# Word-Level Multi-Fix Rectifiability of Finite Field Arithmetic Circuits

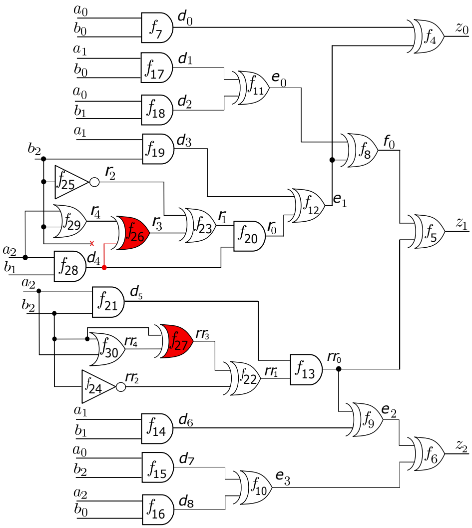
**Vikas Rao**[1], Irina Ilioaea[2], Haden Ondricek[1], Priyank Kalla[1], and Florian Enescu[3]

[1]Electrical & Computer Engineering, University of Utah
[2]Department of Mathematics, Louisiana State University Shreveport
[3]Mathematics & Statistics, Georgia State University

# Outline

- Problem Description
- Preliminaries
- Problem Statement and Objective
- Single-fix Application
- Multi-Fix setup
  - Mathematical Challenges
- Rectifiability Check
- Experimental Results
- Summary and Future work

# Problem Description:Rectification




A faulty implementation of a 3-bit modulo multiplier ($Z = A \cdot B \mod P(x)$)

- Agnostic to the fault model, check for rectification at particular targets

isQED® Symposium 2021

# Problem Description: Rectification

- Agnostic to the fault model, check for rectification at particular targets
  - Single-fix Rectification (SFR)
    - Correct circuit by changing function at a single net

# Problem Description: Rectification

- Agnostic to the fault model, check for rectification at particular targets
  - Single-fix Rectification (SFR)
    - Correct circuit by changing function at a single net
- In a general setting, SFR might not be desired or may not exist

# Problem Description: Rectification

- Agnostic to the fault model, check for rectification at particular targets
  - Single-fix Rectification (SFR)
    - Correct circuit by changing function at a single net
- In a general setting, SFR might not be desired or may not exist
  - Multi-fix Rectification (MFR)
    - Correct circuit by changing functions at multiple nets
    - Contribution: Multi-fix rectifiability setup and check

# Preliminaries: Finite fields

- Fields - set of elements over which operations $(+, \cdot, /)$ can be performed
  - Ex. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$

# Preliminaries: Finite fields

- Fields - set of elements over which operations $(+, \cdot, /)$ can be performed
  - Ex. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$
- Finite fields (Galois fields) - Finite set of elements
  - Ex. $\mathbb{F}_q$, where $q = p^n, \; p \; = \; prime, \; n \in \mathbb{Z}_{\geq 1}$
    - With $n = 1, \; and \; p = 2$, $\mathbb{F}_2 = \mathbb{B} = \{0, 1\}$
  - On circuits, $p = 2$, $n =$ data-operand width

# Preliminaries: Finite fields

- Fields - set of elements over which operations $(+, \cdot, /)$ can be performed
  - Ex. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$
- Finite fields (Galois fields) - Finite set of elements
  - Ex. $\mathbb{F}_q$, where $q = p^n,\ p\ =\ prime,\ n \in \mathbb{Z}_{\geq 1}$
    - With $n = 1,\ and\ p = 2$, $\mathbb{F}_2 = \mathbb{B} = \{0, 1\}$
  - On circuits, $p = 2$, $n =$ data-operand width
- Hardware cryptography extensively based on $\mathbb{F}_{2^n}$ (we use $\mathbb{F}_{2^n}$)

# Preliminaries: Circuit polynomials over $\mathbb{F}_{2^n}$

- Boolean logic gates in $\mathbb{F}_2$ $(\mathbb{F}_2 \subset \mathbb{F}_{2^n})$. Over $\mathbb{F}_2$, $-1 = +1 \pmod 2$

$$
\begin{aligned}
z &= \;\sim a && \implies z + a + 1 && \pmod 2 \\
z &= a \wedge b && \implies z + a \cdot b && \pmod 2 \\
z &= a \vee b && \implies z + a \cdot b + a + b && \pmod 2 \\
z &= a \oplus b && \implies z + a + b && \pmod 2
\end{aligned}
$$

# Preliminaries: Circuit polynomials over $\mathbb{F}_{2^n}$

- Boolean logic gates in $\mathbb{F}_2$ $(\mathbb{F}_2 \subset \mathbb{F}_{2^n})$. Over $\mathbb{F}_2$, $-1 = +1 \pmod 2$

$$
\begin{aligned}
z &= \ \sim a & &\implies z + a + 1 & &\pmod 2 \\
z &= a \wedge b & &\implies z + a \cdot b & &\pmod 2 \\
z &= a \vee b & &\implies z + a \cdot b + a + b & &\pmod 2 \\
z &= a \oplus b & &\implies z + a + b & &\pmod 2
\end{aligned}
$$

- Word-level polynomials [$\gamma$ = primitive element of $\mathbb{F}_{2^n}$]

$$
\begin{aligned}
\mathit{Output} &: Z + z_0 + \gamma \cdot z_1 + \cdots + \gamma^{n-1} \cdot z_{n-1} \\
\mathit{Input} &: A + a_0 + \gamma \cdot a_1 + \cdots + \gamma^{n-1} \cdot a_{n-1}
\end{aligned}
$$

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width
- A faulty circuit implementation $C$ for specification $f$
  - Model gates as polynomials over $\mathbb{F}_{2^n}$

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width
- A faulty circuit implementation $C$ for specification $f$
  - Model gates as polynomials over $\mathbb{F}_{2^n}$
- A primitive polynomial $P_n(x)$ used to construct $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width
- A faulty circuit implementation $C$ for specification $f$
  - Model gates as polynomials over $\mathbb{F}_{2^n}$
- A primitive polynomial $P_n(x)$ used to construct $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
- A set of $m$ targets from $C$
  - Model it over $\mathbb{F}_{2^m}$ - challenges?

# Problem Statement and Objective

- A multivariate specification polynomial $f \in \mathbb{F}_{2^n}$
  - $n$ is the operand width
- A faulty circuit implementation $C$ for specification $f$
  - Model gates as polynomials over $\mathbb{F}_{2^n}$
- A primitive polynomial $P_n(x)$ used to construct $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
- A set of $m$ targets from $C$
  - Model it over $\mathbb{F}_{2^m}$ - challenges?
- **Check if $C$ is rectifiable at these $m$ targets**

- Let $R = \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{f_1, \ldots, f_s\} \in R$

- Let $R = \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{f_1, \ldots, f_s\} \in R$
- In our context
  - $x_1, \ldots, x_d$: Variables (nets of the circuit)
  - $Z$: bit-vector representation for variables
  - $f_1, \ldots, f_s$: Polynomials from the circuit (logic gate relations)

# Preliminaries: Variables, Polynomials and Monomial order

- Let $R = \mathbb{F}_{2^n}[x_1, \dots, x_d, Z]$
  - $\{f_1, \dots, f_s\} \in R$
- In our context
  - $x_1, \dots, x_d$: Variables (nets of the circuit)
  - $Z$: bit-vector representation for variables
  - $f_1, \dots, f_s$: Polynomials from the circuit (logic gate relations)
- Multivariate polynomials: Impose monomial order "$>$" on $R$
  - We utilize lexicographic term order

