Opinion Formation with Imperfect Information

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Abstract. We study opinion formation games based on the famous model proposed by Friedkin and Johsen. In today's huge social networks the assumption that in each round agents update their opinions by taking into account the opinions of *all* their friends could be unrealistic. Therefore, we assume that in each round each agent gets to meet with only one random friend of hers. Since it is more likely to meet some friends than others we assume that agent i meets agent j with probability p_{ij}. Specifically, we define an opinion formation game, where at round t, agent i with intrinsic opinion $s_i \in [0,1]$ and expressed opinion $x_i(t) \in [0,1]$ meets with probability p_{ij} neighbor j with opinion $x_j(t)$ and suffers a disagreement cost that is a convex combination of $(x_i(t) - s_i)^2$ and $(x_i(t) - x_j(t))^2$.

For a dynamics in the above setting to be considered as natural it must be simple, converge to the equilibrium x^*, and perhaps most importantly, it must be a rational choice for selfish agents. In this work we show that an intuitive game play, is a natural dynamics for the above game. We prove that, after $O(1/\varepsilon^2)$ rounds, the opinion vector is within error ε of the equilibrium. Moreover, to show that selfish agents would choose to update their opinions according to our update rule, we show that it has no-regret.

The classical Friedkin-Johsen dynamics converges to the equilibrium within error ε after only $O(\log(1/\varepsilon))$ rounds whereas, in our imperfect information setting, our update rule needs $\widetilde{O}(1/\varepsilon^2)$ rounds. We ask whether there exists a different natural dynamics for our problem with better rate of convergence. We answer this question in the negative by showing that dynamics based on no-regret algorithms cannot converge with less than $\text{poly}(1/\varepsilon)$ rounds.

1   Introduction

The study of opinion formation dynamics has a long history (see e.g. [Jac08]). Today, the advent of social media makes the study of opinion formation in such large social networks even more important. Apart from purely theoretical interest, realistic models of how people form their opinions by interacting with each other in such social networks are also of practical interest for prediction, advertisement etc. Opinion formation is based on information exchange, between socially connected people (e.g. family, friends, colleagues), who interact often and affect each other's opinion. Moreover, opinion formation is often *dynamic* in the sense that discussions and interactions lead to changes in the expressed opinions. The wide use of the internet and social media have made the dynamic aspects of opinion formation even more dominant. To capture opinion formation on a formal level, several models have been proposed (see e.g., [DeG74,FJ90,HK02,BKO11] for continuous opinions and [FGV16a,YOA+13,BFM16] for discrete ones). A common assumption, that dates back to DeGroot [DeG74], is that opinions evolve through a form of repeated averaging of information collected from the agent social neighborhoods.

Our work builds on the model proposed by Friedkin and Johnsen [FJ90](FJ model). The FJ model is a variation on the DeGroot model that captures the fact consensus on the opinions of a social group is not generally reached. According to FJ model there exists a set of n agents. Each agent i holds an internal opinion $s_i \in [0, 1]$, which is private and invariant over time and a public opinion $x_i \in [0, 1]$. Initially, agents start with their internal opinion and at each round $t \geq 1$, update their public opinion $x_i(t)$ to a weighted average of public opinion of their social neighbors and their internal opinion,

$$x_i(t) = \frac{\sum_{j \neq i} w_{ij} x_j(t-1) + w_{ii} s_i}{\sum_{j \neq i} w_{ij} + w_{ii}} \tag{1}$$

where the weights $w_{ij} \geq 0$ quantify the influence between the agents and $w_{ii} > 0$ the self confidence of the agents towards their internal belief.

The FJ model is one of most influential models for opinion formation. Its has a very simple update rule, making it plausible for modeling natural behavior and its basic assumptions are aligned with empirical findings on the way opinions are formed [AFH+05,Kra09]. At the same time, it admits a unique equilibrium point $x^* \in [0, 1]^n$ to which it converges exponentially fast [GS14]. The FJ model has also been studied under a game theoretic view point. Bindel et al. considered its update rule as the minimizer of a quadratic disagreement cost function and based on it they defined the following one shot opinion formation game [BKO11]. The strategy of each agent i is her public opinion x_i, incurring her a disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_{ii}(x_i - s_i)^2 \tag{2}$$

Note that the FJ model is the *simultaneous best response dynamics* in the repeated version of this one shot game and that the Nash equilibrium of the above game is x^*. In [BKO11] they quantified the inefficiency of the equilibrium with respect to the total disagreement cost. Namely, the Price of Anarchy is less than $9/8$ when $w_{ij} = w_{ji}$.

We remark in [BKO11], Bindel et al. also introduced an alternative, game-theoretic, framework for studying the dynamics of opinions. Instead of modeling the way opinions evolve by precise dynamical processes, agents can be considered to repeatedly play a suitable one shot opinion formation game and update their opinions so as to minimize their individual disagreement cost. This framework allows for different aspects of the opinion formation process to be captured by suitable one shot games. For example, in [BGM13,EFHS17] variants of the above opinion formation game were introduced and the convergence properties of the *best response dynamics* were studied. Conceptually, this framework, also allows for the agents to update their opinions in more abstract ways than a specific update rule of a precise dynamical process. For example, in the above repeated game, agents instead of adopting *best response*, could update their opinions according to a *no-regret* learning algorithm.

1.1   Motivation and our setting

The equilibrium point x^* of the FJ model is generally considered to be a good estimation of the final opinions formed in a social network. For example, Gionis et al. [GTT13] considered the problem of identifying k

agents in an network to set their internal opinion equal to 1 so as to maximize the sum of the opinions in the equilibrium point x^* ($\|x^*\|_1$). This work belongs to a recent line of research concerning influence maximization problems with respect to x^*, [GTT13,AKPT18,MMT17]. The reason for modeling such problems by the equilibrium x^* is evident: the FJ model is one of the standard models for opinion formation and it converges exponentially fast to x^*.

Our work is motivated by the fact that the FJ model assumes a large amount of information exchange between the agents. At each round, its update rule requires that every agent learns all the opinions of her social neighbors. In today's large social networks where users usually have several hundreds of friends it is unlikely that, each day, they learn the opinions of all their social neighbors. In such environments it is far more reasonable to assume that individuals randomly meet a small subset of their acquaintances and these are the only opinions that they learn. Information exchange constraints consist the FJ model unsuitable for modeling the opinion formation process in such large networks and therefore, it is not clear whether x^* captures the limiting behavior of the opinions. In this work we ask:

Question 1. Is the equilibrium x^* an adequate way to model the final formed opinions in a large social network? Namely, are there simple variants of the FJ model that require limited information exchange and converge fast to x^*? Can they be justified as natural behavior for selfish agents under a game-theoretic solution concept?

To capture the fact that, in large networks, agents meet a small random subset of their friends each day, we consider an imperfect information variant of the opinion formation game defined in [BKO11]. In equation (2) the coefficient w_{ij} measures the influence that j poses on i. Following the common belief that "the more often we interact with someone, the more we are influenced by her", w_{ij} can be interpreted as a measure on how frequently i meets j. In our imperfect information variant, each agent i randomly meets an agent j (with respect to her weights) and experiences disagreement cost based on the distance of their opinions and on the distance of the expressed opinion of i and her internal opinion. More formally, we consider the following one shot game.

Definition 1. *For a given opinion vector $x \in [0,1]^n$, the disagreement cost of agent i is the random variable $C_i(x_i, x_{-i})$ defined as follows:*

- *Agent i meets one of her neighbors j with probability $p_{ij} = w_{ij}/\sum_{j \in N_i} w_{ij}$.*
- *Agent i suffers cost $C_i(x_i, x_{-i}) = (1-a_i)(x_i - x_j)^2 + a_i(x_i - s_i)^2$, where $\alpha_i = w_i/(\sum_{j \in N_i} w_{ij} + w_i)$.*

Our one shot game departs from the original opinion formation game of [BKO11]. In the game of [BKO11] an opinion vector $x \in [0,1]^n$ defines *deterministically* the cost $C_i(x)$ of each agent i, whereas in our variant of Definition 1 it defines a probability distribution on the cost $C_i(x)$ that i suffers. Notice that the expected disagreement cost of each agent i (scaled by the constant $\sum_{j \neq i} w_{ij} + w_{ii}$) is the same as the disagreement cost defined in equation (2). As a result, the Nash equilibrium with respect to the expected cost of the agents, is the equilibrium point x^*.

In order to model the opinion formation process in limited information exchange environments, we consider that the agents are engaged in the above one-shot game. At round t, each agent i selects an opinion $x_i(t)$ and suffers a disagreement cost based on the opinion of the neighbor that she randomly met. At the end of the round she is informed only about the opinion of this neighbor and may use this information to update her opinion. Information exchange is now vastly reduced: at each round each agent learns the opinion of *just one* of her neighbors. Question 1 now takes the following more concrete form.

Question 2. Can the agents update their opinions according to the limited information that they receive such that the produced opinion vector $x(t)$ convergences fast to the equilibrium x^* and the total disagreement cost that they experience is minimal?

1.2   Contribution

We introduce a simple and intuitive update rule that and we show that if the agents adopt it to update their opinions, the resulting opinion vector $x(t)$ converges to x^*. Our update rule is a *Follow the Leader algorithm* meaning that at round t, each agent updates her opinion to the minimizer of total disagreement cost that

she experienced until $t-1$. It also has a very simple form: it is roughly the time average of the opinions that the agent observes. In Section 3, we bound its convergence rate and we show that in order to achieve ε distance from x^*, poly$(1/\varepsilon)$ rounds are needed. The main technical difficulty lies on stochastic nature of our process, we deal it with by using elegant concentration arguments. In Section 4, we show that this rule ensures *no-regret* to any agent that adopts it. Namely, the average disagreement cost (that the agent experiences) per round approaches that of expressing the best opinion in hindsight. The latter makes our algorithm a natural behavioral assumption for agents that selfishly want to minimize their incurred disagreement cost. The no regret property of our update rule can be derived by the more general results in [HAK07]. However, we present a short and simple proof that may have some interest. Our results imply that even if agents are subjected to information exchange constraints, there exists natural dynamics that converge to equilibrium x^*, making it a robust choice for modeling the limiting behavior of the opinions.

