Progressive Generation of Canonical Sums of Products Using a SAT Solver

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Sum of Products (SOP) Representation

- **Sum of products (SOP)** is a two-level representation that can represent a Boolean function as a sum (OR, +) of cubes, $S = c_1 + \cdots + c_m$
- **Cube** c is a Boolean product (AND, $\cdot$) of literals, $c_i = l_1 \cdot \ldots \cdot l_k$
- **Literal** l is a variable v or its negation $\bar{v}$

Truth table

x_1	x_2	x_3	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

cubes

$f = x_2 x_3$

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1	1	0	1
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minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
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cubes

$$f = x_2x_3 + x_1x_2$$

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minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

$f = x_2x_3 + x_1x_2$

Off-set SOP

x_1	x_2	x_3	f
0	0	-	0
0	-	0	0
1	0	-	0

cubes

$\bar{f} = \bar{x}_1\,\bar{x}_2$

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minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

$f = x_2x_3 + x_1x_2$

Off-set SOP

x_1	x_2	x_3	f
0	0	-	0
0	-	0	0
1	0	-	0

cubes

$\bar{f} = \bar{x}_1\,\bar{x}_2 + \bar{x}_1\,\bar{x}_3$

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1	1	1	1

minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

$f = x_2x_3 + x_1x_2$

Off-set SOP

x_1	x_2	x_3	f
0	0	-	0
0	-	0	0
1	0	-	0

cubes

$\bar{f} = \bar{x}_1\,\bar{x}_2 + \bar{x}_1\,\bar{x}_3 + x_1\bar{x}_2$

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1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

minterms

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

$f = x_2x_3 + x_1x_2$

SOP size: 2 cubes, 4 literals

smaller SOP

Off-set SOP

x_1	x_2	x_3	f
0	0	-	0
0	-	0	0
1	0	-	0

cubes

$\bar{f} = \bar{x}_1\,\bar{x}_2 + \bar{x}_1\,\bar{x}_3 + x_1\bar{x}_2$

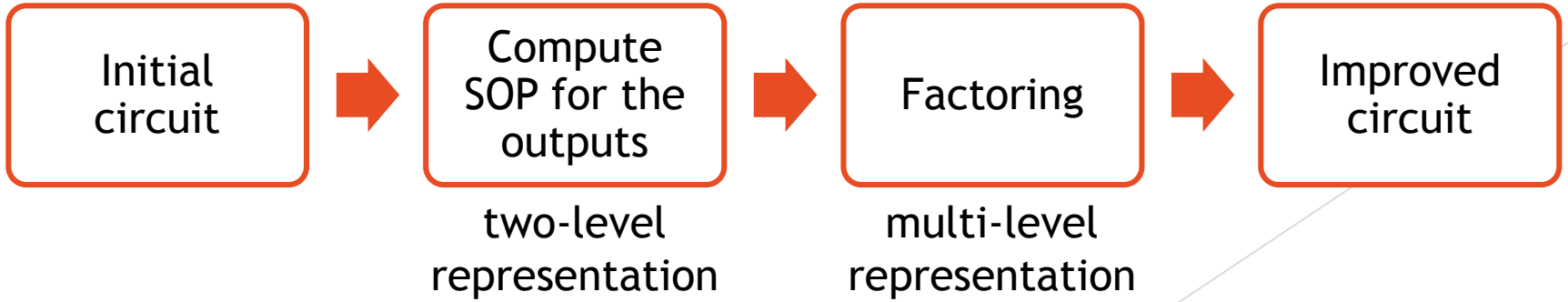
SOP size: 3 cubes, 6 literals

Use of SOPs

- Mapping into Programmable Logic Arrays (PLAs)

Supported in many tools for logic optimization (in academia and industry)

- Delay optimization in technology independent synthesis and technology mapping
- Test generation
- Multi-level logic synthesis for global circuit restructuring (case-study)



Existing SOP Generation and Minimization

- From Binary Decision Diagrams (BDDs)
  - **Problem 1**: BDD generation is not scalable for some functions
    - The BDD size is exponential in the number of input variables → BDD memory explosion problem
  - **Problem 2**: Incompatible with incremental applications
    - The complete BDD should be build before converting it to SOP
- ESPRESSO style minimizers: version enhanced by SAT solving
  - **Problem 1**: Requires as input a computed SOP: do not generate a new SOP from a multi-level representation
  - **Problem 2**: The existing SOP generation is based on enumeration on satisfying assignments

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  - **Problem 2**: The existing SOP generation is based on enumeration on satisfying assignments

Solution?

SAT-based SOP Generation and Minimization

- From Binary Decision Diagrams (BDDs)

Feasibility of computation

BDD generation is not scalable for some functions

size is exponential in the number of input variables → BDD memory explosion problem

Progressive generation

Incompatible with incremental applications

Partial SOPs

plete BDD should be build before converting it to SOP

- ESPRESSO style minimizers: version enhanced by SAT solving

Completely based on SAT solving

Requires as input a computed SOP: do not generate a new SOP from a multi-level
tion

The existing SOP generation is based on enumeration on satisfying assignments

Solution?

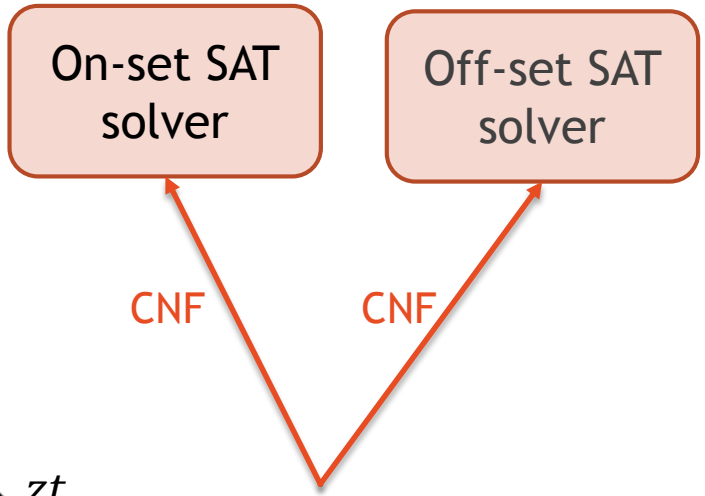
SAT-based SOP Generation and Minimization

- Completely based on SAT solvers
- **Progressive generation**: cube by cube (generated and minimized)
  - Generation of a **partial SOP**
  - Prediction of **feasibility of computation**
- SAT-based generation of **canonical SOPs**
  - Unique SOP for *a given function* and *variable order*, independent of
    - The initial structure of the circuit
    - CNF generation algorithm
    - The used SAT solver
    - Operating system
- The generated SOPs are **irredundant**
  - No literal and no cube can be removed without changing the function

Outline

- SAT-based SOP Generation
- Methods for Runtime Improvement
- Experimental Results
- Conclusion