# Preliminaries: Variables, Polynomials and Monomial order

- Let $R = \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{f_1, \ldots, f_s\} \in R$
- In our context
  - $x_1, \ldots, x_d$: Variables (nets of the circuit)
  - $Z$: bit-vector representation for variables
  - $f_1, \ldots, f_s$: Polynomials from the circuit (logic gate relations)
- Multivariate polynomials: Impose monomial order "$>$" on $R$
  - We utilize lexicographic term order
- Vanishing Polynomials: $F_0 = \langle x_1^2 + x_1, \ldots, x_d^2 + x_d, Z^{2^n} + Z \rangle$
  - Restrict solutions to $x_i$ in $\mathbb{F}_2$
  - Restrict solutions to $Z$ in $\mathbb{F}_{2^n}$

- $J = \langle F \rangle = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{h_1 f_1 + \cdots + h_s f_s : \; h_i \in R\}$

- $J = \langle F \rangle = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{h_1 f_1 + \cdots + h_s f_s : \ h_i \in R\}$
- Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{F}_{2^n}^d$ $s.t.$ $f_1(\boldsymbol{a}) = \cdots = f_s(\boldsymbol{a}) = 0$

$$V(J) = \text{Set of all } \{\boldsymbol{a}\} \text{ s.t. } \begin{cases} f_1(\boldsymbol{a}) = 0, \\ f_2(\boldsymbol{a}) = 0, \\ \vdots \\ f_s(\boldsymbol{a}) = 0 \end{cases}$$

- $J = \langle F \rangle = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z]$
  - $\{h_1 f_1 + \cdots + h_s f_s : \; h_i \in R\}$
- Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{F}_{2^n}^d$ $s.t.$ $f_1(\boldsymbol{a}) = \cdots = f_s(\boldsymbol{a}) = 0$

$$V(J) = \text{Set of all } \{\boldsymbol{a}\} \text{ s.t. } \begin{cases} f_1(\boldsymbol{a}) = 0, \\ f_2(\boldsymbol{a}) = 0, \\ \vdots \\ f_s(\boldsymbol{a}) = 0 \end{cases}$$

- $V(J)$ correspond to function mappings (Truth tables)

# Gröbner Basis and Ideal membership

- An ideal can have many generators.
  - $J = \langle f_1, \ldots, f_s \rangle = \langle p_1, \ldots, p_m \rangle = \cdots = \langle g_1, \ldots, g_t \rangle$
  - Gröbner Basis (GB) is one such set with special properties

# Gröbner Basis and Ideal membership

- An ideal can have many generators.
  - $J = \langle f_1, \ldots, f_s \rangle = \langle p_1, \ldots, p_m \rangle = \cdots = \langle g_1, \ldots, g_t \rangle$
  - Gröbner Basis (GB) is one such set with special properties
- Let $J = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ and $G = \{g_1, \ldots, g_t\}$.
  - $G$ is a Gröbner basis of $J$ $\iff$ $\forall f \in J, f \xrightarrow{g_1, \ldots, g_t}_+ 0$
  - Ideal membership: Let $f$ be a polynomial in $R$:
    - if $f \xrightarrow{g_1, \ldots, g_t}_+ 0$, then $f$ is a member of $J$.

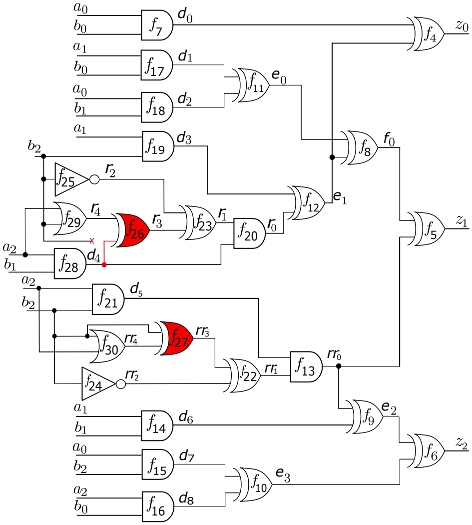
# Gröbner Basis and Ideal membership

- An ideal can have many generators.
  - $J = \langle f_1, \ldots, f_s \rangle = \langle p_1, \ldots, p_m \rangle = \cdots = \langle g_1, \ldots, g_t \rangle$
  - Gröbner Basis (GB) is one such set with special properties

- Let $J = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ and $G = \{g_1, \ldots, g_t\}$.
  - $G$ is a Gröbner basis of $J \iff \forall f \in J, f \xrightarrow{g_1, \ldots, g_t}_+ 0$
  - Ideal membership: Let $f$ be a polynomial in $R$:
    - if $f \xrightarrow{g_1, \ldots, g_t}_+ 0$, then $f$ is a member of $J$.

# Preliminaries: Term order

- Reverse Topological Term Order (RTTO)
  - Exploit circuit structure to avoid expensive GB computation
  - Standard practice to order variables topologically from POs to PIs




A faulty implementation of a 3-bit modulo multiplier ($Z = A \cdot B \mod P_3(x)$)

# Preliminaries: Term order and Polynomials

- RTTO $>$: $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \cdots > \{d_1 > d_2 > d_3 > r_0 > d_5 > rr_1\} > \{r_1 > rr_3 > rr_2\} > \{r_2 > r_3 > rr_4\} > \{r_4 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$

iSQED® Symposium 2021

- RTTO $>$: $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \cdots > \{d_1 > d_2 > d_3 > r_0 > d_5 > rr_1\} > \{r_1 > rr_3 > rr_2\} > \{r_2 > r_3 > rr_4\} > \{r_4 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$
- Circuit polynomials under RTTO $>$:

$$
\begin{aligned}
f_1 &: Z + z_0 + \gamma \cdot z_1 + \gamma^2 \cdot z_2; & f_{22} &: rr_1 + rr_3 + rr_2; \\
f_2 &: A + a_0 + \gamma \cdot a_1 + \gamma^2 \cdot a_2; & f_{23} &: r_1 + r_2 + r_3; \\
f_3 &: B + b_0 + \gamma \cdot b_1 + \gamma^2 \cdot b_2; & f_{26} &: r_3 + r_4 + d_4; \\
f_4 &: z_0 + d_0 + e_1; & f_{27} &: rr_3 + rr_4 + b_2; \\
f_5 &: z_1 + f_0 + rr_0; & & \ldots \\
& \ldots & f_{30} &: rr_4 + a_2 + b_2 + a_2 b_2;
\end{aligned}
$$

- RTTO $>$: $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \cdots > \{d_1 > d_2 > d_3 > r_0 > d_5 > rr_1\} > \{r_1 > rr_3 > rr_2\} > \{r_2 > r_3 > rr_4\} > \{r_4 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$
- Circuit polynomials under RTTO $>$:

$$\begin{array}{rl} f_1 : Z + z_0 + \gamma \cdot z_1 + \gamma^2 \cdot z_2; & f_{22} : rr_1 + rr_3 + rr_2; \\ f_2 : A + a_0 + \gamma \cdot a_1 + \gamma^2 \cdot a_2; & f_{23} : r_1 + r_2 + r_3; \\ f_3 : B + b_0 + \gamma \cdot b_1 + \gamma^2 \cdot b_2; & f_{26} : r_3 + r_4 + d_4; \\ f_4 : z_0 + d_0 + e_1; & f_{27} : rr_3 + rr_4 + b_2; \\ f_5 : z_1 + f_0 + rr_0; & \ldots \\ \ldots & f_{30} : rr_4 + a_2 + b_2 + a_2 b_2; \end{array}$$

- $F = \{f_1, \ldots, f_{30}\}$, $F_0 = \{a_0^2 - a_0, \ldots, z_2^2 - z_2, A^8 - A, \ldots, Z^8 - Z\}$.
  - Ideal $J + J_0 = \langle F \cup F_0 \rangle$ models $C$.

