Contribution Title*

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1 Introduction

1.1 Friedkin-Johsen Model and Opinion Formation Games

In the Friedkin-Johsen model (FJ-model) an undirected graph $G(V,E)$ with n nodes, is assumed, where V denotes the agents and E the social relations between them. Each agent i poses an internal opinion $s_i \in [0,1]$ and a self confidence coefficient $\alpha_i \in [0,1]$. At each round $t \geq 1$, agent i updates her opinion as follows:

$$x_i(t) = (1-\alpha_i)\frac{\sum_{j \in N_i} x_j(t-1)}{|N_i|} + \alpha_i s_i, \tag{1}$$

where N_i is the set of her neighbors. The simplicity of the update rule (1) makes the FJ-model plausible, because in real social networks it is very unlikely that agents change their opinions according to complex rules. Based on the FJ-model, in [?], they propose an *opinion formation game* in which the strategy that each agent i plays, is the opinion $x_i \in [0,1]$ that she publicly expresses incurring her a cost

$$C_i(x_i, x_{-i}) = (1-a_i)\sum_{j \in N_i}(x_i - x_j)^2 + a_i(x_i - s_i)^2. \tag{2}$$

In [?] they prove that it always admits a unique Nash Equilibrium x^* and has Price of Anarchy $9/8$ if G is undirected and $O(n)$ in the directed case. We denote an instance of this game as $I = (G,s,a)$. The FJ-model can be viewed as the best-response dynamics of the of the opinion formation game. More precisely, each agent i has an opinion $x_1(0), \ldots, x_n(0)$ and at round $t \geq 1$, suffers the cost (3) and updates her opinion so as to minimize her individual cost

$$x_i(t) = \operatorname*{argmin}_{x \in [0,1]}(1-a_i)\sum_{j \in N_i}(x - x_j(t-1))^2 + a_i(x - s_i)^2 \tag{3}$$

In [?] it is proved that for any instance $I = (G,s,a)$ the opinion vector $x(t) = (x_1(t), \ldots, x_n(t))$ converges to the unique equilibrium point x^*. Moreover, this is achieved by an update rule that is very simple but more importantly it is *rational*, in the sense that each agent i adopts this rule in order to minimize her individual cost.

1.2 Opinion Formation Games with Random Payoffs

In the game defined by (2), agent's i cost $C_i(x_i, x_{-i})$ is a deterministic function of the opinion vector x. Many recent works (see e.g. [?]) study games with random payoffs, that is agent's i cost $(C_i(x_i, x_{-i}))$ is a random variable. The random payoff setting can be much more realistic, since randomness may naturally occur because of incomplete information, noise or other stochastic factors. Motivated by this line of research we introduce a random payoff variant of the opinion formation game (2).

Definition 1. *Let $I = (G,s,a)$ an instance of the opinion formation game and x the opinion vector. Each agent i,*

- *picks uniformly at random one of her neighbors $j \in N_i$*
- *suffers cost $C_i(x_i, x_{-i}) = (1-a_i)\sum_{j \in N_i}(x_i - x_j)^2 + a_i(x_i - s_i)^2$*

This game is more compatible to a realistic settings since in real world networks (e.g. Facebook, Twitter e.t.c.), each agent may have several hundreds of friends. As a result is far more reasonable to assume that every day, each agent meets a random small subset of her acquaintances and suffers a cost based on how much she disagrees with them. Since the expected cost of our random payoff variant equals the cost of game (2), $\mathbf{E}[C(x_i, x_{-i})] = (1-a_i)\sum_{j \in N_i}(x_i - x_j)^2 + a_i(x_i - s_i)^2$, the results about the existence of a unique equilibrium, and the PoA bounds also hold in our variant. However, the best-response dynamics in the random payoff game does not converge to the equilibrium. In this work, we investigate whether there exists natural dynamics that leads the system to the equilibrium point x^*.

1.3 Our Results and Techniques

In this work we propose the following dynamics for the random payoff game of Definition 1.

- Initially, each agent i has an opinion $x_1(0), \dots, x_n(0)$.
- At round $t \geq 1$, each agent i meets uniformly at random one of her friends $j \in N_i$, suffers cost $C_i^t(x_i(t-1), x_j(t-1)) = (1-a_i)(x_i(t-1) - x_j(t-1))^2 + a_i(x_i(t-1) - s_i)^2$ and updates her opinion

$$x_i(t) = \operatorname*{argmin}_{x \in [0,1]} \sum_{\tau=1}^{t} C_i^\tau(x, x_{\pi_i(\tau)}(\tau - 1)) \tag{4}$$

  where $\pi_i(\tau)$ denotes the index of the index of the neighbor that i selected at round τ.

The above dynamics is the *fictitious play* in the game defined by the instance $I = (G, s, a)$. Generally speaking *fictitious play* does guarantees convergence to the equilibrium. We show that for our game with random payoffs fictitious play converges to equilibrium, giving the following bound for its convergence rate

Theorem 1. *For every instance $I = (G(V, E), s, a)$, if $x(t)$ is the opinion vector generated by the update rule (4) then*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq C\sqrt{\log|V|}\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

The update rule (4) guarantees convergence while vastly reducing the information exchange between the agents at each round. Namely, each agent learns the opinion of only one of her neighbors whereas in the classical FJ-model (1) each agent learns the opinions of all of her neighbors. In terms of total communication needed to get within distance ε of the equilibrium, the update rule (4) needs $O(|V| \log |V|)$ communication while (1) needs $O(|E|)$. Of course for this difference to be significant we need each agent to have at least $O(\log |V|)$ neighbors. A large social network like Facebook has approximately 2 billion users and each user has usually more that 100 friends which far more than $\log(2\ 10^9)$.

Apart from converging to the equilibrium, our update rule (4) is also a rational behavioral assumption since it ensures *no regret* for the agents. Having no-regret means that the average cost for each agent i after T rounds is close to the average cost that she would suffer by expressing any fixed opinion. This is a very important feature of our update rule because even players that selfishly will to minimize their incurred cost, could choose to play according to it.