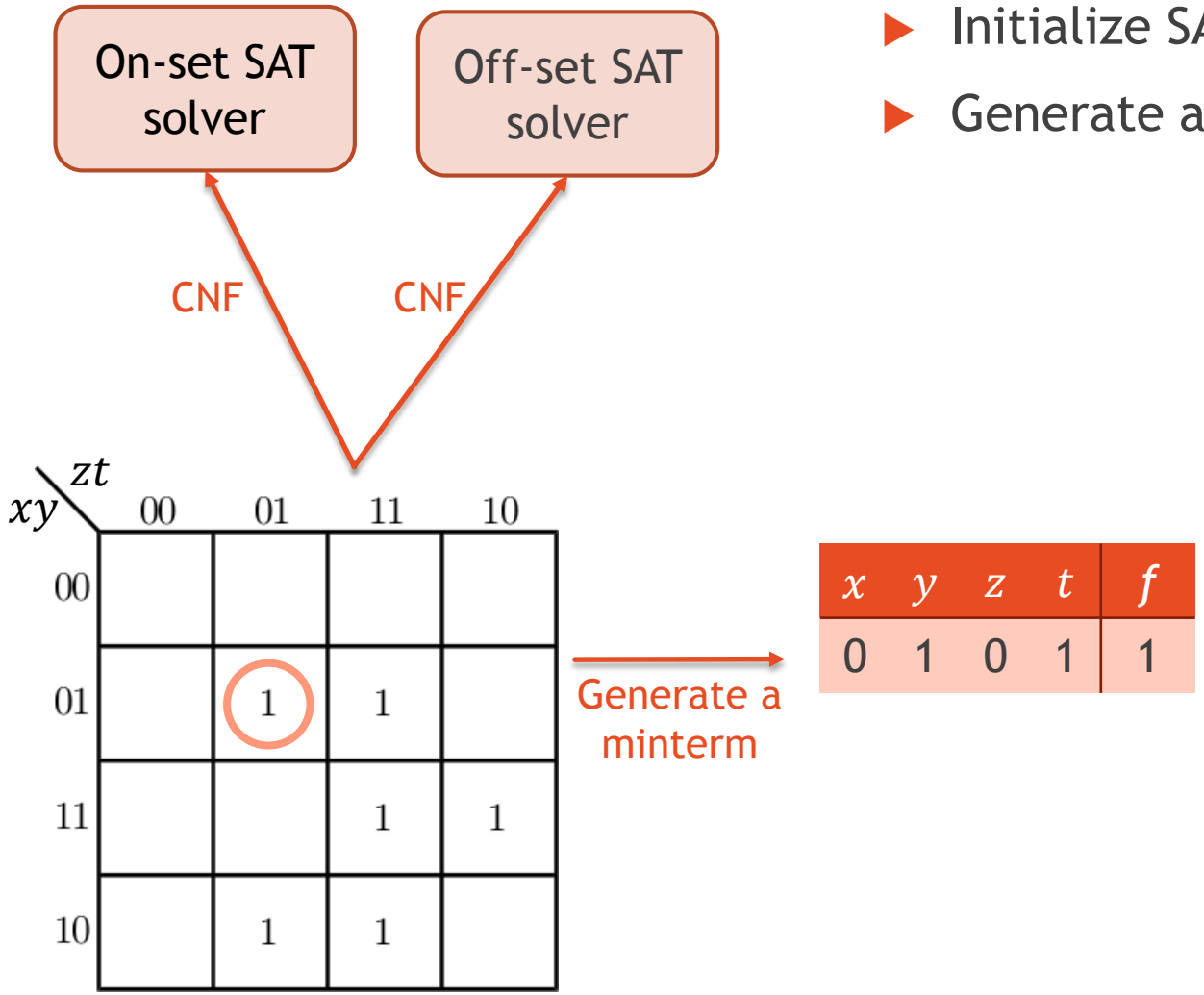
Progressive SAT-based SOP Generation: Global view



xy \ zt	00	01	11	10
00				
01		1	1	
11			1	1
10		1	1	

- Initialize SAT solvers

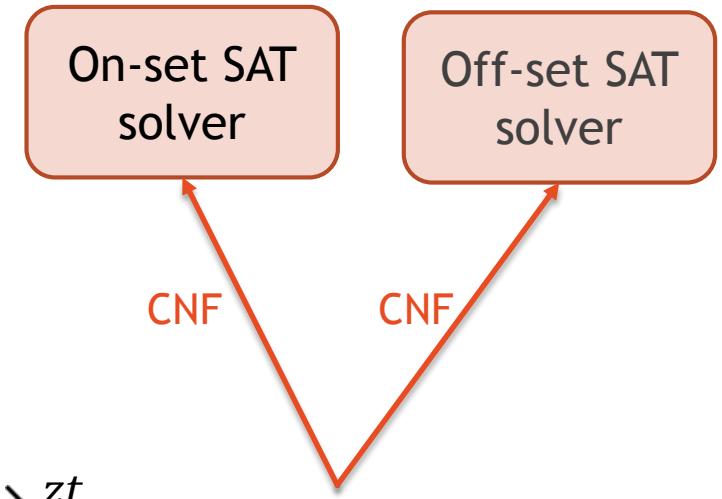
Progressive SAT-based SOP Generation: Global view



x	y	z	t	f
0	1	0	1	1

- Initialize SAT solvers
- Generate a minterm

Progressive SAT-based SOP Generation: Global view



- Initialize SAT solvers
- Generate a minterm
- Expand the minterm to a cube

xy \ zt	00	01	11	10
00				
01		1	1	
11			1	1
10		1	1	

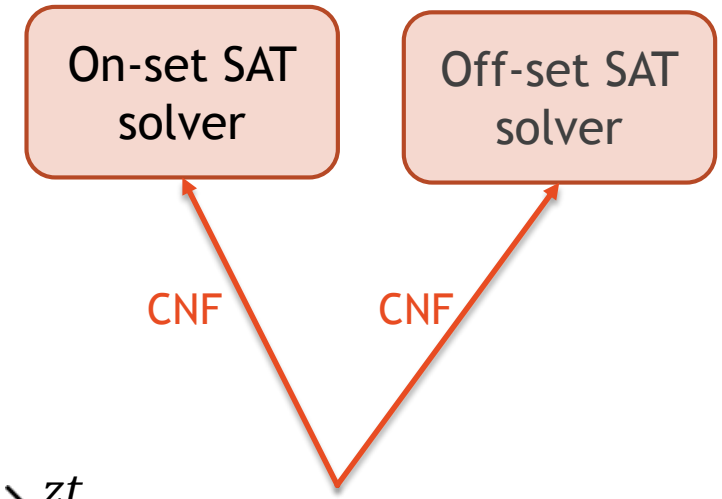
Generate a minterm

x	y	z	t	f
0	1	0	1	1

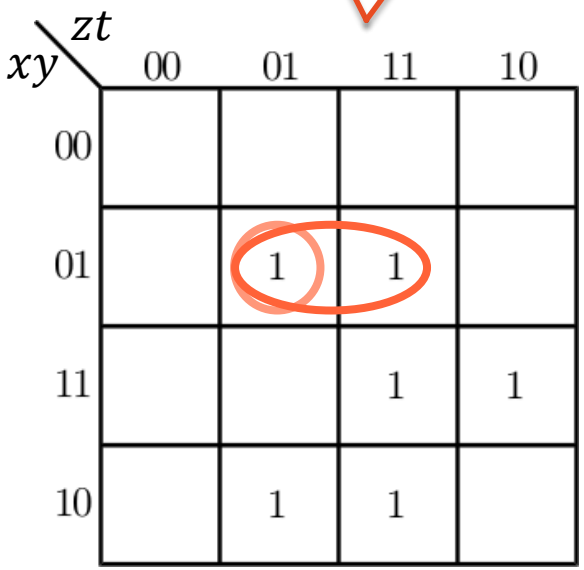
Expand to a cube

x	y	z	t	f
0	1	-	1	1

Progressive SAT-based SOP Generation: Global view



- Initialize SAT solvers
- Generate a minterm
- Expand the minterm to a cube
- Add the cube as a blocking clause



