-regret algorithm A_i for the OCO problem with $\mathcal{F}_{s_i,\alpha_i}$, since by Definition 4,

$$\frac{1}{t}\sum_{\tau=0}^{t} f_i(x_i(\tau), x_{W_i^\tau}(\tau)) \leq \frac{1}{t}\min_{x\in[0,1]} \sum_{\tau=0}^{t} f_i(x, x_{W_i^\tau}(\tau)) + \frac{o(t)}{t}$$

The latter means that the time averaged total disagreement cost that she suffers is similar to the time averaged cost by expressing the best fixed opinion and this holds no matter the opinions of the agents that i meets. Theorem 2 ensures that *fictitious play* is a no-regret algorithm for any OCO problem $\mathcal{F}_{s_i,\alpha_i}$ and explains why (4) is a rational update rule for the agents.

The rest of the section is dedicated to prove Theorem 2. We first prove that a similar strategy that also takes into account the value b_t admits no-regret (Lemma 3). Obviously knowing the value b_t before selecting x_t is in direct contrast with the OCO framework, however proving the no-regret property for this algorithm easily extends to establishing the no-regret property of fictitious play.

Lemma 3. *Let $\{b_t\}_{t=0}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. Let $y_t = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t} f(x,b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$$

Proof. By definition of y_t, $\sum_{\tau=0}^{t} f(y_t, b_\tau) = \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$, so

$$\begin{aligned}
\sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) &= \sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t} f(y_t, b_\tau) \\
&= \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_t, b_\tau) \\
&\leq \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_{t-1}, b_\tau)
\end{aligned}$$

The last inequality follows by the fact that $y_{t-1} = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x,b_\tau)$ Inductuvely, we prove that $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.

Now we can understand reason why *fictitious play* admits no regret. Since the cost incurred by the sequence y_t is at most that of the best fixed strategy, we can compare the cost incurred by x_t with that of y_t. However, the functions in $\mathcal{F}_{\alpha,s}$ are Lipschitz-continuous and more specifically quadratic. These functions are all "similar" to each other, so the extra term $f(x_t, b_t)$ that y_t takes into account doesn't change dramatically the minimum point of the sum. Thus, for each t the numbers x_t and y_t are quite close and as a result the difference in their cost must be quite small. The above are formally stated and proved in Lemma 4.

Lemma 4. *For all $t \geq 0$, $f(x_t, b_t) \leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.*

Proof. We first prove that for all t,

$$|x_t - y_t| \leq \frac{1-\alpha}{t+1}. \tag{9}$$

By definition $x_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} b_\tau}{t}$ and $y_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t} b_\tau}{t+1}$.

$$\begin{aligned}
|x_t - y_t| &= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau}{t} - \frac{\sum_{\tau=0}^{t} b_\tau}{t+1}\right| \\
&= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau - t b_t}{t(t+1)}\right| \\
&\leq \frac{1-\alpha}{t+1}
\end{aligned}$$

The last inequality follows from the fact that $b_\tau \in [0,1]$. We now use inequality (9) to bound the difference $f(x_t, b_t) - f(y_t, b_t)$.

$$\begin{aligned} f(x_t, b_t) &= \alpha(x_t - s)^2 + (1-\alpha)(x_t - y_t)^2 \\ &\le \alpha(y_t - s)^2 + 2\alpha\,|y_t - s|\,|x_t - y_t| + \alpha\,|x_t - y_t|^2 \\ &\quad + (1-\alpha)(y_t - y_t)^2 + 2(1-\alpha)\,|y_t - y_t|\,|x_t - y_t| + (1-\alpha)\,|x_t - y_t|^2 \\ &\le f(y_t, b_t) + 2\,|x_t - y_t| + |y_t - x_t|^2 \\ &\le f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2} \end{aligned}$$

Theorem 2 easily follows since

$$\begin{aligned} \sum_{\tau=0}^{t} f(x_\tau, b_\tau) &\le \sum_{\tau=0}^{t} f(y_\tau, b_\tau) + \sum_{\tau=0}^{T} 2\frac{1-\alpha}{\tau+1} + \sum_{\tau=0}^{t} \frac{(1-\alpha)^2}{(\tau+1)^2} \\ &\le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + 2(1-\alpha)(\log t + 1) + (1-\alpha)\frac{\pi^2}{6} \\ &\le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + O(\log t) \end{aligned}$$

5 Lower Bound for Opinion Dependent Dynamics

In the previous sections we saw that if each agent i updates her opinion according to update rule 4 the resulting opinion vector $x(t)$ produced for an instance $I = (P, s, \alpha)$ converges to the unique equilibrium point x^* and ensures no-regret for the OCO problem with F_{s_i,α_i}. In this section we investigate whether we can establish exponential convergence while keeping the no-regret property for the agents. More precisely, can we select for each $(s,\alpha) \in [0,1]^2$ a no-regret algorithm $A_{s,\alpha}$ for the OCO problem with $\mathcal{F}_{s,\alpha}$ such that if for each instance $I = (P, s, \alpha)$ each agent i updates her opinion according to A_{s_i,α_i}, the resulting opinion vector $x(t)$ converges exponentially fast to x^*?

Definition 5 (opinion dependent update rule). *An opinion dependent update rule A is a sequence of functions $(A_t)_{t=0}^{\infty}$ where $A_t : [0,1]^{t+2} \mapsto [0,1]$.*

Definition 6 (opinion dependent Dynamics). *Let an opinion depedent update rule A. For a given instance $I = (P, s, \alpha)$ the rule A produces an opinion depedent dynamics $x_A(t)$ defined as follows:*

- *Initially each agent i has opinion $x_i^A(0) = A_0(s_i, \alpha_i)$*
- *At each round $t \ge 1$, each agent i updates her opinion as follows:*

$$x_i^A(t) = A_t(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), \alpha_i, s_i)$$

where W_i^t is the neighbors that i meets at round t.