# Application: Single-Fix Rectification

- Denote polynomial $f : Z + A \cdot B$ as the design specification.

# Application: Single-Fix Rectification

- Denote polynomial $f : Z + A \cdot B$ as the design specification.
- Ideal $J + J_0 = \langle F \cup F_0 \rangle$ representing circuit $C$.

- Denote polynomial $f : Z + A \cdot B$ as the design specification.
- Ideal $J + J_0 = \langle F \cup F_0 \rangle$ representing circuit $C$.
- Circuit designed over $\mathbb{F}_{2^n} = \mathbb{F}_{2^3} (n = 3)$ using $P_n(x) = P_3(x) = x^3 + x + 1$ with $P_3(\gamma) = 0$

- Denote polynomial $f : Z + A \cdot B$ as the design specification.
- Ideal $J + J_0 = \langle F \cup F_0 \rangle$ representing circuit $C$.
- Circuit designed over $\mathbb{F}_{2^n} = \mathbb{F}_{2^3} (n = 3)$ using $P_n(x) = P_3(x) = x^3 + x + 1$ with $P_3(\gamma) = 0$
- **Is this circuit rectifiable at net $r_3$?**

# SFR Application: Rectification Check

1. Rectification check at net $r_3$:
   - $J_1 = \langle F_1 \rangle$, where $F_1 = \{f_1, \dots, f_{26} = r_3 + 0, \dots, f_{30}\}$
   - $J_2 = \langle F_2 \rangle$, where $F_2 = \{f_1, \dots, f_{26} = r_3 + 1, \dots, f_{30}\}$

# SFR Application: Rectification Check

1. Rectification check at net $r_3$:
   - $J_1 = \langle F_1 \rangle$, where $F_1 = \{f_1, \ldots, f_{26} = r_3 + 0, \ldots, f_{30}\}$
   - $J_2 = \langle F_2 \rangle$, where $F_2 = \{f_1, \ldots, f_{26} = r_3 + 1, \ldots, f_{30}\}$
2. Compute $rem_1$ and $rem_2$:
   - $rem_1 = f \xrightarrow{J_1, J_0}_{+} (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma^2 + \gamma) \cdot a_2 b_2$
   - $rem_2 = f \xrightarrow{J_2, J_0}_{+} (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma + 1) \cdot a_2 b_1 + (\gamma^2 + \gamma) \cdot a_2 b_2$

# SFR Application: Rectification Check

1. Rectification check at net $r_3$:
   - $J_1 = \langle F_1 \rangle$, where $F_1 = \{f_1, \dots, f_{26} = r_3 + 0, \dots, f_{30}\}$
   - $J_2 = \langle F_2 \rangle$, where $F_2 = \{f_1, \dots, f_{26} = r_3 + 1, \dots, f_{30}\}$
2. Compute $rem_1$ and $rem_2$:
   - $rem_1 = f \xrightarrow{J_1, J_0}_+ (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma^2 + \gamma) \cdot a_2 b_2$
   - $rem_2 = f \xrightarrow{J_2, J_0}_+ (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma + 1) \cdot a_2 b_1 + (\gamma^2 + \gamma) \cdot a_2 b_2$
3. SFR possible **iff** $V(rem_1) \cup V(rem_2) = \mathbb{F}_{2^3}^{|X_{PI}|} = V(J_0)$

# SFR Application: Rectification Check

1. Rectification check at net $r_3$:
   - $J_1 = \langle F_1 \rangle$, where $F_1 = \{f_1, \dots, f_{26} = r_3 + 0, \dots, f_{30}\}$
   - $J_2 = \langle F_2 \rangle$, where $F_2 = \{f_1, \dots, f_{26} = r_3 + 1, \dots, f_{30}\}$
2. Compute $rem_1$ and $rem_2$:
   - $rem_1 = f \xrightarrow{J_1, J_0}_+ (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma^2 + \gamma) \cdot a_2 b_2$
   - $rem_2 = f \xrightarrow{J_2, J_0}_+ (\gamma + 1) \cdot a_2 b_1 b_2 + (\gamma + 1) \cdot a_2 b_1 + (\gamma^2 + \gamma) \cdot a_2 b_2$
3. SFR possible **iff** $V(rem_1) \cup V(rem_2) = \mathbb{F}_{2^3}^{|X_{PI}|} = V(J_0)$
   - Compute $G = GB(rem_1 \cdot rem_2, J_0)$ and check if $G = J_0$
   - In this example, target $r_3$ doesn't admit SFR

iSQED® Symposium 2021

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}, 1 \mid n, \mathbb{F}_2 \subset \mathbb{F}_{2^n}$,

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}, 1 \mid n, \mathbb{F}_2 \subset \mathbb{F}_{2^n},$
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}$, $1 \mid n$, $\mathbb{F}_2 \subset \mathbb{F}_{2^n}$,
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$
  - Ex. For $m = 2, n = 3$, $2 \nmid 3$, $\mathbb{F}_{2^2} \not\subset \mathbb{F}_{2^3}$

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}, 1 \mid n, \mathbb{F}_2 \subset \mathbb{F}_{2^n},$
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$
  - Ex. For $m = 2, n = 3$, $2 \nmid 3$, $\mathbb{F}_{2^2} \not\subset \mathbb{F}_{2^3}$
- Composite field $\mathbb{F}_{2^k}$
  - $\mathbb{F}_{2^m} \subset \mathbb{F}_{2^k}$ and $\mathbb{F}_{2^n} \subset \mathbb{F}_{2^k}$

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}, 1 \mid n, \mathbb{F}_2 \subset \mathbb{F}_{2^n},$
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$
  - Ex. For $m = 2, n = 3$, $2 \nmid 3$, $\mathbb{F}_{2^2} \not\subset \mathbb{F}_{2^3}$
- Composite field $\mathbb{F}_{2^k}$
  - $\mathbb{F}_{2^m} \subset \mathbb{F}_{2^k}$ and $\mathbb{F}_{2^n} \subset \mathbb{F}_{2^k}$
  - What are the mathematical challenges?