Generate a minterm

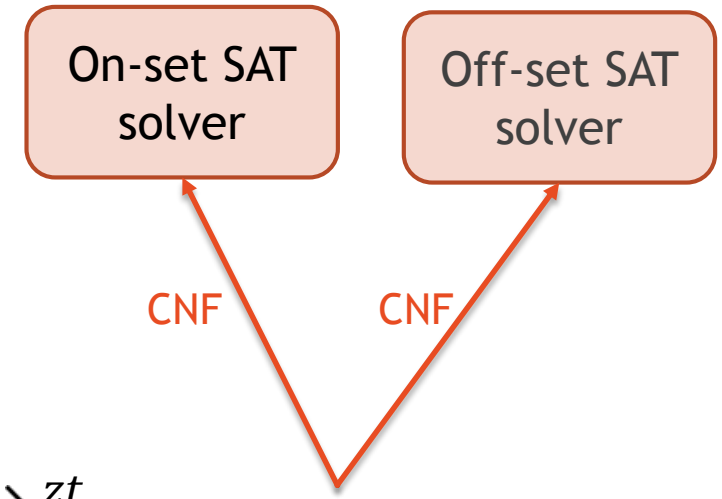
x	y	z	t	f
0	1	0	1	1

Expand to a cube

x	y	z	t	f
0	1	-	1	1

$\bar{x}yt = (x + \bar{y} + \bar{t})$

Progressive SAT-based SOP Generation: Global view



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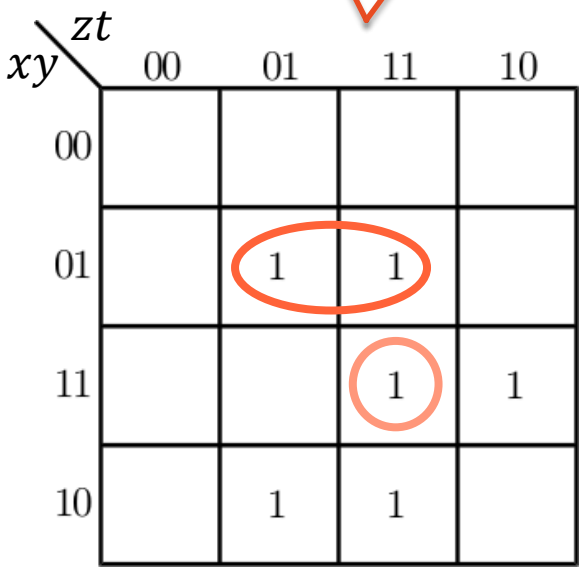

Generate a minterm

x	y	z	t	f
0	1	0	1	1
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Expand to a cube

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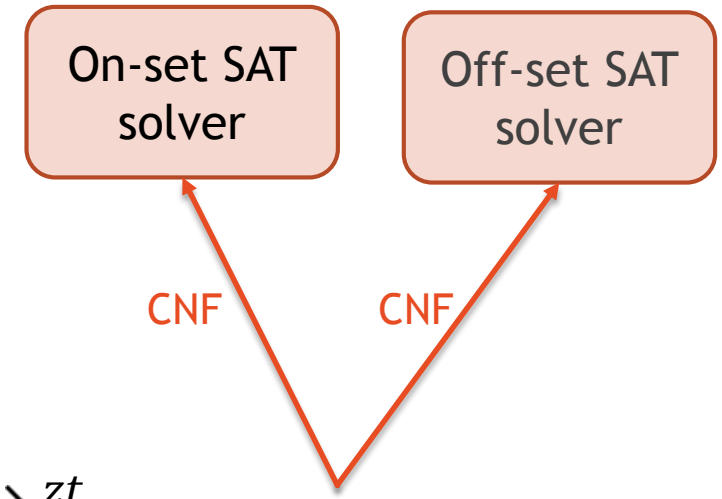

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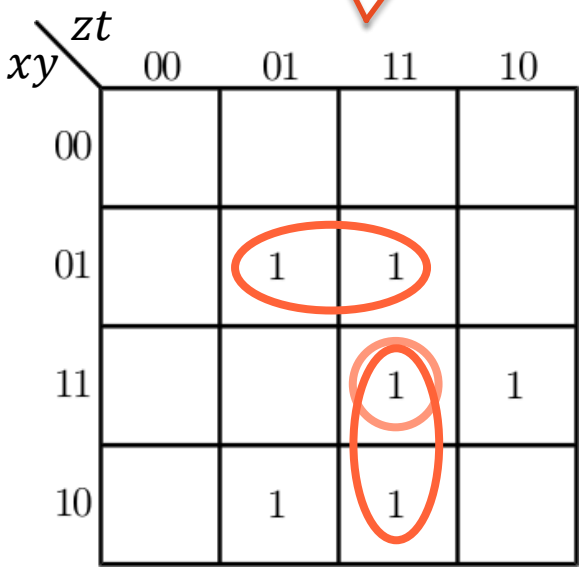

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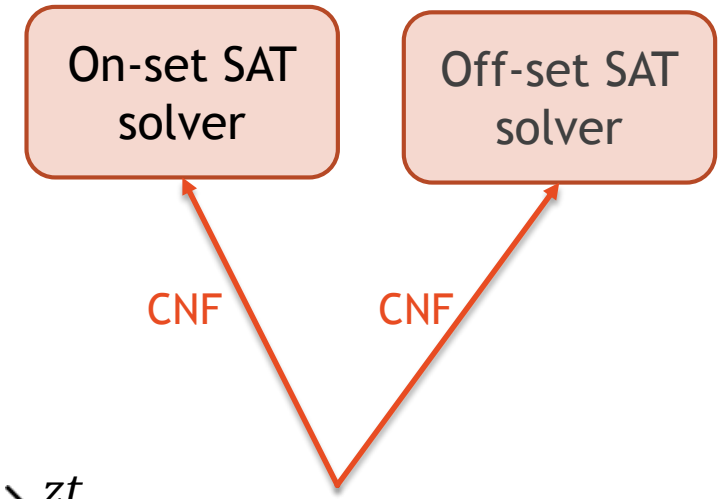