By Definition 3 any collection of no-regret algorithms $A_{s,\alpha}$ for all $s, \alpha \in [0,1]^2$, can be encoded as opinion dependent update rule $(A_t(b_0, \ldots, b_{t-1}, s, \alpha) = A_{s,\alpha}^t(b_0, \ldots, b_{t-1}))$. As a result, a lower bound for the opinion dependent dynamics answers our initial question. An *opinion dependent* update rule A produces different *opinion dependent dynamics* $x_A(t)$ for different instances $I = (P, s, \alpha)$, since I determines the s_i, α_i for each agent and the probability distribution according to which the random meetings take place. An example of an opinion depedent update rule is (4), since $x_i(t) = (1-\alpha_i)\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i$. For an instance $I = (P, s, \alpha)$, rule (4) produces the *opinion dependent dynamics* $x(t)$ whose convergence properties to x^* were studied in Section 3.

We are interested in lower bounds about the convergence rate of *opinion dependent dynamics* because these bounds also hold for the convergence rate of the *no-regret dynamics* for our repeated game. Assume that each agent i updates $x_i(t)$ according to a no-regret algorithm A^i for the OCO problem where the adversary selects the functions $(1-\alpha_i)(x-b_t)^2+\alpha_i(x-s_i)^2$. By Definition 4, we have that $x_i(t)=A^i_t(b_0,\ldots,b_{t-1})$. Notice that each agent i selects a no-regret algorithm for a different OCO problem defined by her α_i, s_i. Also, b_t is the neigboring opinion that i learns at time t, so $b_t = x_{W_i^t}(t)$. Hence, the respective *no-regret dynamics* $\{x(t)\}_{t=1}^{\infty}$ are *opinion dependent* since

$$x_i(t) = A^i_t\left(x_{W_i^0}(0),\ldots,x_{W_i^{t-1}}(t-1)\right)$$

where $A_t : \{0,1\}^{t+2} \mapsto [0,1]$.

Theorem 3. *Let A be an* opinion dependent *update rule, which all agents use to update their opinions. For any $c>0$ there exists an instance $I=(P,s,a)$ such that*

$$\mathbf{E}\left[\|x_A(t)-x^*\|_\infty\right] = \Omega(1/t^{1+c}),$$

where $x_A(t)$ denotes the opinion vector produced by A for the instance $I=(P,s,\alpha)$.

At first we show that any opinion dependent A, achieving the previous convergence rate, can be used as an estimator of the parameter $p\in[0,1]$ of Bernoulli random variable with the same asymptotic error rate. This reduction is formally stated in Lemma 5. Since we prove Theorem 3 using a reduction to an estimation problem we shall first briefly introduce some definitions and notation. For simplicity we will restrict the following definitions of estimators and risk to the case of estimating the mean of Bernoulli random variables. Given t independent samples from a Bernoulli random variable $B(p)$ an estimator is an algorithm that takes these samples as inputs and outputs an answer in $[0,1]$.

Definition 7. *An estimator $\theta=(\theta_t)_{t=1}^{\infty}$ is a sequence of functions, $\theta_t : \{0,1\}^t \mapsto [0,1]$.*

Perhaps the first estimator that comes to one's mind is the *sample mean*, that is $\theta_t = (1/t)\sum_{i=1}^t X_i$. Of course for an estimator to be efficient we would like its answer to be close to the mean p of the Bernoulli that generated the samples. To measure the efficiency of an estimator we define the *risk* which corresponds to the expected loss of an estimator.

Definition 8. *For an estimator $\theta=(\theta_t)_{t=1}^{\infty}$ we define its risk $E_p[|\theta_t(X_1,\ldots,X_t)-p|]$, where*

$$E_p[|\theta_t(X_1,\ldots,X_t)-p|] = \sum_{(y_1,\ldots,y_t)\in\{0,1\}^t} |\theta_t(y_1,\ldots,y_t)-p| \; p^{\sum_{i=1}^t y_i} \; (1-p)^{t-\sum_{i=1}^t y_i}$$

The risk $E_p[|\theta_t(Y_1,\ldots,Y_t)-p|]$ is the expected distance of the estimated value θ_t from the parameter p, when the distribution that generated the samples is $B(p)$. For convenience we also write it as $E_p[|\theta_t-p|]$. The risk quantifies the error rate of the estimated value $\hat{p}=\theta_t(Y_1,\ldots,Y_t)$ to the real parameter p as the number of samples t grows. Since p is unknown, any meaningful estimator $\theta=(\theta_t)_{t=1}^{\infty}$ must guarantee that $\lim_{t\to\infty} E_p[|\theta_t-p|]=0$ for all p. For example, *sample mean* has error rate $E_p[|\theta_t-p|]\le\frac{1}{2\sqrt{t}}$.

We show now that any opinion dependent update rule A, achieving the convergence rate of Theorem 3, can be used as an estimator of the parameter $p\in[0,1]$ of a Bernoulli random variable with asymptotically the same error rate. The reduction is formally stated and in Lemma 5.

Lemma 5. *Let A an opinion dependent update rule such that for all instances I, $\lim\limits_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t)-x^*\|_\infty\right]=0$. Then there exists an estimator $\theta_A=(\theta^A_t)_{t=1}^{\infty}$ such that for all $p\in[0,1]$,*

$$\lim_{t\to\infty} t^{1+c} E_p[|\theta^A_t-p|]=0$$

Proof. We sketch here the main idea. For a full proof see Section B of the Appendix. For a given $p\in[0,1]$, we construct an instance I_p such that $x^*_c=p$ for an agent c. Moreover, agent c must receive only values 1 or 0 with probability p and $1-p$ respectively. This can be easily done using the directed star graph $K_{1,2}$. The agent corresponding to the center node, c, has $\alpha_c=1/2$ and whereas the leaf nodes have $a_{1,2}=1$, $s_1=0$, $s_2=1$, as shown in Figure 1. It follows that the estimator θ_t with $\theta^A_t=2x^A_c(t)$ has error $E_p[|\theta^A_t-p|]=\frac{1}{2}\mathbf{E}_{I_p}\left[\|x_A(t)-x^*\|_\infty\right]$. Meaning that $\lim\limits_{t\to\infty} t^{1+c}E_p[|\theta^A_t-p|]=0$ for all $p\in[0,1]$.