Theorem 2. *For every instance $I = (G, s, a)$, for every agent i*

$$\sum_{\tau=1}^{t} C_i^\tau(x_i(\tau)) \leq min_{x \in [0,1]} \sum_{\tau=1}^{t} C_i^\tau(x) + O\left(\log t\right)$$

Even though our update rule (4) has the above desired properties it only achieves convergence rate of $\widetilde{O}(1/\sqrt{t})$ for a fixed instance I with $\rho > 1/2$. For such an instance the original FJ-model outperforms our update rule since it achieves convergence rate $O(1/2^t)$. We investigate whether this gap is due to an inherent characteristic of the random payoff setting i.e. learning the opinion of just one random neighbor vs learning all of them or we could find another natural dynamics that converges exponentially fast. Can the agents select another no regret algorithm that guarantees an asymptotically faster convergence rate? In Section 4 we investigate the following question

Question 1. Is there a no regret algorithm that the agents can choose such that for any instance $I = (G, s, a)$, $\mathbf{E}\left[\|x_t - x^*\|_\infty\right] = O(1/t)$?

To answer this question we first show that the existence of a no regret algorithm (that achieves this convergence rate to the equilibrium) implies the existence of an algorithm that uses i.i.d samples from a Bernoulli random variable $B(p)$ to estimate its success probability p with the same asymptotic error rate. We use sample complexity lower bound techniques from statistics to show that such a no regret algorithm is unlikely to exist. We show that updating the opinion according to no regret algorithms leads to update rules that are totally ignorant of the specific graph structure G. Perhaps surprisingly, in Section 5 we present an update rule that takes into account only the maximum degree of the graph and for any fixed instance I achieves convergence rate $O(1/2^{\sqrt{t}})$.

1.4 Related Work

2 Constant Memory with full window

In this work we investigate an variant of the above process. We denote by V the set of agents, and $N_i \subset V$ the set of neighbors of agent i. Let $x(t) \in [0,1]^n$ be the opinion vector at round t. We denote by $x_i(t)$ the opinion of agent i at round t. At round $t+1$, $x(t+1)$ is constructed in the following way. At first, each agent i gets to see the opinion $x_j(t)$, where $j \in N_i$ is picked uniformly at random from the set of her neighbors. Let W_i^t be the random variable corresponding to the selected neighbor. Now each agent $i \in V$ updates her opinion as follows

$$x_i(t+1) = (1-\alpha_i)\frac{\sum_{\tau=1}^{t+1} x_{W_i^\tau}(\tau-1)}{t+1} + \alpha_i s_i.$$

We remark that each agent i get to see *only* the opinion $x_{W_i^t}$ and not the label W_i^t of her neighbor.

Let $x^* \in [0,1]^n$ be the unique equilibrium point of the given instance I. We prove that the above stochastic process has the following convergence rate to x^*.

Theorem 3.

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

Proof. Setting $p = 1/\sqrt{t}$ in Lemma 3 yields the result.

We start by stating the standard Hoeffding bound

Lemma 1 (Hoeffding's Inequality). *Let $X = (X_1 + \cdots + X_t)/t$, $X_i \in [0,1]$. Then,*

$$\mathbf{P}\left[|X - \mathbf{E}\left[X\right]| > \lambda\right] < 2e^{-2t\lambda^2}.$$

Lemma 2. *With probability at least $1-p$, $\|x^t - x^*\|_\infty \leq e(t)$, where $e(t)$ satisfies the following recursive relation*

$$e(t) = \delta(t) + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \text{ and } e(0) = \|x^0 - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 nt^2/(6p))}{t}}$.

Corollary 1. *The function $e(t)$ satisfies the following recursive relation*

$$e(t+1) - e(t) + \alpha\frac{e(t)}{t+1} = \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1}$$

Proof. As we have already mentioned for any instance I there exists a unique equilibrium vector x^*. Since $W_i^\tau \sim \mathrm{U}(N_i)$ we have that $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \frac{\sum_{j\in N_i} x_j^*}{|N_i|}$. Since W_i^τ are independent random variables, we can use Hoeffding's inequality (Lemma 1) to get

$$\mathbf{P}\left[\left|\frac{\sum_{\tau=1}^t x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] < \frac{6}{\pi^2}\frac{p}{nt^2},$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Therefore, by the union bound

$$\begin{aligned}
&\mathbf{P}\left[\text{for all } t \geq 1 : \max_{i\in V}\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \leq \\
&\sum_{t=1}^{\infty}\mathbf{P}\left[\max_{i\in V}\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| > \delta(t)\right] \leq \\
&\sum_{i=1}^{\infty}\frac{6}{\pi^2}\frac{1}{t^2}\sum_{i=1}^{n}\frac{p}{n} = p
\end{aligned}$$

As a result with probability $1-p$ we have that for all t

$$\left|\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} - \frac{\sum_{j\in N_i} x_j^*}{|N_i|}\right| \leq \delta(t) \tag{5}$$

We will our claim by induction. We assume that $\|x^\tau - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x_{W_i^\tau}(\tau-1)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau} + \sum_{\tau=1}^{t} e(\tau-1)}{t} + \alpha_i s_i \qquad (6)\\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=1}^{t} x^*_{W_i^\tau}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\frac{\sum_{j\in N_i} x_j^*}{|N_i|} + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \qquad (7)\\
&\leq x_i^* + \delta(t) + (1-\alpha)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}$$

We get (6) from the induction step and (7) from inequality (5). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\alpha)\frac{\sum_{\tau=1}^{t} e(\tau)}{t}$. As a result $||x_i(t) - x^*||_\infty \leq e(t)$.

In order to bound the convergence time of the system, we just need to bound the convergence rate of the function $e(t)$. The following lemma provides us with a simple upper bound for the convergence rate of our process.