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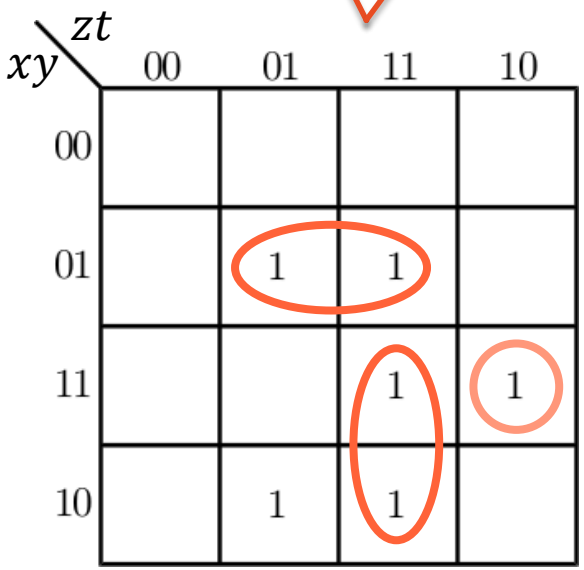

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Expand to a cube

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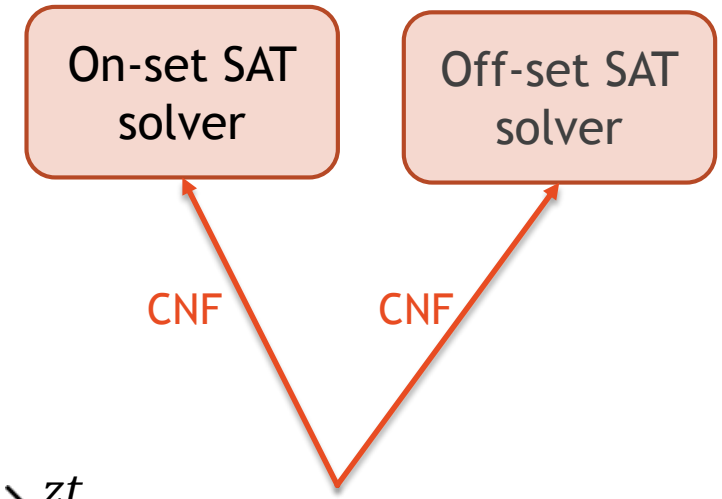

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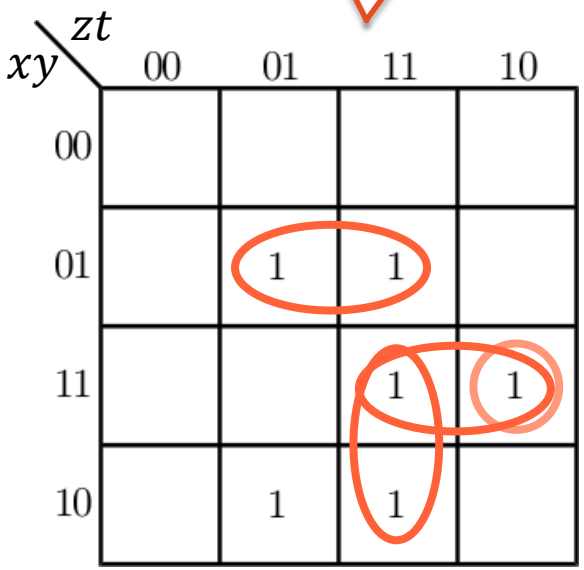

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1	1	1	1	1
1	1	1	0	1

Expand to a cube

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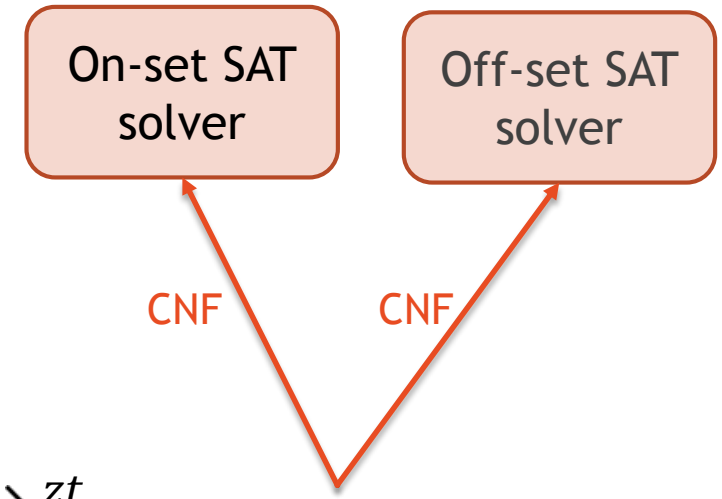
1	1	1	-	1

Progressive SAT-based SOP Generation: Global view



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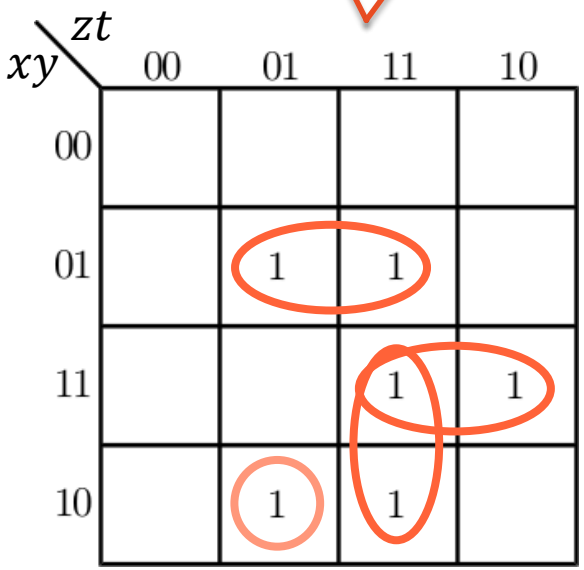

Generate a minterm

x	y	z	t	f
0	1	0	1	1
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1	1	1	0	1
1	0	0	1	1

Expand to a cube

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1	-	1	1	1
1	1	1	-	1

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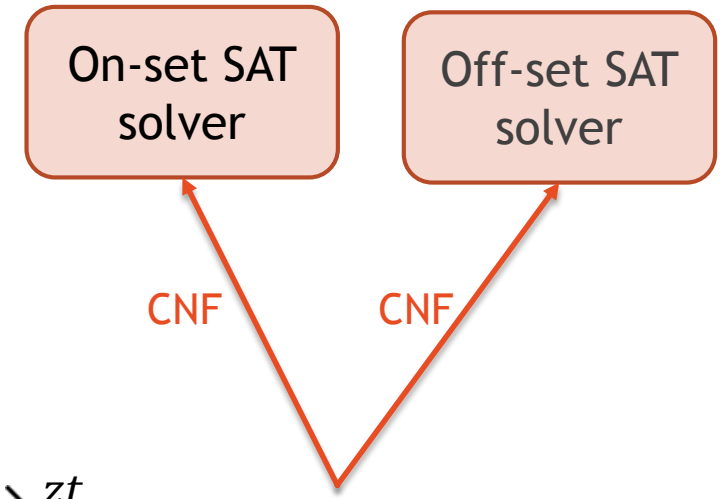

