# Word-Level Multi-Fix Rectifiability of Finite Field Arithmetic Circuits

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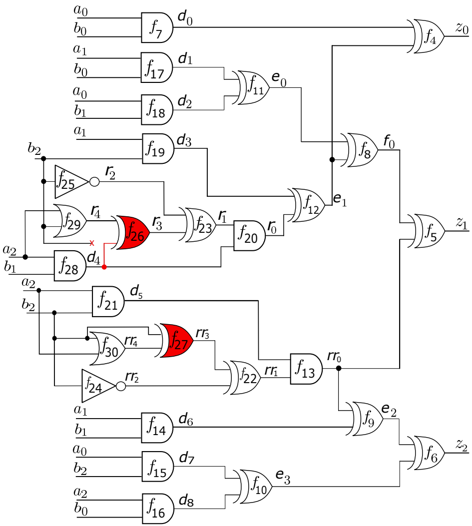
# Outline

- Problem Description
- Motivation and Application
- Preliminaries
- Multi-Fix setup
  - Mathematical challenges
- Rectifiability check
- Experimental results
- Conclusion and Future work

# Problem Description: Multi-error logic rectification




A faulty implementation of a 3-bit modulo multiplier

# Problem Description

- Agnostic to the fault model, check for rectification at particular targets
  - Single-fix Rectification (SFR)
    - Correct circuit by changing function at a single net
- In a general setting, SFR might not be desired or may not exist
  - Multi-fix Rectification (MFR)
    - Correct circuit by changing functions at multiple nets
    - Contribution: Multi-fix rectifiability setup and check

# Preliminaries: Finite field basics

- Fields - set of elements over which operations $(+, \cdot, /)$ can be performed
  - Ex. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$
- Finite fields (Galois fields) - Finite set of elements
  - Ex. $\mathbb{F}_q$, where $q = p^n, \ p \ = \ prime, \ n \in \mathbb{Z}_{\geq 1}$
    - When $n = 1$, $\mathbb{F}_p = \mathbb{Z}_p \pmod{p}$
    - With $p = 2$, $\mathbb{F}_2 = \mathbb{B} = \{0, 1\}$
  - On circuits, $p = 2$, $n =$ data-operand width
- Hardware cryptography extensively based on $\mathbb{F}_{2^n}$ (we use $\mathbb{F}_{2^n}$)

# Preliminaries: Modeling circuit polynomials over $\mathbb{F}_{2^n}$

- Boolean logic gates in $\mathbb{F}_2$ $(\mathbb{F}_2 \subset \mathbb{F}_{2^n})$. Over $\mathbb{F}_2$, $-1 = +1 \pmod 2$

$$
\begin{aligned}
z &= \ \sim a && \implies z + a + 1 && \pmod 2 \\
z &= a \wedge b && \implies z + a \cdot b && \pmod 2 \\
z &= a \vee b && \implies z + a \cdot b + a + b && \pmod 2 \\
z &= a \oplus b && \implies z + a + b && \pmod 2
\end{aligned}
$$

- Word-level polynomials [$\gamma$ = primitive element of $\mathbb{F}_{2^n}$]

$$\mathit{Output} : Z + z_0 + \gamma \cdot z_1 + \cdots + \gamma^{n-1} \cdot z_{n-1}$$

$$\mathit{Input} : A + a_0 + \gamma \cdot a_1 + \cdots + \gamma^{n-1} \cdot a_{n-1}$$

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width
  - Ex. $Z = A \cdot B \pmod{P_n(x)}$ over $\mathbb{F}_{2^n}$
- A faulty circuit implementation $C$ for specification $f$
  - Model gates as polynomials over $\mathbb{F}_{2^n}$
- A primitive polynomial $P_n(x)$ used to construct $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n}$ constructed as $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
  - Let $\gamma$ be one of the roots of $P_n(x)$, i.e. $P_n(\gamma) = 0$
- A set of $m$ targets from $C$ (modeled over $\mathbb{F}_{2^m}$)
- Check if $C$ is rectifiable at these $m$ targets

# Algebraic Geometry: Ideals

- Let $R = \mathbb{F}_{2^n}[x_1, \dots, x_d, Z]$
  - $\{f_1, \dots, f_s\} \in R$
- In our context
  - $x_1, \dots, x_d$: Variables (nets of the circuit)
  - $Z$: bit-vector representation for variables
  - $f_1, \dots, f_s$: Polynomials from the circuit (logic gate relations)
- $J = \langle F \rangle = \langle f_1, \dots, f_s \rangle \subseteq R$
  - $\{h_1 f_1 + \cdots + h_s f_s : \ h_i \in R\}$
  - Polynomials $f_1, \dots, f_s$: *basis* or *generators* of $J$
- Vanishing Ideal: $J_0 = \langle F_0 \rangle = \langle x_1^2 + x_1, \dots, x_d^2 + x_d, Z^{2^n} + Z \rangle$
  - Restrict solutions to $x_i$ in $\mathbb{F}_2$, and solutions to $Z$ in $\mathbb{F}_{2^n}$

- $J = \langle F \rangle = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
- Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{F}_{2^n}^d$ $s.t.$ $f_1(\boldsymbol{a}) = \cdots = f_s(\boldsymbol{a}) = 0$

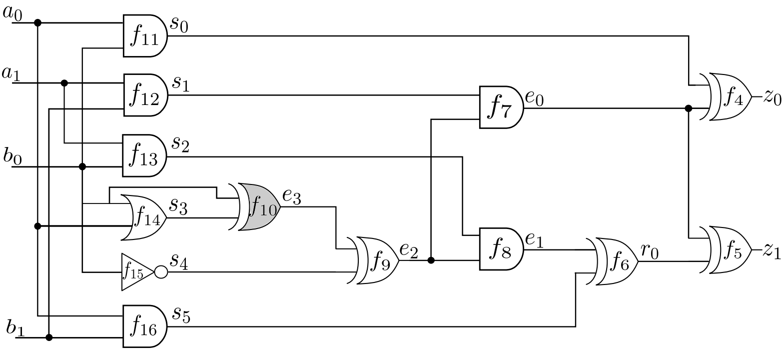
$$V(J) = \text{Set of all } \{\boldsymbol{a}\} \text{ s.t. } \begin{cases} f_1(\boldsymbol{a}) = 0, \\ f_2(\boldsymbol{a}) = 0, \\ \vdots \\ f_s(\boldsymbol{a}) = 0 \end{cases}$$

- $V(J)$ correspond to function mappings (Truth tables)

# Gröbner Basis and Ideal membership

- An ideal $J = \langle f_1, \ldots, f_s \rangle \subseteq R$ can have many generators.
  - $J = \langle p_1, \ldots, p_m \rangle = \cdots = \langle g_1, \ldots, g_t \rangle$
  - Gröbner Basis (GB) is one such set with special properties
- Let $J = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ and $G = GB(J) = \{g_1, \ldots, g_t\}$.
  - $G$ is a Gröbner basis of $J \iff \forall f \in J, f \xrightarrow{g_1, \ldots, g_t}_+ 0$
  - Ideal membership: Let $f$ be a polynomial in $R$:
    - if $f \xrightarrow{g_1, \ldots, g_t}_+ 0$, then $f$ is a member of $J$.