Lemma 3. *Let $e(t)$ be a function satisfying the recursion of Corollary 1. Then with probability at least $1-p$ we have that*

$$e(t) = \begin{cases} O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^\alpha}\right) & \text{if } \alpha \leq 1/2 \\ O\left(\sqrt{\log(\frac{n}{p})}\frac{(\log t)^{3/2}}{t^{1/2}}\right) & \text{if } \alpha > 1/2 \end{cases}$$

Proof. At first, since $\frac{\delta(t)}{t}$ is a decreasing function for $p \leq 1/4$, we have that $e(t) \leq (1-\frac{\alpha}{t}) + g(t)$, where $g(t) = \frac{\delta(t)}{t}$.

$$\begin{aligned}
e(t) &\leq (1-\frac{\alpha}{t})e(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})(1-\frac{\alpha}{t-1})e(t-2) + (1-\frac{\alpha}{t})g(t-1) + g(t) \\
&\leq (1-\frac{\alpha}{t})\cdots(1-\alpha)e(0) + \sum_{\tau=1}^{t} g(\tau)\prod_{i=\tau+1}^{t}(1-\frac{\alpha}{i}) \\
&\leq \frac{e(0)}{t^{\alpha}} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^{\alpha}} + \sum_{\tau=1}^{t} g(\tau)e^{-\alpha(H_t - H_\tau)} \\
&\leq \frac{e(0)}{t^{\alpha}} + e^{-\alpha H_t}\sum_{\tau=1}^{t} g(\tau)e^{\alpha H_\tau} \\
&\leq \frac{e(0)}{t^{\alpha}} + \frac{O\left(\sqrt{\log(\frac{n}{p})}\right)}{t^{\alpha}}\sum_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}
\end{aligned}$$

We observe that

$$\sum_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}} \leq \int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau,$$

since $\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}$

- If $\alpha \leq 1/2$ then

$$\int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau \leq \int_{\tau=1}^{t}\tau^{1/2}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau = O((\log t)^{3/2}))$$

- If $\alpha > 1/2$ then

$$\begin{aligned}
\int_{\tau=1}^{t}\tau^{\alpha}\frac{\sqrt{\log\tau}}{\tau^{3/2}}d\tau &= \int_{\tau=1}^{t}\tau^{\alpha-1/2}\frac{\sqrt{\log\tau}}{\tau}d\tau \\
&= \frac{2}{3}\int_{\tau=1}^{t}\tau^{\alpha-1/2}((\log\tau)^{3/2})'d\tau \\
&= \frac{2}{3}(\log t)^{3/2} - (\alpha-1/2)\frac{2}{3}\int_{\tau=1}^{t}\tau^{\alpha-3/2}(\log\tau)^{3/2}d\tau \\
&= O\left((\log t)^{3/2}\right)
\end{aligned}$$

3 No Regret

We consider the following online convex optimization problem. At each time step t, the player i selects a real number x^t and a function $f^t(x)$ arrives. The player then suffers $f^t(x^t)$ cost. The functions $f^t(x)$ have the following form:

$$f^t(x) = \alpha(x-s_i)^2 + (1-\alpha)(x-a_t)^2$$

where $s_i, \alpha \in [0,1]$ and are independent of t and $a_t \in [0,1]$. In other words, the function f^t is uniquely determined by the number a_t.

We show that for this class of functions *fictitious play* admits no regret.

Theorem 4. *Let f^t be a sequence of functions, where each function has the form: $f^t(x) = \alpha(x - s_i)^2 + (1-\alpha)(x - a_t)^2$, where $s_i, \alpha \in [0,1]$ and are independent of t and $a_t \in [0,1]$. If we define $x^t = \arg min_{x\in[0,1]} \sum_{\tau=1}^{t-1} f^\tau(x)$ then for each time T*

$$\sum_{t=1}^{T} f^t(x^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + O(\log T)$$

In order to prove this claim, we will first define a similar rule y^t that takes into account the function f^t for the "prediction" at time t. Intuitively, this guarrantees that the rule admits no regret.

Lemma 4. *Let $y^t = argmin_{x\in[0,1]} \sum_{\tau=1}^{t} f^\tau(x)$ then*

$$\sum_{t=1}^{T} f^t(y^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x)$$

Proof. By definition of y^t, $min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) = \sum_{t=1}^{T} f^t(y^T)$, so

$$\begin{aligned}
\sum_{t=1}^{T} f^t(y^t) - min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) &= \sum_{t=1}^{T} f^t(y^t) - \sum_{t=1}^{T} f^t(y^T) \\
&= \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^T) \\
&\leq \sum_{t=1}^{T-1} f^t(y^t) - \sum_{t=1}^{T-1} f^t(y^{T-1}) \\
&= \sum_{t=1}^{T-2} f^t(y^t) - \sum_{t=1}^{T-2} f^t(y^{T-1})
\end{aligned}$$

Continuing in the same way, we get $\sum_{t=1}^{T} f^t(y^t) \leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x)$.

Now we can derive some intuition for the reason that *fictitious play* admits no regret. Since the cost incurred by the sequence y^t is at most that of the best fixed strategy, we can compare the cost incurred by x^t with that of y^t. However, for each t the numbers x^t and y^t are quite close and as a result the difference in their cost must be quite small.

Lemma 5. *For all t, $f^t(x^t) \leq f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-a)^2}{t^2}$.*

Proof. We first prove that for all t,

$$\left|x^t - y^t\right| \leq \frac{1-\alpha}{t} \tag{8}$$

.By definition $x^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1}$ and $y^t = \alpha s_i + (1-\alpha)\frac{\sum_{\tau=1}^{t} a_\tau}{t}$.

$$\begin{aligned}
\left|x^t - y^t\right| &= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau}{t-1} - \frac{\sum_{\tau=1}^{t} a_\tau}{t}\right| \\
&= (1-\alpha)\left|\frac{\sum_{\tau=1}^{t-1} a_\tau - (t-1)a_t}{t(t-1)}\right| \\
&\leq \frac{1-\alpha}{t}
\end{aligned}$$

The last inequality follows from the fact that $a_\tau \in [0,1]$. We now use inequality (8) to bound the difference $f^t(x^t) - f^t(y^t)$.