# Unified framework motivation

- For Single-fix, $m = 1$
  - Rectification patch modeled over $\mathbb{F}_{2^m} = \mathbb{F}_{2^1} = \mathbb{F}_2$
  - Circuit modeled over $\mathbb{F}_{2^n}$
    - $\forall n \in \mathbb{Z}_{>1}, 1 \mid n, \mathbb{F}_2 \subset \mathbb{F}_{2^n},$
- For Multi-fix, since $m > 1$, $\mathbb{F}_{2^m}$ might not be contained in $\mathbb{F}_{2^n}$
  - Ex. For $m = 2, n = 3, 2 \nmid 3, \mathbb{F}_{2^2} \not\subset \mathbb{F}_{2^3}$
- Composite field $\mathbb{F}_{2^k}$
  - $\mathbb{F}_{2^m} \subset \mathbb{F}_{2^k}$ and $\mathbb{F}_{2^n} \subset \mathbb{F}_{2^k}$
  - What are the mathematical challenges?
  - What $P_K(x)$ should be used for constructing $\mathbb{F}_{2^k}$

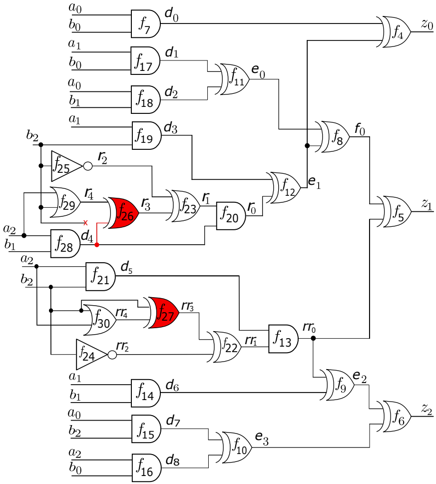
# Multi-fix Rectification: Prior work

- Craig interpolation and/or iterative SAT solving [*Huang. et al*, DAC'11][*Huang. et al*, DATE'12]
  - Iteratively and incrementally patch the circuit
  - Compute multiple partial single-fix functions at the given $m$ targets
- Resource aware ECO patch generation [*Jiang. et al*, DAC'18][*Mishchenko. et al*, DAC'18] [*Fujita. et al*, ISCAS'19]
- Symbolic sampling technique [*Jiang. et al*, DAC'19]
  - Enumerate rectification points functionally and match the circuitry of patches implicitly
  - Scalability achieved by modeling computations in symbolic sampling domain

# Multi-fix Rectification: Prior work

- Craig interpolation and/or iterative SAT solving [*Huang. et al*, DAC'11][*Huang. et al*, DATE'12]
  - Iteratively and incrementally patch the circuit
  - Compute multiple partial single-fix functions at the given *m* targets
- Resource aware ECO patch generation [*Jiang. et al*, DAC'18][*Mishchenko. et al*, DAC'18] [*Fujita. et al*, ISCAS'19]
- Symbolic sampling technique [*Jiang. et al*, DAC'19]
  - Enumerate rectification points functionally and match the circuitry of patches implicitly
  - Scalability achieved by modeling computations in symbolic sampling domain

- Approaches infeasible on arithmetic circuits




A faulty implementation of a 3-bit ($n$=3) Mastrovito multiplier



isQED Symposium 2021

# MFR Notation: Field setup

- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$

# MFR Notation: Field setup

- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$

# MFR Notation: Field setup

- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - Word-level polynomials for $Z, A$:
    - $f_Z : Z + \sum_{i=0}^{n-1} \gamma^i z_i, \ f_A : A + \sum_{i=0}^{n-1} \gamma^i a_i$

# MFR Notation: Field setup

- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - Word-level polynomials for $Z, A$:
    - $f_Z : Z + \sum_{i=0}^{n-1} \gamma^i z_i,\ f_A : A + \sum_{i=0}^{n-1} \gamma^i a_i$
- Patch size $m$ modeled over $\mathbb{F}_{2^m}$

# MFR Notation: Field setup

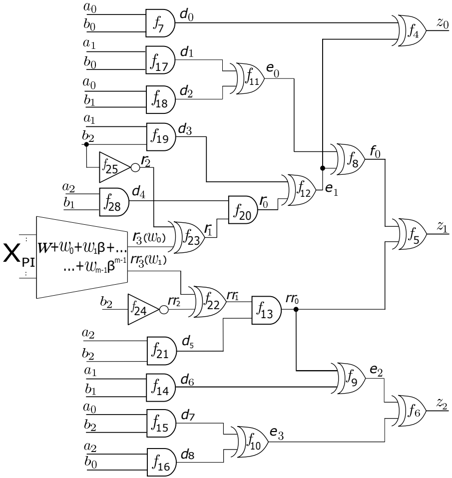
- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - Word-level polynomials for $Z, A$:
    - $f_Z : Z + \sum_{i=0}^{n-1} \gamma^i z_i, \ f_A : A + \sum_{i=0}^{n-1} \gamma^i a_i$

- Patch size $m$ modeled over $\mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^m} = \mathbb{F}_2[x] \pmod{P_m(x)}$
    - We select a degree-$m$ primitive polynomial $P_m(x) \in \mathbb{F}_2[x]$; $P_m(\beta) = 0$

# MFR Notation: Field setup

- Circuit with data-path size $n$ modeled over $\mathbb{F}_{2^n}$
  - $\mathbb{F}_{2^n} = \mathbb{F}_2[x] \pmod{P_n(x)}$
    - $P_n(x) \in \mathbb{F}_2[x]$ is a given degree-$n$ primitive polynomial; $P_n(\gamma) = 0$
  - Word-level polynomials for $Z, A$:
    - $f_Z : Z + \sum_{i=0}^{n-1} \gamma^i z_i,\ f_A : A + \sum_{i=0}^{n-1} \gamma^i a_i$
- Patch size $m$ modeled over $\mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^m} = \mathbb{F}_2[x] \pmod{P_m(x)}$
    - We select a degree-$m$ primitive polynomial $P_m(x) \in \mathbb{F}_2[x]$; $P_m(\beta) = 0$
  - Word-level polynomial for $W$:
    - $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$
    - $\{w_0, \ldots, w_{m-1}\} \subset \{x_1, \ldots, x_d\}$

# Application: Word-level representation




Patch function modeled as a 2-bit-vector word ($m$=2), $f_W : W + r_3 + \beta \cdot rr_3$

## MFR Challenges: $\mathbb{F}_{2^k}$ and $P_k(x)$

- Smallest $k$ is $LCM(n, m)$
  - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P_k(x)}$
    - $P_k(x)$ is a degree-$k$ primitive polynomial; $P_k(\alpha) = 0$

# MFR Challenges: $\mathbb{F}_{2^k}$ and $P_k(x)$

- Smallest $k$ is $LCM(n, m)$
  - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P_k(x)}$
    - $P_k(x)$ is a degree-$k$ primitive polynomial; $P_k(\alpha) = 0$
- Mathematical challenge: Given $P_n(x)$ and $P_m(x)$, compute $P_k(x)$ such that $P_n(\gamma) = P_m(\beta) = P_k(\alpha) = 0$
  - $\gamma = \alpha^{(2^k-1)/(2^n-1)} = \alpha^\lambda$
  - $\beta = \alpha^{(2^k-1)/(2^m-1)} = \alpha^\mu$

# MFR Challenges: $\mathbb{F}_{2^k}$ and $P_k(x)$

- Smallest $k$ is $LCM(n, m)$
  - $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^n}$ and $\mathbb{F}_{2^k} \supset \mathbb{F}_{2^m}$
  - $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P_k(x)}$
    - $P_k(x)$ is a degree-$k$ primitive polynomial; $P_k(\alpha) = 0$
- Mathematical challenge: Given $P_n(x)$ and $P_m(x)$, compute $P_k(x)$ such that $P_n(\gamma) = P_m(\beta) = P_k(\alpha) = 0$
  - $\gamma = \alpha^{(2^k-1)/(2^n-1)} = \alpha^\lambda$
  - $\beta = \alpha^{(2^k-1)/(2^m-1)} = \alpha^\mu$
- Solved using factorization of univariate polynomials over finite fields

# Contribution: Computing $P_k(x)$

- Obtain UPFs of $P_n(x^\lambda)$ and $P_m(x^\mu)$ in $\mathbb{F}_2[x]$
- Then, $\exists P_k(x) \in \mathbb{F}_2[x]$ as a common factor of $P_n(x^\lambda)$ and $P_m(x^\mu)$, such that:
  - $P_k(x)$ is a degree-$k$ primitive polynomial in $\mathbb{F}_2[x]$ with $P_k(\alpha) = 0$