- Circuit designed over $\mathbb{F}_{2^n} = \mathbb{F}_{2^2} (n = 2)$ using irreducible polynomial $P_n(x) = P_2(x) = x^2 + x + 1$ with $P_2(\gamma) = 0$




A 2-bit faulty modulo multiplier implementation.

- Denote polynomial $f : Z + A \cdot B$ as the design specification.
- Impose RTTO $>$

$f_1 : Z + z_0 + \gamma \cdot z_1;$
$f_2 : A + a_0 + \gamma \cdot a_1;$
$f_3 : B + b_0 + \gamma \cdot b_1;$
$f_4 : z_0 + s_0 + e_0;$
$f_5 : z_1 + e_0 + r_0;$
$f_6 : r_0 + e_1 + s_5;$
$f_7 : e_0 + s_1 e_2;$
$f_8 : e_1 + s_2 e_2;$
$f_9 : e_2 + e_3 + s_4;$
$f_{10} : e_3 + b_0 + s_3;$
$f_{11} : s_0 + a_0 b_0;$
$f_{12} : s_1 + a_1 b_1;$
$f_{13} : s_2 + a_1 b_0;$
$f_{14} : s_3 + a_0 + b_0 + a_0 b_0;$
$f_{15} : s_4 + b_0 + 1;$
$f_{16} : s_5 + a_0 b_1;$

- $F = \{f_1, \ldots, f_{16}\}$, $F_0 = \{a_0^2 - a_0, a_1^2 - a_1, b_0^2 - b_0, b_1^2 - b_1\}$
- Ideal Membership Test: $f \xrightarrow{F,F_0}_+ \gamma^1 \cdot (a_0 a_1 b_1 b_0 + a_0 a_1 b_1 + a_1 b_1 b_0 + a_1 b_0) + \gamma^0 \cdot (a_0 a_1 b_1 b_0 + a_0 a_1 b_1 + a_1 b_1 b_0)$

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# SFR Application: Rectification Check

1. Rectification check at net $e_3$: $W = \{e_3\}$
   - $J_1 = \langle F_1 \rangle$, where $F_1 = \{f_1, \ldots, f_{10} = e_3 + 0, \ldots, f_{16}\}$
   - $J_2 = \langle F_2 \rangle$, where $F_2 = \{f_1, \ldots, f_{10} = e_3 + 1, \ldots, f_{16}\}$
2. Compute $r_1$ and $r_2$:
   - $r_1 = f \xrightarrow{J_1, J_0}_+ (\gamma + 1)a_1 b_1 b_0 + (\gamma + 1)a_1 b_1$
   - $r_2 = f \xrightarrow{J_2, J_0}_+ (\gamma + 1)a_1 b_1 b_0 + (\gamma)a_1 b_0$
3. Single-fix rectification possible iff $V(r_1) \cup V(r_2) = \mathbb{F}_{2^3}^{|X_{PI}|} = V(J_0)$
   - Compute $G = GB(r1 \cdot r2, J_0)$ and check if $G = J_0$
   - In this example, target $e_3$ admits SFR

# Unified framework motivation

- Single-fix is a special case of MFR with $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - Since $m = 1$ divides $n$, $\mathbb{F}_2 \subset \mathbb{F}_{2^n}, \forall n \in \mathbb{Z}_{>1}$
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$
  - Ex. $\mathbb{F}_{2^2}$ is not contained in $\mathbb{F}_{2^3}$, $m = 2, n = 3$
- Need a higher composite field $\mathbb{F}_{2^k}$ such that
  - $\mathbb{F}_{2^m} \subset \mathbb{F}_{2^k}$ and $\mathbb{F}_{2^n} \subset \mathbb{F}_{2^k}$
  - What are the mathematical challenges?
  - What primitive polynomial $P_K(x)$ should be used for constructing $\mathbb{F}_{2^k}$

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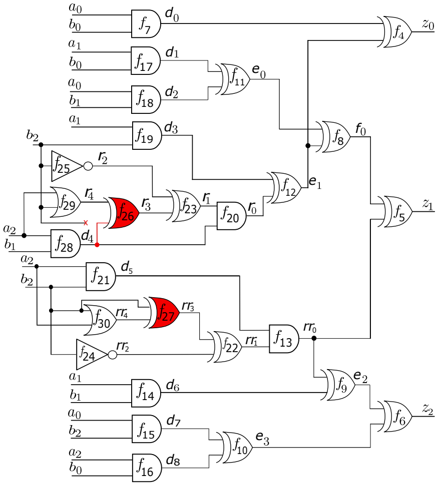
# Multi-fix Rectification: Prior work

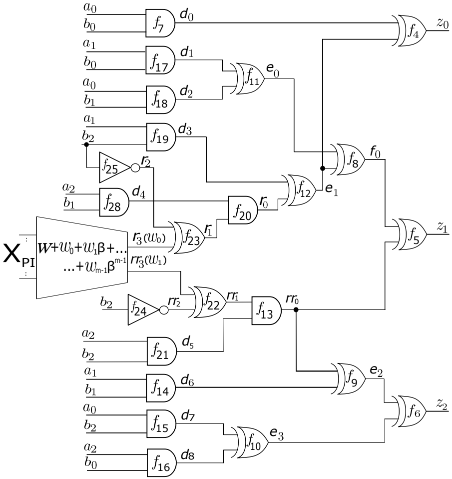
- Craig interpolation and/or iterative SAT solving [*Huang. et al*, DAC'11][*Huang. et al*, DATE'12]
  - Iteratively and incrementally patch the circuit
  - Compute multiple partial single-fix functions at the given *m* targets
- Resource aware ECO patch generation [*Jiang. et al*, DAC'18][*Mishchenko. et al*, DAC'18] [*Fujita. et al*, ISCAS'19]
- Approaches infeasible on arithmetic circuits
- Symbolic sampling technique [*Jiang. et al*, DAC'19]
  - Enumerate rectification points functionally and match the circuitry of patches implicitly
  - Scalability achieved by modeling computations in symbolic sampling domain