$$\begin{aligned}
f^t(x^t) &= \alpha(x^t - s_i)^2 + (1-\alpha)(x^t - a_t)^2 \\
&\leq \alpha(y^t - s_i)^2 + 2\alpha \left|y^t - s_i\right| \left|x^t - y^t\right| + \alpha \left|x^t - y^t\right|^2 \\
&\quad + (1-\alpha)(y_t - a_t)^2 + 2(1-\alpha)\left|y^t - a_t\right| \left|x^t - y^t\right| + (1-\alpha)\left|x^t - y^t\right|^2 \\
&\leq f^t(y^t) + 2\left|x^t - y^t\right| + \left|y^t - x^t\right|^2 \\
&\leq f^t(y^t) + 2\frac{1-\alpha}{t} + \frac{(1-\alpha)^2}{t^2}
\end{aligned}$$

Theorem 4 easily follows since:

$$\begin{aligned}
\sum_{t=1}^{T} f^t(x^t) &\leq \sum_{t=1}^{T} f^t(y^t) + \sum_{t=1}^{T} 2\frac{1-\alpha}{t} + \sum_{t=1}^{T} \frac{(1-\alpha)^2}{t^2} \\
&\leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + 2(1-\alpha)(\log T + 1) + (1-\alpha)\frac{\pi^2}{6} \\
&\leq min_{x\in[0,1]} \sum_{t=1}^{T} f^t(x) + O(\log T)
\end{aligned}$$

4 Lower Bound

Theorem 5. *Suppose $\hat{\theta}^t$ is an estimator for the probability p of a Bernoulli random variable that takes t samples. Then, for every interval $[a,b]$ which is contained in $[0,1]$ and for every $c > 0$, it holds that:*

$$\lim_{t\to\infty} t^{1+c} \int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = \infty$$

In order to prove this, we need the following result:

Lemma 6 (Fano's Inequality). *Suppose we have an estimator $\hat{\theta}^t$ for the probability of a Bernoulli distribution. The unknown distribution belongs to a family of distributions $\mathcal{P}$ with the property that the probabilities of every two distributions in this family differ by at least 2δ, where δ is a positive real number. Then, for every subset $P_1, P_2, ..., P_n$ of distributions from this family, where p_i is the probability of distribution P_1, the following inequality holds:*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right] \geq \delta\left(1 - \frac{I(X;V) + \log 2}{\log n}\right)$$

We are now going to use Theroem (6) to prove the following lemma, which provides a lower bound for the mean of the expectation of errors of the estimator.

Lemma 7. *Suppose we choose t Bernoulli distributions $\{P_i\}_{i=0}^{t-1}$ with the following probabilities:*

$$p_i = a + \frac{b-a}{2t} + i\frac{b-a}{t}, i = 0, ..., t-1$$

. Then:

$$\frac{1}{t}\sum_{i=1}^{t} \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right] \geq \frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right)$$

where $c_1, c_2 > 0$ and $c_1 > \frac{1}{2}$.

Proof. Without loss of generality, we can assume that t is a multiple of 5. For simplicity, we set $E_i = \mathbf{E}\left[\left|\hat{\theta}^t - p_i\right|\right]$. By the choice of p_i, we have that

$$\left|p_i - p_j\right| \geq \frac{b-a}{t}$$

, for every distinct i, j in the family. This means that for the family of t distributions that we picked, the property of Lemma (6) is satisfied for $\delta = \frac{b-a}{2t}$. Thus, by dividing the t distributions into groups of 5 and applying Lemma (6) to each one, we obtain:

$$\begin{aligned}\frac{\sum k = i^{i+4}}{5} &\geq \frac{b-a}{2t}\left(1 - \frac{I(X;V) + \log 2}{\log n}\right) \\ &= \frac{b-a}{2t}\left(1 - \frac{\log 2}{\log 5} - \frac{I(X;V)}{\log 5}\right) \\ &= \frac{b-a}{2t}\left(c_1 - \frac{I(X;V)}{\log 5}\right) \qquad (9)\end{aligned}$$

where $c_1 = 1 - \frac{\log 2}{\log 5} > \frac{1}{2}$. We are now going to upper bound the mutual information $I(X;V)$ for every group of 5 distributions. We denote by P_i^t the distribution on vectors with t coordinates, where each coordinate is sampled independently from P_i. Let $U = \{i, i+1, i+2, i+3, i+4\}$ be a family of 5 distributions. Using a well known inequality REFERENCE NEEDED HERE:

$$\begin{aligned}I(X;V) &\leq \frac{1}{5^2} \sum_{\substack{i,j \in U \\ i \neq j}} D_{kl}\left(P_i^t, P_j^t\right) \\ &\leq D_{kl}\left(P_i^t, P_{i+4}^t\right) \\ &= tD_{kl}\left(P_i, P_{i+4}\right) \\ &\leq t\frac{\left(p_i - p_{i+4}\right)^2}{p_i(1-p_i)} \\ &\leq t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\left(p_i - p_{i+4}\right)^2 \\ &= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\left(\frac{b-a}{2t} + i\frac{b-a}{t} - \frac{b-a}{t} - (i+4)\frac{b-a}{t}\right)^2 \\ &= t\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\frac{16(b-a)^2}{t^2} \\ &= \frac{16(b-a)^2}{t}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)\end{aligned}$$

So, if we set

$$c_2 = 16\frac{(b-a)^2}{\log 5}\left(\frac{1}{a(1-a)} + \frac{1}{b(1-b)}\right)$$

then by equation (9) we obtain:

$$\sum_{k=i}^{i+4} \geq \frac{5(b-a)}{2t}\left(c_1 - \frac{c_2}{t}\right) \qquad (10)$$

Inequality (10) holds for every group of 5 distributions. By summing up for all groups:

$$\begin{aligned}\sum_{i=0}^{t-1} E_i &\geq t\frac{b-a}{2t}\left(c_1 - \frac{c_2}{t}\right) \\ &= \frac{b-a}{2}\left(c_1 - \frac{c_2}{t}\right)\end{aligned}$$

which is what we wanted to prove.