In Section 5, we show that for any update rule that ensures *no-regret* to the agents, the resulting opinion vector $x(t)$ cannot converge to x^* faster than polynomially. We prove the latter for a larger class of update rules, *opinion dependent update rules*, which includes all the update rules that only depend on the observed opinions. We show that the existence of opinion dependent update rules with fast convergence rate implies the existence of Bernoulli estimators with small sample complexity rate and we present a novel information theoretic argument to rule out their existence. Interestingly, polynomial convergence is not a generic property our limited information exchange setting. In Section 6, we present an update rule that apart from the observed opinions, also uses the weights w_{ij} and converges exponentially fast to x^*. Our results provide a clear and intuitive characterization of the update rules that can achieve exponential convergence and serve as an "algorithmic guide" for future limited information exchange variants of the FJ model. Moreover, they indicate the fundamental reason underlying the exponentially fast convergence of the FJ model. It has little to do with the "large" information exchange that it requires and much more to do with the fact that the it uses the exact values of the weights w_{ij}.

1.3   Related Work

Apart from the aforementioned results there exists a large amount of literature concerning the FJ model. Many recent works [BGM13,CKO13,BFM16,EFHS17] bound the inefficiency of equilibrium in variants of opinion formation game defined in [BKO11]. In [GS14] they bound that the convergence time of the FJ model in in special graph topologies. In [BFM16], a variant of the opinion formation game in which social relations depend on the expressed opinions, is studied. They prove that, the discretized version of the above game admits a potential function and thus best-response converges to the Nash equilibrium. Convergence results in other discretized variants of the FJ model can be found in [YOA$^+$13,FGV16b]. In [FPS16] the convergence properties of limited information variants of the Heglesmann-Krause model [HK02] and the FJ model, are examined.

Other works, that relate to ours, concern the convergence properties of dynamics based on no-regret learning algorithms. In [FV97,FS99,SA00,SALS15] it is proved that in a finite n-person game if each agent updates her mixed strategy according to a no-regret algorithm the resulting *time-averaged* strategy vector converges to Coarse Correlated Equilibrium. The convergence properties of no-regret dynamics for games with infinite strategy spaces were considered in [EMN09]. They proved that for a large class of games with concave utility function (socially concave games), the time-averaged strategy vector converges to the PNE. More recent work investigate a stronger notion of convergence of no-regret dynamics. In [CHM17] they show that, in n-person finite generic games that admit unique Nash equilibrium, the strategy vector converges *locally* and exponentially fast to it. They also provide conditions for *global* convergence. Our results fit in this line of research since we show that for a game with *infinite* strategy space, the strategy vector (and not the time-averaged) converges to the unique Nash equilibrium.

Imperfect information settings have recently received substantial attention from the scientific community since they provide realistic models for the practical applications of game theory. Perfect payoff information is rare in practise; agents act based on random or noisy past payoff observations. Kleinberg et al. in [KPT09] treated load-balancing in distributed systems as a repeated game and analyzed the convergence properties no-regret online learning algorithms under the *full information assumption* that each agent learns the load of every machine. In a subsequent work [KPT11], the same authors consider the same problem in a *limited information setting* ("bulletin board model") in which each agent learns the load of just the machine that

served him. In [HCM17,MS17] they examine the convergence properties of online learning algorithms in case the payoffs observed by the agents are contaminated with random noise. Finally, in [BM17] they show stochastic dynamics that lead to no regret regardless of the amount of noise in the player's observations and show that in specific zero-sum games time averages converge to the NE for any noise level.

2   Our Results and Techniques

For simplicity we adopt the following notation for an instance of the game of Definition 1.

Definition 2. *We denote an instance of the opinion formation game of Definition 1 as* (P, s, α).

- P *is a* $n \times n$ *matrix with non-negative elements* p_{ij}, *with* $p_{ii} = 0$ *and* $\sum_{j=1}^{n} p_{ij}$ *is either* 0 *or* 1.
- $s \in [0, 1]^n$ *is the internal opinion vector.*
- $\alpha \in (0, 1]^n$ *the self confidence coefficient vector.*

We remind that $p_{ij} = w_{ij}/(\sum_{j \in N_i} w_{ij})$ and $\alpha_i = w_{ii}/(\sum_{j \in N_i} w_{ij} + w_{ii})$. The coefficient $\alpha_i \in (0, 1]$ since by the definition of FJ-model $w_{ii} > 0$ and measures the reluctance of agent i to adopt opinion other than s_i. Note that a given instance $I = (P, s, \alpha)$ also defines an instance of the FJ-model and we denote as $x^* \in [0, 1]^n$ its equilibrium point. We remind that x^* is also the Nash equilibrium of one shot game of Definition 1 defined by instance $I = (P, s, \alpha)$. Another parameter of an instance $I = (P, s, \alpha)$ that we often use is $\rho = \min_{i \in V} \alpha_i$. Throughout the paper we consider that the agents repeatedly play the one shot game of Definition 1. We denote as W_i^t the neighbor that agent i met at round t, which is a random variable whose probability distribution is determined by the instance $I = (P, s, \alpha)$ of the game, $\mathbf{P}\left[W_i^t\right] = p_{ij}$.

In Section 3, we study the convergence properties of the dynamics $x(t)$ when all agents update their opinions according to the "follow the leader" principle. Since each agent i must select $x_i(t)$, before knowing which of her neighbors she will meet and what opinion her neighbor will express, their update rule says "*play the best according to what you have observed*". For a given instance (P, s, a) of the game the Follow the Leader dynamics $x(t)$ is defined in 1 and Theorem 1 shows its convergence rate to x^*.

Algorithm 1 Follow the Leader dynamics

1: Initially $x_i(0) = s_i$ for all agents i.
2: At round $t \geq 0$ each agent i:
    3: Meets neighbor with index W_i^t, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$.
    4: Suffers cost $(1 - \alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$ and learns the opinion $x_{W_i^t}$.
    4: Updates her opinion

$$x_i(t + 1) = \underset{x \in [0,1]}{\operatorname{argmin}} \sum_{\tau=0}^{t} (1 - \alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i(x - s_i)^2 \tag{3}$$

Theorem 1. *Let* $I = (P, s, \alpha)$ *be an instance of the opinion formation game of Definition 1 with equilibrium* $x^* \in [0, 1]^n$. *The opinion vector* $x(t) \in [0, 1]^n$ *produced by update rule (3) after* t *rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\, \frac{(\log t)^2}{t^{\min(1/2, \rho)}},$$

where $\rho = \min_{i \in V} a_i$ *and* C *is a universal constant.*

In Section 4 we argue that, apart from its simplicity, update rule (3) ensures no-regret for the agents, and therefore is a *rational game play* for selfish agents. Since each agent i selfishly wants to minimizing the disagreement cost that she experiences, it is natural to assume that she selects $x_i(t)$ according to an *no-regret algorithm* for the *online convex optimization problem* where the adversary chooses a function $f_t(x) = (1 - \alpha_i)(x - b_t)^2 + \alpha_i(x - s_i)^2$ at each round t. In Theorem 2 we prove that "Follow the Leader" is a no-regret algorithm for the above OCO problem. We remark that this does not hold, if the adversary can pick functions from a different class (see e.g. chapter 5 in [Haz16]).

Theorem 2. *Consider the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Let $\{b_t\}_{t=1}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. If $x_t = \mathrm{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O\left(\log t\right)$$

On the positive side, the FTL dynamics is a simple and natural stochastic process that ensures convergence to x^* while vastly reducing the information exchange between the agents per round; it only requires each agent to learn the opinion of just one neighbor. In terms of total communication exchange needed to get within distance ε of the equilibrium x^*, the FTL dynamics requires $\widetilde{O}(n)$ communication exchange while the FJ model needs $O(|E|)$. On the negative side, its convergence rate is outperformed by the rate of FJ model. For a fixed instance $I = (P, s, \alpha)$, the FTL dynamics converges with rate $\widetilde{O}(1/t^{\min(\rho,1/2)})$ while FJ-model converges with rate $O(e^{-\rho t})$ [GS14]. As a result the following question arises

Question 3. Can the agents adopt other no-regret algorithms such that the resulting dynamics $x(t)$ converges exponentially fast to x^*?

In Section 5 we answer this question in the negative. The reason that FTL dynamics converges slowly is that update rule (3) only depends on the opinions of the agents that agent i meets, α_i, and s_i. This is also true for any update rule that ensures no-regret to the agents (see Section 5). We call such update rules "*opinion dependent*" and the produced dynamics *opinion dependent dynamics.*

Definition 3 (opinion dependent update rule). *An opinion dependent update rule A is a sequence of functions $(A_t)_{t=0}^{\infty}$ where $A_t : [0,1]^{t+2} \mapsto [0,1]$.*

Definition 4 (opinion dependent dynamics). *Let an opinion dependent update rule A. For a given instance $I = (P, s, \alpha)$ the rule A produces an opinion depedent dynamics $x_A(t)$ defined as follows:*

- *Initially each agent i selects her opinion $x_i^A(0) = A_0(s_i, \alpha_i)$*
- *At each round $t \geq 1$, each agent i selects her opinion as: $x_i^A(t) = A_t(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), \alpha_i, s_i)$, where W_i^t is the neighbors that i meets at round t.*

Note that FTL dynamics is an opinion dependent dynamics since update rule 3 can be written equivalently, $x_i(0) = s_i$ and $x_i(t) = (1-\alpha_i) \sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i$. In Theorem 3 we show that for any opinion dependent update rule there exists an instance $I = (P, s, \alpha)$ where poly$(1/\varepsilon)$ rounds are required to achieve convergence within error ε.