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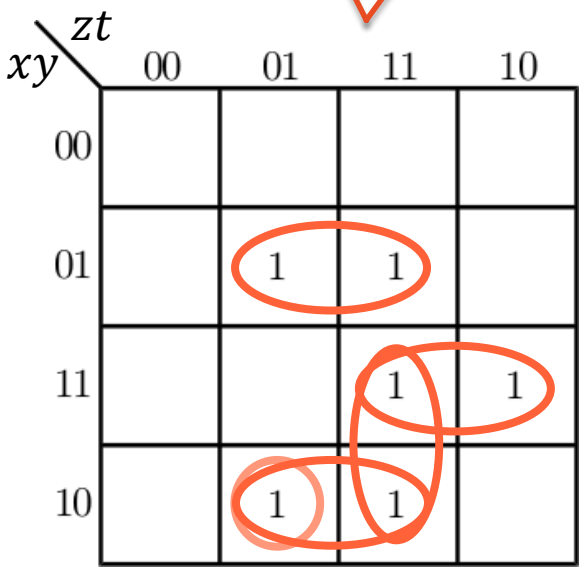

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0	1	0	1	1
1	1	1	1	1
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1	0	0	1	1

Expand to a cube

x	y	z	t	f
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1	-	1	1	1
1	1	1	-	1
1	0	-	1	1

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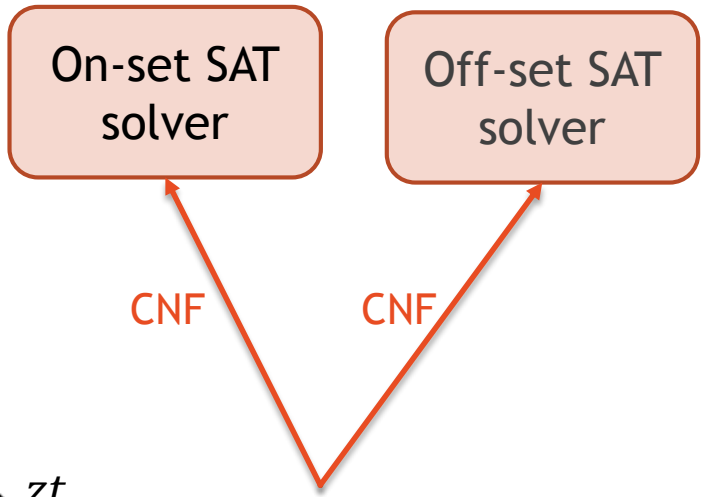
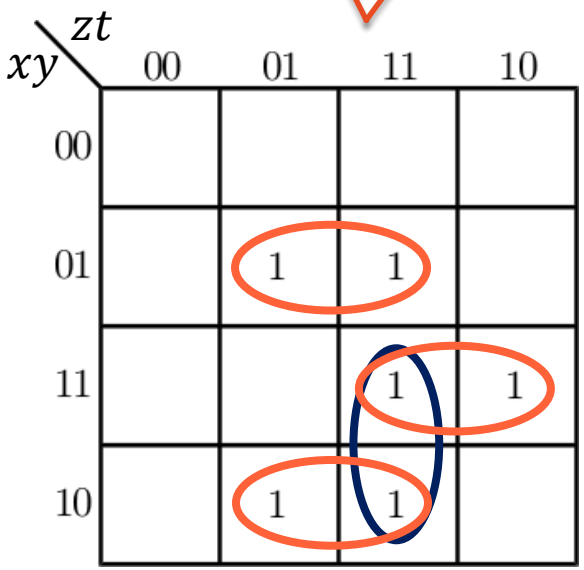


- Initialize SAT solvers
- Generate a minterm
- Expand the minterm to a cube
- Add the cube as a blocking clause
- Remove redundant cubes



Generate a minterm

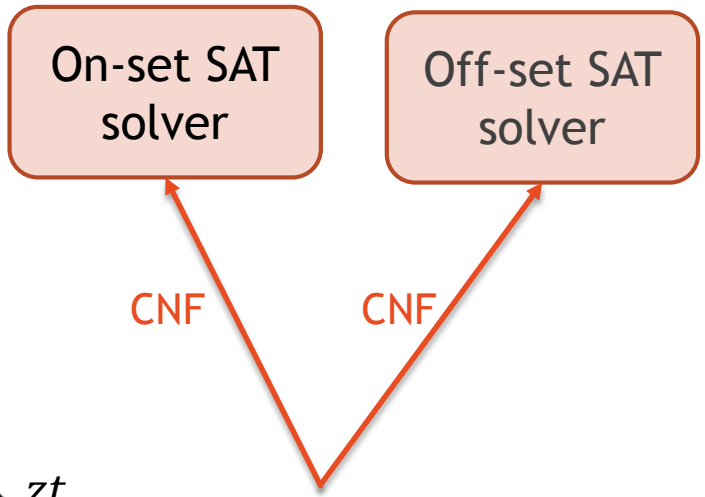
x	y	z	t	f
0	1	0	1	1
1	1	1	1	1
1	1	1	0	1
1	0	0	1	1

Expand to a cube

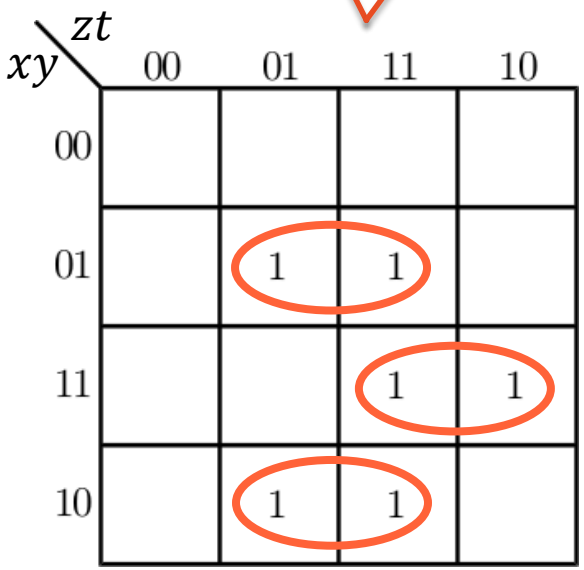
(Redundant?) On-set SOP

x	y	z	t	f
0	1	-	1	1
1	-	1	1	1
1	1	1	0	1
1	0	-	1	1

Progressive SAT-based SOP Generation: Global view



- Initialize SAT solvers
- Generate a minterm
- Expand the minterm to a cube
- Add the cube as a blocking clause
- Remove redundant cubes