Next, we state a lemma regarding an upper bound on the expectation of the error in each p_i.

Lemma 8. *For every $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$:*

$$\mathbf{E}\left[|\hat{\theta}^t - p_i|\right]_{p_i} \leq c_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p + |p - p_i|$$

where c_a^b is a constant that depends on a, b.

Now, we find an upper bound for the quantity $\sum_{i=0}^{t-1} E_i$. Our goal is to find an upper bound where the coefficient of $(b-a)/t$ is less than $1/4$.

Lemma 9.

$$\frac{1}{t}\sum_{i=0}^{t-1} E_i \leq \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right] dp + \frac{b-a}{4t}$$

Proof. By Lemma (8) we get that for every $p \in [-(b-a)/2t + p_i, p_i + (b-a)/2t]$:

$$E_i \leq c_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p + |p_i - p|$$

We integrate both sides of the equation with respect to p and get:

$$\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} E_i dp \leq c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} |p_i - p|\, dp$$

Hence:

$$\begin{aligned}\frac{b-a}{t} E_i &\leq c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + 2\int_{p_i - \frac{b-a}{2t}}^{p_i} |p_i - p|\, dp \\ &= c_a^b \int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + \frac{1}{4}\frac{(b-a)^2}{t^2}\end{aligned}$$

Therefore:

$$\frac{E_i}{t} \leq \frac{c_a^b}{b-a}\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + \frac{b-a}{4t^2} \tag{11}$$

By summing up Equation (11) for all t intervals, we get:

$$\begin{aligned}\frac{\sum_{i=0}^{t-1} E_i}{t} &\leq \frac{c_a^b}{b-a}\sum_{i=0}^{t-1}\int_{p_i - \frac{b-a}{2t}}^{p_i + \frac{b-a}{2t}} \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + \frac{b-a}{4t} \\ &= \frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp + \frac{b-a}{4t}\end{aligned}$$

Now, by combining Lemmas (7) and (9) we get:

$$\frac{c_a^b}{b-a}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp \geq \frac{b-a}{2t}\left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right)$$

By multiplying by t^{1+c} we get:

$$\frac{c_a^b}{b-a} t^{1+c}\int_a^b \mathbf{E}\left[|\hat{\theta}^t - p|\right]_p dp \geq \frac{b-a}{2} t^c\left(\left(c_1 - \frac{1}{2}\right) - \frac{c_2}{t}\right) \tag{12}$$

The coefficient of t^c in the right hand side of (12) is positive, so by sendig $t \to \infty$ we get:

$$\lim_{t\to\infty} t^{1+c}\int_a^b \mathbf{E}\left[\|\hat{\theta}^t - p\|_\infty\right] dp = \infty$$

5 Lower bound

Theorem 6. *Let P be a protocol run by each agent that produces vector output x^t after t iterations. Suppose that for every graph G and for every choice of parameters $s_i, \alpha_i \in [0,1]$ the error of the output is:*

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] = f(n,t)$$

where n is the number of nodes in the graph. Then,

$$\lim_{t\to\infty} \sqrt{t} f(t,t) > 0$$

Proof. Let G be a graph that consists of 2 star topologies. That is, we have 2 center nodes v_1, v_2. Each of the two center nodes has exactly t neighbours, which have degree 1. We will call these nodes the leaves of the graph. This graph has $2t+2$ nodes. We set $\alpha_i = 1$ for all the leaves and $\alpha_i = 1/2$ for the two centers. For exactly $t/2 + \sqrt{t}/2$ leaves of v_1 we set $s_i = 1$, for the rest $s_i = 0$. Similarly, for exactly $t/2 - \sqrt{t}/2$ leaves of v_2 we set $s_i = 1$, for the rest $s_i = 0$. By direct calculation, we get that the equilibrium point for this graph is the following:

$$x_i^* = \begin{cases} s_i & \text{if i is a leaf node} \\ \frac{1}{4} + \frac{1}{4\sqrt{t}} & \text{if } i = v_1 \\ \frac{1}{4} - \frac{1}{4\sqrt{t}} & \text{if } i = v_2 \end{cases}$$

Since the graph has $2t+2$ nodes:

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] = f(2t+2,t) \tag{13}$$

On the other hand, since $\|x^t - x^*\|_\infty \geq \max\{|x_{v_1} - x_{v_1}^*|, |x_{v_2} - x_{v_2}^*|\}$ we get:

$$\mathbf{E}\left[\|x^t - x^*\|_\infty\right] \geq \frac{1}{2}\mathbf{E}\left[|x_{v_1}^t - x_{v_1}^*|\right] + \frac{1}{2}\mathbf{E}\left[|x_{v_2}^t - x_{v_2}^*|\right] \tag{14}$$

Suppose we have a star graph with center node c. At each time t, the oracle returns to the center node either the value of a leaf with $s_i = 1$ (e.g.$f(1,1,t)$) with probability :

$$p = \frac{\#\text{ leaves with } s_i = 1}{\#\text{ leaves}}$$

or the value of a leaf with $s_i = 0$ (e.g.$f(1,0,t)$) with probability $1-p$. As a result, knowing the 0-1 choice of the oracle at time step t, one can compute x_c^t, meaning that $\hat{\theta}^t = 2x_c^t$ is an estimator for a Bernoulli random variable with probability p. This obviously holds for every choice of $s_i \in \{0,1\}$. This means that in our graph, $\hat{\theta}_1 = 2x_{v_1}^t$ is an estimator for the Bernoulli with probability $p_1 = \frac{1}{2} + \frac{1}{2\sqrt{t}}$ and $\hat{\theta}_2 = 2x_{v_2}^t$ is an estimator for the Bernoulli with probability $p_2 = \frac{1}{2} - \frac{1}{2\sqrt{t}}$. To be more precise, since the protocol is executed in the same way in all nodes, $\hat{\theta}_1$ and $\hat{\theta}_2$ are the ouputs of one estimator $\hat{\theta}$, depending on the distribution from which this estimator gets its input. By combining inequalities (13) and (14) we get:

$$f(2t+2,t) \geq \frac{1}{4}E_{p_1}\{|\hat{\theta} - p_1|\} + \frac{1}{4}E_{p_2}\{|\hat{\theta} - p_2|\} \tag{15}$$