Theorem 3. *Let A be an* opinion dependent *update rule, which all agents use to update their opinions. For any $c > 0$ there exists an instance $I = (P, s, a)$ such that $\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = \Omega(1/t^{1+c})$, where $x_A(t)$ denotes the opinion vector produced by A for the instance $I = (P, s, \alpha)$.*

To prove Theorem 3, we show that opinion dependent rules with "small round complexity", imply the existence of estimators for Bernoulli distributions with "small" sample complexity. Then with a simple argument presented in Lemma 6, we show that such estimators cannot exist. In Section 5 we also briefly discuss two well-known sample complexity lower bounds from the statistics literature and explain why they do not work in our case.

In Section 6, we present a simple update rule that is not opinion dependent and achieves error rate $e^{-O(\sqrt{t})}$. This update rule is a function of the opinions and the indices of the agents that i met, α_i, s_i and the i-th row the matrix P. We mention that the lower bound presented in Theorem 3 applies for "opinion dependent rules" that also depend on the agents' indices that i met. Therefore, the dependency on the row P_i is inevitable in order to obtain exponential convergence. Although, the assumption that the agents are aware of the influence matrix P is up to discussion, this update rule reveals that the slow convergence of *opinion dependent* update rules is not due to the reduced information exchange (learning the opinion of only one agent), but due to the fact that the agents are "oblivious" to the influence matrix P of the game and they learn it during the game play.

3   The Convergence Rate of FTL Dynamics

In this section that we prove that FTL dynamics converges to the unique equilibrium point x^*. Notice that for an instance $I = (P, s, \alpha)$, the opinion vector $x(t) \in [0,1]^n$ of the FTL dynamics (see **??**) can be written equivalently as follows:

- Initially all agents adopt their internal opinion, $x_i(0) = s_i$
- At round $t \geq 1$, each agent i updates her opinion as follows:

$$x_i(t) = (1 - \alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \tag{4}$$

where W_i^τ is the neighbor that i met at round t. Since the opinion vector $x(t)$ is a random vector, our convergence metric is $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$ where the expectation is taken over the random meeting of the agents. Our convergence result is stated in Theorem 1 and it is the main result of the section.

Theorem 1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^2}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

At first we present a high level idea of the proof. Notice that the unique equilibrium point $x^* \in [0,1]^n$ of the instance $I = (P, s, \alpha)$ satisfies the following equations for each agent $i \in V$,

$$x_i^* = (1 - \alpha_i)\sum_{j \in N_i} p_{ij} x_j^* + \alpha_i s_i$$

Since we are interested in bounding the $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$, we can use the above equations to bound $|x_i(t) - x_i^*|$.

$$|x_i(t) - x_i^*| = (1 - \alpha_i)|\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*|$$

$$= (1 - \alpha_i)|\sum_{j \in N_i}\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]x_j(\tau)}{t} - \sum_{j \in N_i} p_{ij} x_j^*|$$

$$\leq (1 - \alpha_i)\sum_{j \in N_i}|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]x_j(\tau)}{t} - p_{ij} x_j^*|$$

Now assume that $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}| = 0$ for all $t \geq 1$, then with simple algebraic manipulations one can prove that $\|x(t) - x^*\|_\infty \leq e(t)$ where $e(t)$ satisfies the recursive equation $e(t) = (1 - \rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. It follows that $\|x(t) - x^*\|_\infty \leq 1/t^\rho$ meaning that $x(t)$ converges to x^*. Obviously the latter assumption does not hold, however since W_i^τ are independent random variables with $\mathbf{P}\left[W_i^\tau\right] = p_{ij}$, $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}|$ tends to 0 with probability 1. In Lemma 1 we use this fact to obtain a similar recursive relation for $e(t)$ and then in Lemma 2 we upper bound the solution of this recursive equation.

Lemma 1. *Let $e(t)$ the solution of the following recursion,*

$$e(t) = \delta(t) + (1 - \rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$$

where $e(0) = \|x(0) - x^*\|_\infty$, $\delta(t) = \sqrt{\frac{\ln(\pi^2 n t^2/6p)}{t}}$ and $\rho = \min_{i \in V} \alpha_i$. Then,

$$\mathbf{P}\left[\text{for all } t \geq 1, \ \|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - p$$

Proof. At first we prove that with probability at least $1 - p$, for all $t \geq 1$ and all agents i:

$$\left| \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^*}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| \leq \delta(t) \tag{5}$$

where $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2 / (6p))}{t}}$.

Since W_i^τ are independent random variables with $\mathbf{P}[W_i^\tau = j] = p_{ij}$ and $\mathbf{E}\left[x_{W_i^\tau}^*\right] = \sum_{j \in N_i} p_{ij} x_j^*$. By the Hoeffding's inequality we get

$$\mathbf{P}\left[\left| \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^*}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| > \delta(t) \right] < 6p/(\pi^2 n t^2).$$

To bound the probability of error for all rounds $t = 1$ to ∞ and all agents i, we apply the union bound

$$\sum_{t=1}^\infty \mathbf{P}\left[\max_i \left| \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^*}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| > \delta(t) \right] \leq \sum_{t=1}^\infty \frac{6}{\pi^2} \frac{1}{t^2} \sum_{i=1}^n \frac{p}{n} = p$$

As a result with probability at least $1 - p$ we have that for all $t \geq 1$ and all agents i,

$$\left| \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^*}{t} - \sum_{j \in N_i} p_{ij} x_j^* \right| \leq \delta(t) \tag{6}$$

Now we can prove our claim by induction. Assume that $\|x(\tau) - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t - 1$. Then

$$\begin{aligned}
x_i(t) &= (1 - \alpha_i) \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \\
&\leq (1 - \alpha_i) \frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^* + \sum_{\tau=0}^{t-1} e(\tau)}{t} + \alpha_i s_i \tag{7} \\
&\leq (1 - \alpha_i) \left(\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}^*}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \right) + \alpha_i s_i \\
&\leq (1 - \alpha_i) \left(\sum_{j \in N_i} p_{ij} x_j^* + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \right) + \alpha_i s_i \tag{8} \\
&\leq x_i^* + \delta(t) + (1 - \rho) \left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \right)
\end{aligned}$$

We get (7) from the induction step and (8) from inequality (6). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. As a result $\|x(t) - x^*\|_\infty \leq e(t)$ and the induction is complete. Therefore, we have that with probability at least $1 - p$, $\|x(t) - x^*\|_\infty \leq e(t)$ for all $t \geq 1$.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion*

$$e(t) = \delta(t) + (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \quad \text{and} \quad e(0) = \|x(0) - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\ln(Dt^{2.5})}{t}}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2.5 \ln(D)} \frac{(\ln t)^{3/2}}{t^{\min(\rho, 1/2)}}$.

The proof of Lemma 2 and Theorem 1 can be found in the appendix. Theorem 1 is proved by a direct application of Lemma 1 and 2.

4   FTL Dynamics has no-regret

In this section we explain why the update rule 3 ("Follow the Leader") ensures no-regret to the agents that play the repeated version of the one shot game of Definition 1. Based on the disagreement cost that each agent experiences, we consider an appropriate *Online Convex Optimization* problem. This problem can be viewed as the following "game" played between an adversary and a player. *At round $t \geq 0$,*

1. *the player selects a value $x_t \in [0,1]$.*
2. *the adversary observes the x_t and selects a $b_t \in [0,1]$*
3. *the player receives cost $f(x_t, b_t) = (1-\alpha)(x_t - b_t)^2 + \alpha(x_t - s)^2$.*

where α, s are constants in $[0,1]$. The goal of the player is to pick x_t based on the history $(b_0, \ldots, b_{t-1})$ in a way that minimizes her total cost. Generally, different OCO problems can be defined by a set of functions $\mathcal{F}$ that the adversary chooses from and a feasibility set $\mathcal{K}$ from which the player picks her value (see [Haz16] for an introduction to the OCO framework). In our case the feasibility set is $\mathcal{K} = [0,1]$ and the set of functions is $\mathcal{F}_{\alpha,s} = \{(1-\alpha)(x-b)^2 + \alpha(x-s)^2, \text{ for all } b \in [0,1]\}$. As a result, each selection of the constants s, α lead to a different OCO problem.

Definition 5. *An algorithm A for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0,1]$ is a sequence of functions $(A_t)_{t=1}^{\infty}$ where $A_t : [0,1]^t \mapsto [0,1]$.*

Definition 6. *An algorithm A is no-regret for the OCO problem with $\mathcal{F}_{a,s}$ and $\mathcal{K} = [0,1]$ if and only if for all sequences $(b_t)_{t=0}^{\infty}$ that the adversary may choose, if $x_t = A_t(b_0, \ldots, b_{t-1})$ then for all t*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + o(t)$$

Informally speaking if the player selects the value x_t according to a *no-regret algorithm* then she does not regret for not playing any fixed value no matter what the choices of the adversary are. We prove that the "Follow the Leader" i.e. $x_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ is a no-regret algorithm for all the OCO problems with $\mathcal{F}_{\alpha,s}$. This is formally stated in Theorem 2 and is the main result of this section.

