APPENDIX

A. Lifting a Pose To Landmark Measurement

With the robot frame denoted $\mathcal{F}_b$, the world frame denoted $\mathcal{F}_a$, and the landmark denoted l, a relative pose-landmark measurement is derived using

$$\mathbf{r}_b^{lb} = \mathbf{C}_{ba}\left(\mathbf{r}_a^{la} - \mathbf{r}_a^{ba}\right). \tag{17}$$

The quantity $\mathbf{r}_b^{lb}$ is measured. The error used should be quadratic in the quantities of interest to be manageable in a QCQP, therefore (17) is manipulated to yield

$$\mathbf{C}_{ab}\mathbf{r}_b^{lb} = \mathbf{r}_a^{la} - \mathbf{r}_a^{ba}, \tag{18}$$

and with a measurement $\check{\mathbf{r}}_b^{lb}$ the error term given the measurement is formed as

$$e(\mathbf{C}_{ab}, \mathbf{r}_a^{la}, \mathbf{r}_a^{ba}; \check{\mathbf{r}}_b^{lb}) = \left\|(\mathbf{r}_a^{la} - \mathbf{r}_a^{ba}) - \mathbf{C}_{ab}\check{\mathbf{r}}_b^{lb}\right\|_{\mathrm{F}}^2. \tag{19}$$

By renaming the variables, the same error may be written as

$$\begin{aligned} e(\mathbf{C}_{ab}, \boldsymbol{\ell}, \mathbf{r}; \mathbf{y}) &= \|(\boldsymbol{\ell} - \mathbf{r}) - \mathbf{C}\mathbf{y}\|_2^2 && (20)\\ &= \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{r}^\mathsf{T}\mathbf{r} + \mathbf{y}^\mathsf{T}\cancel{\mathbf{C}^\mathsf{T}\mathbf{C}}\mathbf{y} + 2\mathbf{r}^\mathsf{T}\mathbf{C}\mathbf{y} - 2\mathbf{r}^\mathsf{T}\boldsymbol{\ell} - 2\boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y}. && (21) \end{aligned}$$

This needs to be lifted into a factor of the form $\left\langle \mathbf{Q}, \boldsymbol{\Xi}^\mathsf{T}\boldsymbol{\Xi}\right\rangle$, with $\boldsymbol{\Xi} = \begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\end{bmatrix}$, which is done as

$$\begin{aligned} e(\mathbf{C}_{ab}, \boldsymbol{\ell}, \mathbf{r}; \mathbf{y}) &= \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{r}^\mathsf{T}\mathbf{r} + \mathbf{y}^\mathsf{T}\cancel{\mathbf{C}^\mathsf{T}\mathbf{C}}\mathbf{y} + 2\mathbf{r}^\mathsf{T}\mathbf{C}\mathbf{y} - 2\mathbf{r}^\mathsf{T}\boldsymbol{\ell} - 2\boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y} && (22)\\ &= \left\langle \mathbf{Q}, \boldsymbol{\Xi}^\mathsf{T}\boldsymbol{\Xi}\right\rangle && (23)\\ &= \left\langle \mathbf{Q}, \begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\end{bmatrix}\right\rangle && (24)\\ &= \left\langle \mathbf{Q}, \begin{bmatrix}\mathbf{1}\\ \mathbf{C}^\mathsf{T}\\ \mathbf{r}^\mathsf{T}\\ \boldsymbol{\ell}\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\end{bmatrix}\right\rangle && (25)\\ &= \left\langle \mathbf{Q}, \begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\\ & \mathbf{1} & \mathbf{C}^\mathsf{T}\mathbf{r} & \mathbf{C}^\mathsf{T}\boldsymbol{\ell}\\ & & \mathbf{r}^\mathsf{T}\mathbf{r} & \mathbf{r}^\mathsf{T}\boldsymbol{\ell}\\ & & & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell}\end{bmatrix}\right\rangle && (26)\\ &= \left\langle \begin{bmatrix}\operatorname{diag}([\mathbf{y}^\mathsf{T}\mathbf{y}\mathbf{e}_1]) & & & \\ & & 2\mathbf{y} & -2\mathbf{y}\\ & & 1 & -2\\ & & & 1\end{bmatrix}, \begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r} & \boldsymbol{\ell}\\ & \mathbf{1} & \mathbf{C}^\mathsf{T}\mathbf{r} & \mathbf{C}^\mathsf{T}\boldsymbol{\ell}\\ & & \mathbf{r}^\mathsf{T}\mathbf{r} & \mathbf{r}^\mathsf{T}\boldsymbol{\ell}\\ & & & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell}\end{bmatrix}\right\rangle. && (27) \end{aligned}$$