We will now try to bound the right hand side of equation (15). We first notice that:

$$\begin{aligned} E_{p_1}|\hat{\theta} - p_1| &\geq \delta\mathbf{E}\left[\mathbb{1}\{|\hat{\theta} - p_1| > \delta\}\right] \\ &= \delta\mathbf{P}\left[|\hat{\theta} - p_1| > \delta\right] \end{aligned}$$

The same holds for p_2. It suffices now to provide a lower bound for this probability. In order to do this, we use a standard reduction from estimation problems to testing problems(see lecam). In a testing problem, there exist two Bernoulli distributions P_1 and P_2. First, nature selects with equal probability one of them and let $V \in \{1,2\}$ denote this random choice. Then, we draw a random vector $(Y_1, \ldots, Y_t)$ from the t-fold product distribution P_V^t. The goal is to decide whether $V = 1$ or $V = 2$. More precisely, we have to select a suitable function $\psi : \{0,1\}^t \mapsto \{1,2\}$ that minimizes $\mathbf{P}\left[\psi(Y_1, \ldots, Y_t) \neq V\right]$ First, we state a Lemma that provides a lower bound for this probability.

Lemma 10. *For every function* $\psi : \{0,1\}^t \mapsto \{1,2\}$*:*

$$\mathbf{P}\left[\psi(Y_1,\ldots,Y_t) \neq V\right] \geq \frac{1}{2}\left(1 - D_{TV}(P_1^t, P_2^t)\right)$$

where t *is the number of samples.*

We have:

$$\begin{aligned}\frac{1}{4}E_{p_1}\{|\hat{\theta} - p_1|\} + \frac{1}{4}E_{p_2}\{|\hat{\theta} - p_2|\} &= \frac{1}{4}E\{|\hat{\theta} - p_1| \, |V = 1\} + \frac{1}{4}E\{|\hat{\theta} - p_2| \, |V = 2\} \\ &\geq \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} > \frac{1}{2}|V = 1\right] + \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} < \frac{1}{2}|V = 2\right]\end{aligned}$$

We define :

$$\psi = \begin{cases} 1 & \hat{\theta} < \frac{1}{2} \\ 2 & \hat{\theta} > \frac{1}{2} \end{cases}$$

As a result,

$$\begin{aligned}\frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} > \frac{1}{2}|V = 1\right] + \frac{1}{4\sqrt{t}}\mathbf{P}\left[\hat{\theta} < \frac{1}{2}|V = 2\right] &\geq \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\hat{\theta} > \frac{1}{2}, V = 1\right] + \mathbf{P}\left[\hat{\theta} > \frac{1}{2}, V = 2\right]\right) \\ &= \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\psi = 2, V = 1\right] + \mathbf{P}\left[\psi = 1, V = 2\right]\right) \\ &= \frac{1}{2\sqrt{t}}\left(\mathbf{P}\left[\psi \neq V\right]\right) \\ &\geq \frac{1}{4\sqrt{t}}(1 - D_{TV}(P_1^t, P_2^t)) \\ &\geq \frac{1}{24\sqrt{t}}\end{aligned}$$

So

$$\lim_{t \to \infty} \sqrt{t} f(2t + 2, t) > 0$$

5.1 Estimating Bernoulli Distributions

The problem definition is the following:

- There exists an unkown parameter $p \in [0,1]$
- We receive i.i.d. observations X_i , where $X_i \sim B(p)$
- We output an estimate value $\hat{p} = \hat{\theta}_t(X_1,\ldots,X_t)$ according to some function $\hat{\theta}_t : \{0,1\}^t \mapsto [0,1]$.

The goal is to appropriately select $\hat{\theta}_t$ such that $\hat{p}$ and p are relatively close. For example a very known selection is

$$\hat{\theta}_t(X1,\ldots,X_t) = \frac{\sum_{i=1}^t X_i}{t}$$

which is the sample mean estimator

Definition 2. *An estimator* $\hat{\theta}$ *is a colection of functions* $\{\hat{\theta}_t\}_{t=1}^{\infty}$*, where* $\hat{\theta}_t : \{0,1\}^t \mapsto [0,1]$.

As the number of samples grows we would expect that an effiencient estimator $\hat{\theta}$ produces more and more accurate values for p. The following definition provides us the metric to evaluate the quality of an estimator $\hat{\theta}$.

Definition 3. *For any estimator* $\hat{\theta}_t$ *we define*

$$E_p[|\hat{\theta}_t(X_1,\ldots,X_t)| - p|] = \sum_{(x_1,\ldots,x^t)\in\{0,1\}^t} |\hat{\theta}_t(x^1,\ldots,x^t) - p| \cdot p^{\sum_{i=1}^t x^i}(1-p)^{t-\sum_{i=1}^t x^i}$$

The quantity $E_p[|\hat{\theta}_t(X_1,\ldots,X_t)| - p|]$ is the expected distance of the estimated value $\hat{\theta}_t$ from the parameter p, when the distribution of the samples is $B(p)$. To simplify notation we also denote it as $E_p[|\hat{\theta}_t - p|]$.

Notice that $E_p[|\hat{\theta}_t - p|]$ does not serve as a good metric since an effiencient estimator $\hat{\theta}$ must garantee good estimates for any parameter p. Consider an estimator $\hat{\theta}$ that outputs $1/2$ for all t. In this case, $E_p[|\hat{\theta}_t - p|] = |p - 1/2|$ meaning that this estimator performs optimally when $p = 1/2$, but extremely bad in any other case. As a result the standard metric in the statistics literature is the following.