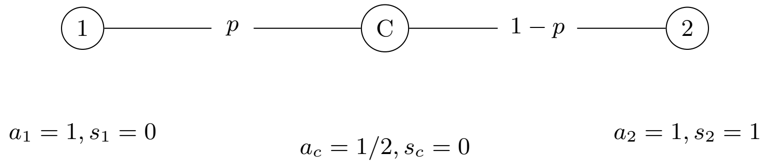

Fig. 1: The Lower Bound Instance

It follows by Lemma 5 that in order to prove Theorem 3 we just need to prove the following claim.

Claim. For any estimator $\theta = (\theta_t)_{t=1}^{\infty}$ there exists a fixed $p \in [0, 1]$ such that

$$\lim_{t\to\infty} t^{1+c}\mathbf{E}_p \left[|\theta_t - p\right] > 0.$$

The above claim states that for any estimator $\theta = (\theta_t)_{t=1}^{\infty}$, we can inspect the functions $\theta_t : \{0, 1\}^t \mapsto [0, 1]$ and then choose a $p \in [0, 1]$ such that the function $E_p[|\theta_t - p|] = \Omega(1/t^{1+c})$. As a result, we have reduced the construction of a lower bound concerning the round complexity of a dynamical process to a lower bound concerning the sample complexity of estimating the parameter p of a Bernoulli distribution.

At this point we should mention that it is known that $\Omega(1/\varepsilon^2)$ samples are needed to estimate the parameter p of a Bernoulli random variable within additive error ε. Another well-known result is that taking the average of the samples is the *best* way to estimate the mean of a Bernoulli random variable. These results would indicate that the best possible rate of convergence for an *opinion dependent dynamics* would be $O(1/\sqrt{t})$. However, there is some fine print in these results which does not allow us to use them. In order to explain the various limitations of these methods and results we will briefly discuss some of them.

Before presenting Lemma 6 we briefly discuss some fundamental results concerning sample complexity lower bounds for statistical estimation. Perhaps the oldest sample complexity lower bound for estimation problems is the well-known Cramer-Rao inequality. Let the function $\theta_t : \{0, 1\}^t \mapsto [0, 1]$ such that $E_p[\theta_t] = p$ for all $p \in [0, 1]$, then

$$\mathbf{E}_p \left[(\theta_t - p)^2\right] \geq \frac{p(1-p)}{t}. \tag{10}$$

Since $\mathbf{E}_p \left[|\theta_t - p|\right]$ can be lower bounded by $\mathbf{E}_p \left[(\theta_t - p)^2\right]$ we can apply the Cramer-Rao inequality and prove Claim 5 for the *unbiased* estimators $\theta = (\theta_t)_{t=1}^{\infty}$. An estimator $\theta = (\theta_t)_{t=1}^{\infty}$ is *unbiased* if $E_p[\theta_t] = p$ for all $p \in [0, 1]$. Obviously, we need to prove the claim for any estimator θ, however this is a first indication that our claim holds.

To the best of our knowledge, sample complexity lower bounds without assumptions about the estimator are given as lower bounds for the *minimax risk*, which was defined [3] by Wald in [Wal39] as

$$\min_{\theta_t} \max_{p\in[0,1]} \mathbf{E}_p \left[|\theta_t - p|\right].$$

Minimax risk captures the idea that after we pick the best possible algorithm, an adversary inspects it and picks the worst possible $p \in [0, 1]$ to generate the samples that our algorithm will get as input. The methods of Le'Cam, Fano, and Assouad are well-known information-theoretic methods to establish lower bounds for the minimax risk. For more on these methods see [Yu97,**?**] and the very good lecture notes of Duchi, [Duc]. As we stated before, it is well known that the minimax risk for the case of estimating the mean of a Bernoulli is lower bounded by $\Omega(1/\sqrt{t})$ and this lower bound can be established by Le Cam's method. In order to show why such arguments do no work for our purposes we shall sketch how one would apply Le Cam's method to get this lower bound. To apply Le Cam's method, one typically chooses two Bernoulli distributions whose means are far but their total variation distance is small. Le Cam showed that when two distributions are

[3] Although the minimax risk is defined for any estimation problem and loss function, for simplicity, we write the minimax risk for estimating the mean of a Bernoulli random variable.

close in total variation then given a sequence of samples $X_1, \ldots, X_t$ it is hard to tell whether these samples were produced by P_1 or P_2. The hardness of this *testing* problem implies the hardness of *estimating* the parameters of a family of distribution. For our problem the two distributions would be $B(1/2 - 1/\sqrt{t})$ and $B(1/2 + 1/\sqrt{t})$. It is not hard to see that their total variation distance is at most $O(1/t)$, which implies a lower bound $\Omega(1/\sqrt{t})$ for the minimax risk. The problem here is that the parameters of the two distributions depend on the number of samples t. The more samples the algorithm gets to see, the closer the adversary takes the 2 distributions to be. For our problem we would like to *fix* an instance and then argue about the rate of convergence of any algorithm on this instance. Namely, having an instance that depends on t does not work for us.