Generate a minterm

x	y	z	t	f
0	1	0	1	1
1	1	1	1	1
1	1	1	0	1
1	0	0	1	1

Expand to a cube

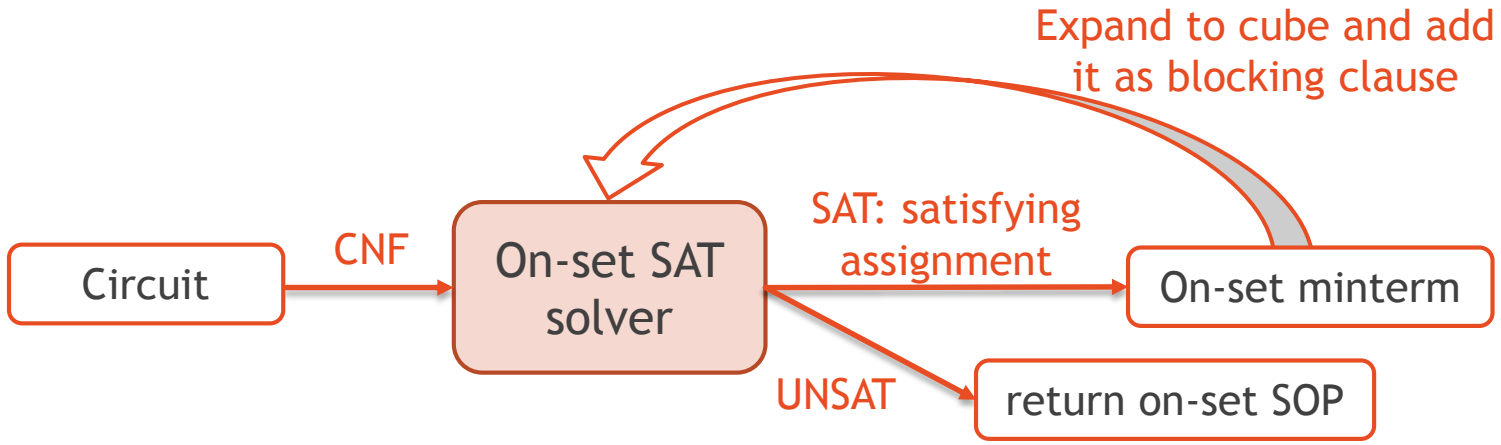
Irredundant On-set SOP

x	y	z	t	f
0	1	-	1	1
1	-	1	1	1
1	0	-	1	1

Outline

- SAT-based SOP Generation
  - Generation of Minterms
  - Expansion of Minterms to Cubes
  - Removing Redundant Cubes
- Methods for Runtime Improvement
- Experimental Results
- Conclusion

Generation of Minterms

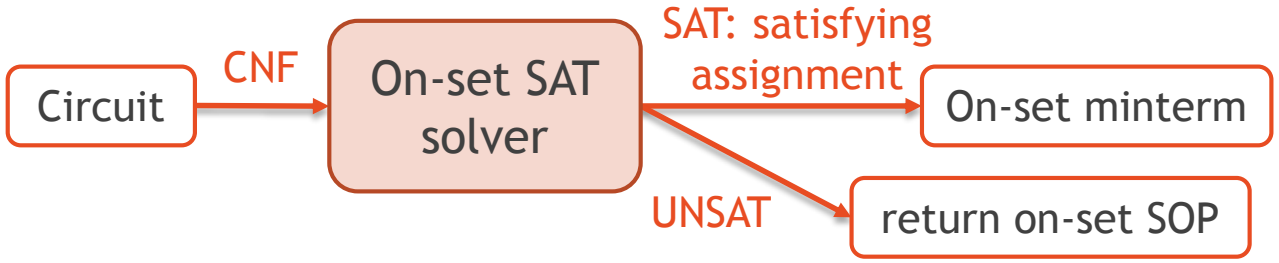


- *Non-canonical SOPs*: A SAT assignment
- *Canonical SOPs*: The lexicographically smallest SAT assignment under the variable order
  - Use the LEXSAT algorithm
  - Deterministic algorithm: generate the same minterms in the same order

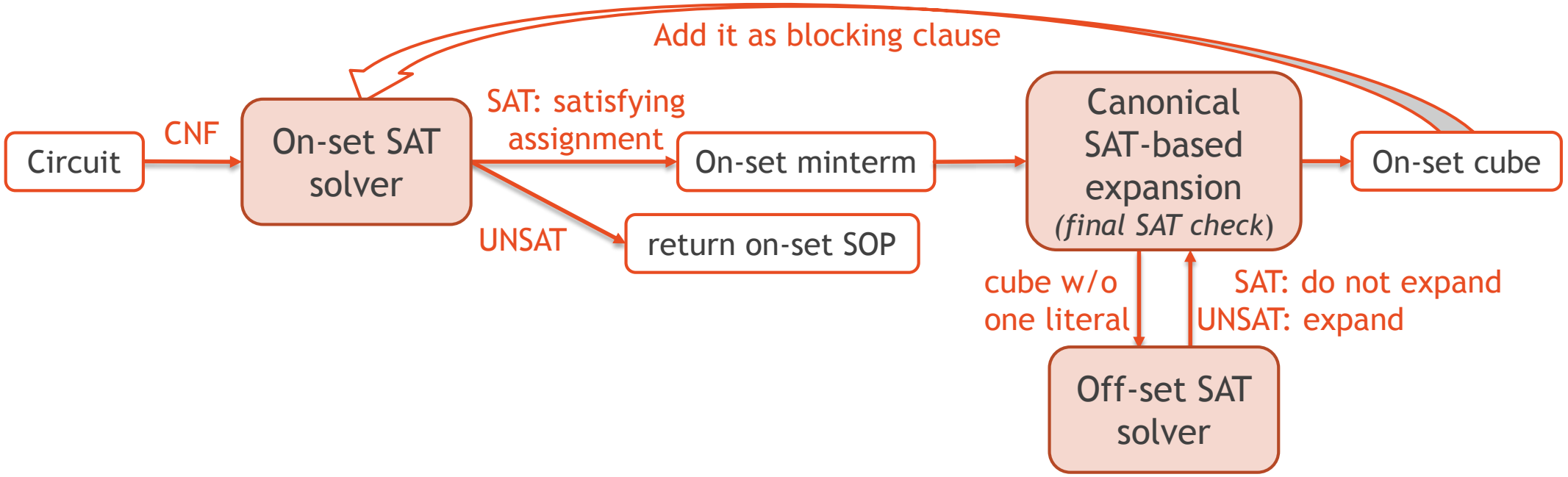
Outline

- SAT-based SOP Generation
  - Generation of Minterms
  - Expansion of Minterms to Cubes
  - Removing Redundant Cubes
- Methods for Runtime Improvement
- Experimental Results
- Conclusion