# Application: Multi-fix Rectification




A faulty implementation of a 3-bit ($n$=3) Mastrovito multiplier

# Application: Word-level representation




Patch function modeled as a 2-bit-vector word ($m$=2)

$$W = \{r_3, rr_3\} = r_3 + \beta \cdot rr_3, \; (w_0 = r_3, w_1 = rr_3).$$

# MFR Notation: Field setup

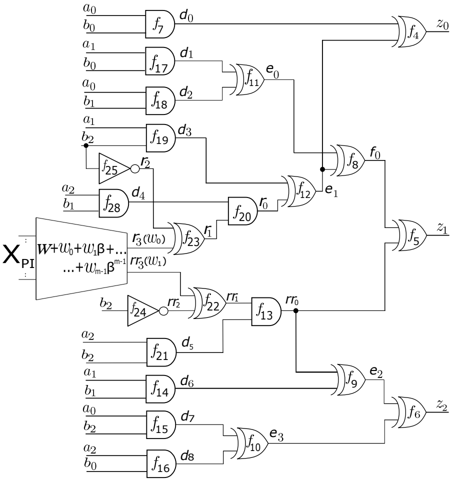
- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n}$ is constructed as $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - The word-level polynomials for $Z, A$ are modeled as:
    - $f_Z : Z + \sum_{i=0}^{n-1} \gamma^i z_i, \; f_A : A + \sum_{i=0}^{n-1} \gamma^i a_i$
- Patch size $m$ modeled over $\mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^m}$ is constructed as $\mathbb{F}_{2^m} = \mathbb{F}_2[x] \pmod{P_m(x)}$
    - We select a degree-$m$ primitive polynomial $P_m(x) \in \mathbb{F}_2[x]$; $P_m(\beta) = 0$
  - The word-level polynomial for $W$ is modeled as:
    - $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$
    - $\{w_0, \ldots, w_{m-1}\} \subset \{x_1, \ldots, x_d\}$

# MFR Challenges: $\mathbb{F}_{2^k}$ and $P_k(x)$

- Smallest $k$ is $LCM(n, m)$
  - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^k}$ is constructed as $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P_k(x)}$
    - $P_k(x)$ is a degree-$k$ primitive polynomial; $P_k(\alpha) = 0$
- Mathematical challenge: Given $P_n(x)$ and $P_m(x)$, compute $P_k(x)$ such that $P_n(\gamma) = P_m(\beta) = P_k(\alpha) = 0$
  - $\gamma = \alpha^{(2^k-1)/(2^n-1)} = \alpha^\lambda$
  - $\beta = \alpha^{(2^k-1)/(2^m-1)} = \alpha^\mu$
- Solved using factorization of univariate polynomials over finite fields

- Obtain UPFs of $P_n(x^\lambda)$ and $P_m(x^\mu)$ in $\mathbb{F}_2[x]$
- Then, $\exists P_k(x) \in \mathbb{F}_2[x]$ as a common factor of $P_n(x^\lambda)$ and $P_m(x^\mu)$, such that:
  - $P_k(x)$ is a degree-$k$ primitive polynomial in $\mathbb{F}_2[x]$ with $P_k(\alpha) = 0$

# Application: Word-level representation




Patch function modeled as a 2-bit-vector word (*m*=2)

$W = \{r_3, rr_3\} = r_3 + \beta \cdot rr_3$, $(w_0 = r_3, w_1 = rr_3)$.

# Application: Computing $P_k(x)$

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = al^9, \;\; \beta = \alpha^{21}$
- Composite field: $k = LCM(2,3) = 6$
  - $UPF(P_3(x^9)) = (x^9)^3 + (x^9) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^4 + x^2 + x + 1)(x^3 + x + 1);$
  - $UPF(P_2(x^{21})) = (x^{21})^2 + (x^{21}) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^5 + x^3 + x^2 + 1)(x^6 + x^5 + x^4 + x + 1)(x^6 + x + 1)(x^6 + x^3 + 1);$
  - We choose $P_6(x) = x^6 + x^5 + 1$ as the required $P_k(x)$.

# MFR Notation: Incorrect $P_k(x)$

- Note that if we incorrectly choose $P_k(x) = x^6 + x^3 + 1$
- For its root $\alpha$, we have

$$
\begin{aligned}
\alpha^6 + \alpha^3 + 1 &= 0 \\
(\alpha^3)(\alpha^6 + \alpha^3 + 1) &= 0 \text{ (multiply by } \alpha^3\text{)} \\
\alpha^9 + \alpha^6 + \alpha^3 &= 0 \\
\gamma + 1 &= 0
\end{aligned} \tag{1}
$$

- However, $\gamma \neq 1$, as $\gamma$ is a primitive element of $\mathbb{F}_{2^n}$
- Selecting arbitrary $P_k(x)$ leads to erroneous results

# MFR Notation: Word-level Formulation steps

- Modify field to $\mathbb{F}_{2^k}$ and compute $P_k(x)$
- Update ring by adding word-level target representation $W$
- Construct a polynomial set $F'$ as follows:
  - Start with $F' = F$
  - Remove polynomials with $w_i$'s as leading terms
  - Substitute for $w_i$'s the respective word-level polynomials
  - Add $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$

- 2-bit rectification patch over the 3-bit circuit can be performed over the field $\mathbb{F}_{2^6}$
  - Field $\mathbb{F}_{2^6} = \mathbb{F}_2[X] \pmod{P_6(X)}$
- Update polynomial set $F$ to $F'$ as:

$$F' = \{f_1, \ldots, f_3, f_4', f_5', f_6, f_7', f_8', f_9, f_w, f_{11}, f_{13} \ldots, f_{20}\}$$

$$\begin{aligned} f_4' &: z_0 + (\beta W^2 + \beta^2 W) + d_0; \quad & f_5' &: z_1 + f_0 + (W^2 + W); \\ f_7' &: f_0 + (\beta W^2 + \beta^2 W) + e_1; \quad & f_8' &: e_2 + (W^2 + W) + d_6; \\ f_w &: W + e_0 + \beta d_5; \quad & \beta &= \alpha^{21}; \gamma = \alpha^9; \end{aligned}$$

