Word-Level Multi-Fix Rectifiability of Finite Field Arithmetic Circuits

isQED Symposium 2021

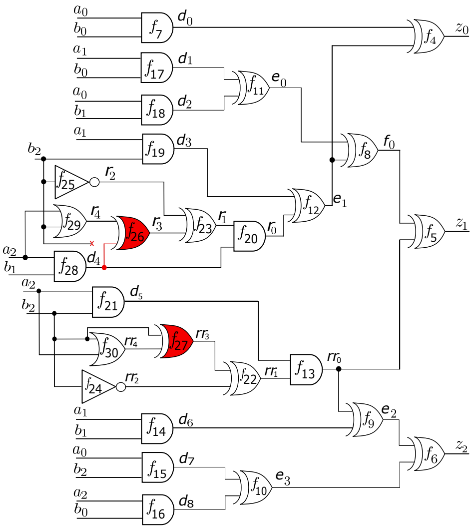
Vikas Rao[1], Irina Ilioaea[2], Haden Ondricek[1], Priyank Kalla[1], and Florian Enescu[3]

[1]Electrical & Computer Engineering, University of Utah
[2]Department of Mathematics, Louisiana State University Shreveport
[3]Mathematics & Statistics, Georgia State University

Outline

- Problem Description and Motivation
- Preliminaries
- Unified Framework
  - Mathematical Challenges
- Rectifiability Check
- Implementation
- Experimental Results
- Summary and Future work

Problem Description: Rectification



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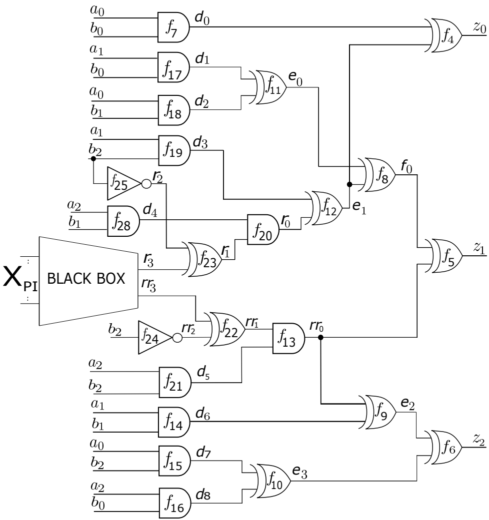

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- Applications:
  - Cryptography: RSA, Ellyptic Curve Cryptography (ECC)
  - Error Correcting Codes, Digital Signal Processing, RFID, etc.

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- Rectification Motivation:
  - Automated debugging
  - Synthesize sub-functions as opposed to complete redesign

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- Hardware cryptography extensively based on $\mathbb{F}_{2^n}$ (we use $\mathbb{F}_{2^n}$)

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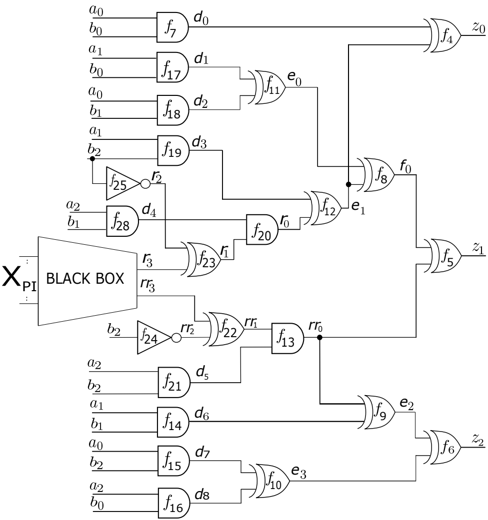
    - Root of one is not the same as the other

Problem Description: Field Containment



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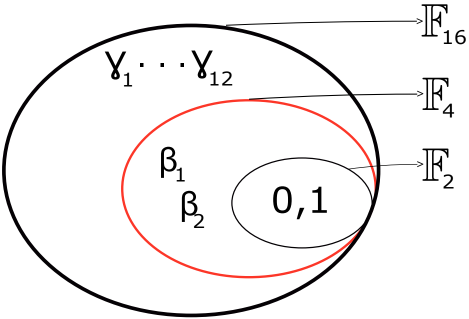

$$\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{16}$$

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Unified framework motivation