Theorem 2. *Consider the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Let $\{b_t\}_{t=1}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. If $x_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$$

Returning in our repeated game, it is reasonabe to assume that each agent i selects $x_i(t)$ according to no-regret algorithm A_i for the OCO problem with $\mathcal{F}_{s_i, \alpha_i}$, since by Definition 6,

$$\frac{1}{t} \sum_{\tau=0}^{t} f_i(x_i(\tau), x_{W_i^\tau}(\tau)) \leq \frac{1}{t} \min_{x \in [0,1]} \sum_{\tau=0}^{t} f_i(x, x_{W_i^\tau}(\tau)) + \frac{o(t)}{t}$$

The latter means that the time averaged total disagreement cost that she suffers is similar to the time averaged cost by expressing the best fixed opinion and this holds no matter the opinions of the agents that i meets. Theorem 2 ensures that *fictitious play* is a no-regret algorithm for any OCO problem $\mathcal{F}_{s_i, \alpha_i}$ and explains why (3) is a rational update rule for the agents.

The rest of the section is dedicated to prove Theorem 2. We first prove that a similar strategy that also takes into account the value b_t admits no-regret (Lemma 3). Obviously knowing the value b_t before selecting x_t is in direct contrast with the OCO framework, however proving the no-regret property for this algorithm easily extends to establishing the no-regret property of fictitious play. We give the proof of the following simple lemma in Appendix B.

Lemma 3. *Let $\{b_t\}_{t=0}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. Let $y_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$$

Now we can understand reason why "Follow the Leader" admits no regret. Since the cost incurred by the sequence y_t is at most that of the best fixed strategy, we can compare the cost incurred by x_t with that of y_t. However, the functions in $\mathcal{F}_{\alpha,s}$ are Lipschitz-continuous and more specifically quadratic. These functions are all similar to each other, so the extra term $f(x_t, b_t)$ that y_t takes into account doesn't change dramatically the minimum point of the sum. Thus, for each t the numbers x_t and y_t are quite close and as a result the difference in their cost must be quite small. The above are formally stated in Lemma 4. For the proof see Appendix B.

Lemma 4. *For all $t \geq 0$, $f(x_t, b_t) \leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.*

Theorem 2 easily follows since

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \sum_{\tau=0}^{t} f(y_\tau, b_\tau) + \sum_{\tau=0}^{T} 2\frac{1-\alpha}{\tau+1} + \sum_{\tau=0}^{t} \frac{(1-\alpha)^2}{(\tau+1)^2}$$

$$\leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + 2(1-\alpha)(\log t + 1) + (1-\alpha)\frac{\pi^2}{6}$$

$$\leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + O(\log t)$$

5   Lower Bound for Opinion Dependent Dynamics

In the previous sections we saw that if each agent i updates her opinion according to update rule 3 the resulting opinion vector $x(t)$ produced for an instance $I = (P, s, \alpha)$ converges to the unique equilibrium point x^* and ensures no-regret for the OCO problem with F_{s_i,α_i}. In this section we investigate whether we can select for each $(s, \alpha) \in [0,1]^2$ a no-regret algorithm $A_{s,\alpha}$ for the OCO problem with $\mathcal{F}_{s,\alpha}$ such that if for each instance $I = (P, s, \alpha)$ each agent i updates her opinion according to A_{s_i,α_i}, the resulting opinion vector $x(t)$ always converges exponentially fast to x^*. We answer this question in the negative, because such a selection of no-regret algorithms yields an opinion dependent update rule. Specifically, the function $A_t : \{0,1\}^{t+2} \mapsto [0,1]$ is defined as $A_t(b_0, \ldots, b_{t-1}, a, s) = A_{a,s}^t(b_0, \ldots, b_{t-1})$.

The next theorem is our fundamental result about lower bounds in *opinion dependent* update rules.

Theorem 3. *Let A be an* opinion dependent *update rule, which all agents use to update their opinions. For any $c > 0$ there exists an instance $I = (P, s, a)$ such that $\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = \Omega(1/t^{1+c})$, where $x_A(t)$ denotes the opinion vector produced by A for the instance $I = (P, s, \alpha)$.*

At first we show that any opinion dependent A, achieving the previous convergence rate, can be used as an estimator of the parameter $p \in [0,1]$ of Bernoulli random variable with the same asymptotic error rate. This reduction is formally stated in Lemma 5. Since we prove Theorem 3 using a reduction to an estimation problem we shall first briefly introduce some definitions and notation. For simplicity we will restrict the following definitions of estimators and risk to the case of estimating the mean of Bernoulli random variables. Given t independent samples from a Bernoulli random variable $B(p)$ an estimator is an algorithm that takes these samples as inputs and outputs an answer in $[0,1]$.

Definition 7. *An estimator $\theta = (\theta_t)_{t=1}^\infty$ is a sequence of functions, $\theta_t : \{0,1\}^t \mapsto [0,1]$.*

Perhaps the first estimator that comes to one's mind is the *sample mean*, that is $\theta_t = (1/t)\sum_{i=1}^t X_i$. Of course for an estimator to be efficient we would like its answer to be close to the mean p of the Bernoulli that generated the samples. To measure the efficiency of an estimator we define the *risk* which corresponds to the expected loss of an estimator.

Definition 8. *For an estimator $\theta = (\theta_t)_{t=1}^\infty$ we define its risk $E_p[|\theta_t(X_1, \ldots, X_t) - p|]$, where*

$$E_p[|\theta_t(X_1, \ldots, X_t) - p|] = \sum_{(y_1,\ldots,y_t) \in \{0,1\}^t} |\theta_t(y_1, \ldots, y_t) - p|\ p^{\sum_{i=1}^t y_i}\ (1-p)^{t-\sum_{i=1}^t y_i}$$

The risk $E_p[|\theta_t(Y_1, \ldots, Y_t) - p|]$ is the expected distance of the estimated value θ_t from the parameter p, when the distribution that generated the samples is $B(p)$. For convenience we also write it as $E_p[|\theta_t - p|]$. The risk quantifies the error rate of the estimated value $\hat{p} = \theta_t(Y_1, \ldots, Y_t)$ to the real parameter p as the number of samples t grows. Since p is unknown, any meaningful estimator $\theta = (\theta_t)_{t=1}^{\infty}$ must guarantee that $\lim_{t \to \infty} E_p[|\theta_t - p|] = 0$ for all p. For example, *sample mean* has error rate $E_p[|\theta_t - p|] \leq \frac{1}{2\sqrt{t}}$.

We show now that any opinion dependent update rule A, achieving the convergence rate of Theorem 3, can be used as an estimator of the parameter $p \in [0, 1]$ of a Bernoulli random variable with asymptotically the same error rate. The reduction is formally stated and in Lemma 5.

Lemma 5. *Let A an opinion dependent update rule such that for all instances I, $\lim\limits_{t \to \infty} t^{1+c}\mathbf{E}\left[\|x_A(t) - x^*\|_{\infty}\right] = 0$. Then there exists an estimator $\theta_A = (\theta_t^A)_{t=1}^{\infty}$ such that for all $p \in [0, 1]$, $\lim_{t \to \infty} t^{1+c}E_p[|\theta_t^A - p|] = 0$.*

Proof. We sketch here the main idea. For a full proof see Section C of the Appendix. We will employ the graph of Figure 1. By straightforward computation, we get that the equilibrium point of the graph is $x_c^* = q/3, x_1 = q/6 + 1/2, x_2 = q/6$. For a given $p \in [0, 1]$, we construct an instance I_q of Figure 1 with $q = p$ and simulate the update rule for all three nodes at the same time. For the center node, the choice of neighbor is given by the value of the bernoulli random variable that the estimator takes as input. Then, the value of the center node converges to $p/3$ with rate $1/t^{1+c}$. It follows that the estimator θ_t with $\theta_t^A = 3x_c^A(t)$ has
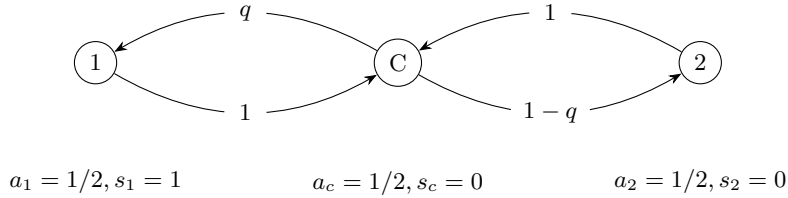

$$a_1 = 1/2, s_1 = 1 \qquad a_c = 1/2, s_c = 0 \qquad a_2 = 1/2, s_2 = 0$$

Fig. 1: The Lower Bound Instance

error $E_p[|\theta_t^A - p|] = 3\mathbf{E}_{I_p}\left[\|x_A(t) - x^*\|_{\infty}\right]$. Meaning that $\lim_{t \to \infty} t^{1+c}E_p[|\theta_t^A - p|] = 0$ for all $p \in [0, 1]$.

It follows by Lemma 5 that in order to prove Theorem 3 we just need to prove the following claim.

Claim. For any estimator $\theta = (\theta_t)_{t=1}^{\infty}$ there exists a fixed $p \in [0, 1]$ such that $\lim_{t \to \infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0$.

The above claim states that for any estimator $\theta = (\theta_t)_{t=1}^{\infty}$, we can inspect the functions $\theta_t : \{0, 1\}^t \mapsto [0, 1]$ and then choose a $p \in [0, 1]$ such that the function $E_p[|\theta_t - p|] = \Omega(1/t^{1+c})$. As a result, we have reduced the construction of a lower bound concerning the round complexity of a dynamical process to a lower bound concerning the sample complexity of estimating the parameter p of a Bernoulli distribution.