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# MFR Contribution: Rectification Check

- Multi-fix rectification at target $W$
  - Construct the following ideals:
    - $J_i = \langle F_i' \rangle = \{f_1', \dots, f_w = W + \delta(i), \dots, f_s'\}$ :$1 \leq i \leq 2^m$,
      $\delta(0) = 0, \delta(1) = 1, \delta(2) = \beta, \dots, \delta(2^m) = \beta^{2^m - 2}$
  - Performing the reductions for all $1 \leq i \leq 2^m$:
    - $f \xrightarrow{F_i', F_0}_+ r_i$
  - Let $V_{\mathbb{F}_q}(r_i)$ denote the varieties of the respective $r_i$'s
  - Multi-fix rectification exists at target $W$:

$$\textbf{if and only if } \bigcup_{i=1}^{2^m} V_{\mathbb{F}_q}(r_i) = \mathbb{F}_q^{|X_{PI}|} = V(J_0)$$

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# MFR Application: Rectification Check

- Constructing the $J_i$ ideals:
  - $J_1 = \langle F_1' \rangle$, where $F_1'[f_w] = W + \delta(1) = W$,
  - $J_2 = \langle F_2' \rangle$, where $F_2'[f_w] = W + \delta(2) = W + 1$,
  - $J_3 = \langle F_3' \rangle$, where $F_3'[f_w] = W + \delta(3) = W + \beta$,
  - $J_4 = \langle F_4' \rangle$, where $F_4'[f_w] = W + \delta(4) = W + \beta^2$
- Reducing the specification $f : Z + A \cdot B$ modulo these ideals, we get:
  - $rem_1 = f \xrightarrow{F_1' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2) + \alpha^{36}(a_2 b_2)$
  - $rem_2 = f \xrightarrow{F_2' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1) + \alpha^{36}(a_2 b_2)$
  - $rem_3 = f \xrightarrow{F_3' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2)$
  - $rem_4 = f \xrightarrow{F_4' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1)$
- Computing $GB(r_1 \cdot r_2 \cdot r_3 \cdot r_4, F_0) = F_0$
- Target $W$ with nets $r_3$ and $rr_3$ admits MFR

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# Future work: Rectification function

- A polynomial which can be computed to rectify the circuit
  - $W = a_2 b_1 b_2 + \beta \cdot a_2 b_2$
  - $r_3 = (a_2 \wedge b_1 \wedge b_2), \quad rr_3 = (a_2 \wedge b_2)$

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# Focus: Finite Field Arithmetic Circuits

- Applications:
  - RSA, ECC, Error correcting codes, RFID, etc.
    - Crypto-system bugs can leak secret keys [*Biham. et al*, Crypto'08]
    - RFID tag cloning could cause counterfeiting [*Batina. et al*, Security'09]
  - Large datapath sizes in ECC crypto systems
    - In $\mathbb{F}_{2^n}$, $n = 163, 233, 283, 409, 571$ (NIST standard)
- Rectification Motivation:
  - Synthesize sub-functions as opposed to complete redesign
  - Automated debugging

# MFR Experiments: SINGULAR Implementation

Table: Word-level multi-fix rectifiability check against word level specification. Time is in seconds; rows marked '*' indicates $m \nmid n$; Benchmark = Mastrovito architecture, $n$ = Datapath Size, #Gates = No. of gates, K = $10^3$, $m$ = patch size, $k$ = encompassing composite field size, PF = time for polynomial factorization and computing minpoly for the composite field, RC = time for rectification check

| n | #Gates | m | k | PF | RC |
|---|---|---|---|---|---|
| 12 | 0.45K | 2 | 12 | NA | 0.4 |
| 16 | 0.8K | 2 | 16 | NA | 3.2 |
| *16 | 0.8K | 3 | 48 | – | – |
| *20 | 0.0 | 3 | 60 | – | – |
| 32 | 2.8K | 2 | 32 | NA | 184 |
| 48 | 6.4K | 3 | 48 | NA | – |
| 64 | 11.2K | 2 | 64 | NA | – |

# MFR Experiments: Custom software

Table: Word-level multi-fix rectifiability check against word level specification. Time is in seconds; Benchmark = Mastrovito architecture, $n$ = Datapath Size, #Gates = No. of gates, K = $10^3$, $m$ = word length of patch function, $k$ = encompassing composite field size (degree of primpoly used), PF = time for polynomial factorization and computing minpoly for the composite field, PBS = PolyBori setup (ring declaration/poly collection/spec collection), VF = time for verification, MFS = Multi-fix check setup, MFRC = time for multi-fix rectification check, TE = Total execution time

| $n$ | **#Gates** | $m$ | $k$ | **PF** | **PBS** | *VF* | *MFS* | *MFRC* | *TE* |
|---|---|---|---|---|---|---|---|---|---|
| 12 | 0.45K | 2 | 12 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 |
| 12 | 0.45K | 3 | 12 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 |
| 16 | 0.8K | 2 | 16 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 |
| 16 | 0.8K | 3 | 48 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 | $<$0.01 |
| 32 | 2.8K | 2 | 32 | $<$0.01 | 0.1 | $<$0.01 | $<$0.01 | $<$0.01 | 0.15 |
| 64 | 11.2K | 2 | 64 | $<$0.1 | 0.5 | $<$0.01 | $<$0.01 | 0.2 | 0.9 |
| 96 | 24.5K | 2 | 96 | $<$0.1 | 1.4 | 0.1 | $<$0.01 | $<$0.01 | 1.7 |
| 128 | 43.2K | 2 | 128 | $<$0.3 | 3.1 | 0.3 | $<$0.1 | $<$0.1 | 3.6 |
| 163 | 69.8K | 2 | 326 | $<$0.4 | 6.2 | 2.0 | $<$0.1 | 0.4 | 7.5 |
| 233 | 119K | 2 | 466 | $<$1 | 13.0 | 0.9 | 0.15 | $<$0.1 | 14.3 |
| 283 | 190K | 2 | 566 | $<$2 | 39.0 | 2.1 | 0.2 | $<$0.1 | 41.3 |
| 409 | 384K | 2 | 818 | $<$2 | 190 | 3.5 | 0.5 | 0.1 | 195.4 |
| 571 | 827K | 2 | 1042 | $<$3 | 2170 | 9.1 | 1.1 | $<$0.1 | 2183 |