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- Circuit modeled over $\mathbb{F}_{2^n}$
  - $P_n(x)$ is given, and let $P_n(\gamma) = 0$
- Rectification patch modeled over $\mathbb{F}_{2^m}$
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- Multi-fix: $m > 1$, $\mathbb{F}_{2^m} \not\subset \mathbb{F}_{2^n}$
  - Ex. For $m = 2$ *and* $n = 3$, $\mathbb{F}_{2^2} \not\subset \mathbb{F}_{2^3}$

MFR Challenges: $\mathbb{F}_{2^k}$ and $P_k(x)$

- Smallest k is $LCM(n, m)$
    - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
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  - How are elements α, β, and γ related?

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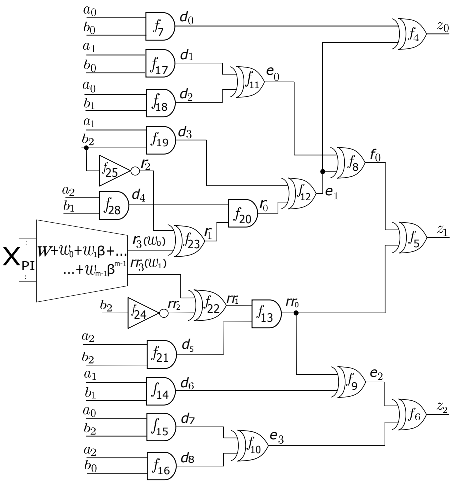
- Univariate Polynomial Factorization (UPF)
  - Obtain UPFs of $P_n(x^{\lambda})$ and $P_m(x^{\mu})$ in $\mathbb{F}_2[x]$
- Then, $\exists P_k(x) \in \mathbb{F}_2[x]$ as a common factor of $P_n(x^{\lambda})$ and $P_m(x^{\mu})$, such that:
  - $P_k(x)$ is a degree-k primitive polynomial in $\mathbb{F}_2[x]$ with $P_k(\alpha) = 0$

Application: Word-level representation



Patch function modeled as a 2-bit-vector word (m=2), $f_W : W + r_3 + \beta \cdot rr_3$

Application: Computing $P_k(x)$

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  - $UPF(P_2(x^{21})) = (x^{21})^2 + (x^{21}) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^5 + x^3 + x^2 + 1)(x^6 + x^5 + x^4 + x + 1)(x^6 + x + 1)(x^6 + x^3 + 1);$

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Circuit polynomials over $\mathbb{F}_{2^k}$

- Boolean logic gates in $\mathbb{F}_2$ $(\mathbb{F}_2 \subset \mathbb{F}_{2^k})$. Over $\mathbb{F}_2$, $-1 = +1 \pmod 2$

$$
\begin{aligned}
z &= \ \sim a & &\implies z + a + 1 & &\pmod 2 \\
z &= a \wedge b & &\implies z + a \cdot b & &\pmod 2 \\
z &= a \vee b & &\implies z + a \cdot b + a + b & &\pmod 2 \\
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- Word-level polynomials with $\gamma = \alpha^\lambda$ and $\beta = \alpha^\mu$

$$
\begin{aligned}
\mathit{Output} &: Z + z_0 + \gamma \cdot z_1 + \cdots + \gamma^{n-1} \cdot z_{n-1} \\
\mathit{Input} &: A + a_0 + \gamma \cdot a_1 + \cdots + \gamma^{n-1} \cdot a_{n-1} \\
\mathit{Patch} &: W + w_0 + \beta \cdot w_1 + \cdots + \beta^{m-1} \cdot w_{m-1}
\end{aligned}
$$

Circuit Polynomials and Setup

- Ring $R = \mathbb{F}_{2^k}[Z, A, B, \ldots, W, r_3, rr_3, \ldots, a_0, a_1, \ldots, b_1, b_2]$