B. Lifting a Pose to Known Landmark Measurement

The error is the same as for the unknown landmark position measurement,

$$\begin{aligned} e(\mathbf{C}_{ab}, \mathbf{r}; \boldsymbol{\ell}, \mathbf{y}) &= \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{r}^\mathsf{T}\mathbf{r} + \mathbf{y}^\mathsf{T}\cancel{\mathbf{C}^\mathsf{T}\mathbf{C}}\mathbf{y} + 2\mathbf{r}^\mathsf{T}\mathbf{C}\mathbf{y} - 2\mathbf{r}^\mathsf{T}\boldsymbol{\ell} - 2\boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y} && (28)\\ &= \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{r}^\mathsf{T}\mathbf{r} + \mathbf{y}^\mathsf{T}\mathbf{y} + 2\mathbf{r}^\mathsf{T}\mathbf{C}\mathbf{y} - 2\mathbf{r}^\mathsf{T}\boldsymbol{\ell} - 2\boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y}, && (29) \end{aligned}$$

where subscripts are dropped for the sake of brevity. The last term bears a bit more consideration to include it in the matrix inner product formulation such that

$$\begin{aligned} \boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y} &= \operatorname{tr}(\mathbf{C}\mathbf{y}\boldsymbol{\ell}^\mathsf{T}) && (30)\\ &= \operatorname{tr}(\mathbf{C}(\boldsymbol{\ell}\mathbf{y}^\mathsf{T})^\mathsf{T}) && (31)\\ &= \left\langle \mathbf{C}, \boldsymbol{\ell}\mathbf{y}^\mathsf{T}\right\rangle. && (32) \end{aligned}$$

However, now the landmark $\boldsymbol{\ell}$ is known, such that the error is a function of $\mathbf{C}_{ab}, \mathbf{r}$ and parametrized using $\boldsymbol{\ell}, \mathbf{y}$, where $\mathbf{y}$ is the measured relative position of the landmark. Using the state $\begin{bmatrix}\mathbf{1} & \mathbf{C} & \mathbf{r}\end{bmatrix}$, the error may be written

$$
\begin{aligned}
e(\mathbf{C}_{ab}, \mathbf{r}; \boldsymbol{\ell}, \mathbf{y}) &= \boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{r}^\mathsf{T}\mathbf{r} + \mathbf{y}^\mathsf{T}\mathbf{y} + 2\mathbf{r}^\mathsf{T}\mathbf{C}\mathbf{y} - 2\mathbf{r}^\mathsf{T}\boldsymbol{\ell} - 2\boldsymbol{\ell}^\mathsf{T}\mathbf{C}\mathbf{y} && (33)\\
&= \langle \mathbf{Q}, \mathbf{X}^\mathsf{T}\mathbf{X} \rangle && (34)\\
&= \left\langle \mathbf{Q}, \begin{bmatrix} \mathbf{1} & \mathbf{C} & \mathbf{r} \\ & \mathbf{1} & \mathbf{C}^\mathsf{T}\mathbf{r} \\ & & \mathbf{r}^\mathsf{T}\mathbf{r} \end{bmatrix} \right\rangle && (35)\\
&= \left\langle \begin{bmatrix} \operatorname{diag}([(\boldsymbol{\ell}^\mathsf{T}\boldsymbol{\ell} + \mathbf{y}^\mathsf{T}\mathbf{y})\mathbf{e}_1]) & -2\boldsymbol{\ell}\mathbf{y}^\mathsf{T} & -2\boldsymbol{\ell} \\ & & 2\mathbf{y} \\ & & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{C} & \mathbf{r} \\ & \mathbf{1} & \mathbf{C}^\mathsf{T}\mathbf{r} \\ & & \mathbf{r}^\mathsf{T}\mathbf{r} \end{bmatrix} \right\rangle && (36)
\end{aligned}
$$