Trying to get a lower bound without assumptions about the estimators while respecting our need for a fixed (independent of t) p we prove Lemma 6. In fact, we show something stronger: for *almost all* $p \in [0,1]$, any estimator θ cannot achieve rate $o(1/t^{1+c})$. More precisely, suppose we select a p uniformly at random in $[0,1]$ and run the estimator θ with samples from the distribution $B(p)$, then with probability 1 the error rate $\mathbf{E}_p\left[|\theta_t - p|\right] \in \Omega(1/t^{1+c})$. Although we do not show the sharp lower bound $\Omega(1/\sqrt{t})$ we prove that no exponential convergence rate is possible and we remark that our proof is fairly simple, intuitive, and could be of independent interest.

Lemma 6. *Let a Bernoulli estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *with error rate* $\mathbf{E}_p\left[|\theta_t - p|\right]$. *For any* $c > 0$, *if we select* p *uniformly at random in* $[0,1]$ *then*

$$\mathbf{P}\left[\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \ > 0\right] = 1$$

Proof. Let an estimator $\theta = \{\theta_t\}_{t=1}^{\infty}$, where $\theta_t : \{0,1\}^t \mapsto [0,1]$. The function θ_t can have at most 2^t different values. Without loss of generality we assume that θ_t takes the same value $\theta_t(x)$ for all $x \in \{0,1\}^t$ with the same number of 1's. For example, $\theta_3(\{1,0,0\}) = \theta_3(\{0,1,0\}) = \theta_3(\{0,0,1\})$. This is due to the fact that for any $p \in [0,1]$,

$$\sum_{0 \le i \le t} \sum_{\|x\|_1 = i} |\theta_t(x) - p|\, p^i (1-p)^{t-i} \ge \sum_{0 \le i \le t} \binom{t}{i} \left| \frac{\sum_{\|x\|_1 = i} \theta_t(x)}{\binom{t}{i}} - p \right| p^i (1-p)^{t-i}.$$

For any estimator θ with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$ there exists another estimator θ' that satisfies the above property and $\mathbf{E}_p\left[|\theta'_t - p|\right] \le \mathbf{E}_p\left[|\theta_t - p|\right]$ for all $p \in [0,1]$. Thus we can assume that θ_t takes at most $t+1$ different values. Let A denote the set of p for which the estimator has error rate $o(1/t^{1+c})$, that is

$$A = \{p \in [0,1] : \lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] = 0\}$$

We show that if we select p uniformly at random in $[0,1]$ then $\mathbf{P}\left[p \in A\right] = 0$. We also define the set

$$A_k = \{p \in [0,1] : \text{for all } t \ge k,\ t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \le 1\}$$

Observe that if $p \in A$ then there exists t_p such that $p \in A_{t_p}$, meaning that $A \subseteq \cup_{k=1}^{\infty} A_k$. As a result,

$$\mathbf{P}\left[p \in A\right] \le \mathbf{P}\left[p \in \bigcup_{k=1}^{\infty} A_k\right] \le \sum_{k=1}^{\infty} \mathbf{P}\left[p \in A_k\right]$$

To complete the proof we show that $\mathbf{P}\left[p \in A_k\right] = 0$ for all k. Notice that $p \in A_k$ implies that for $t \ge k$, the estimator θ must always have a value $\theta_t(i)$ close to p. Using this intuition we define the set

$$B_k = \{p \in [0,1] : \text{for all } t \ge k,\ t^{1+c} \min_{0 \le i \le t} |\theta_t(i) - p| \le 1\}$$

We now show that $A_k \subseteq B_k$. Since $p \in A_k$ we have that for all $t \ge k$

$$t^{1+c} \min_{0 \le i \le t} |\theta_t(i) - p| \sum_{i=0}^{t} \binom{t}{i} p^i (1-p)^{t-i} \le t^{1+c} \sum_{i=0}^{t} \binom{t}{i} |\theta_t(i) - p|\, p^i (1-p)^{t-i} = t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \le 1/2.$$

Thus, $\mathbf{P}[p \in A_k] \leq \mathbf{P}[p \in B_k]$. At first we write the set B_k in the following equivalent form

$$B_k = \cap_{t=k}^{\infty} \{p \in [0,1] : \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\}$$

As a result,

$$\mathbf{P}[p \in B_k] \leq \mathbf{P}\left[\min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\right], \text{for all } t \geq k$$

Each value $\theta_t(i)$ "covers" length $1/t^{1+c}$ from its left and right, as shown in Figure 2, and since there are at

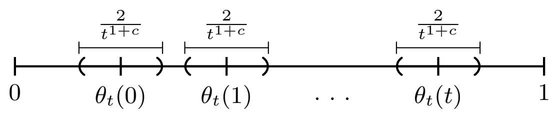

Fig. 2: Estimator output at time t

most $t+1$ such values we have for all $t \geq k$ the set

$$\{p \in [0,1] \ : \ \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\} = \bigcup_{i=0}^{t} \left(\theta_t(i) - \frac{1}{t^{1+c}},\ \theta_t(i) + \frac{1}{t^{1+c}}\right).$$

For each interval in the above union we have that $\mathbf{P}\left[|\theta_t(i) - p| \leq 1/t^{1+c}\right] \leq 2/t^{1+c}$ and by the union bound we get $\mathbf{P}[p \in B_k] \leq 2(t+1)/t^{1+c}$, for all $t \geq k$. We conclude that $\mathbf{P}[p \in B_k] = 0$.

Remark 1. The only point that we use that the update rules are *opinion dependent* is in Lemma 5. It is not diffcult to see that the reduction still holds if the update rules also depend on the indices of the neighbors that an agent meets or if agents i, j with $s_i = s_j$ adopt different update rules. As a result, Theorem 3 still applies.