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- Circuit polynomials under a term order $>$:

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\begin{array}{rl}
f_1 : Z + z_0 + \gamma \cdot z_1 + \gamma^2 \cdot z_2; & f_{22} : rr_1 + rr_3 + rr_2; \\
f_2 : A + a_0 + \gamma \cdot a_1 + \gamma^2 \cdot a_2; & f_{23} : r_1 + r_2 + r_3; \\
f_3 : B + b_0 + \gamma \cdot b_1 + \gamma^2 \cdot b_2; & f_{26} : r_3 + r_4 + d_4; \\
f_4 : z_0 + d_0 + e_1; & f_{27} : rr_3 + rr_4 + b_2; \\
f_5 : z_1 + f_0 + rr_0; & \ldots \\
\ldots & f_{30} : rr_4 + a_2 + b_2 + a_2 b_2; \\
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f_4 : z_0 + d_0 + e_1; & f_{27} : rr_3 + rr_4 + b_2; \\
f_5 : z_1 + f_0 + rr_0; & \ldots \\
\ldots & f_{30} : rr_4 + a_2 + b_2 + a_2 b_2; \\
\ldots & f_W : W + r_3 + \beta \cdot rr_3;
\end{array}
$$

- $F = \{f_1, \ldots, f_{30}, f_W\}$
- $F_0 = \{a_0^2 - a_0, \ldots, z_2^2 - z_2, A^8 - A, \ldots, Z^8 - Z, W^4 - W\}$.

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MFR Contribution: Rectification Check

- Multi-fix rectification at target W
  - Construct the following polynomial sets:

$$F'_l = \langle f_1, \ldots, f_W = W + \delta[l], \ldots, f_s \rangle, 1 \leq l \leq 2^m,$$
$$\textit{Where } (\delta[1], \ldots, \delta[2^m]) = (0, 1, \beta, \ldots, \beta^{2^m-2}).$$

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  - Reduce the specification $f : Z + A \cdot B$ modulo these sets:
    - $f \xrightarrow{F'_l, F_0}_+ rem_l, \forall 1 \leq l \leq 2^m$

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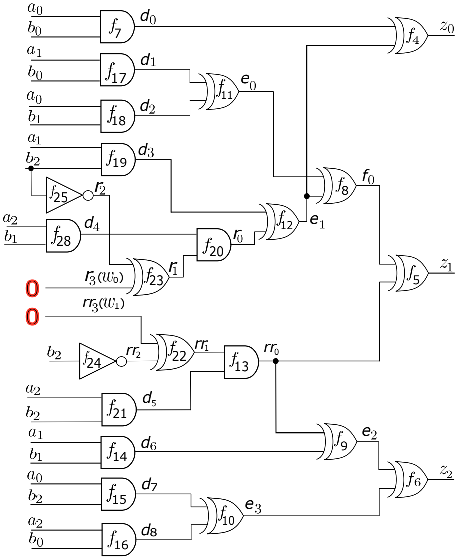
    - $f \xrightarrow{F_l', F_0}_+ rem_l, \forall 1 \leq l \leq 2^m$
  - Multi-fix rectification exists at target W:

if and only if $\prod_{l=1}^{2^m} rem_l \xrightarrow{F_0}_+ 0$


a0
b0
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f4
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b0
f17
d1
f11
e0
a0
b1
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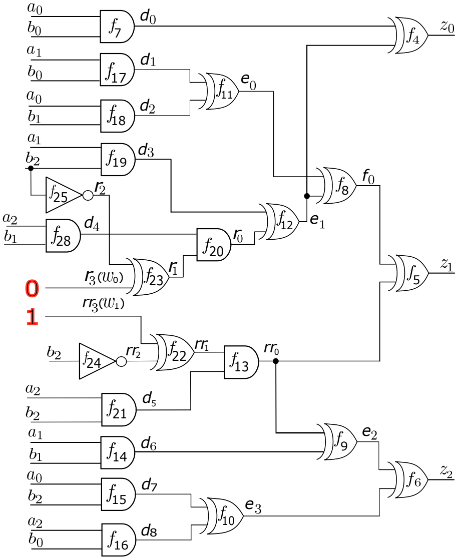
a1
b2
f19
d3
f8
f0
f25
r2
f12
e1
a2
b1
f28
d4
f20
r0
f23
r1
f5
z1
0
r3(w0)
0
rr3(w1)
f22
rr1
f13
rr0
b2
f24
rr2
a2
b2
f21
d5
a1
b1
f14
d6
f9
e2
f6
z2
a0
b2
f15
d7
f10
e3
a2
b0
f16
d8