C. Lifting a Relative Pose Measurement

Given a relative pose measurment such that

$$
\begin{aligned}
\mathbf{T}_{k+1} &= \mathbf{T}_k \Delta\mathbf{T} && (37)\\
\begin{bmatrix} \mathbf{C}_{k+1} & \mathbf{r}_{k+1} \\ \mathbf{0} & 1 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_k\Delta\mathbf{C} & \mathbf{r}_k + \mathbf{C}_k\Delta\mathbf{r} \\ \mathbf{0} & 1 \end{bmatrix}, && (38)
\end{aligned}
$$

the first error corresponds to the translation error and is given by

$$
\begin{aligned}
J_r &= \|\mathbf{r}_{k+1} - \mathbf{r}_k - \mathbf{C}_k\Delta\mathbf{r}\|_2^2 && (39)\\
&= (\mathbf{r}_{k+1} - \mathbf{r}_k - \mathbf{C}_k\Delta\mathbf{r})^\mathsf{T}(\mathbf{r}_{k+1} - \mathbf{r}_k - \mathbf{C}_k\Delta\mathbf{r}) && (40)\\
&= \mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1} + \mathbf{r}_k^\mathsf{T}\mathbf{r}_k + \Delta\mathbf{r}^\mathsf{T}\Delta\mathbf{r} + 2\mathbf{r}_k^\mathsf{T}\mathbf{C}_k\Delta\mathbf{r} - 2\mathbf{r}_{k+1}^\mathsf{T}\mathbf{C}_k\Delta\mathbf{r} - 2\mathbf{r}_{k+1}\mathbf{r}_k. && (41)
\end{aligned}
$$

For $\mathbf{X} = \begin{bmatrix}\mathbf{C}_k & \mathbf{r}_k & \mathbf{r}_{k+1}\end{bmatrix}$, this yields the following cost,

$$
\begin{aligned}
J_r &= \langle \mathbf{Q}, \mathbf{X}^\mathsf{T}\mathbf{X} \rangle && (42)\\
&= \left\langle \mathbf{Q}, \begin{bmatrix} \mathbf{C}_k^\mathsf{T} \\ \mathbf{r}_k^\mathsf{T} \\ \mathbf{r}_{k+1} \end{bmatrix} \begin{bmatrix}\mathbf{C}_k & \mathbf{r}_k & \mathbf{r}_{k+1}\end{bmatrix} \right\rangle && (43)\\
&= \left\langle \mathbf{Q}, \begin{bmatrix} \mathbf{C}_k^\mathsf{T}\mathbf{C}_k & \mathbf{C}_k^\mathsf{T}\mathbf{r}_k & \mathbf{C}_k^\mathsf{T}\mathbf{r}_{k+1} \\ \mathbf{r}_k^\mathsf{T}\mathbf{C}_k & \mathbf{r}_k^\mathsf{T}\mathbf{r}_k & \mathbf{r}_k^\mathsf{T}\mathbf{r}_{k+1} \\ \mathbf{r}_{k+1}^\mathsf{T}\mathbf{C}_k & \mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_k & \mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1} \end{bmatrix} \right\rangle && (44)\\
&= \left\langle \begin{bmatrix} \mathbf{0} & 2\Delta\mathbf{r} & -2\Delta\mathbf{r} \\ & 1 & -2 \\ & & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{C}_k & \mathbf{r}_k & \mathbf{r}_{k+1} \\ \mathbf{r}_k^\mathsf{T}\mathbf{C}_k & \mathbf{r}_k^\mathsf{T}\mathbf{r}_k & \mathbf{r}_k^\mathsf{T}\mathbf{r}_{k+1} \\ \mathbf{r}_{k+1}^\mathsf{T}\mathbf{C}_k & \mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_k & \mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1} \end{bmatrix} \right\rangle. && (45)
\end{aligned}
$$