Definition 4. *For any estimator $\hat{\theta}_t$ we define its risk with t samples as,*

$$R^t(\hat{\theta}) = max_{p\in[0,1]} E_p[|\hat{\theta}_t - p|]$$

Now our goal is clear, we need to design estimators $\hat{\theta}_t$ such that the risk $R^t(\hat{\theta}_t)$ is as small as possible. For example one can easily prove that the risk rate of the *sample mean estimator* ($\hat{\theta}_t = \frac{\sum_{i=1}^t X_i}{t}$) $R^t(\hat{\theta}_t) \leq \frac{1}{\sqrt{2t}}$. Obviously the next question is whether we can find another (probably more complex estimator) that acheives risk smaller than that of $\frac{1}{\sqrt{2t}}$.

Theorem 7. *Any estimator $\hat{\theta}$ has risk $R^t(\hat{\theta}) \geq \frac{1}{16\sqrt{t}}$*

The above theorem can be easily derived by standard techniques (e.g. Le Cam or Fano method) that have been develloped in the statistics literature to provide lower bounds on the risk of estimators. Especially, proving this lower bound for the Bernoulli case is requires a simple applications of these methods. The above theorem also tells us that the risk of the *sample mean estimator* is up to constant factors optimal. Are we done?

Theorem 8. *There exists an estimator $\hat{\theta}$ such that for all $p \in [0,1]$,*

$$E_p[|\hat{\theta}_t - p|] = O\left(\frac{1}{\sqrt{t}}\right)$$

Can this rate be imporoved? For example, is there a an estimator $\hat{\theta}$ such that for all $p \in [0,1]$, $E_p[|\hat{\theta}_t - p|] = O\left(\frac{1}{t}\right)$?

It seems that the latter question can be answered negatively using Theorem 7. Unfortunately, this is not the case. The difference is subtle but very important. The definition of the risk $R^t = \max_{p\in[0,1]} E_p[|\hat{\theta}_t - p|]$, allows the maximum to be attained in a p that depends on t (e.g. $p = 1/2 + 1/t$). As a result, Theorem 7 does not provide us with information about the above question.

Theorem 9. *For any estimator $\hat{\theta}$ there exists a fixed $p \in [0,1]$ such that*

$$E_p[|\hat{\theta}_t - p|] = \Omega(\frac{1}{t})$$

To the best of our knowledge this question is not answered in the statistics literature. However, the same question has been asked for more the estimator of more general and complex distribution. These results ensure the above statement for estimators that satisfy certain propetries (unbaised estimators, locally regular, e.t.c.). In this section we provide a theorem that holds for any estimator and indicates that estimators that don't satisfy the above statement are very unlikely to exists.

Theorem 10. *For any Bernoulli estimator $\hat{\theta}$, for all $[a,b] \subseteq [0,1]$,*

$$\lim_{t\to\infty} t \int_a^b E_p[|\hat{\theta}_t - p|]dp = +\infty$$

In case that statement .. is not true than there exists an estimator $\hat{\theta}$ that simultaneous satisfies the following two properties:

1. for all $p \in [0,1]$, $\lim_{t\to\infty} tE_p[|\hat{\theta}_t - p|] = 0$
2. for all $[a,b] \in [0,1]$, $\lim_{t\to\infty} t\int_a^b E_p[|\hat{\theta}_t - p|] = +\infty$

We conjecture that there does not exists a function that satisfies both of these properties. Even in case that this is not true, such a function has a very degenerate form making it very difficult to be the risk function of an estimator.

5.2 Final Lower Bound

Theorem 11. *Let a graph oblivious protocol P such that for any instance I, $E[||x^t - x^*||_\infty] \leq f(I)g(t)$, where $f : I \mapsto R_+$ and $g : N \mapsto R_+$. Then $lim_{t\to\infty} tf(t) \neq 0$.*

Proof. We use the protocol P to construct an estimator $\hat{\theta}$ and then we use the theorem ... For any $k, n \in N$ with $k \leq n$ we construct the following instance.

- A star graph with $n+1$ nodes
- k leaves have $a_i = 1$ and $s_i = 1$
- $n-k$ leaves have $a_i = 1$ and $s_i = 0$
- the central node has $a_{n+1} = 1/2$ and $s_{n+1} = 0$
- directed edges from the central node to all the leaves.

Observe that for all $i \neq n+1$, $x_i^* = s_i$ and $x_{n+1}^* = \frac{k}{2n}$. For all $i \neq n+1$, $x_i^t = P(1,1,t)$ if $s_i = 1$ and $x_i^t = P(0,1,t)$ if $s_i = 0$. Notice that these are deterministic functions with respect to t and are independent of the values of k, n. Now, $x_{n+1}^t = P(P(y_1,1,1), P(y_2,1,2), \ldots, P(y_t,1,t), 1/2, 0, t)$, where $x_i \in 0,1$. As a result, $x_{n+1}^t = g(y_1, \ldots, y_t)$, where $g : \{0,1\}^t \mapsto [0,1]$. As a result, the estimator $\hat{\theta}$ with $\hat{\theta}_t = \frac{x_c^t}{2}$ is valid estimator. Now let $p = k/n$ we have that

$$\begin{aligned} E_p[||x^t - x^*||] &\geq E_p[|x_{n+1}^t - \frac{p}{2}|] \\ &= E_p[|\frac{\hat{\theta}_t}{2} - \frac{p}{2}|] \\ &= \frac{1}{2} E_p[|\hat{\theta}_t - p|] \end{aligned}$$

Finally, for all $p \in [0,1]$ $E_p[|\hat{\theta}_t - p|] \leq 2f(I_{k,n})g(t) = f'(p)g(t)$. By theorem .., $\lim_{t\to\infty} tf(t) \neq 0$.

6 Graph Aware Protocol - Frequencies

Let P be a discrete distribution over $[d]$, and let $S_1, \ldots, S_t$ be t i.i.d samples drawn from P, i.e. $(S_1, \ldots, S_t) \sim P^t$. The empirical distribution $\hat{P}_t$ is the following estimator of the density of P.

$$\hat{P}_N(A) = \frac{\sum_{i=1}^N \mathbb{1}_{[S_i \in A]}}{N}, \tag{16}$$

where $A \subseteq [d]$. In words, $\hat{P}_N$ simply counts how many times the value i appeared in the samples $S_1, \ldots S_N$. We will use the following version of the classical Vapnik and Chervonenkis inequality.