## Application: Computing $P_k(x)$

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = \alpha^9, \;\; \beta = \alpha^{21}$

# Application: Computing $P_k(x)$

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = \alpha^9, \;\; \beta = \alpha^{21}$
- Composite field: $k = LCM(2, 3) = 6$

## Application: Computing $P_k(x)$

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = \alpha^9, \;\; \beta = \alpha^{21}$
- Composite field: $k = LCM(2,3) = 6$
  - $UPF(P_3(x^9)) = (x^9)^3 + (x^9) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^4 + x^2 + x + 1)(x^3 + x + 1);$

iSQED Symposium 2021

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = \alpha^9, \;\; \beta = \alpha^{21}$
- Composite field: $k = LCM(2,3) = 6$
  - $UPF(P_3(x^9)) = (x^9)^3 + (x^9) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^4 + x^2 + x + 1)(x^3 + x + 1);$
  - $UPF(P_2(x^{21})) = (x^{21})^2 + (x^{21}) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^5 + x^3 + x^2 + 1)(x^6 + x^5 + x^4 + x + 1)(x^6 + x + 1)(x^6 + x^3 + 1);$

# Application: Computing $P_k(x)$

- $P_3(x) = x^3 + x + 1, \;\; P_2(x) = x^2 + x + 1, \;\; \gamma = \alpha^9, \;\; \beta = \alpha^{21}$
- Composite field: $k = LCM(2, 3) = 6$
  - $UPF(P_3(x^9)) = (x^9)^3 + (x^9) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^4 + x^2 + x + 1)(x^3 + x + 1)$;
  - $UPF(P_2(x^{21})) = (x^{21})^2 + (x^{21}) + 1 = (x^6 + x^5 + x^2 + x + 1)(x^6 + x^5 + 1)(x^6 + x^4 + x^3 + x + 1)(x^6 + x^5 + x^3 + x^2 + 1)(x^6 + x^5 + x^4 + x + 1)(x^6 + x + 1)(x^6 + x^3 + 1)$;
  - We choose $P_6(x) = x^6 + x^5 + 1$ as the required $P_k(x)$.

# MFR Notation: Incorrect $P_k(x)$

- If we incorrectly choose $P_k(x) = x^6 + x^3 + 1$

# MFR Notation: Incorrect $P_k(x)$

- If we incorrectly choose $P_k(x) = x^6 + x^3 + 1$
- For its root $\alpha$, we have

# MFR Notation: Incorrect $P_k(x)$

- If we incorrectly choose $P_k(x) = x^6 + x^3 + 1$
- For its root $\alpha$, we have

$$\begin{aligned}
\alpha^6 + \alpha^3 + 1 &= 0 \\
(\alpha^3)(\alpha^6 + \alpha^3 + 1) &= 0 \text{ (multiply by } \alpha^3\text{)} \\
\alpha^9 + \alpha^6 + \alpha^3 &= 0 \\
\gamma + 1 &= 0
\end{aligned}$$

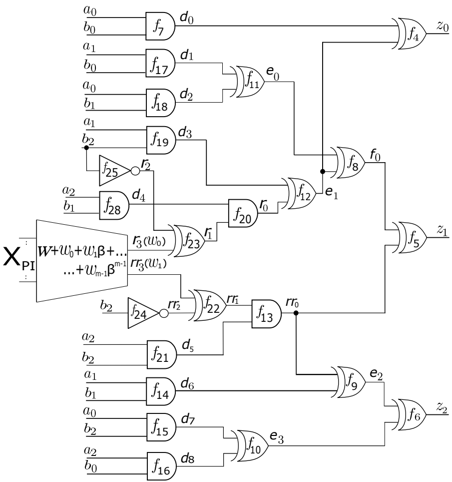
# MFR Notation: Incorrect $P_k(x)$

- If we incorrectly choose $P_k(x) = x^6 + x^3 + 1$
- For its root $\alpha$, we have

$$\begin{aligned} \alpha^6 + \alpha^3 + 1 &= 0 \\ (\alpha^3)(\alpha^6 + \alpha^3 + 1) &= 0 \text{ (multiply by } \alpha^3) \\ \alpha^9 + \alpha^6 + \alpha^3 &= 0 \\ \gamma + 1 &= 0 \end{aligned}$$

- However, $\gamma \neq 1$, as $\gamma$ is a primitive element of $\mathbb{F}_{2^n}$
- Selecting arbitrary $P_k(x)$ leads to erroneous results

# Application: Word-level representation




Patch function modeled as a 2-bit-vector word ($m$=2), $f_W : W + r_3 + \beta \cdot rr_3$

# MFR Notation: Word-level reasoning

- Obtain each $w_i$ as a polynomial function in $W, \beta$
  - $\forall i \in 1, \ldots, m, \quad w_i = \mathcal{F}_i(W, \beta)$

# MFR Notation: Word-level reasoning

- Obtain each $w_i$ as a polynomial function in $W, \beta$
  - $\forall i \in 1, \ldots, m, \quad w_i = \mathcal{F}_i(W, \beta)$

$$
\begin{aligned}
W &= w_0 + \cdots + \beta^{m-1} \cdot w_{m-1} \\
W^2 &= w_0^2 + \cdots + \beta^{2(m-1)} \cdot w_{m-1}^2 \\
&\ldots \\
W^{2^{m-1}} &= w_0 + \cdots + \beta^{2^{m-1}(m-1)} \cdot w_{m-1}
\end{aligned}
$$

# MFR Notation: Word-level reasoning

- Obtain each $w_i$ as a polynomial function in $W, \beta$
  - $\forall i \in 1, \ldots, m, \quad w_i = \mathcal{F}_i(W, \beta)$

$$
\begin{aligned}
W &= w_0 + \cdots + \beta^{m-1} \cdot w_{m-1} \\
W^2 &= w_0^2 + \cdots + \beta^{2(m-1)} \cdot w_{m-1}^2 \\
&\ldots \\
W^{2^{m-1}} &= w_0 + \cdots + \beta^{2^{m-1}(m-1)} \cdot w_{m-1}
\end{aligned}
$$

- Solved using Gaussian elimination

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \ldots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \ldots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \ldots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:
   - Start with $F' = F$

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \dots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:
   - Start with $F' = F$
   - Remove polynomials with $w_i$'s as leading terms

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \dots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:
   - Start with $F' = F$
   - Remove polynomials with $w_i$'s as leading terms
   - Substitute $\forall i \in 1, \dots, m, \quad w_i = \mathcal{F}_i(W, \beta)$

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \ldots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:
   - Start with $F' = F$
   - Remove polynomials with $w_i$'s as leading terms
   - Substitute $\forall i \in 1, \ldots, m, \quad w_i = \mathcal{F}_i(W, \beta)$
   - Add $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$