The second error corresponds to the rotation error and is given by

$$
\begin{aligned}
J_c &= \|\mathbf{C}_{k+1} - \mathbf{C}_k\Delta\mathbf{C}\|_F^2 && (46)\\
&= \operatorname{tr}\left((\mathbf{C}_{k+1} - \mathbf{C}_k\Delta\mathbf{C})(\mathbf{C}_{k+1} - \mathbf{C}_k\Delta\mathbf{C})^\mathsf{T} \right) && (47)\\
&= 2\operatorname{tr}(\mathbf{1}) - 2\operatorname{tr}(\mathbf{C}_{k+1}^\mathsf{T}\mathbf{C}_k\Delta\mathbf{C}) && (48)\\
&= 2\operatorname{tr}(\mathbf{1} - \Delta\mathbf{C}\mathbf{C}_{k+1}^\mathsf{T}\mathbf{C}_k) && (49)\\
& && (50)\\
& && (51)
\end{aligned}
$$

Making use of $\operatorname{tr}(\mathbf{A}\mathbf{B}^\mathsf{T}) = \sum_i \sum_j \mathbf{A}_{ij}\mathbf{B}_{ij} = \langle \mathbf{A}, \mathbf{B} \rangle$ allows to write

$$
\begin{aligned}
J_c &= 2\operatorname{tr}(\mathbf{1} - \Delta\mathbf{C}\mathbf{C}_{k+1}^\mathsf{T}\mathbf{C}_k) && (52)\\
&= 2\left(\operatorname{tr}\mathbf{1} - \langle \Delta\mathbf{C}, \mathbf{C}_k^\mathsf{T}\mathbf{C}_{k+1} \rangle\right) && (53)\\
&= \left\langle \mathbf{Q}, \begin{bmatrix} \mathbf{1} \\ \mathbf{C}_k^\mathsf{T} \\ \mathbf{C}_{k+1}^\mathsf{T} \end{bmatrix} \begin{bmatrix}\mathbf{1} & \mathbf{C}_k & \mathbf{C}_{k+1}\end{bmatrix} \right\rangle && (54)\\
&= \left\langle \begin{bmatrix} 2\mathbf{1} & & \\ & \mathbf{0} & -2\Delta\mathbf{C} \\ & & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{C}_k & \mathbf{C}_{k+1} \\ \mathbf{C}_k^\mathsf{T} & \mathbf{C}_k^\mathsf{T}\mathbf{C}_k & \mathbf{C}_k^\mathsf{T}\mathbf{C}_{k+1} \\ \mathbf{C}_{k+1}^\mathsf{T} & \mathbf{C}_{k+1}^\mathsf{T}\mathbf{C}_k & \mathbf{C}_{k+1}^\mathsf{T}\mathbf{C}_{k+1} \end{bmatrix} \right\rangle. && (55)
\end{aligned}
$$

D. Lifting a Prior

Given a prior such that the error is given by

$$
\begin{aligned}
J_p &= \|\mathbf{x} - \check{\mathbf{x}}\|_2^2 && (56)\\
&= (\mathbf{x} - \check{\mathbf{x}})^{\mathsf{T}}(\mathbf{x} - \check{\mathbf{x}}) && (57)\\
&= \mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\check{\mathbf{x}}^{\mathsf{T}}\mathbf{x} + \check{\mathbf{x}}^{\mathsf{T}}\check{\mathbf{x}} && (58)
\end{aligned}
$$

for $\mathbf{X} = \begin{bmatrix}\mathbf{1} & \mathbf{x}\end{bmatrix}$, this yields the following cost

$$
\begin{aligned}
J_p &= \mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\check{\mathbf{x}}^{\mathsf{T}}\mathbf{x} + \check{\mathbf{x}}^{\mathsf{T}}\check{\mathbf{x}} && (59)\\
&= \left\langle \mathbf{Q}, \mathbf{X}^{\mathsf{T}}\mathbf{X} \right\rangle && (60)\\
&= \left\langle \begin{bmatrix}\check{\mathbf{x}}^{\mathsf{T}}\check{\mathbf{x}} & -2\check{\mathbf{x}}^{\mathsf{T}} \\ & 1\end{bmatrix}, \begin{bmatrix}\mathbf{1} & \mathbf{x} \\ \mathbf{x} & \mathbf{x}^{\mathsf{T}}\mathbf{x}\end{bmatrix} \right\rangle && (61)
\end{aligned}
$$