At this point we should mention that it is known that $\Omega(1/\varepsilon^2)$ samples are needed to estimate the parameter p of a Bernoulli random variable within additive error ε. Another well-known result is that taking the average of the samples is the *best* way to estimate the mean of a Bernoulli random variable. These results would indicate that the best possible rate of convergence for an *opinion dependent dynamics* would be $O(1/\sqrt{t})$. However, there is some fine print in these results which does not allow us to use them. In order to explain the various limitations of these methods and results we will briefly discuss some of them.

Before presenting Lemma 6 we briefly discuss some fundamental results concerning sample complexity lower bounds for statistical estimation. Perhaps the oldest sample complexity lower bound for estimation problems is the well-known Cramer-Rao inequality. Let the function $\theta_t : \{0, 1\}^t \mapsto [0, 1]$ such that $E_p[\theta_t] = p$ for all $p \in [0, 1]$, then

$$\mathbf{E}_p\left[(\theta_t - p)^2\right] \geq \frac{p(1-p)}{t}. \tag{9}$$

Since $\mathbf{E}_p\left[|\theta_t - p|\right]$ can be lower bounded by $\mathbf{E}_p\left[(\theta_t - p)^2\right]$ we can apply the Cramer-Rao inequality and prove Claim 5 for the *unbiased* estimators $\theta = (\theta_t)_{t=1}^{\infty}$. An estimator $\theta = (\theta_t)_{t=1}^{\infty}$ is *unbiased* if $E_p[\theta_t] = p$ for all $p \in [0, 1]$. Obviously, we need to prove the claim for any estimator θ, however this is a first indication that our claim holds.

To the best of our knowledge, sample complexity lower bounds without assumptions about the estimator are given as lower bounds for the *minimax risk*, which was defined [3] by Wald in [Wal39] as

$$\inf_{\theta_t} \sup_{p \in [0,1]} \mathbf{E}_p\left[|\theta_t - p|\right].$$

Minimax risk captures the idea that after we pick the best possible algorithm, an adversary inspects it and picks the worst possible $p \in [0,1]$ to generate the samples that our algorithm will get as input. The methods of Le'Cam, Fano, and Assouad are well-known information-theoretic methods to establish lower bounds for the minimax risk. For more on these methods see [Yu97,Tsy08] and the very good lecture notes of Duchi, [Duc]. As we stated before, it is well known that the minimax risk for the case of estimating the mean of a Bernoulli is lower bounded by $\Omega(1/\sqrt{t})$ and this lower bound can be established by Le Cam's method. In order to show why such arguments do no work for our purposes we shall sketch how one would apply Le Cam's method to get this lower bound. To apply Le Cam's method, one typically chooses two Bernoulli distributions whose means are far but their total variation distance is small. Le Cam showed that when two distributions are close in total variation then given a sequence of samples $X_1, \ldots, X_t$ it is hard to tell whether these samples were produced by P_1 or P_2. The hardness of this *testing* problem implies the hardness of *estimating* the parameters of a family of distribution. For our problem the two distributions would be $B(1/2 - 1/\sqrt{t})$ and $B(1/2 + 1/\sqrt{t})$. It is not hard to see that their total variation distance is at most $O(1/t)$, which implies a lower bound $\Omega(1/\sqrt{t})$ for the minimax risk. The problem here is that the parameters of the two distributions depend on the number of samples t. The more samples the algorithm gets to see, the closer the adversary takes the 2 distributions to be. For our problem we would like to *fix* an instance and then argue about the rate of convergence of any algorithm on this instance. Namely, having an instance that depends on t does not work for us.

Trying to get a lower bound without assumptions about the estimators while respecting our need for a fixed (independent of t) p we prove Lemma 6. In fact, we show something stronger: for *almost all* $p \in [0,1]$, any estimator θ cannot achieve rate $o(1/t^{1+c})$. More precisely, suppose we select a p uniformly at random in $[0,1]$ and run the estimator θ with samples from the distribution $B(p)$, then with probability 1 the error rate $\mathbf{E}_p\left[|\theta_t - p|\right] \in \Omega(1/t^{1+c})$. Although we do not show the sharp lower bound $\Omega(1/\sqrt{t})$ we prove that no exponential convergence rate is possible and we remark that our proof is fairly simple, intuitive, and could be of independent interest.

Lemma 6. *Let a Bernoulli estimator $\theta = (\theta_t)_{t=1}^\infty$ with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$. For any $c > 0$, if we select p uniformly at random in $[0,1]$ then*

$$\mathbf{P}\left[\lim_{t \to \infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0\right] = 1$$

Proof. Let an estimator $\theta = \{\theta_t\}_{t=1}^\infty$, where $\theta_t : \{0,1\}^t \mapsto [0,1]$. The function θ_t can have at most 2^t different values. Without loss of generality we assume that θ_t takes the same value $\theta_t(x)$ for all $x \in \{0,1\}^t$ with the same number of 1's. For example, $\theta_3(\{1,0,0\}) = \theta_3(\{0,1,0\}) = \theta_3(\{0,0,1\})$. This is due to the fact that for any $p \in [0,1]$,

$$\sum_{0 \leq i \leq t} \sum_{\|x\|_1 = i} |\theta_t(x) - p| \, p^i(1-p)^{t-i} \geq \sum_{0 \leq i \leq t} \binom{t}{i} \left|\frac{\sum_{\|x\|_1 = i} \theta_t(x)}{\binom{t}{i}} - p\right| p^i(1-p)^{t-i}.$$

For any estimator θ with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$ there exists another estimator θ' that satisfies the above property and $\mathbf{E}_p\left[|\theta_t' - p|\right] \leq \mathbf{E}_p\left[|\theta_t - p|\right]$ for all $p \in [0,1]$. Thus we can assume that θ_t takes at most $t+1$ different values. Let A denote the set of p for which the estimator has error rate $o(1/t^{1+c})$, that is

$$A = \{p \in [0,1] : \lim_{t \to \infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] = 0\}$$

We show that if we select p uniformly at random in $[0,1]$ then $\mathbf{P}\left[p \in A\right] = 0$. We also define the set

$$A_k = \{p \in [0,1] : \text{for all } t \geq k, \ t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] \leq 1\}$$

---

[3] Although the minimax risk is defined for any estimation problem and loss function, for simplicity, we write the minimax risk for estimating the mean of a Bernoulli random variable.

Observe that if $p \in A$ then there exists t_p such that $p \in A_{t_p}$, meaning that $A \subseteq \cup_{k=1}^{\infty} A_k$. As a result,

$$\mathbf{P}\left[p \in A\right] \leq \mathbf{P}\left[p \in \bigcup_{k=1}^{\infty} A_k\right] \leq \sum_{k=1}^{\infty} \mathbf{P}\left[p \in A_k\right]$$

To complete the proof we show that $\mathbf{P}\left[p \in A_k\right] = 0$ for all k. Notice that $p \in A_k$ implies that for $t \geq k$, the estimator θ must always have a value $\theta_t(i)$ close to p. Using this intuition we define the set

$$B_k = \{p \in [0,1] : \text{for all } t \geq k, \ t^{1+c} \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1\}$$

We now show that $A_k \subseteq B_k$. Since $p \in A_k$ we have that for all $t \geq k$

$$t^{1+c} \min_{0 \leq i \leq t} |\theta_t(i) - p| \sum_{i=0}^{t} \binom{t}{i} p^i (1-p)^{t-i} \leq t^{1+c} \sum_{i=0}^{t} \binom{t}{i} |\theta_t(i) - p| \, p^i (1-p)^{t-i} = t^{1+c} \mathbf{E}_p\left[|\theta_t - p|\right] \leq 1/2.$$

Thus, $\mathbf{P}\left[p \in A_k\right] \leq \mathbf{P}\left[p \in B_k\right]$. At first we write the set B_k as

$$B_k = \bigcap_{t=k}^{\infty} \{p \in [0,1] : \ \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\}.$$

As a result, $\mathbf{P}\left[p \in B_k\right] \leq \mathbf{P}\left[\min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\right]$, for all $t \geq k$. Each value $\theta_t(i)$ "covers" length
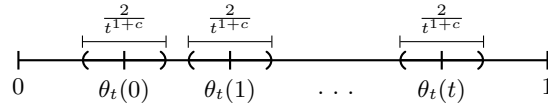

Fig. 2: Estimator output at time t

$1/t^{1+c}$ from its left and right, as shown in Figure 2, and since there are at most $t+1$ such values we have for all $t \geq k$ the set

$$\{p \in [0,1] \ : \ \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\} = \bigcup_{i=0}^{t} \left(\theta_t(i) - \frac{1}{t^{1+c}}, \ \theta_t(i) + \frac{1}{t^{1+c}}\right).$$

For each interval in the above union we have that $\mathbf{P}\left[|\theta_t(i) - p| \leq 1/t^{1+c}\right] \leq 2/t^{1+c}$ and by the union bound we get $\mathbf{P}\left[p \in B_k\right] \leq 2(t+1)/t^{1+c}$, for all $t \geq k$. We conclude that $\mathbf{P}\left[p \in B_k\right] = 0$.

Remark 1. The only point that we use that the update rules are *opinion dependent* is in Lemma 5. It is not difficult to see that the reduction still holds if the update rules also depend on the indices of the neighbors that an agent meets or if agents i, j with $s_i = s_j$ adopt different update rules. As a result, Theorem 3 still applies.