# Rectification function computation

- SFR of finite field arithmetic circuits [*Rao. et al*, FMCAD'18][*Rao. et al*, IWLS'18]
  - Quantification based computation
  - Alternate to Craig Interpolation

- Currently addressing function computation at a word-level for finite field arithmetic circuits:
  - Rectification function computation at multiple nets in terms of primary inputs [Due notification GLSVLSI'21]
    - Define and formulate existence of don't cares
    - Devise algorithms to explore don't cares for logic optimization
  - Formulate rectification setup in terms of internal nets of the circuit.
    - Explore word-level don't care formulation in terms of internal nets.
  - Extend the multi-fix approach to integer arithmetic circuits and address the associated challenges.

# Publications


[1] V. Rao, U. Gupta, I. Ilioaea, A. Srinath, P. Kalla, and F. Enescu, "Post-Verification Debugging and Rectification of Finite Field Arithmetic Circuits using Computer Algebra Techniques," in *Formal Methods in Computer Aided Design (FMCAD)*, Oct 2018, pp. 1–9.

[2] V. Rao, U. Gupta, I. Ilioaea, P. Kalla, and F. Enescu, "Resolving Unknown Components in Arithmetic Circuits using Computer Algebra Methods - poster presentation," in *International Workshop on Logic and Synthesis(IWLS)*, 2018.

[3] U. Gupta, I. Ilioaea, V. Rao, A. Srinath, P. Kalla, and F. Enescu, "Rectification of Arithmetic Circuits with Craig Interpolants in Finite Fields," in *VLSI-SoC: Design and Engineering of Electronics Systems Based on New Computing Paradigms*. Springer International Publishing, June 2019, vol. 561, ch. 5, pp. 79–106.

# Publications

[4] ——, "On the Rectifiability of Arithmetic Circuits using Craig Interpolants in Finite Fields," in *IFIP/IEEE Intl. Conf. on VLSI (VLSI-SoC)*, Oct 2018, pp. 49–54.

[5] U. Gupta, P. Kalla, and V. Rao, "Boolean Gröbner Basis Reductions on Finite Field Datapath Circuits Using the Unate Cube Set Algebra," *IEEE Trans. on CAD of Integrated Circuits and Systems*, vol. 38, no. 3, pp. 576–588, Mar 2019.

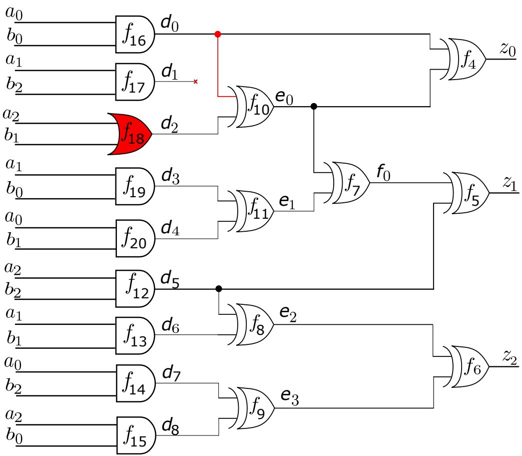
[6] U. Gupta, I. Ilioaea, P. Kalla, F. Enescu, V. Rao, and A. Srinath, "Craig Interpolants in Finite Fields using Algebraic Geometry: Theory and Applications," in *Intl. Workshop on Logic and Synthesis (IWLS)*, June 2018, pp. 70–77.

[7] U. Gupta, P. Kalla, and V. Rao, "Boolean Gröbner Basis Reductions on Datapath Circuits Using the Unate Cube Set Algebra," in *International Workshop on Logic & Synthesis*, pp. 124–131.

**THANK YOU!**

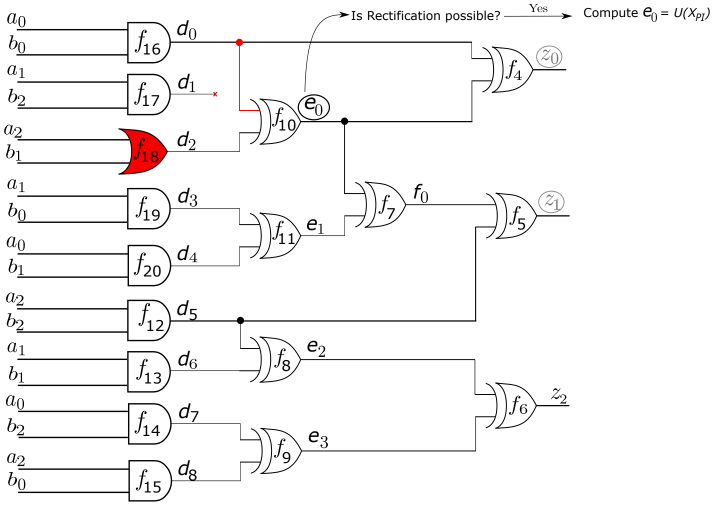
Questions?

$$Z = A \cdot B \pmod{P(X)}$$




A buggy implementation of a 3-bit modulo multiplier

# Problem Description: Rectification



# Finite Field Notations

- Finite (Galois) Field $\mathbb{F}_q$:
  - Set of $q$ finitely many elements. $q = p^n$, $p = prime$
- $\mathbb{F}_2 = \mathbb{B} = \{0, 1\}$
- On circuits, $p = 2$, $n =$ data-operand width
- Hardware cryptography extensively based on $\mathbb{F}_{2^n}$ (we use $\mathbb{F}_{2^n}$)
- $\mathbb{F}_2 \subset \mathbb{F}_{2^n}$, $n > 1$
- Contribution: Application to integer arithmetic circuits
  - Infinite sets: More investigation needed

# Modeling Circuits using Polynomials

- Circuit $C$ modeled as polynomials
- Boolean logic gates in $\mathbb{F}_2$ ($\mathbb{F}_2 \subset \mathbb{F}_{2^n}$); Over $\mathbb{F}_2$ $-1 = +1 \pmod 2$)

$$
\begin{aligned}
z &= \neg a \rightarrow z + a + 1 \pmod 2 \\
z &= a \wedge b \rightarrow z + a \cdot b \pmod 2 \\
z &= a \vee b \rightarrow z + a + b + a \cdot b \pmod 2 \\
z &= a \oplus b \rightarrow z + a + b \pmod 2
\end{aligned}
$$