Lemma 11. *Let $\mathcal{A}$ be a collection of subsets of $\{0, \ldots, d\}$ and Let $S_{\mathcal{A}}(N)$ be the Vapnik-Chervonenkis shatter coefficient, defined by*

$$S_{\mathcal{A}(N)} = \max_{x_1,\ldots,x_t \in [d]} |\{\{x_1, \ldots, x_t\} \cap A : A \in \mathcal{A}\}|.$$

Then

$$\mathbf{E}_{P^N}\left[\max_{A\in\mathcal{A}} |\hat{P}_t(A) - P(A)|\right] \leq 2\sqrt{\frac{\log 2S_{\mathcal{A}}(N)}{N}}$$

An interesting question is whether poly$(1/\varepsilon)$ rounds are necessary when protocols have access to the unlabeled oracle. We show that this is not true by showing a protocol that needs $\ln^2(1/\varepsilon)$ rounds to be within error ε from the solution x^*. Our protocol is graph-aware in the sense that it is allowed to depend on the graph G. Therefore, we show that without any constraints on the protocols it is hard to prove strong lower bounds for our problem. The problem in this case is that all agents could agree to stop updating their public opinions for enough rounds so that everybody learns, with high probability, exactly the average of the opinions of their neighbors. Having the exact average they can perform a step of the original update of the FJ-model which results in a similar and fast convergence rate. Of course if the opinions of the neighbors are all different we can still use an argument similar to that of the protocol of section COUPONS COLLECTOR. Since some neighbors may share opinions we have to come up with a different solution for this problem. Let $L \le d$ be the number of different opinions and k_i be the number of neighbors who have opinion ξ_i. Then the average of the opinions of the neighbors is $(\sum_{j=1}^{L} k_i \xi_i)/d_i$. Instead of using the average like we did in section SAMPLE MEAN, we will instead find *exactly* the frequencies k_i and then compute their average exactly with high probability. We next describe our protocol. Since it is the same for all agents, For simplicity we describe it for agent i.

Algorithm 1 Graph Aware Update Rule

$x_i(0) \leftarrow s_i$, $t \leftarrow 0$.
for $l = 1, \dots, \infty$ **do**
  $\varepsilon \leftarrow 1/2^l$.
  for $t = 1, \dots, \ln(1/\varepsilon))$ **do**
    Keep a map h from $[0, 1]$ to $[d_i]$ and array of counters A of length d_i and an array B of length M_1.
    $M_1 = O(\ln(1/\varepsilon))$, $M_2 = O(d^2 \ln(d))$
    for $j = 1, \dots, M_1$ **do**
      for $k = 1, \dots, M_2$ **do**
        Call the unlabeled oracle $O_i^u(t)$ to get an opinion X_i.
        if X_i is not in h **then**
          Insert X_i to h.
        else
          $A(h(X_i)) \leftarrow A(h(X_i)) + 1$.
        $t \leftarrow t + 1$
      Divide all entries of A by M_2.
      Round all entries of h to the closest multiple of $1/d$.
      $B(j) = \sum_{i=1}^{d_i} A(i)$.
    $x_i(t) \leftarrow \mathrm{maj}_j\, B(j)$.

Theorem 12. *The update rule 1 for $a > 1/2$ achieves convergence rate*

$$\mathbf{E}\left[\|x_t - x^*\|_\infty\right] \le C e^{-\sqrt{t}/(d\sqrt{\log d})},$$

where C is a universal constant.

Proof. According to the update rule 1 all agents fix their opinions $x_i(t)$ for $M_1 \times M_2$ rounds. To estimate the sum of the opinions each agent estimates the frequencies k_j/d_i. Since the neighbors have at most d_i different opinions we can think of the opinions as natural numbers in $[d_i]$. The oracle returns the opinion of a random neighbor and therefore the samples X_i returned by the oracle O_i^u are drawn from a discrete distribution P supported on $[d_i]$. The probability $P(j)$ of the j-th opinion is the number of neighbors having this opinion k_j/d_i. To learn the probabilities $P(j)$ using samples from P. Letting $\mathcal{A} = \{\{1\}, \{2\}, \dots, \{d_i\}\}$ we have from Lemma 16 that

$$\mathbf{E}_{P^m}\left[\max_{j\in[d_i]} \left|\hat{P}_m(j) - P(j)\right|\right] \le 2\sqrt{\frac{\log 2d_i}{m}},$$

since $S_{\mathcal{A}} \leq d_i$. Therefore, an agent can draw $m = 100d^2 \log(2d)$ to learn the frequencies k_j/d_i within expected error $1/(5d)$. Notice now that the array A after line 15 corresponds to the empirical distribution of equation (20). Notice that if the agents have estimations of the frequencies k_j/d_i with error smaller than $1/d$ then by rounding them to the closest multiple of $1/d_i$ they learn the frequencies exactly. By Markov's inequality we have that with probability at least $4/5$ the rounded frequencies are exactly correct. By standard Chernoff bounds we have that if the agents repeat the above procedure $\ln(1/\delta)$ times and keep the most frequent of the answers $B(j)$ then they will obtain the correct answer with probability at least $1-\delta$. From Lemma **??** we have that to achieve error ε we need $\log(1/\varepsilon)/(1/(1-\alpha))$ rounds. Since we need all nodes to succeed at computing the exact averages for $\log(1/\varepsilon)/\log(1/(1-\alpha))$ rounds we have from the union bound that for $\delta < \frac{\varepsilon \ln(1/(1-\alpha))}{n \ln(1/\varepsilon)}$, with probability at least $1-\varepsilon$ the error is at most ε. For the expected error after $T = O(d^2 \log d(\log(n/\varepsilon) + \log\log(1/\varepsilon) + \log\log(1-a))$ rounds we have that $\mathbf{E}\left[\|x_T - x^*\|_\infty\right] = O(\varepsilon)$.