6   An Update Rule with Exponential Convergence Rate

We already discussed that the reason that opinion dependent dynamics suffer slow convergence is that the update rule depends only on the expressed opinions. In this section we provide an update rule showing that information about the graph G combined with agents that do not act selfishly, can restore the exponential convergence rate. Our update rule, depends not only on the expressed opinions of the agents but also on their indices and matrix P. Having this knowledge, one could try to come up with an update rule resembling the original update rule of the FJ model. In update rule 2, each agent could store the *most recent* opinions of the random neighbors that she meets in an array and then update her opinion according to their weighted sum (each agent knows row i of P). The problem with this approach is that the opinions of the neighbors that she keeps in her array are *outdated*, i.e. the opinion of neighbor of agent i is different than what she expressed in their last meeting. The good news are that as long as this outdatedness is bounded we can still achieve exponential convergence to the equilibrium. By bounded outdatedness we mean that there exists a number of rounds B such that all agents have met all their neighbors at least once from t to $t + B$.

Remark 2. It is necessary to know the matrix P in order for this update rule to work. We first observe that the lower bound of Section 5 also holds in case the algorithm learns the index of the chosen neighbor. The reason is that the reduction involves only two neighbors with different opinions, so they are distinguishable. Therefore, if we tried to learn P by observing the frequencies of the indices of the neighbors and run update rule 2 with the empirical frequencies instead of the p_{ij}, our lower bound ensures that the rate of convergence would not be $O(1/t^{1+c})$ for any $c > 0$. Intuitively, if we know P then the algorithm converges exponentially, since the slow part of the process is learning the probabilities p_{ij} precisely.

Algorithm 2 Asynchronous Update Rule

1: Initially $x_i(0) = s_i$ for all agent i.
2: Each agent i keeps an array M_i of length d_i.
3: At round $t \geq 1$ each agent i:
    4: $x_i(t) = (1 - \alpha_i) \sum_{j=1}^{d_i} p_{ij} M_i[j] + \alpha_i s_i$
    5: Meets neighbor W_i^t and learns the opinion $x_{W_i^t}(t)$.
    6: $M_i[W_i^t] \leftarrow x_{W_i^t}(t-1)$.

In [BT97], they show a convergence rate guarantee for 2 assuming that there exists a such a window B. In the following we briefly summarize their result. For completeness we give here a prove tailored for our purposes. Using a simple induction we get that bounded outdatedness preserves the exponential convergence. Its simple proof can be found in Appendix E.

Lemma 7. *Let* $\rho = \min_i a_i$, *and* $\pi_{ij}(t) \in \mathbf{N}$ *be the most recent round before round* t, *that agent* i *met agent* j. *If for all* $t \geq B$, $t - B \leq \pi_{ij}(t) \leq t - 1$ *then, for all* $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1 - \rho)^k$.

In our randomized setting there does not exist fixed length window is not true but we can easily adapt this to hold with high probability. To do this observe that agent i simply needs to wait to meet the neighbor j with the smallest weight p_{ij}. Therefore, after $\log(1/\delta)/\min_j p_{ij}$ rounds we have that with probability at least $1 - \delta$ agent i met all her neighbors at least once. Since we want this to be true for all agents we shall roughly take $B = 1/\min_{p_{ij}>0} p_{ij}$. In Section E of the Appendix we give the detailed argument that leads to the Lemma 8, showing that the convergence rate of update rule 2 is exponential.

Lemma 8. *Let* $x(t)$ *be the dynamics corresponding to update rule 2. We have*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2 \exp\left(-\rho \min_{ij} p_{ij} \frac{\sqrt{t}}{4 \ln(nt)}\right)$$

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A The Convergence Rate of FTL Dynamics

We give here the proof of the following technical lemma that we used to derive an upper bound on the rate of convergence of the fictitious play dynamics in Section 3. We restate Lemma 2 for completeness.

Lemma 2. *Let $e(t)$ be a function satisfying the recursion*

$$e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t} \ \text{ and } e(0) = \|x(0) - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\frac{\ln(Dt^{2.5})}{t}}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2.5\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\,1/2)}}$.

Proof. Observe that for all $t \geq 0$ the function $e(t)$ the following recursive relation

$$e(t+1) = e(t)\left(1 - \frac{\rho}{t+1}\right) + \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \tag{10}$$

For $t = 0$ we have that

$$e(1) = (1-\rho)e(0) + \delta(1) = (1-\rho)e(0) + \sqrt{\ln D} \tag{11}$$

Observe that for $D > \mathrm{e}^2$, $\delta(t)$ is decreasing for all $t \geq 1$. Therefore, $\delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \leq \frac{\delta(t)}{t+1}$ and from equations (10) and (11) we get that for all $t \geq 0$

$$e(t+1) \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{\ln(D(t+1)^2)}}{(t+1)^{3/2}} \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{2\ln(D(t+1))}}{(t+1)^{3/2}}$$

Now let $g(t) = \frac{\sqrt{2\ln(Dt)}}{t^{3/2}}$ to obtain for all $t \geq 1$

$$
\begin{aligned}
e(t) &\leq (1 - \frac{\rho}{t})e(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})(1 - \frac{\rho}{t-1})e(t-2) + (1 - \frac{\rho}{t})g(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})\cdots(1-\rho)e(0) + \sum_{\tau=1}^{t} g(\tau) \prod_{i=\tau+1}^{t}(1 - \frac{\rho}{i}) \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho(H_t - H_\tau)} \\
&\leq \frac{e(0)}{t^\rho} + e^{-\rho H_t}\sum_{\tau=1}^{t} g(\tau)e^{\rho H_\tau} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2}}{t^\rho}\sum_{\tau=1}^{t}\tau^\rho\frac{\sqrt{\ln(D\tau)}}{\tau^{3/2}} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2\ln D}}{t^\rho}\sum_{\tau=1}^{t}\frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}
\end{aligned}
$$

We observe that

$$\sum_{\tau=1}^{t}\frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}} \leq \int_{\tau=1}^{t}\frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}\mathrm{d}\tau \tag{12}$$

since, $\tau \mapsto \frac{\sqrt{\ln\tau}}{\tau^{3/2-\rho}}$ is a decreasing function of τ for all $\rho \in [0,1]$.

– If $\rho \leq 1/2$ then

$$\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} d\tau \leq \sqrt{\ln t} \int_{\tau=1}^{t} \frac{1}{\tau} d\tau = (\ln t)^{3/2}$$

– If $\rho > 1/2$ then

$$\begin{aligned}
\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} d\tau &= \int_{\tau=1}^{t} \tau^{\rho-1/2} \frac{\sqrt{\ln \tau}}{\tau} d\tau \\
&= \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-1/2} ((\ln \tau)^{3/2})' d\tau \\
&= \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2} - (\rho-1/2) \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-3/2} (\ln \tau)^{3/2} d\tau \\
&\leq \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2}
\end{aligned}$$

Theorem 7. *Let $I = (P, s, \alpha)$ be any instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t)$ produced by update rule 3 after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(1/2, \rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

Proof. By Lemma 1 we have that for all $t \geq 1$ and $p \in [0,1]$,

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e_p(t)\right] \geq 1 - p$$

where $e_p(t)$ is the solution of the recursion, $e_p(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Setting $p = \frac{1}{12\sqrt{t}}$ we have that

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - \frac{1}{12\sqrt{t}}$$

where $e(t)$ is the solution of the recursion $e(t) = \delta(t) + (1-\rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(2\pi^2 n t^{2.5})}{t}}$. Since $2\pi^2 \geq e^{2.5}$, Lemma 2 applies and $e(t) \leq C\sqrt{\log n} \frac{\log t^{3/2}}{t^{\min(\rho, 1/2)}}$ for some universal constant C. Finally,

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq \frac{1}{12\sqrt{t}} + (1 - \frac{1}{12\sqrt{t}}) C\sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho, 1/2)}} \leq (C + \frac{1}{12}) \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho, 1/2)}}$$

B   FTL dynamics has no-regret

We prove here Lemma 3, which we restate for convenience.

Lemma 10. *Let $\{b_t\}_{t=0}^{\infty}$ be an arbitrary sequence with $b_t \in [0,1]$. Let $y_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$ then for all t,*

$$\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$$

Proof. By definition of y_t, $\sum_{\tau=0}^{t} f(y_t, b_\tau) = \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$, so

$$
\begin{aligned}
\sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) &= \sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t} f(y_t, b_\tau) \\
&= \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_t, b_\tau) \\
&\leq \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_{t-1}, b_\tau)
\end{aligned}
$$

The last inequality follows by the fact that $y_{t-1} = \text{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ Inductively, we prove that $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.

We next prove Lemma 4 which we combine with Lemma 3 to show that the update rule 3 ensures no regret for the agents. We first restate the Lemma for convenience.

Lemma 11. *For all $t \geq 0$, $f(x_t, b_t) \leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.*

Proof. We first prove that for all t,

$$|x_t - y_t| \leq \frac{1-\alpha}{t+1}. \tag{13}$$

By definition $x_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} b_\tau}{t}$ and $y_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t} b_\tau}{t+1}$.

$$
\begin{aligned}
|x_t - y_t| &= (1-\alpha) \left| \frac{\sum_{\tau=0}^{t-1} b_\tau}{t} - \frac{\sum_{\tau=0}^{t} b_\tau}{t+1} \right| \\
&= (1-\alpha) \left| \frac{\sum_{\tau=0}^{t-1} b_\tau - t b_t}{t(t+1)} \right| \\
&\leq \frac{1-\alpha}{t+1}
\end{aligned}
$$

The last inequality follows from the fact that $b_\tau \in [0, 1]$. We now use inequality (13) to bound the difference $f(x_t, b_t) - f(y_t, b_t)$.

$$
\begin{aligned}
f(x_t, b_t) &= \alpha(x_t - s)^2 + (1-\alpha)(x_t - y_t)^2 \\
&\leq \alpha(y_t - s)^2 + 2\alpha|y_t - s||x_t - y_t| + \alpha|x_t - y_t|^2 \\
&\quad + (1-\alpha)(y_t - y_t)^2 + 2(1-\alpha)|y_t - y_t||x_t - y_t| + (1-\alpha)|x_t - y_t|^2 \\
&\leq f(y_t, b_t) + 2|x_t - y_t| + |y_t - x_t|^2 \\
&\leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}
\end{aligned}
$$

C   Lower bound for no-regret Dynamics

In the following Lemma we show how we can use algorithm A to construct an estimator $\hat{\theta}_A$ for Bernoulli distributions. We restate 5 for completeness.