- Specification in $\mathbb{F}_{2^n}$, $f_{spec} : Z + AB$
- Word level polynomials [$\gamma =$ Primitive element of $\mathbb{F}_{2^n}$]
  - Output: $Z + z_0 + \gamma z_1 + \gamma^2 z_2 + \cdots + \gamma^{n-1} z_{n-1}$,
  - Input: $A + \sum_{i=0}^{n-1} \gamma^i a_i$, and so on

# Polynomial Ring

- Given $\{x_1, \ldots, x_d\}$
  - Monomial $X = x_1^{e_1} \cdot x_2^{e_2} \cdots x_d^{e_d}$, where $e_i \in \mathbb{Z}_{\geq 0}, i \in \{1, \ldots, d\}$
  - Polynomial $f = c_1 X_1 + c_2 X_2 + \cdots + c_t X_t$; $c_i \in \mathbb{F}_{2^n}$
- All such $f$ form the ring $R = \mathbb{F}_{2^n}[x_1, \ldots, x_d]$
- Multivariate polynomials: need to order the monomials
- Impose monomial order "$>$" on $R$
  - We utilize *lex* term order
- $f = c_1 X_1 + c_2 X_2 + \cdots + c_t X_t$ (with lex order)
  - $lt(f) = c_1 X_1, \; lm(f) = X_1, \; lc(f) = c_1$

# Sum, Product, and Quotient of Ideals

Given $J_1 = \langle f_1, \ldots, f_s \rangle \in R$ and $J_2 = \langle h_1, \ldots, h_r \rangle \in R$

- Sum of ideals:
  - $J_1 + J_2 = \langle f_1, \ldots, f_s, h_1 \ldots, h_r \rangle$
- Product of ideals:
  - $J_1 \cdot J_2 = \langle f_i \cdot h_j : 1 \leq i \leq s, 1 \leq j \leq r \rangle$
- Ideal quotient of $J_1$ by $J_2$:
  - $J_1 : J_2 = \{ f \in R \mid f \cdot h \in J_1, \forall h \in J_2 \}$
- Ideals and varieties are dual concepts
  - $V(J_1 + J_2) = V(J_1) \cap V(J_2)$
  - $V(J_1 \cdot J_2) = V(J_1) \cup V(J_2)$
  - $V(J_1 : J_2) = V(J_1) - V(J_2)$

# Vanishing Ideals

- For variables in circuit ideals:
  - Bit-level $x_i$: $x_i^2 - x_i$ or $x_i^2 + x_i$ as $-1 = +1 \pmod 2$ over $\mathbb{F}_{2^n}$
  - Word-level $Z$, $A$: $Z^{2^n} - Z$, $A^{2^n} - A$
- Vanishing Ideal:
  $J_0 = \langle F_0 \rangle = \langle x_1^2 + x_1, \ldots, x_d^2 + x_d, Z^{2^n} + Z, A^{2^n} + A \rangle$
- Vanishing Ideal purpose:
  - Restrict solutions to $x_i$ in $\mathbb{F}_2$
  - Restrict solutions to $Z, A$ in $\mathbb{F}_{2^n}$
- For circuits [*Lv. et al*, TCAD'13]
  - Only need $J_0^{X_{PI}} = \langle F_0^{X_{PI}} \rangle = \langle x_i^2 + x_i : x_i \in X_{PI} \rangle$ added to $J$

# MFR Notations: Composite Field

- For a given circuit with data-path size $n$
  - Polynomials modeled over $R = \mathbb{F}_{2^n}[Z, A, x_1, \ldots, x_d]$
    - $\{x_1, \ldots, x_d\}$ are all the bit-level variables (nets) in the circuit
    - $Z$ and $A$ are the word-level output and input, respectively
  - $\mathbb{F}_{2^n}$ is constructed as $\mathbb{F}_{2^n} = \mathbb{F}_2[X] \pmod{P_n(X)}$
    - $P_n(X) \in \mathbb{F}_2[X]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - The word-level polynomials for $Z, A$ are modeled as:
    - $f_z : Z + \sum_{i=0}^{n-1} \gamma^i z_i$; $f_a : A + \sum_{i=0}^{n-1} \gamma^i a_i$;
- Patch $W$ for $m$ targets is computed as a polynomial function in the field $\mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^m}$ is constructed as $\mathbb{F}_{2^m} = \mathbb{F}_2[X] \pmod{P_m(X)}$
    - We select a degree-$m$ primitive polynomial $P_m(X) \in \mathbb{F}_2[X]$; $P_m(\beta) = 0$
  - The word-level polynomial for $W$ is modeled as:
    - $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$
    - $\{w_0, \ldots, w_{m-1}\} \subset \{x_1, \ldots, x_d\}$

## MFR Notations: Composite Field

- Determine the smallest single field ($\mathbb{F}_{2^k}$) to operate both circuit ($\mathbb{F}_{2^n}$) and patch ($\mathbb{F}_{2^m}$)
- Smallest $k$ is $LCM(n, m)$
  - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^k}$ is constructed as $\mathbb{F}_{2^k} = \mathbb{F}_2[X] \pmod{P_k(X)}$
    - $P_k(X)$ is a degree-$k$ primitive polynomial; $P_k(\alpha) = 0$
- Mathematical challenge: Given $P_n(X)$ and $P_m(X)$, compute $P_k(X)$ such that $P_n(\gamma) = P_m(\beta) = P_k(\alpha) = 0$
  - $\gamma = \alpha^{(2^k-1)/(2^n-1)} = \alpha^\lambda$
  - $\beta = \alpha^{(2^k-1)/(2^m-1)} = \alpha^\mu$
- Solved using factorization of univariate polynomials over finite fields

## MFR Notations: Univariate Polynomial factorization (UPF)

- Given a monic univariate polynomial $f \in \mathbb{F}_q[X]$, where $\mathbb{F}_q$ is any finite field
    - Find a complete factorization $f = f_1^{e_1} \cdot f_2^{e_2} \cdots f_l^{e_l}$
        - Where $f_1, f_2, \ldots, f_l$ are pairwise distinct monic irreducible polynomials in $\mathbb{F}_q[X]$ and $e_1, \ldots, e_l$ are positive integers.