Application: Remainder generation



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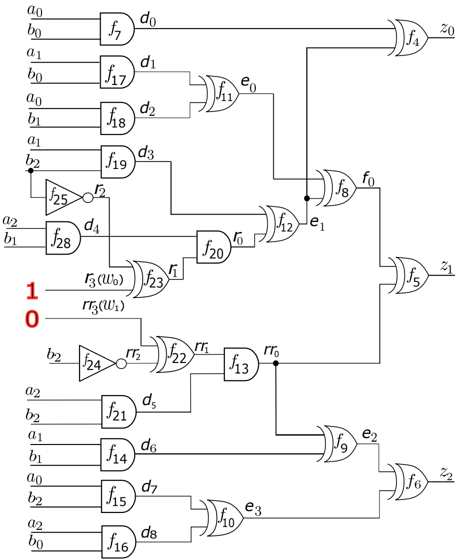

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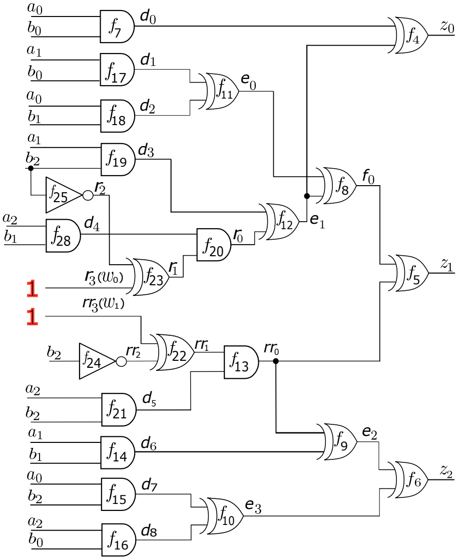

- Constructing the F_l':
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  - F_2', where $F_2'[f_W] = W + \delta(2) = W + 1$,
  - F_3', where $F_3'[f_W] = W + \delta(3) = W + \beta$,
  - F_4', where $F_4'[f_W] = W + \delta(4) = W + \beta^2$

MFR Application: Rectification Check

- Constructing the F_l':
  - F_1', where $F_1'[f_W] = W + \delta(1) = W$,
  - F_2', where $F_2'[f_W] = W + \delta(2) = W + 1$,
  - F_3', where $F_3'[f_W] = W + \delta(3) = W + \beta$,
  - F_4', where $F_4'[f_W] = W + \delta(4) = W + \beta^2$
- Reducing the specification $f : Z + A \cdot B$:
  - $rem_1 = f \xrightarrow{F_1' \cup F_0}_+ \alpha^{27}(a_2 b_1 b_2) + \alpha^{36}(a_2 b_2)$
  - $rem_2 = f \xrightarrow{F_2' \cup F_0}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1) + \alpha^{36}(a_2 b_2)$
  - $rem_3 = f \xrightarrow{F_3' \cup F_0}_+ \alpha^{27}(a_2 b_1 b_2)$
  - $rem_4 = f \xrightarrow{F_4' \cup F_0}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1)$

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- $rem_1 \cdot rem_2 \cdot rem_3 \cdot rem_4 \xrightarrow{F_0}_+ 0$
- Target W with nets r_3 and rr_3 admits MFR