# MFR Notation: MFR setup steps

1. Setup a new ring $R' = \mathbb{F}_{2^k}[x_1, \ldots, x_d, Z, A, W]$
   - $\mathbb{F}_{2^k}$ is constructed using $P_k(x)$
   - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $w_i$
2. Construct a polynomial set $F'$ as follows:
   - Start with $F' = F$
   - Remove polynomials with $w_i$'s as leading terms
   - Substitute $\forall i \in 1, \ldots, m, \quad w_i = \mathcal{F}_i(W, \beta)$
   - Add $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$
   - Substitute $\beta = \alpha^\mu, \gamma = \alpha^\lambda$

iSQED® Symposium 2021

# Circuit Polynomials and Setup

- Ring $R = \mathbb{F}_{2^n}[x_1, \ldots, x_d, Z, A]$
  - $\mathbb{F}_{2^n}$ is constructed using $P_n(x)$
- Circuit polynomials under RTTO $>$:

$$
\begin{aligned}
f_1 &: Z + z_0 + \gamma \cdot z_1 + \gamma^2 \cdot z_2; & f_{22} &: rr_1 + rr_3 + rr_2;\\
f_2 &: A + a_0 + \gamma \cdot a_1 + \gamma^2 \cdot a_2; & f_{23} &: r_1 + r_2 + r_3;\\
f_3 &: B + b_0 + \gamma \cdot b_1 + \gamma^2 \cdot b_2; & f_{26} &: r_3 + r_4 + d_4;\\
f_4 &: z_0 + d_0 + e_1; & f_{27} &: rr_3 + rr_4 + b_2;\\
f_5 &: z_1 + f_0 + rr_0; & &\ldots\\
&\ldots & f_{30} &: rr_4 + a_2 + b_2 + a_2 b_2;
\end{aligned}
$$

- $F = \{f_1, \ldots, f_{30}\}$, $F_0 = \{a_0^2 - a_0, \ldots, z_2^2 - z_2, A^8 - A, \ldots, Z^8 - Z\}$.
  - Ideal $J + J_0 = \langle F \cup F_0 \rangle$ models $C$.

# MFR Application: Word-level Formulation

- ring $R' = \mathbb{F}_{2^6}[x_1, \ldots, x_d, Z, A, W]$
  - $\mathbb{F}_{2^6} = \mathbb{F}_2[x] \pmod{P_6(x)}$, $P_6(x) = x^6 + x^5 + 1$

# MFR Application: Word-level Formulation

- ring $R' = \mathbb{F}_{2^6}[x_1, \ldots, x_d, Z, A, W]$
  - $\mathbb{F}_{2^6} = \mathbb{F}_2[x] \pmod{P_6(x)}$, $P_6(x) = x^6 + x^5 + 1$
  - WRTO $>$: $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \cdots > \{d_1 > d_2 > d_3 > r_0 > d_5 > rr_1\} > \{r_1\} > \{\mathbf{W}\} > \{\mathbf{rr_3} > rr_2\} > \{r_2 > \mathbf{r_3} > rr_4\} > \{r_4 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$

iSQED® Symposium 2021

# MFR Application: Word-level Formulation

- ring $R' = \mathbb{F}_{2^6}[x_1, \ldots, x_d, Z, A, W]$
  - $\mathbb{F}_{2^6} = \mathbb{F}_2[x] \pmod{P_6(x)}$, $P_6(x) = x^6 + x^5 + 1$
  - WRTO $>$: $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \cdots > \{d_1 > d_2 > d_3 > r_0 > d_5 > rr_1\} > \{r_1\} > \{\mathbf{W}\} > \{\mathbf{rr_3} > rr_2\} > \{r_2 > \mathbf{r_3} > rr_4\} > \{r_4 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$
- Update polynomial set $F$ to $F'$ as:

$$rr_3 = W^2 + W, \quad r_3 = \beta W^2 + \beta^2 W$$

$$f'_{22} : rr_1 + (W^2 + W) + rr_2$$

$$f'_{23} : r_1 + r_2 + (\beta W^2 + \beta^2 W)$$

$$f_W : W + r_3 + \beta \cdot rr_3$$

$$\beta = \alpha^{21} \text{ and } \gamma = \alpha^9$$

$$F' = \{f_1, \ldots, f_{21}, \ f'_{22}, f'_{23}, f_W, \ldots, f_{30}\} - \{f_{26}, f_{27}\}$$

# MFR Contribution: Rectification Check

- Multi-fix rectification at target $W$
  - Construct the following ideals:

$$J'_l = \langle F'_l \rangle = \langle f_1, \dots, f'_W = W + \delta[l], \dots, f_s \rangle, 1 \leq l \leq 2^m,$$
$$\textit{Where } (\delta[1], \dots, \delta[2^m]) = (0, 1, \beta, \dots, \beta^{2^m-2}).$$

- Multi-fix rectification at target $W$
  - Construct the following ideals:

$$J'_l = \langle F'_l \rangle = \langle f_1, \ldots, f'_W = W + \delta[l], \ldots, f_s \rangle, 1 \leq l \leq 2^m,$$
$$\textit{Where } (\delta[1], \ldots, \delta[2^m]) = (0, 1, \beta, \ldots, \beta^{2^m-2}).$$

  - Perform the reductions:
    - $f \xrightarrow{F'_l, F_0}_+ rem_l, \forall 1 \leq l \leq 2^m$
  - Let $V_{\mathbb{F}_q}(rem_l)$ denote the varieties of the respective $rem_l$'s

- Multi-fix rectification at target $W$
  - Construct the following ideals:

$$J'_l = \langle F'_l \rangle = \langle f_1, \ldots, f'_W = W + \delta[l], \ldots, f_s \rangle, 1 \leq l \leq 2^m,$$
$$\textit{Where } (\delta[1], \ldots, \delta[2^m]) = (0, 1, \beta, \ldots, \beta^{2^m-2}).$$

  - Perform the reductions:
    - $f \xrightarrow{F'_l, F_0}_+ rem_l, \forall 1 \leq l \leq 2^m$
  - Let $V_{\mathbb{F}_q}(rem_l)$ denote the varieties of the respective $rem_l$'s
  - Multi-fix rectification exists at target $W$:

$$\textbf{if and only if} \bigcup_{l=1}^{2^m} V_{\mathbb{F}_q}(rem_l) = \mathbb{F}_q^{|X_{PI}|} = V(J_0)$$

# MFR Application: Rectification Check

- Constructing the $J_i$ ideals:
  - $J_1 = \langle F_1' \rangle$, where $F_1'[f_w] = W + \delta(1) = W$,
  - $J_2 = \langle F_2' \rangle$, where $F_2'[f_w] = W + \delta(2) = W + 1$,
  - $J_3 = \langle F_3' \rangle$, where $F_3'[f_w] = W + \delta(3) = W + \beta$,
  - $J_4 = \langle F_4' \rangle$, where $F_4'[f_w] = W + \delta(4) = W + \beta^2$

- Constructing the $J_i$ ideals:
  - $J_1 = \langle F_1' \rangle$, where $F_1'[f_w] = W + \delta(1) = W$,
  - $J_2 = \langle F_2' \rangle$, where $F_2'[f_w] = W + \delta(2) = W + 1$,
  - $J_3 = \langle F_3' \rangle$, where $F_3'[f_w] = W + \delta(3) = W + \beta$,
  - $J_4 = \langle F_4' \rangle$, where $F_4'[f_w] = W + \delta(4) = W + \beta^2$
- Reducing the specification $f : Z + A \cdot B$ modulo these ideals, we get:
  - $rem_1 = f \xrightarrow{F_1' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2) + \alpha^{36}(a_2 b_2)$
  - $rem_2 = f \xrightarrow{F_2' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1) + \alpha^{36}(a_2 b_2)$
  - $rem_3 = f \xrightarrow{F_3' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2)$
  - $rem_4 = f \xrightarrow{F_4' \cup F_0'}_+ \alpha^{27}(a_2 b_1 b_2 + a_2 b_1)$