6 Faster Update Rules

We already discussed that the reason that opinion dependent dynamics suffer slow convergence is that the update rule depends only on the expressed opinions. In this section we provide an update rule showing that information about the graph G combined with agents that do not act selfishly, can restore the exponential convergence rate. Our update rule, depends not only on the expressed opinions of the agents but also on their indices and matrix P. Having this knowledge, one could try to come up with an update rule resembling the original update rule of the FJ model. In update rule 2, each agent could store the *most recent* opinions of the random neighbors that she meets in an array and then update her opinion according to their weighted sum (each agent knows row i of P). The problem with this approach is that the opinions of the neighbors that she keeps in her array are *outdated*, i.e. the opinion of neighbor of agent i is different than what she expressed in their last meeting. The good news are that as long as this outdatedness is bounded we can still achieve exponential convergence to the equilibrium. By bounded outdatedness we mean that there exists a number of rounds B such that all agents have met all their neighbors at least once from t to $t + B$.

Remark 2. It is necessary to know the matrix P in order for this update rule to work. We first observe that the lower bound of Section 5 also holds in case the algorithm learns the index of the chosen neighbor. The reason is that the reduction involves only two neighbors with different opinions, so they are distinguishable. Therefore, if we tried to learn P by observing the frequencies of the indices of the neighbors and run update rule 2 with the empirical frequencies instead of the p_{ij}, our lower bound ensures that the rate of convergence would not be $O(1/t^{1+c})$ for any $c > 0$. Intuitively, if we know P then the algorithm converges exponentially, since the slow part of the process is learning the probabilities p_{ij} precisely.

In [BT97], they show a convergence rate guarantee for 2 assuming that there exists a such a window B. In the following we briefly summarize their result. For completeness we give here a prove tailored for our purposes. Using a simple induction we get that bounded outdatedness preserves the exponential convergence.

Algorithm 2 Asynchronous Update Rule

1: Initially $x_i(0) = s_i$ for all agent i.
2: Each agent i keeps an array M_i of length d_i.
3: At round $t \geq 1$ each agent i:
  4: $x_i(t) = (1 - \alpha_i) \sum_{j=1}^{d_i} p_{ij} M_i[j] + \alpha_i s_i$
  5: Meets neighbor W_i^t and learns the opinion $x_{W_i^t}(t)$.
  6: $M_i[W_i^t] \leftarrow x_{W_i^t}(t-1)$.

Lemma 7. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t) \in \mathbf{N}$ be the most recent round before round t, that agent i met agent j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t) \leq t - 1$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1-\rho)^k$.*

Proof. To prove our claim we use induction on k. For the induction base $k = 1$,

$$\begin{aligned}|x_i(t) - x_i^*| &= |(1-\alpha_i) \sum_{j \in N_i} p_{ij}(x_j(\pi_{ij}(t)) - x_j^*)| \\ &\leq (1-\alpha_i) \sum_{j \in N_i} p_{ij} |(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1-\rho)\end{aligned}$$

From the induction hypothesis we have for $\pi_{ij}(t) \geq (k-1)B$, that $|x_j(\pi_{ij}(t)) - x_j^*| \leq (1-\rho)^{k-1}$. For $k \geq 2$, we again have that $|x_i(t) - x_i^*| \leq (1-\rho) \sum_{j \in N_i} p_{ij} |(x_j(\pi_{ij}(t)) - x_j^*)|$. Since $t - B \leq \pi_{ij}(t)$ and $t \geq kB$, we have that $\pi_{ij}(t) \geq (k-1)B$ and the induction hypothesis applies.

In our randomized setting there does not exist fixed length window is not true but we can easily adapt this to hold with high probability. To do this observe that agent i simply needs to wait to meet the neighbor j with the smallest weight p_{ij}. Therefore, after $\log(1/\delta)/\min_j p_{ij}$ rounds we have that with probability at least $1 - \delta$ agent i met all her neighbors at least once. Since we want this to be true for all agents we shall roughly take $B = 1/\min_{p_{ij}>0} p_{ij}$. In Section D of the Appendix we give the detailed argument that leads to the Lemma 8, showing that the convergence rate of update rule 2 is exponential.

Lemma 8. *Let $x(t)$ be the dynamics corresponding to update rule 2. We have*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2 \exp\left(-\rho(1-\rho) \min_{ij} p_{ij} \frac{\sqrt{t}}{4 \ln(nt)}\right)$$

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A Fictitious Play Convergence Rate

We give here the proof of the following technical lemma that we used to derive an upper bound on the rate of convergence of the fictitious play dynamics in Section 3. We restate Lemma 2 for completeness.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion*

$$e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \text{ and } e(0) = \|x(0) - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\ln(Dt^{2.5})}{t}}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2.5\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\, 1/2)}}$.

Proof. Observe that for all $t \geq 0$ the function $e(t)$ the following recursive relation

$$e(t+1) = e(t)\left(1 - \frac{\rho}{t+1}\right) + \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \tag{11}$$

For $t = 0$ we have that

$$e(1) = (1-\rho)e(0) + \delta(1) = (1-\rho)e(0) + \sqrt{\ln D} \tag{12}$$

Observe that for $D > \mathrm{e}^2$, $\delta(t)$ is decreasing for all $t \geq 1$. Therefore, $\delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \leq \frac{\delta(t)}{t+1}$ and from equations (11) and (12) we get that for all $t \geq 0$

$$e(t+1) \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{\ln(D(t+1)^2)}}{(t+1)^{3/2}} \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{2\ln(D(t+1))}}{(t+1)^{3/2}}$$

Now let $g(t) = \frac{\sqrt{2\ln(Dt)}}{t^{3/2}}$ to obtain for all $t \geq 1$

$$\begin{aligned}
e(t) &\leq (1 - \frac{\rho}{t})e(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})(1 - \frac{\rho}{t-1})e(t-2) + (1 - \frac{\rho}{t})g(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})\cdots(1-\rho)e(0) + \sum_{\tau=1}^{t} g(\tau) \prod_{i=\tau+1}^{t}(1 - \frac{\rho}{i}) \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau) e^{-\rho\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau) e^{-\rho(H_t - H_\tau)} \\
&\leq \frac{e(0)}{t^\rho} + e^{-\rho H_t}\sum_{\tau=1}^{t} g(\tau) e^{\rho H_\tau} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2}}{t^\rho}\sum_{\tau=1}^{t} \tau^\rho \frac{\sqrt{\ln(D\tau)}}{\tau^{3/2}} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2\ln D}}{t^\rho}\sum_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}}
\end{aligned}$$

We observe that

$$\sum_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}} \leq \int_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}} \mathrm{d}\tau \tag{13}$$

since, $\tau \mapsto \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}}$ is a decreasing function of τ for all $\rho \in [0, 1]$.