# MFR Notations: Finding Primitive Polynomial $P_k(X)$

- Obtain UPFs of $P_n(X^\lambda)$ and $P_m(X^\mu)$
  - Coefficients will be in $\mathbb{F}_2$ and degrees will be less than $\lambda$ and $\mu$, respectively.
    - $P_n(X^\lambda) = P_{n1}^{a1} \cdot P_{n2}^{a2} \cdots P_{nl}^{al}$, and
    - $P_m(X^\mu) = P_{m1}^{b1} \cdot P_{m2}^{b2} \cdots P_{mg}^{bg}$
- Conjecture: $\exists P_{ni}(X) \in \{P_{n1}, P_{n2}, \ldots, P_{nl}\}$ and $\exists P_{mj}(X) \in \{P_{m1}, P_{m2}, \ldots, P_{mg}\}$, such that:
  - $P_k(X) = P_{ni}(X) = P_{mj}(X)$,
  - $P_k(X)$ is a degree-$k$ primitive polynomial in $\mathbb{F}_2[X]$ such that $P_k(\alpha) = 0$

# MFR Application: Verification

- Circuit designed using irreducible polynomial $P(X) = X^3 + X + 1$ with $P(\gamma) = 0$
- Denote polynomial $f : Z + A \cdot B$ as the design specification.
- Impose RTTO $>$

$$
\begin{array}{ll}
f_1 : Z + z_0 + \gamma z_1 + \gamma^2 z_2; & f_{11} : e_1 + d_3 + d_4; \\
f_2 : A + a_0 + \gamma a_1 + \gamma^2 a_2; & f_{12} : d_5 + a_2 + b_2; \\
f_3 : B + b_0 + \gamma b_1 + \gamma^2 b_2; & f_{13} : d_6 + a_1 b_1; \\
f_4 : z_0 + e_0 + d_0; & f_{14} : d_7 + a_0 b_2; \\
f_5 : z_1 + f_0 + d_5; & f_{15} : d_8 + a_2 b_0; \\
f_6 : z_2 + e_2 + e_3; & f_{16} : d_0 + a_0 b_0; \\
f_7 : f_0 + e_0 + e_1; & f_{17} : d_1 + a_1 b_2; \\
f_8 : e_2 + d_5 + d_6; & f_{18} : d_2 + a_2 + b_1 + a_2 b_1; \\
f_9 : e_3 + d_7 + d_8; & f_{19} : d_3 + a_1 b_0; \\
f_{10} : e_0 + d_0 + d_2; & f_{20} : d_4 + a_0 b_1;
\end{array}
$$

- Polynomial Set
  - $F = \{f_1, \dots, f_{20}\}$
  - $F_0^{PI} = \{a_0^2 - a_0, a_1^2 - a_1, a_2^2 - a_2, b_0^2 - b_0, b_1^2 - b_1, b_2^2 - b_2\}$
- $f \xrightarrow{F, F_0^{PI}}_+ = \gamma^2(a_2b_2 + a_2 + b_2) + \gamma(a_0b_0 + a_1b_2 + b_1 + a_2b_2 + b_2) + (1)(a_0b_0 + a_1b_2 + b_1 + a_2)$
- Set of affected outputs: $\mathcal{O}_a = \{z_0, z_1, z_2\}$
- Intersection of set of nets in fan-in cones of $\mathcal{O}_a$ is $\emptyset$
  - Implies no SFR points
- We select *m*=2 and see if the circuit can be rectified by changing functions at two nets

# MFR Application: Selecting $m$ Targets

- Since all the outputs are affected, all the nets in the circuit are initial candidate targets
  - $\mathcal{I}_n = \{z_0, z_1, z_2, f_0, e_2, e_3, e_0, e_1, d_5, d_6, d_7, d_8, d_0, d_2, d_3, d_4\}$
  - Associate a cost for each net driven by synthesis constraints
    - Nets which lie in the intersection of multiple outputs are assigned lowest cost
    - Rest of the nets are assigned cost based on their topological level in the design
    - $\mathcal{I}_c = \{4, 4, 4, 3, 2, 2, -2, 2, -2, 1, 1, 1, -2, -2, 1, 1\}$
- Solved as weighted set cover problem
  - Partition $\mathcal{O}_a$ into $m$ distinct non-empty subsets such that
    - Intersection of fan-in cones of output bits within a subset is non-empty
  - If such a cover $\mathcal{M}$ exists ($|\mathcal{M}| = m$), each of the $m$ targets are selected from the $m$ distinct covers
    - $\mathcal{O}_a = \{\{z_0, z_1\}, \{z_2\}\}$
    - $\mathcal{M} = \{\mathcal{M}_0 : \{e_0, d_0, d_2\}, \mathcal{M}_1 : \{d_5, d_6, d_7, d_8, e_2, e_3, z_2\}\}$

# MFR Application: Word-level Formulation

- Update ring properties
  - $R = \mathbb{F}_q[x_1, \ldots, x_d, Z, A, W]$
  - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $e_0$
    - $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \{f_0 > e_2 > e_3\} > \{\mathbf{W} > e_0 > e_1 > d_5 > d_6 > d_7 > d_8\} > \{d_0 > d_1 > d_2 > d_3 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$.
- Update polynomial set $F$ to $F'$:
  - Delete polynomials for $w_i$'s
  - Delete polynomials in the transitive fan-in of $w_i$'s only
  - Transitive fan-outs of $w_i$'s need to be replaced with their equivalent word-level representations in terms of $W$
  - Add $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$

- Composite field: $k = LCM(2,3) = 6$
  - $UPF(P_3(X^9)) = \{\mathbf{X^6+X^4+X^3+X+1}, X^6+X^4+X^2+X+1, \mathbf{X^6+X^5+1}, X^6+X^5+X^2+X+1\}$
  - $UPF(P_2(X^{21})) = \{\mathbf{X^6+X^4+X^3+X+1}, \mathbf{X^6+X^5+1}, X^6+X^3+1, X^6+X^5+X^2+X+1, X^6+X^5+X^3+X^2+1, X^6+X+1, X^6+X^5+X^4+X+1\}$
  - We will pick $P_6(X) = X^6+X^4+X^3+X+1$ as the primitive polynomial to setup the unified framework.