# MFR Application: Rectification Check

- Constructing the $J_i$ ideals:
  - $J_1 = \langle F_1' \rangle$, where $F_1'[f_w] = W + \delta(1) = W$,
  - $J_2 = \langle F_2' \rangle$, where $F_2'[f_w] = W + \delta(2) = W + 1$,
  - $J_3 = \langle F_3' \rangle$, where $F_3'[f_w] = W + \delta(3) = W + \beta$,
  - $J_4 = \langle F_4' \rangle$, where $F_4'[f_w] = W + \delta(4) = W + \beta^2$
- Reducing the specification $f : Z + A \cdot B$ modulo these ideals, we get:
  - $rem_1 = f \xrightarrow{F_1' \cup F_0'}_+ \alpha^{27}(a_2b_1b_2) + \alpha^{36}(a_2b_2)$
  - $rem_2 = f \xrightarrow{F_2' \cup F_0'}_+ \alpha^{27}(a_2b_1b_2 + a_2b_1) + \alpha^{36}(a_2b_2)$
  - $rem_3 = f \xrightarrow{F_3' \cup F_0'}_+ \alpha^{27}(a_2b_1b_2)$
  - $rem_4 = f \xrightarrow{F_4' \cup F_0'}_+ \alpha^{27}(a_2b_1b_2 + a_2b_1)$
- Compute $GB(r_1 \cdot r_2 \cdot r_3 \cdot r_4, F_0) = F_0$
- Target $W$ with nets $r_3$ and $rr_3$ admits MFR

# Implementation

- Custom software:

## Implementation

- Custom software:
  - PolyBori reduction procedure for remainder generation

# Implementation

- Custom software:
  - PolyBori reduction procedure for remainder generation
  - Singular to compute $P_k(x)$ and model composite field

# Implementation

- Custom software:
  - PolyBori reduction procedure for remainder generation
  - Singular to compute $P_k(x)$ and model composite field
  - Custom high level finite field engine

# Implementation

- Custom software:
  - PolyBori reduction procedure for remainder generation
  - Singular to compute $P_k(x)$ and model composite field
  - Custom high level finite field engine
    - Bit-vector and coefficient computations
    - Decision procedure

# Implementation

- Custom software:
  - PolyBori reduction procedure for remainder generation
  - Singular to compute $P_k(x)$ and model composite field
  - Custom high level finite field engine
    - Bit-vector and coefficient computations
    - Decision procedure
- Experiments performed on a 3.5GHz Intel(R) Core$^{\text{TM}}$ i7-4770K Quad-Core CPU with 32 GB RAM

# Focus: Finite Field Arithmetic Circuits

- Applications:
  - RSA, ECC, Error correcting codes, RFID, etc.
    - Crypto-system bugs can leak secret keys [*Biham. et al*, Crypto'08]
    - RFID tag cloning could cause counterfeiting [*Batina. et al*, Security'09]
  - Large datapath sizes in ECC crypto systems
    - In $\mathbb{F}_{2^n}$, $n = 163, 233, 283, 409, 571$ (NIST standard)

iSQED® Symposium 2021

## Focus: Finite Field Arithmetic Circuits

- Applications:
  - RSA, ECC, Error correcting codes, RFID, etc.
    - Crypto-system bugs can leak secret keys [*Biham. et al*, Crypto'08]
    - RFID tag cloning could cause counterfeiting [*Batina. et al*, Security'09]
  - Large datapath sizes in ECC crypto systems
    - In $\mathbb{F}_{2^n}$, $n = 163, 233, 283, 409, 571$ (NIST standard)
- Rectification Motivation:
  - Synthesize sub-functions as opposed to complete redesign
  - Automated debugging

# MFR Experiments: Mastrovito

Table: Time is in seconds; $n$ = Datapath Size, $m$ = target word size, $k$ = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

| *n* | *m* | *k* | AM | #G | #BO | PBS | VMS | RC | TE |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 5 | 80 | 100 | 0.8 | 6 | 0.04 | 0.06 | 0.12 | 0.22 |
| 32 | 5 | 160 | 120 | 2.8 | 8 | 0.13 | 0.12 | 0.4 | 0.65 |
| 64 | 3 | 192 | 160 | 11.2 | 5 | 0.57 | 0.45 | 227 | 228 |
| 96 | 2 | 96 | 240 | 24.5 | 5 | 1.47 | 0.26 | 0.83 | 2.56 |
| 128 | 2 | 128 | 370 | 43.2 | 5 | 3.23 | 0.5 | 2.03 | 5.76 |
| 163 | 5 | 815 | 550 | 69.8 | 6 | 6.04 | 3.36 | 11.9 | 21.3 |
| 233 | 2 | 466 | 750 | 119 | 3 | 13 | 1.2 | 0.01 | 14.2 |
| 283 | 2 | 566 | 1300 | 190 | 2 | 38 | 4.2 | 0.1 | 42.3 |
| 409 | 2 | 818 | 2400 | 384 | 2 | 190 | 5 | 0.1 | 195 |
| 571 | 2 | 1042 | 5000 | 827 | 5 | 2150 | 12 | 0.1 | 2162 |

Table: Time is in seconds; $n$ = Datapath Size, $m$ = target word size, $k$ = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

| $n$ | $m$ | $k$ | AM | #G | #BO | PBS | VMS | RC | TE |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 5 | 80 | 100 | 0.9 | 16 | 0.04 | 0.56 | 35.6 | 36 |
| 32 | 5 | 160 | 120 | 2.8 | 32 | 0.13 | 0.57 | 27.6 | 28.3 |
| 64 | 3 | 192 | 160 | 9.6 | 47 | 0.52 | 0.32 | 1.79 | 2.63 |
| 96 | 2 | 96 | 240 | 21 | 96 | 1.36 | 1.27 | 13.3 | 16 |
| 128 | 2 | 128 | 370 | 35.8 | 128 | 2.8 | 1.4 | 64.2 | 68.4 |
| 163 | 5 | 815 | 550 | 57.5 | 128 | 5.2 | 6.8 | 262 | 274 |
| 233 | 2 | 466 | 750 | 112 | 233 | 11.5 | 3.5 | 360 | 375 |
| 283 | 2 | 566 | 1300 | 171 | 283 | 35 | 11 | 1503 | 1549 |
| 409 | 2 | 818 | 2400 | 340 | 409 | 134 | 10 | 4920* | 5064 |
| 571 | 2 | 1042 | 5000 | 663 | 12 | 1313 | 82 | 0.2 | 1395 |

# MFR Experiments: Point Addition

Table: Time is in seconds; $n$ = Datapath Size, $m$ = target word size, $k$ = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

| $n$ | $m$ | $k$ | AM | #G | #BO | PBS | VMS | RC | TE |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 5 | 80 | 100 | 0.9 | 7 | 0.06 | 0.11 | 1.73 | 1.9 |
| 32 | 5 | 160 | 120 | 2.9 | 13 | 0.18 | 0.8 | 134 | 135 |
| 64 | 3 | 192 | 160 | 10.6 | 64 | 0.84 | 0.56 | 58.1 | 59.5 |
| 96 | 2 | 96 | 240 | 24.8 | 96 | 2.46 | 0.64 | 14.9 | 18 |
| 128 | 2 | 128 | 370 | 43.2 | 128 | 6.45 | 1.55 | 73 | 81 |
| 163 | 5 | 815 | 550 | 71.6 | 22 | 15.7 | 4.7 | 15 | 35.4 |
| 233 | 2 | 466 | 750 | 122 | 233 | 19.2 | 2.15 | 0.15 | 21.5 |
| 283 | 2 | 566 | 1300 | 208 | 4 | 80.4 | 6.1 | 0.1 | 86.6 |
| 409 | 2 | 818 | 2400 | 368 | 409 | 220 | 10 | 2007 | 2237 |
| 571 | 2 | 1042 | 5000 | 813 | 5 | 2583 | 27 | 880 | 3490 |