– If $\rho \leq 1/2$ then

$$\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} \mathrm{d}\tau \leq \sqrt{\ln t} \int_{\tau=1}^{t} \frac{1}{\tau} \mathrm{d}\tau = (\ln t)^{3/2}$$

– If $\rho > 1/2$ then

$$\begin{aligned}
\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} \mathrm{d}\tau &= \int_{\tau=1}^{t} \tau^{\rho-1/2} \frac{\sqrt{\ln \tau}}{\tau} d\tau \\
&= \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-1/2} ((\ln \tau)^{3/2})' \mathrm{d}\tau \\
&= \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2} - (\rho - 1/2) \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-3/2} (\ln \tau)^{3/2} \mathrm{d}\tau \\
&\leq \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2}
\end{aligned}$$

Theorem 7. *Let $I = (P, s, \alpha)$ be any instance of the opinion formation game of Definition* **??** *with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t)$ produced by update rule 4 after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

Proof. By Lemma 1 we have that for all $t \geq 1$ and $p \in [0,1]$,

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e_p(t)\right] \geq 1 - p$$

where $e_p(t)$ is the solution of the recursion, $e_p(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Setting $p = \frac{1}{12\sqrt{t}}$ we have that

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - \frac{1}{12\sqrt{t}}$$

where $e(t)$ is the solution of the recursion $e(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(2\pi^2 n t^{2.5})}{t}}$. Since $2\pi^2 \geq e^{2.5}$, Lemma 2 applies and $e(t) \leq C \sqrt{\log n} \frac{\log t^{3/2}}{t^{\min(\rho,1/2)}}$ for some universal constant C. Finally,

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq \frac{1}{12\sqrt{t}} + (1 - \frac{1}{12\sqrt{t}}) C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho,1/2)}} \leq (C + \frac{1}{12}) \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho,1/2)}}$$

B Lower bound for no-regret Dynamics

In the following Lemma we show how we can use algorithm A to construct an estimator $\hat{\theta_A}$ for Bernoulli distributions. We restate 5 for completeness.

Theorem 8. *Let the no-regret algorithm A such that for all instances I, $\lim_{t\to\infty} t^{1+c} \mathbf{E}_I\left[\|x_A(t) - x^*\|_\infty\right] = 0$. Then there exists an estimator $\hat{\theta_A}$ such that for all $p \in [0,1]$,*

$$\lim_{t\to\infty} t^{1+c} R_p(t) = 0$$

Proof. At first we remind that an estimator $\hat{\theta}$ is a sequence of functions $\{\hat{\theta}_t\}_{t=1}^{\infty}$, where $\theta_t : \{0,1\}^t \mapsto [0,1]$. We construct such a sequence using the algorithm A. We also remind that when an agent i runs algorithm A, she selects $x_i(t)$ according to the cost functions $\{C_i^\tau\}$ that she has already received

$$x_i(t) = A_t(C_i^1, \ldots, C_i^{t-1})$$

Consider an agent i with $a_i = 1$ and $s_i = 0$ that runs A. Then $C_i^t(x) = x^2$ for all t and $x_i(t) = A_t(x^2, \ldots, x^2)$. The latter means that $x_i(t)$ only depends on t, $x_i(t) = h_0(t)$. Equivalently, if $a_i = 1$ and $s_i = 1$ then $x_i(t) = A_t((1-x)^2, \ldots, (1-x)^2)$ and $x_i(t) = h_1(t)$. Finally, consider an agent i with $a_i = 1/2$ and $s_i = 0$. In this case $C_i^t = \frac{1}{2}x^2 + \frac{1}{2}(x - y_t)^2$, where $y_t \in [0,1]$ is the opinion of the neighbor $j \in N_i$ that i met at round t. As a result, $x_i(t) = A_t(\frac{1}{2}x^2 + \frac{1}{2}(x-y_1)^2, \ldots, \frac{1}{2}x^2 + \frac{1}{2}(x - y_{t-1})^2) = f_t(y_1, \ldots, y_{t-1})$. The estimator $\hat{\theta_A}$ is the following sequences $\{\hat{\theta}_t\}_{t=1}^{\infty}$

$$\hat{\theta}_t(Y_1, \ldots, Y_t) = 2f_{t+1}(h_{Y_1}(1), \ldots, h_{Y_t}(t))$$

Observe that $\hat{\theta}_t : \{0,1\}^t \mapsto [0,1]$ meaning that $\hat{\theta_A}$ is a valid estimator for Bernoulli distributions. Now for any $p \in [0,1]$, we construct an appropriate instance I_p s.t. $R_p(t) = \mathbf{E}_p\left[|\hat{\theta}_t - p|\right] \leq 2\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right]$. Consider the graph of Figure 1, which has a central node with $a_c = 1/2$ and $s_c = 0$ and two leaf nodes $1, 2$ with $a_1 = a_2 = 1$, $s_1 = 1$ and $s_2 = 0$. The weights are $p_{c1} = p$ and $p_{c2} = 1 - p$. Obviously, nodes 1 and 2 always have constant opinions, 1 and 0 respectively. Hence, in each round the center node receives either $h_1(1)$ with probability p or $h_2(0)$ with probability $1 - p$.