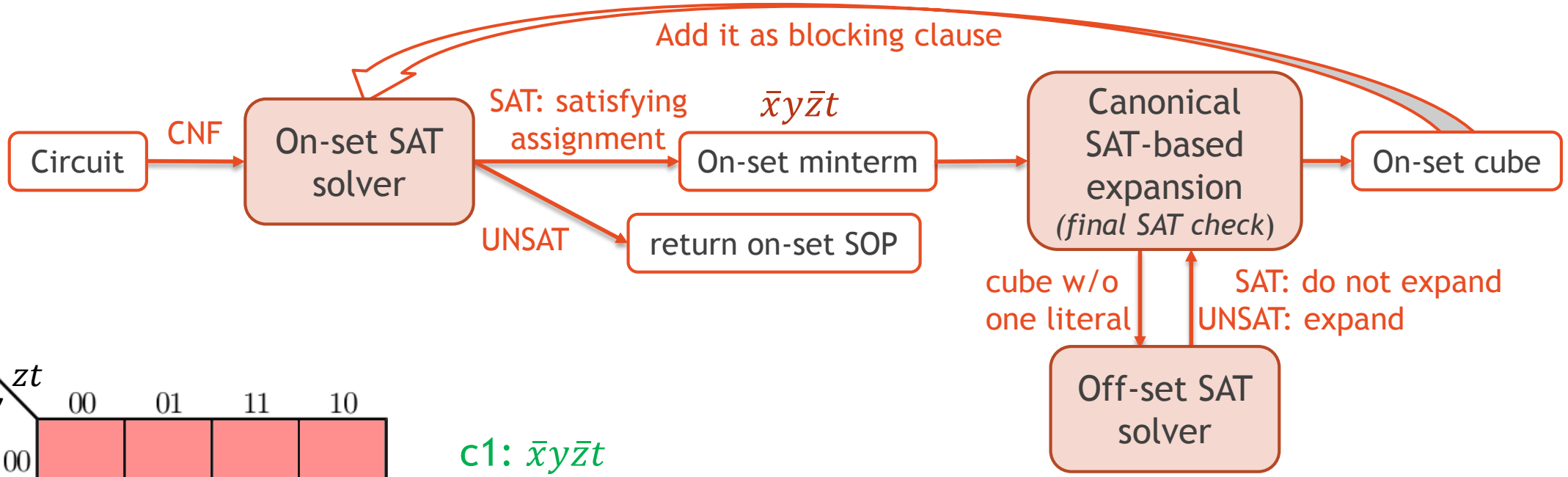
Expansion of Minterms to Cubes

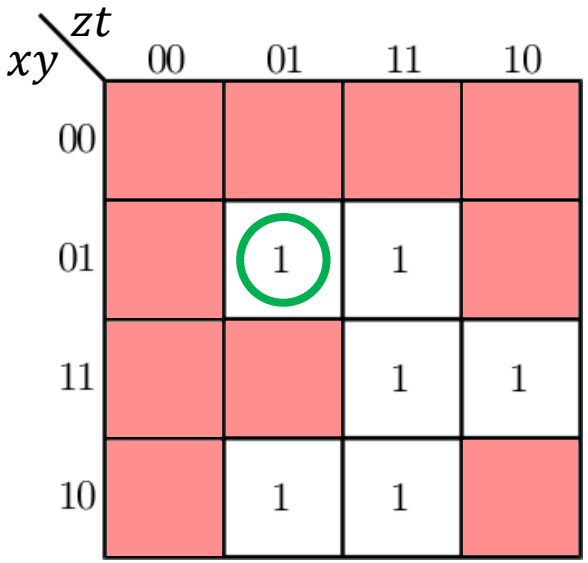


Expansion of Minterms to Cubes



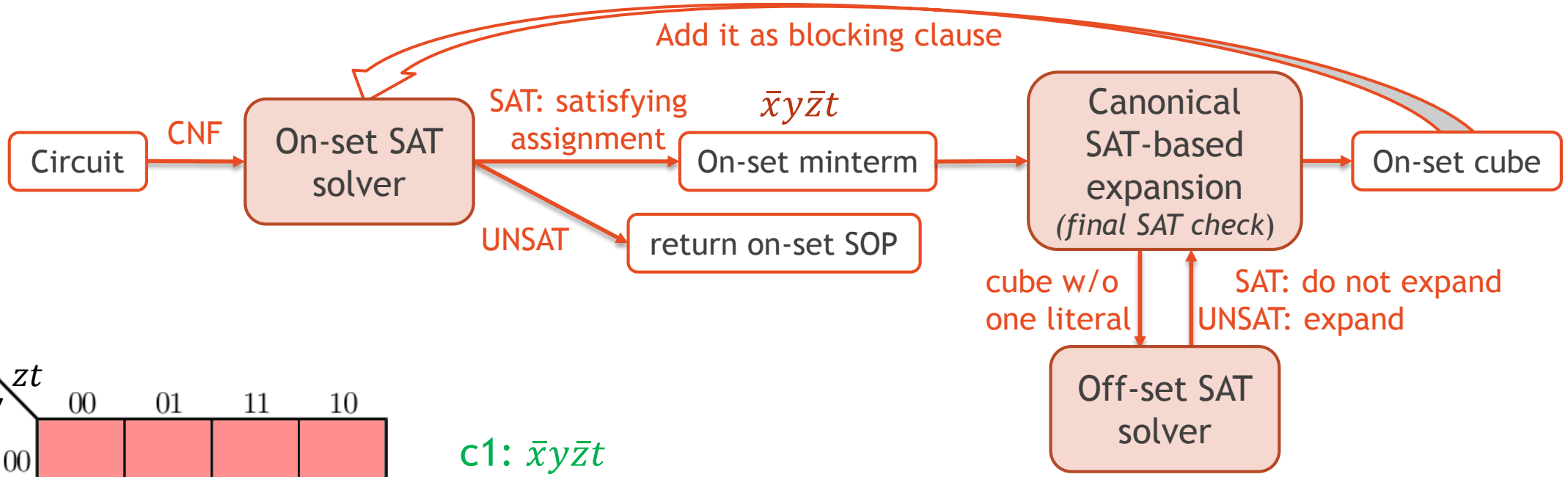
Expansion of Minterms to Cubes

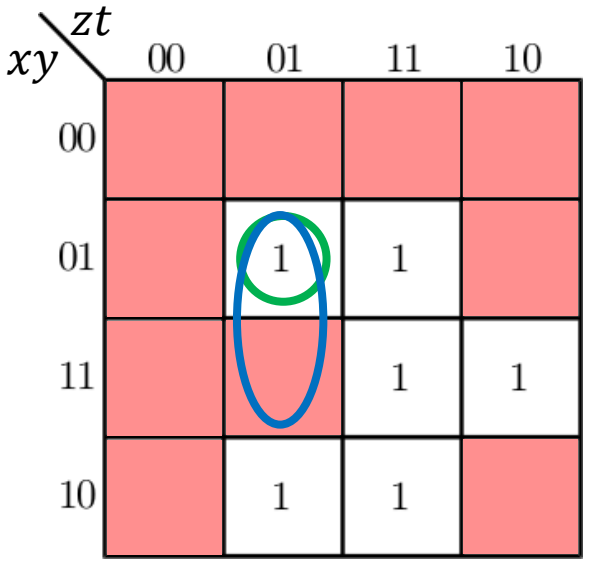




c1: $\bar{x}y\bar{z}t$

Expansion of Minterms to Cubes