Implementation: Boolean Polynomials and ZDDs

- Boolean polynomials as unate cube sets
  - Monomial: a product of positive literals or a cube
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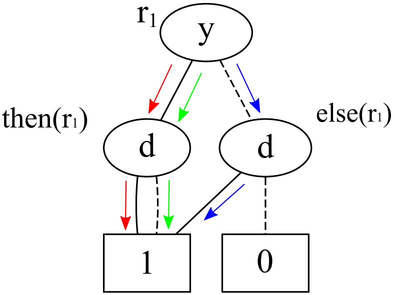
- $r_1 = yd + y + d$ as $\{yd, y, d\}$

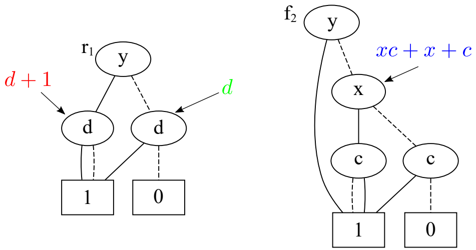


Paths terminating in 1: yd, y, d.

Improved Reduction Using ZDDs

- $r_1 = yd + y + d$, $f_2 = y + xc + x + c$, $r_1 \xrightarrow{f_2}_{+}$

$$
\begin{aligned}
&(yd + y + d) + (d + 1) \cdot (y + xc + x + c) \pmod 2 \\
&= 2 \cdot (yd + y) + d + (d + 1) \cdot (xc + x + c) \pmod 2 \\
&= d + (d + 1) \cdot (xc + x + c) \pmod 2
\end{aligned}
$$



- One step reduction: *else*(r_1) + *then*$(r_1) \cdot$ *else*(f_2) [Algorithm 6]

- Custom software:

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- Custom software:
  - PolyBori reduction procedure for remainder generation
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- Experiments performed on a 3.5GHz Intel(R) Core$^{\text{TM}}$ i7-4770K Quad-Core CPU with 32 GB RAM

Overall Flow

MFR Experiments: Mastrovito

n = Datapath Size, m = target word size, k = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

n	***m***	***k***	**AM**	**#G**	**#BO**	**PBS**	**VMS**	**RC**	**TE**
16	5	80	100	0.8	6	0.04	0.06	0.12	0.22
32	5	160	120	2.8	8	0.13	0.12	0.4	0.65
163	5	815	550	69.8	6	6.04	3.36	11.9	21.3
233	2	466	750	119	3	13	1.2	0.01	14.2
283	2	566	1300	190	2	38	4.2	0.1	42.3
409	2	818	2400	384	2	190	5	0.1	195
571	2	1042	5000	827	5	2150	12	0.1	2162

MFR Experiments: Montgomery

n = Datapath Size, m = target word size, k = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

n	*m*	*k*	AM	#G	#BO	PBS	VMS	RC	TE
16	5	80	100	0.9	16	0.04	0.56	35.6	36
32	5	160	120	2.8	32	0.13	0.57	27.6	28.3
163	5	815	550	57.5	128	5.2	6.8	262	274
233	2	466	750	112	233	11.5	3.5	360	375
283	2	566	1300	171	283	35	11	1503	1549
409	2	818	2400	340	409	134	10	4920	5064
571	2	1042	5000	663	12	1313	82	0.2	1395

MFR Experiments: Point Addition

n = Datapath Size, m = target word size, k = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

n	*m*	*k*	AM	#G	#BO	PBS	VMS	RC	TE
16	5	80	100	0.9	7	0.06	0.11	1.73	1.9
32	5	160	120	2.9	13	0.18	0.8	134	135
163	5	815	550	71.6	22	15.7	4.7	15	35.4
233	2	466	750	122	233	19.2	2.15	0.15	21.5
283	2	566	1300	208	4	80.4	6.1	0.1	86.6
409	2	818	2400	368	409	220	10	2007	2237
571	2	1042	5000	813	5	2583	27	880	3490

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Conclusion and Future work

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- Define and formulate existence of don't cares at the word-level
- Extend the approach to integer arithmetic circuits

THANK YOU

Email: vikas.k.rao@utah.edu