We just need to prove that in I_p, $\mathbf{E}_p\left[|\hat{\theta}_t - p|\right] \leq \frac{1}{2}\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right]$. Notice that $x_c^* = \frac{p}{2}$ and $x_i^* = s_i$ if $i \neq c$. .
At round t, if the oracle returns to the center agent the value $h_1(t)$ of agent-1, then $Y_t = 1$ otherwise $Y_t = 0$. As a result, $\mathbf{P}\left[Y_t = 1\right] = p$ and

$$\begin{aligned}
\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right] &\geq \mathbf{E}_{I_p}\left[|x_c^t - x_c^*|\right] \\
&= \mathbf{E}_{I_p}\left[\left|f_{t+1}(h_{Y_1}(1), \ldots, h_{Y_t}(t)) - \frac{p}{2}\right|\right] \\
&= \mathbf{E}_p\left[\left|\frac{\hat{\theta}_t}{2} - \frac{p}{2}\right|\right] = \frac{1}{2}R_p(t)
\end{aligned}$$

and the result follows.

We next give a rigorous measure-theoretic proof of Theorem 3.

Theorem 9. *Let $\theta_t : \{0,1\}^t \to [0,1]$ be a sequence of estimators for the success probability p of a Bernoulli random variable with distribution P. There exists $p \in [0,1]$ such that*

$$\lim_{t\to\infty} t^2\ \mathbf{E}_{X\sim P^t}\left[|\theta_t(X) - p|\right] > 0.$$

Proof. Observe that $\theta_t(\{0,1\}^t)$ has cardinality at most 2^t. Since

$$\sum_{0\leq i\leq t}\sum_{\|x\|_1 = i} |\theta_t(x) - p|\, p^i(1-p)^{t-i} \geq \sum_{0\leq i\leq t}\binom{t}{i}\left|\frac{\sum_{\|x\|_1=i}\theta_t(x)}{\binom{t}{i}} - p\right| p^i(1-p)^{t-i}.$$

Thus, without loss of generality we assume that $\theta_t(\{0,1\}^t)$ contains at most $t+1$ discrete points.

In the following, we work in the measure space $(\mathbf{R}, \mathcal{M}, \mu)$, where μ is the Lebesgue measure, and $\mathcal{M}$ is the σ-algebra of the Lebesgue measurable sets. Suppose that there exists no such $p \in [0,1]$. Let

$$A = \{p \in [0,1]\ :\ \lim_{t\to\infty} t^2\ \mathbf{E}_{X\sim P^t}\left[|\theta_t(X) - p|\right] = 0\}.$$

Then $A = [0,1]$ and A is measurable as an interval. Notice that,

$$A \subseteq \bigcup_{t=1}^{\infty}\bigcap_{k=t}^{\infty} A_k,$$

where $A_k = \{p \in [0,1]\ :\ R_k(p) < 1/2\}$, and $R_k(p) = k^2\ \mathbf{E}_{X\sim P^k}\left[|\theta_k(X) - p|\right]$. We have that $R_k : [0,1] \to [0, +\infty)$ is polynomial of degree t in p and therefore it is a measurable function. Thus, A_k is measurable. We now show that

$$A_k \subseteq B_k \coloneqq \{p \in [0,1]\ :\ k^2 \min_{0\leq i\leq k} |\theta_k(i) - p| < 1\}.$$

We prove this by contradiction. Suppose that $p \in A_k$ but $p \notin B_k$. Since $p \in A_k$ we have that

$$R_k(p) = k^2 \sum_{i=0}^{k} \binom{k}{i} |\theta_k(i) - p| \, p^i (1-p)^{k-i} \geq k^2 \min_{0 \leq i \leq k} |\theta_k(i) - p| \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} \geq 1.$$

Since the functions $p \mapsto k^2 |\theta_k(i) - p|$ are measurable, their pointwise minimum is measurable and therefore the sets B_k are also measurable. We next proceed to bound $\mu(B_k)$. Since θ_k can only take k different values we have that there exist $k+1$ intervals (a_{k_i}, b_{k_i}) of length at most $2/k^2$ such that $B_k = \bigcup_{i=0}^{k} (a_{k_i}, b_{k_i})$. Since μ is subadditive we have

$$\mu(B_k) \leq \sum_{i=0}^{k} \frac{2}{k^2} = \frac{2(k+1)}{k^2}.$$

Now observe that

$$\mu(A) \leq \mu\left(\bigcup_{t=1}^{\infty} \bigcap_{k=t}^{\infty} A_k\right) \leq \mu\left(\bigcup_{t=1}^{\infty} \bigcap_{k=t}^{\infty} B_k\right) \leq \sum_{t=1}^{\infty} \mu\left(\bigcap_{k=t}^{\infty} B_k\right) \leq \sum_{t=1}^{\infty} \lim_{k \to \infty} \mu(B_k) = 0$$

Which is a contradiction since we assumed $A = [0, 1]$.

C Faster Update Rule

We are now going to state and prove a series of Lemmas that culminate in the proof of Lemma 8. We first turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdateness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 10. *Suppose that the collector picks coupons with mixed probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\textit{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

For convenience of reasoning, we will divide the time in "epochs". The lenght of each epoch is B, so times 1 to B belong to the first epoch etc. The next lemma is a calculation of the appropriate size of B.

Lemma 11. *Suppose we run Algorithm 2 for t rounds. If the size of each epoch is*

$$B = \frac{2}{\min_{ij} p_{ij}} \ln \frac{nt}{p}$$

then with probability at least $1 - p$ all agents pich all their neighbours at least once in every epoch.