Theorem 8. *Let the no-regret algorithm A such that for all instances I, $\lim_{t \to \infty} t^{1+c} \mathbf{E}_I [\|x_A(t) - x^*\|_\infty] = 0$. Then there exists an estimator $\hat{\theta}_A$ such that for all $p \in [0, 1]$,*

$$\lim_{t \to \infty} t^{1+c} R_p(t) = 0$$

Proof. At first we remind that an estimator $\hat{\theta}$ is a sequence of functions $\{\hat{\theta}_t\}_{t=1}^\infty$, where $\theta_t : \{0,1\}^t \mapsto [0,1]$. We construct such a sequence using the algorithm A. We also remind that when an agent i runs algorithm A, she selects $x_i(t)$ according to the cost functions $\{C_i^\tau\}$ that she has already received

$$x_i(t) = A_t(C_i^1, \ldots, C_i^{t-1})$$

Consider an agent i with $a_i = 1$ and $s_i = 0$ that runs A. Then $C_i^t(x) = x^2$ for all t and $x_i(t) = A_t(x^2, \ldots, x^2)$. The latter means that $x_i(t)$ only depends on t, $x_i(t) = h_0(t)$. Equivalently, if $a_i = 1$ and $s_i = 1$ then $x_i(t) = A_t((1-x)^2, \ldots, (1-x)^2)$ and $x_i(t) = h_1(t)$. Finally, consider an agent i with $a_i = 1/2$ and $s_i = 0$. In this case $C_i^t = \frac{1}{2}x^2 + \frac{1}{2}(x - y_t)^2$, where $y_t \in [0,1]$ is the opinion of the neighbor $j \in N_i$ that i met at round t. As a result, $x_i(t) = A_t(\frac{1}{2}x^2 + \frac{1}{2}(x - y_1)^2, \ldots, \frac{1}{2}x^2 + \frac{1}{2}(x - y_{t-1})^2) = f_t(y_1, \ldots, y_{t-1})$. The estimator $\hat{\theta}_A$ is the following sequences $\{\hat{\theta}_t\}_{t=1}^\infty$

$$\hat{\theta}_t(Y_1, \ldots, Y_t) = 2f_{t+1}(h_{Y_1}(1), \ldots, h_{Y_t}(t))$$

Observe that $\hat{\theta}_t : \{0,1\}^t \mapsto [0,1]$ meaning that $\hat{\theta}_A$ is a valid estimator for Bernoulli distributions. Now for any $p \in [0,1]$, we construct an appropriate instance I_p s.t. $R_p(t) = \mathbf{E}_p\left[|\hat{\theta}_t - p|\right] \leq 2\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right]$. Consider the graph of Figure 1, which has a central node with $a_c = 1/2$ and $s_c = 0$ and two leaf nodes $1, 2$ with $a_1 = a_2 = 1$, $s_1 = 1$ and $s_2 = 0$. The weights are $p_{c1} = p$ and $p_{c2} = 1 - p$. Obviously, nodes 1 and 2 always have constant opinions, 1 and 0 respectively. Hence, in each round the center node receives either $h_1(1)$ with probability p or $h_2(0)$ with probability $1 - p$.

We just need to prove that in I_p, $\mathbf{E}_p\left[|\hat{\theta}_t - p|\right] \leq \frac{1}{2}\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right]$. Notice that $x_c^* = \frac{p}{2}$ and $x_i^* = s_i$ if $i \neq c$. .

At round t, if the oracle returns to the center agent the value $h_1(t)$ of agent-1, then $Y_t = 1$ otherwise $Y_t = 0$. As a result, $\mathbf{P}[Y_t = 1] = p$ and

$$
\begin{aligned}
\mathbf{E}_{I_p}\left[\|x^t - x^*\|_\infty\right] &\geq \mathbf{E}_{I_p}\left[|x_c^t - x_c^*|\right] \\
&= \mathbf{E}_{I_p}\left[\left|f_{t+1}(h_{Y_1}(1), \ldots, h_{Y_t}(t)) - \frac{p}{2}\right|\right] \\
&= \mathbf{E}_p\left[\left|\frac{\hat{\theta}_t}{2} - \frac{p}{2}\right|\right] = \frac{1}{2}R_p(t)
\end{aligned}
$$

and the result follows.

We next give a rigorous measure-theoretic proof of Theorem 3.

Theorem 9. *Let $\theta_t : \{0,1\}^t \to [0,1]$ be a sequence of estimators for the success probability p of a Bernoulli random variable with distribution P. There exists $p \in [0,1]$ such that*

$$\lim_{t\to\infty} t^2\, \mathbf{E}_{X\sim P^t}\left[|\theta_t(X) - p|\right] > 0.$$

Proof. Observe that $\theta_t(\{0,1\}^t)$ has cardinality at most 2^t. Since

$$\sum_{0 \leq i \leq t}\sum_{\|x\|_1 = i} |\theta_t(x) - p|\, p^i(1-p)^{t-i} \geq \sum_{0 \leq i \leq t} \binom{t}{i} \left|\frac{\sum_{\|x\|_1 = i}\theta_t(x)}{\binom{t}{i}} - p\right| p^i(1-p)^{t-i}.$$

Thus, without loss of generality we assume that $\theta_t(\{0,1\}^t)$ contains at most $t + 1$ discrete points.

In the following, we work in the measure space $(\mathbf{R}, \mathcal{M}, \mu)$, where μ is the Lebesgue measure, and $\mathcal{M}$ is the σ-algebra of the Lebesgue measurable sets. Suppose that there exists no such $p \in [0,1]$. Let

$$A = \{p \in [0,1]\ :\ \lim_{t\to\infty} t^2\, \mathbf{E}_{X\sim P^t}\left[|\theta_t(X) - p|\right] = 0\}.$$

Then $A = [0,1]$ and A is measurable as an interval. Notice that,

$$A \subseteq \bigcup_{t=1}^\infty \bigcap_{k=t}^\infty A_k,$$

where $A_k = \{p \in [0,1] \ : \ R_k(p) < 1/2\}$, and $R_k(p) = k^2 \, \mathbf{E}_{X \sim P^k} [|\theta_k(X) - p|]$. We have that $R_k : [0,1] \to [0, +\infty)$ is polynomial of degree t in p and therefore it is a measurable function. Thus, A_k is measurable. We now show that

$$A_k \subseteq B_k := \{p \in [0,1] \ : \ k^2 \min_{0 \le i \le k} |\theta_k(i) - p| < 1\}.$$

We prove this by contradiction. Suppose that $p \in A_k$ but $p \notin B_k$. Since $p \in A_k$ we have that

$$R_k(p) = k^2 \sum_{i=0}^k \binom{k}{i} |\theta_k(i) - p| \, p^i (1-p)^{k-i} \ge k^2 \min_{0 \le i \le k} |\theta_k(i) - p| \sum_{i=0}^k \binom{k}{i} p^i (1-p)^{k-i} \ge 1.$$

Since the functions $p \mapsto k^2 |\theta_k(i) - p|$ are measurable, their pointwise minimum is measurable and therefore the sets B_k are also measurable. We next proceed to bound $\mu(B_k)$. Since θ_k can only take k different values we have that there exist $k+1$ intervals (a_{k_i}, b_{k_i}) of length at most $2/k^2$ such that $B_k = \bigcup_{i=0}^k (a_{k_i}, b_{k_i})$. Since μ is subadditive we have

$$\mu(B_k) \le \sum_{i=0}^k \frac{2}{k^2} = \frac{2(k+1)}{k^2}.$$

Now observe that

$$\mu(A) \le \mu\left(\bigcup_{t=1}^\infty \bigcap_{k=t}^\infty A_k\right) \le \mu\left(\bigcup_{t=1}^\infty \bigcap_{k=t}^\infty B_k\right) \le \sum_{t=1}^\infty \mu\left(\bigcap_{k=t}^\infty B_k\right) \le \sum_{t=1}^\infty \lim_{k \to \infty} \mu(B_k) = 0$$

Which is a contradiction since we assumed $A = [0,1]$.

D   An Update Rule with Exponential Convergence Rate

We are now going to state and prove a series of Lemmas that culminate in the proof of Lemma 8. We first turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdateness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 12. *Suppose that the collector picks coupons with mixed probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\text{collector hasn't seen all coupons}\right] \le \frac{1}{e^c}$$

For convenience of reasoning, we will divide the time in "epochs". The length of each epoch is B, so times 1 to B belong to the first epoch etc. The next lemma is a calculation of the appropriate size of B.

Lemma 13. *Suppose we run Algorithm 2 for t rounds. If the size of each epoch is*

$$B = \frac{2}{\min_{ij} p_{ij}} \ln \frac{nt}{p}$$

then with probability at least $1 - p$ all agents pick all their neighbours at least once in every epoch.

Proof. In our setting, coupon i corresponds to the selection of neighbour i. Each node is a collector and wants to gather all $n-1$ coupons during each epoch. Suppose $d = \max_i d_i$ is the maximum degree of the graph. Then, if we set $c = \ln(\frac{nt}{p})$, using Lemma 16 we get that a node hasn't seen at least one neighbour after $c/w + \ln d/w$ samples with probability at most $\frac{p}{nt}$. This means that if we set $D = c/w + \ln d/w = \ln \frac{nt}{p}/w + \ln d/w \ge 2/w \ln \frac{nt}{p}$ when p is small enough, then the probability that a specific agent at a specific epoch hasn't collected all neighbouring opinions at least once is at most $\frac{p}{nt}$. By a simple union bound argument, we get that all agents have seen all their neighbours during all epochs with probability at least $1 - p$.