- Algebraic approach for *m*-target MFR checking
  - Efficiency derived by interpreting targets as a bit-vector

# Conclusion and Future work

- Algebraic approach for *m*-target MFR checking
  - Efficiency derived by interpreting targets as a bit-vector
- New mathematical insights for unified framework
  - Field incompatibility
  - Primitive polynomial computation

# Conclusion and Future work

- Algebraic approach for $m$-target MFR checking
  - Efficiency derived by interpreting targets as a bit-vector
- New mathematical insights for unified framework
  - Field incompatibility
  - Primitive polynomial computation

- Computation of rectification function at the word-level
  - Define and formulate existence of don't cares at the word-level

# Conclusion and Future work

- Algebraic approach for $m$-target MFR checking
  - Efficiency derived by interpreting targets as a bit-vector
- New mathematical insights for unified framework
  - Field incompatibility
  - Primitive polynomial computation

- Computation of rectification function at the word-level
  - Define and formulate existence of don't cares at the word-level
- Extend the approach to integer arithmetic circuits

# MFR Function Example

- Compute a rectification function of the form $W = U(X_{PI})$
  - Here $U$ is the *unknown component* computed as an $m$-bit-vector word
  - It represents the function $W = \sum_{i=0}^{m-1} \beta^i u_i$
    - Where $u_i$'s represent the individual Boolean functions for the respective $w_i$'s.

# MFR Function Example

- Compute a rectification function of the form $W = U(X_{PI})$
  - Here $U$ is the *unknown component* computed as an $m$-bit-vector word
  - It represents the function $W = \sum_{i=0}^{m-1} \beta^i u_i$
    - Where $u_i$'s represent the individual Boolean functions for the respective $w_i$'s.
- A polynomial which can be computed to rectify the circuit
  - $W = a_2 b_1 b_2 + \beta \cdot a_2 b_2$
  - $r_3 = (a_2 \wedge b_1 \wedge b_2), \quad rr_3 = (a_2 \wedge b_2)$

# THANK YOU

Email: vikas.k.rao@utah.edu

# Sum, Product, and Quotient of Ideals

Given $J_1 = \langle f_1, \ldots, f_s \rangle \in R$ and $J_2 = \langle h_1, \ldots, h_r \rangle \in R$

- Sum of ideals:
  - $J_1 + J_2 = \langle f_1, \ldots, f_s, h_1 \ldots, h_r \rangle$
- Product of ideals:
  - $J_1 \cdot J_2 = \langle f_i \cdot h_j : 1 \leq i \leq s, 1 \leq j \leq r \rangle$
- Ideal quotient of $J_1$ by $J_2$:
  - $J_1 : J_2 = \{ f \in R \mid f \cdot h \in J_1, \forall h \in J_2 \}$
- Ideals and varieties are dual concepts
  - $V(J_1 + J_2) = V(J_1) \cap V(J_2)$
  - $V(J_1 \cdot J_2) = V(J_1) \cup V(J_2)$
  - $V(J_1 : J_2) = V(J_1) - V(J_2)$

- Update ring properties
  - $R = \mathbb{F}_q[x_1, \ldots, x_d, Z, A, W]$
  - Modify RTTO $>$ to place the target $W$ before the lowest indexed target $e_0$
    - $\{Z\} > \{A > B\} > \{z_0 > z_1 > z_2\} > \{f_0 > e_2 > e_3\} > \{\mathbf{W} > e_0 > e_1 > d_5 > d_6 > d_7 > d_8\} > \{d_0 > d_1 > d_2 > d_3 > d_4\} > \{a_0 > a_1 > a_2 > b_0 > b_1 > b_2\}$.
- Update polynomial set $F$ to $F'$:
  - Delete polynomials for $w_i$'s
  - Delete polynomials in the transitive fan-in of $w_i$'s only
  - Transitive fan-outs of $w_i$'s need to be replaced with their equivalent word-level representations in terms of $W$
  - Add $f_w : W + \sum_{i=0}^{m-1} \beta^i w_i$

# MFR Experiments: Targets don't admit Rectification

Table: Time is in seconds; $I$ = Index, $n$ = Datapath Size, $m$ = target word size, $k$ = composite field size (degree of $P_k(X)$), AM = Maximum resident memory utilization in Mega Bytes, #G = Number of gates $\times 10^3$, #BO = Number of faulty outputs, PBS = Required time for PolyBori setup (ring declaration/poly collection/spec collection), VMS = Required time for verification, polynomial factorization and computing $P_k(X)$, and MFR setup, RC = Required time for MFR check, TE = Required time for total execution

| | | | | | Mastrovito | | | | | | Montgomery | | | | | | Point Addition | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *I* | *n* | *m* | *k* | AM | #G | #BO | PBS | VMS | RC | TE | #G | #BO | PBS | VMS | RC | TE | #G | #BO | PBS | VMS | RC | TE |
| 1 | 16 | 7 | 112 | 100 | 0.8 | 11 | 0.04 | 0.17 | 4.96 | 5.14 | 0.9 | 13 | 0.05 | 2 | 228 | 230 | 0.9 | 12 | 0.05 | 0.55 | 33 | 33.6 |
| 2 | 32 | 5 | 160 | 120 | 2.8 | 8 | 0.13 | 0.09 | 0.81 | 1.03 | 2.8 | 32 | 0.13 | 0.9 | 100 | 101 | 2.9 | 13 | 0.18 | 0.8 | 244 | 245 |
| 3 | 64 | 3 | 192 | 160 | 11.2 | 5 | 0.58 | 0.23 | 1.64 | 2.45 | 9.6 | 47 | 0.51 | 0.6 | 10.4 | 11.4 | 10.6 | 5 | 0.8 | 0.2 | 4 | 5 |
| 4 | 96 | 2 | 96 | 240 | 24.5 | 5 | 1.48 | 0.25 | 0.04 | 1.77 | 21 | 96 | 1.34 | 2.16 | 87.5 | 91 | 24.8 | 96 | 2.44 | 0.66 | 35.5 | 38.6 |
| 5 | 128 | 2 | 128 | 370 | 43.2 | 5 | 3.21 | 0.53 | 0.1 | 3.84 | 35.8 | 128 | 2.7 | 1.3 | 66 | 70 | 43.2 | 128 | 6 | 2 | 73 | 81 |
| 6 | 163 | 5 | 815 | 550 | 69.8 | 6 | 6.3 | 3.4 | 12 | 21.7 | 57.5 | 128 | 5.3 | 7.7 | 524 | 537 | 71.6 | 22 | 16 | 4.6 | 37 | 57.6 |
| 7 | 409 | 2 | 818 | 2400 | 384 | 2 | 208 | 4 | 0.03 | 212 | 340 | 13 | 127 | 7.9 | 0.13 | 135 | 368 | 3 | 210 | 8 | 928 | 1146 |
| 8 | 571 | 2 | 1042 | 5000 | 827 | 5 | 2246 | 10 | 0.11 | 2256 | 663 | 427 | 1358 | 63.8 | 2.24 | 1424 | 813 | 5 | 2433 | 19 | 5 | 2457 |