Proof. In our setting, coupon i corresponds to the selection of neighbour i. Each node is a collector and wants to gather all $n-1$ coupons during each epoch. Suppose $d = \max_i d_i$ is the maximum degree of the graph. Then, if we set $c = \ln(\frac{nt}{p})$, using Lemma 12 we get that a node hasn't seen at least one neigbour after $c/w + \ln d/w$ samples with probability at most $\frac{p}{nt}$. This means that if we set $D = c/w + \ln d/w = \ln \frac{nt}{p}/w + \ln d/w \geq 2/w \ln \frac{nt}{p}$ when p is small enough, then the probability that a specific agent at a specific epoch hasn't collected all neighbouring opinions at least once is at most $\frac{p}{nt}$. By a simple union bound argument, we get that all agents have seen all their neighbours during all epochs with probability at least $1 - p$.

Since all neigbours are picked at least once during each epoch, the outdateness of each agent is at most twice the length of the epoch. Combining this with Lemma 7 we have the following.

Corollary 1. *If we run Algorithm 2 for t rounds, then with probability at least $1 - p$*

$$\|x^t - x^*\|_\infty \leq (1-\rho)^{\frac{t}{2B}} \leq \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{p})}\right)$$

We now prove Lemma 8 using the previous results.

Proof. Let $u(t) = \|x^t - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 2 we obtain:

$$\mathbf{P}\left[u(t) > \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)\right] \leq p$$

for every probability $p \in [0,1]$. Also, since all the parameters of the problem lie in $[0,1]$, we have

$$\mathbf{E}\left[u(t)|u(t) > r\right] \leq 1$$

Now, by the conditional expectations identity, we get:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &= \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \leq r\right]\mathbf{P}\left[u(t) \leq r\right] \\ &\leq p + r\end{aligned}$$

where $r = \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$. If we set $p = \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \leq \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right) + \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$$

We now evaluate r for our choice of probability p:

$$\begin{aligned} r &= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{p}\right)}\right) \\ &= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{\exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)}\right)}\right) \\ &= \exp\left(-\frac{\rho w t}{2\ln nt + 2\frac{\rho w\sqrt{t}}{2\ln nt}}\right) \\ &\leq \exp\left(-\frac{\rho w t}{4\ln(nt)\sqrt{t}}\right) \\ &= \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \end{aligned}$$

Using the previous calculation, we obtain:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &\leq \exp\left(-\frac{\rho w\sqrt{t}}{2\ln(nt)}\right) + \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\ &\leq 2\exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\ &= 2\exp\left(-\rho\min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)\end{aligned}$$

D Faster Update Rule

We are now going to state and prove a series of Lemmas that culminate in the proof of Lemma 8. We first turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdateness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 12. *Suppose that the collector picks coupons with mixed probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\textit{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

For convenience of reasoning, we will divide the time in "epochs". The lenght of each epoch is B, so times 1 to B belong to the first epoch etc. The next lemma is a calculation of the appropriate size of B.

Lemma 13. *Suppose we run Algorithm 2 for t rounds. If the size of each epoch is*

$$B = \frac{2}{\min_{ij} p_{ij}} \ln \frac{nt}{p}$$

then with probability at least $1-p$ all agents pich all their neighbours at least once in every epoch.

Proof. In our setting, coupon i corresponds to the selection of neighbour i. Each node is a collector and wants to gather all $n-1$ coupons during each epoch. Suppose $d = \max_i d_i$ is the maximum degree of the graph. Then, if we set $c = \ln(\frac{nt}{p})$, using Lemma 12 we get that a node hasn't seen at least one neigbour after $c/w + \ln d/w$ samples with probability at most $\frac{p}{nt}$. This means that if we set $D = c/w + \ln d/w = \ln \frac{nt}{p}/w + \ln d/w \geq 2/w \ln \frac{nt}{p}$ when p is small enough, then the probability that a specific agent at a specific epoch hasn't collected all neighbouring opinions at least once is at most $\frac{p}{nt}$. By a simple union bound argument, we get that all agents have seen all their neighbours during all epochs with probability at least $1-p$.

Since all neigbours are picked at least once during each epoch, the outdateness of each agent is at most twice the length of the epoch. Combining this with Lemma 7 we have the following.

Corollary 2. *If we run Algorithm 2 for t rounds, then with probability at least $1-p$*

$$\|x^t - x^*\|_\infty \leq (1-\rho)^{\frac{t}{2B}} \leq \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{p})}\right)$$

We now prove Lemma 8 using the previous results.

Proof. Let $u(t) = \|x^t - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 2 we obtain:

$$\mathbf{P}\left[u(t) > \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)\right] \leq p$$

for every probability $p \in [0,1]$. Also, since all the parameters of the problem lie in $[0,1]$, we have

$$\mathbf{E}\left[u(t) | u(t) > r\right] \leq 1$$

Now, by the conditional expectations identity, we get:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &= \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \leq r\right]\mathbf{P}\left[u(t) \leq r\right] \\ &\leq p + r\end{aligned}$$

where $r = \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$. If we set $p = \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \leq \exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right) + \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$$

We now evaluate r for our choice of probability p:

$$
\begin{aligned}
r &= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{p}\right)}\right) \\
&= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{\exp\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)}\right)}\right) \\
&= \exp\left(-\frac{\rho w t}{2\ln nt + 2\frac{\rho w\sqrt{t}}{2\ln nt}}\right) \\
&\le \exp\left(-\frac{\rho w t}{4\ln(nt)\sqrt{t}}\right) \\
&= \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}
$$

Using the previous calculation, we obtain:

$$
\begin{aligned}
\mathbf{E}\left[u(t)\right] &\le \exp\left(-\frac{\rho w\sqrt{t}}{2\ln(nt)}\right) + \exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&\le 2\exp\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&= 2\exp\left(-\rho \min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}
$$