Since all neighbours are picked at least once during each epoch, the outdatedness of each agent is at most twice the length of the epoch. We now show Lemma 7 from which we get that bounded outdatedness preserves the exponential convergence.

Lemma 14. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t) \in \mathbf{N}$ be the most recent round before round t, that agent i met agent j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t) \leq t - 1$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1 - \rho)^k$.*

Proof. To prove our claim we use induction on k. For the induction base $k = 1$,

$$|x_i(t) - x_i^*| = |(1 - \alpha_i) \sum_{j \in N_i} p_{ij}(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1 - \alpha_i) \sum_{j \in N_i} p_{ij}|(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1 - \rho)$$

From the induction hypothesis we have for $\pi_{ij}(t) \geq (k - 1)B$, that $|x_j(\pi_{ij}(t)) - x_j^*| \leq (1 - \rho)^{k-1}$. For $k \geq 2$, we again have that $|x_i(t) - x_i^*| \leq (1 - \rho) \sum_{j \in N_i} p_{ij}|(x_j(\pi_{ij}(t)) - x_j^*)|$. Since $t - B \leq \pi_{ij}(t)$ and $t \geq kB$, we have that $\pi_{ij}(t) \geq (k - 1)B$ and the induction hypothesis applies.

Combining this with Lemma 7 we have the following.

Corollary 1. *If we run Algorithm 2 for t rounds, then with probability at least $1 - p$*

$$\|x^t - x^*\|_\infty \leq (1 - \rho)^{\frac{t}{2B}} \leq \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2 \ln(\frac{nt}{p})}\right)$$

We now prove Lemma 8 using the previous results.

Lemma 8. *Let $x(t)$ be the dynamics corresponding to update rule 2. We have*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2 \exp\left(-\rho \min_{ij} p_{ij} \frac{\sqrt{t}}{4 \ln(nt)}\right)$$

Proof. Let $u(t) = \|x^t - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 2 we obtain:

$$\mathbf{P}\left[u(t) > \exp\left(-\frac{\rho w t}{2 \ln(\frac{nt}{p})}\right)\right] \leq p$$

for every probability $p \in [0, 1]$. Also, since all the parameters of the problem lie in $[0, 1]$, we have

$$\mathbf{E}\left[u(t)|u(t) > r\right] \leq 1$$

Now, by the conditional expectations identity, we get:

$$\mathbf{E}\left[u(t)\right] = \mathbf{E}\left[u(t)|u(t) > r\right] \mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \leq r\right] \mathbf{P}\left[u(t) \leq r\right]$$
$$\leq p + r$$

where $r = \exp\left(-\frac{\rho w t}{2 \ln(\frac{nt}{p})}\right)$. If we set $p = \exp\left(-\frac{\rho w \sqrt{t}}{2 \ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \leq \exp\left(-\frac{\rho w \sqrt{t}}{2 \ln nt}\right) + \exp\left(-\frac{\rho w t}{2 \ln(\frac{nt}{p})}\right)$$

We now evaluate r for our choice of probability p:

$$r = \exp\left(-\frac{\rho w t}{2 \ln\left(\frac{nt}{p}\right)}\right)$$

$$= \exp\left(-\frac{\rho w t}{2 \ln\left(\frac{nt}{\exp\left(-\frac{\rho w \sqrt{t}}{2 \ln nt}\right)}\right)}\right)$$

$$= \exp\left(-\frac{\rho w t}{2 \ln nt + 2\frac{\rho w \sqrt{t}}{2 \ln nt}}\right)$$

$$\leq \exp\left(-\frac{\rho w t}{4 \ln(nt)\sqrt{t}}\right)$$

$$= \exp\left(-\frac{\rho w \sqrt{t}}{4 \ln(nt)}\right)$$

Using the previous calculation, we obtain:

$$\mathbf{E}\left[u(t)\right] \leq \exp\left(-\frac{\rho w \sqrt{t}}{2\ln(nt)}\right) + \exp\left(-\frac{\rho w \sqrt{t}}{4\ln(nt)}\right)$$

$$\leq 2\exp\left(-\frac{\rho w \sqrt{t}}{4\ln(nt)}\right)$$

$$= 2\exp\left(-\rho \min_{ij} p_{ij} \frac{\sqrt{t}}{4\ln(nt)}\right)$$

E   An Update Rule with Exponential Convergence Rate

We are now going to state and prove a series of Lemmas that culminate in the proof of Lemma 8. We first turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdateness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 16. *Suppose that the collector picks coupons with mixed probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\text{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

For convenience of reasoning, we will divide the time in "epochs". The length of each epoch is B, so times 1 to B belong to the first epoch etc. The next lemma is a calculation of the appropriate size of B.

Lemma 17. *Suppose we run Algorithm 2 for t rounds. If the size of each epoch is*

$$B = \frac{2}{\min_{ij} p_{ij}} \ln \frac{nt}{p}$$

then with probability at least $1 - p$ all agents pick all their neighbours at least once in every epoch.

Proof. In our setting, coupon i corresponds to the selection of neighbour i. Each node is a collector and wants to gather all $n-1$ coupons during each epoch. Suppose $d = \max_i d_i$ is the maximum degree of the graph. Then, if we set $c = \ln(\frac{nt}{p})$, using Lemma 16 we get that a node hasn't seen at least one neighbour after $c/w + \ln d/w$ samples with probability at most $\frac{p}{nt}$. This means that if we set $D = c/w + \ln d/w = \ln \frac{nt}{p}/w + \ln d/w \geq 2/w \ln \frac{nt}{p}$ when p is small enough, then the probability that a specific agent at a specific epoch hasn't collected all neighbouring opinions at least once is at most $\frac{p}{nt}$. By a simple union bound argument, we get that all agents have seen all their neighbours during all epochs with probability at least $1 - p$.

Since all neighbours are picked at least once during each epoch, the outdatedness of each agent is at most twice the length of the epoch. We now show Lemma 7 from which we get that bounded outdatedness preserves the exponential convergence.

Lemma 18. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t) \in \mathbf{N}$ be the most recent round before round t, that agent i met agent j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t) \leq t - 1$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1 - \rho)^k$.*

Proof. To prove our claim we use induction on k. For the induction base $k = 1$,

$$|x_i(t) - x_i^*| = |(1 - \alpha_i) \sum_{j \in N_i} p_{ij}(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1 - \alpha_i) \sum_{j \in N_i} p_{ij}|(x_j(\pi_{ij}(t)) - x_j^*)| \leq (1 - \rho)$$

From the induction hypothesis we have for $\pi_{ij}(t) \geq (k-1)B$, that $|x_j(\pi_{ij}(t)) - x_j^*| \leq (1 - \rho)^{k-1}$. For $k \geq 2$, we again have that $|x_i(t) - x_i^*| \leq (1 - \rho)\sum_{j \in N_i} p_{ij}|(x_j(\pi_{ij}(t)) - x_j^*)|$. Since $t - B \leq \pi_{ij}(t)$ and $t \geq kB$, we have that $\pi_{ij}(t) \geq (k-1)B$ and the induction hypothesis applies.

Combining this with Lemma 7 we have the following.

Corollary 2. *If we run Algorithm 2 for t rounds, then with probability at least* $1 - p$

$$\|x^t - x^*\|_\infty \le (1 - \rho)^{\frac{t}{2B}} \le \exp\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{p})}\right)$$

We now prove Lemma 8 using the previous results.

Lemma 8. *Let* $x(t)$ *be the dynamics corresponding to update rule 2. We have*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \le 2\exp\left(-\rho\min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)$$

Proof. Let $u(t) = \|x^t - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 2 we obtain:

$$\mathbf{P}\left[u(t) > \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)\right] \le p$$

for every probability $p \in [0,1]$. Also, since all the parameters of the problem lie in $[0,1]$, we have

$$\mathbf{E}\left[u(t)|u(t) > r\right] \le 1$$

Now, by the conditional expectations identity, we get:

$$\mathbf{E}\left[u(t)\right] = \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \le r\right]\mathbf{P}\left[u(t) \le r\right]$$
$$\le p + r$$

where $r = \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$. If we set $p = \exp\left(-\frac{\rho w \sqrt{t}}{2\ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \le \exp\left(-\frac{\rho w \sqrt{t}}{2\ln nt}\right) + \exp\left(-\frac{\rho w t}{2\ln(\frac{nt}{p})}\right)$$

We now evaluate r for our choice of probability p:

$$r = \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{p}\right)}\right)$$

$$= \exp\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{\exp\left(-\frac{\rho w \sqrt{t}}{2\ln nt}\right)}\right)}\right)$$

$$= \exp\left(-\frac{\rho w t}{2\ln nt + 2\frac{\rho w \sqrt{t}}{2\ln nt}}\right)$$

$$\le \exp\left(-\frac{\rho w t}{4\ln(nt)\sqrt{t}}\right)$$

$$= \exp\left(-\frac{\rho w \sqrt{t}}{4\ln(nt)}\right)$$

Using the previous calculation, we obtain:

$$\mathbf{E}\left[u(t)\right] \le \exp\left(-\frac{\rho w \sqrt{t}}{2\ln(nt)}\right) + \exp\left(-\frac{\rho w \sqrt{t}}{4\ln(nt)}\right)$$

$$\le 2\exp\left(-\frac{\rho w \sqrt{t}}{4\ln(nt)}\right)$$

$$= 2\exp\left(-\rho\min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)$$