# MFR Notations: Incorrect Primitive Polynomial

- Note that if we incorrectly choose $P_k(X) = X^6 + X^3 + 1$
- For its root $\alpha$, we have

$$\begin{gathered} \alpha^6 + \alpha^3 + 1 = 0 \\ (\alpha^3)(\alpha^6 + \alpha^3 + 1) = 0 \text{ (multilying by } \alpha^3) \\ \alpha^9 + \alpha^6 + \alpha^3 = 0 \\ \gamma + 1 = 0 \end{gathered}$$

- But we have $\gamma = \alpha^9$
- Selecting arbitrary $P_k(X)$ leads to erroneous results

# MFR Application: Word-level Formulation

- 2-bit rectification patch over the 3-bit circuit can be performed over the field $\mathbb{F}_{2^6}$
  - Field $\mathbb{F}_{2^6} = \mathbb{F}_2[X] \pmod{P_6(X)}$
- Update polynomial set $F$ to $F'$ as:

$$F' = \{f_1, \dots, f_3, f_4', f_5', f_6, f_7', f_8', f_9, f_w, f_{11}, f_{13} \dots, f_{20}\}$$

$$\begin{aligned} f_4' &: z_0 + (\beta W^2 + \beta^2 W) + d_0; & f_5' &: z_1 + f_0 + (W^2 + W); \\ f_7' &: f_0 + (\beta W^2 + \beta^2 W) + e_1; & f_8' &: e_2 + (W^2 + W) + d_6; \\ f_w &: W + e_0 + \beta d_5; & \beta &= \alpha^{21}; \gamma = \alpha^9; \end{aligned}$$

- Multi-fix rectification at target $W$
  - Construct the following ideals:
    - $J_i = \langle F_i' \rangle = \{f_1', \ldots, f_w = W + \delta(i), \ldots, f_s'\}$ :$1 \leq i \leq 2^m$, $\delta(0) = 0, \delta(1) = 1, \delta(2) = \beta, \ldots, \delta(2^m) = \beta^{2^m-2}$
  - Performing the reductions for all $1 \leq i \leq 2^m$:
    - $f \xrightarrow{F_i', F_0^{PI}}_+ r_i$
  - Let $V_{\mathbb{F}_q}(r_i)$ denote the varieties of the respective $r_i$'s
  - Multi-fix rectification exists at target $W$:

$$\textbf{if and only if } \bigcup_{i=1}^{2^m} V_{\mathbb{F}_q}(r_i) = \mathbb{F}_q^{|X_{PI}|} = V(J_0^{PI})$$

# MFR Application: Rectification Check

- Constructing the $J_i$ ideals:
  - $J_1 = \langle F_1' \rangle$, where $F_1'[f_w] = W + \delta(1) = W$,
  - $J_2 = \langle F_2' \rangle$, where $F_2'[f_w] = W + \delta(2) = W + 1$,
  - $J_3 = \langle F_3' \rangle$, where $F_3'[f_w] = W + \delta(3) = W + \beta$,
  - $J_4 = \langle F_4' \rangle$, where $F_4'[f_w] = W + \delta(4) = W + \beta^2$
- Reducing the specification $f : Z + A \cdot B$ modulo these ideals, we get:
  - $r_1 = f \xrightarrow{F_1', F_0^{PI}}_+ a_1 b_2 \gamma^3 + a_2 b_1 \gamma^3 + \gamma^4 a_2 b_2$
  - $r_2 = f \xrightarrow{F_2', F_0^{PI}}_+ a_1 b_2 \gamma^3 + a_2 b_1 \gamma^3 + \gamma^4 a_2 b_2 + \gamma^3$
  - $r_3 = f \xrightarrow{F_3', F_0^{PI}}_+ a_1 b_2 \gamma^3 + a_2 b_1 \gamma^3 + \gamma^4 a_2 b_2 + \gamma^4$
  - $r_4 = f \xrightarrow{F_4', F_0^{PI}}_+ a_1 b_2 \gamma^3 + a_2 b_1 \gamma^3 + \gamma^4 a_2 b_2 + \gamma^6$
- Computing $GB(r_1 \cdot r_2 \cdot r_3 \cdot r_4, F_0^{PI}) = F_0^{PI}$
- Target $W$ with nets $e_0$ and $d_5$ admits MFR

# MFR Notation: Computing Rectification Function

- Compute a rectification function of the form $W = U(X_{PI})$
  - Here $U$ is the *unknown component* computed as an $m$-bit-vector word
  - It represents the function $W = \sum_{i=0}^{m-1} \beta^i u_i$
    - Where $u_i$'s represent the individual Boolean functions for the respective $w_i$'s.
- The *unknown component* problem is then formulated as an ideal membership test and solved using extended Gröbner Basis:

$$W + \beta^0 e_0 + \beta d_5 = W + U = W + \beta^0(a_1 b_2 + a_2 b_1) + \beta a_2 b_2;$$

$$e_0 = a_1 b_2 + a_2 b_1; \quad d_5 = a_2 b_2;$$

# Research Objective: Synthesis of Rectification Function

- Exploring don't cares
  - We computed $U = b_0$, i.e. $f_{10} = e_3 + b_0$
  - We utilized quotient of ideals to compute alternate corrections
    - $U^1 = a_1 * b_0$
    - $U^2 = a_1 * b_1 * b_0 + a_1 * b_1 + a_1$
  - Polynomial $U$ depends on the polynomial $h_i$ (quotient of division by target $x_i$)
  - $h_i$ actually represents the ODCs for the selected target $x_i$
- Algorithmic computation of rectification polynomials
- word-level formulation of don't cares

# Research Objective: Integer arithmetic circuits

- Techniques valid over fields are inapplicable over rings
- Gröbner basis and division algorithms are complicated
- Can be modeled over $\mathbb{Q}$
  - Rectification function computation can result in *fractional coefficients*
  - Extracting Boolean rectification function requires exhaustive simulation
  - No scope of optimization as Extended Gröbner basis technique gives zero control

# Research Objective: Improving Scalability

- Enhance implementation for finite field circuits:
  - Rectification formulation in terms of internal nets
  - Address word-level formulation and the mathematical challenges
  - Devise efficient algorithms based on ZDDs

- Implementation for Integer arithmetic circuits
  - Bit level reduction technique increases the verification time exponentially
    - No monomial cancellations across output bits
  - Need implicit data structure with a word-level representation