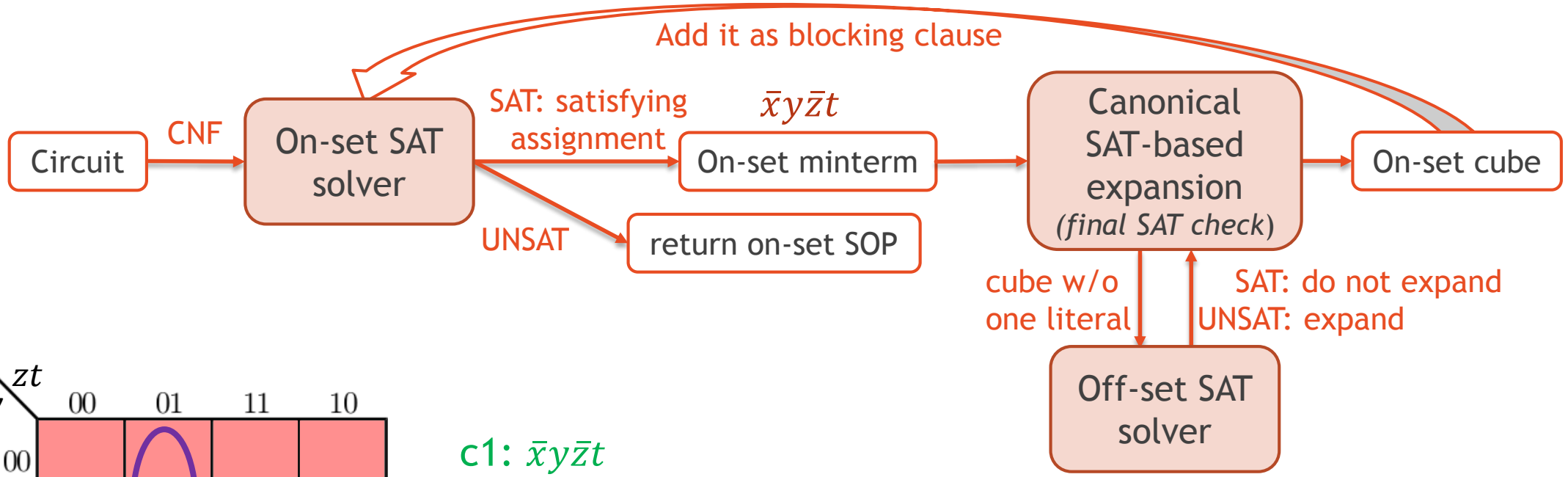


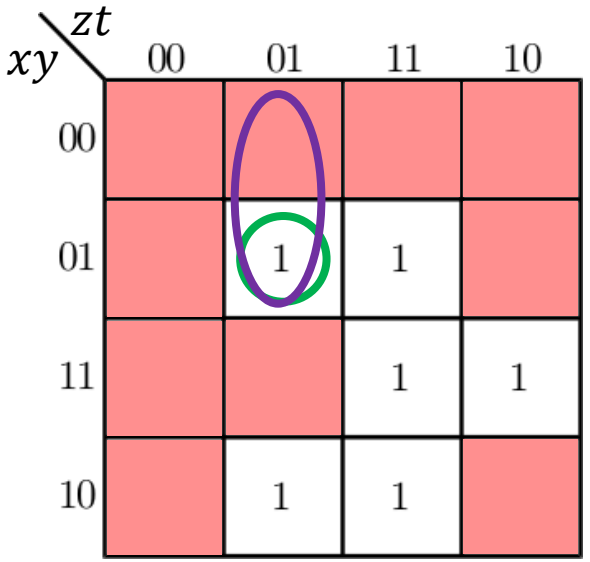


c1: $\bar{x}y\bar{z}t$

c2: $-y\bar{z}t$

Expansion of Minterms to Cubes



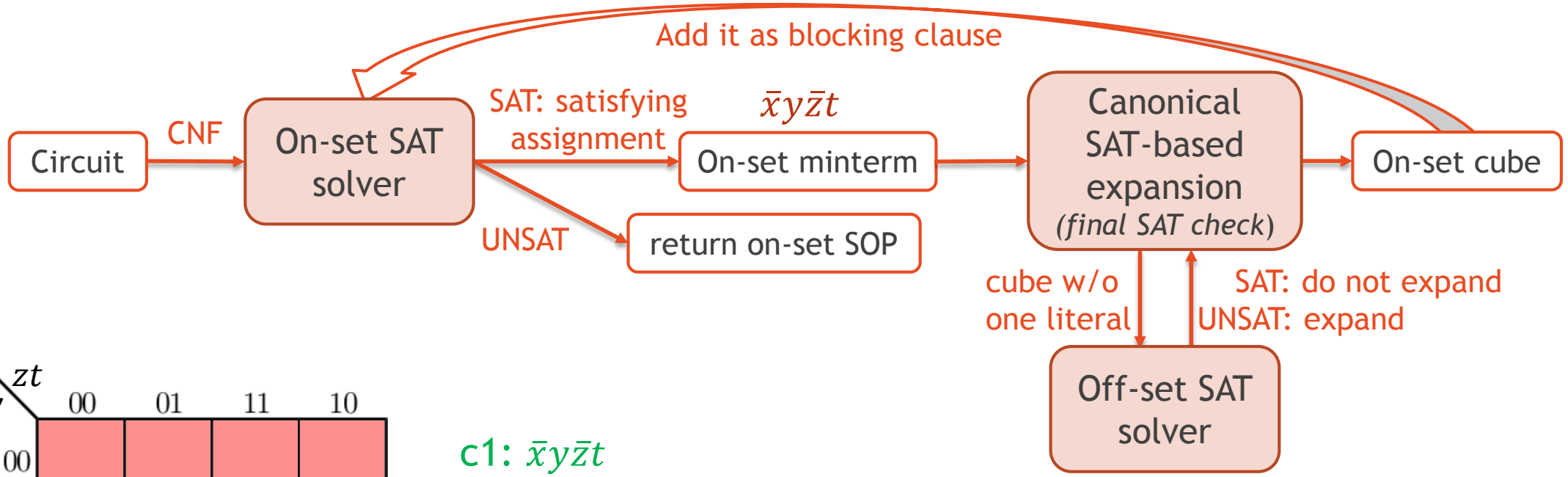


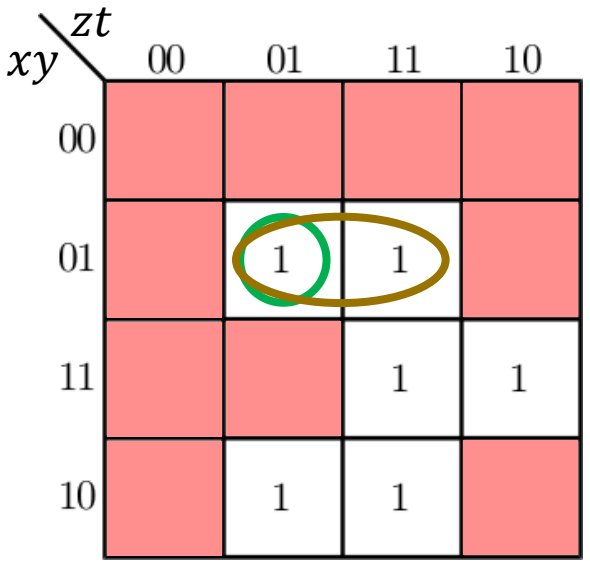
c1: $\bar{x}y\bar{z}t$

c2: $-y\bar{z}t$

c3: $\bar{x}-\bar{z}t$

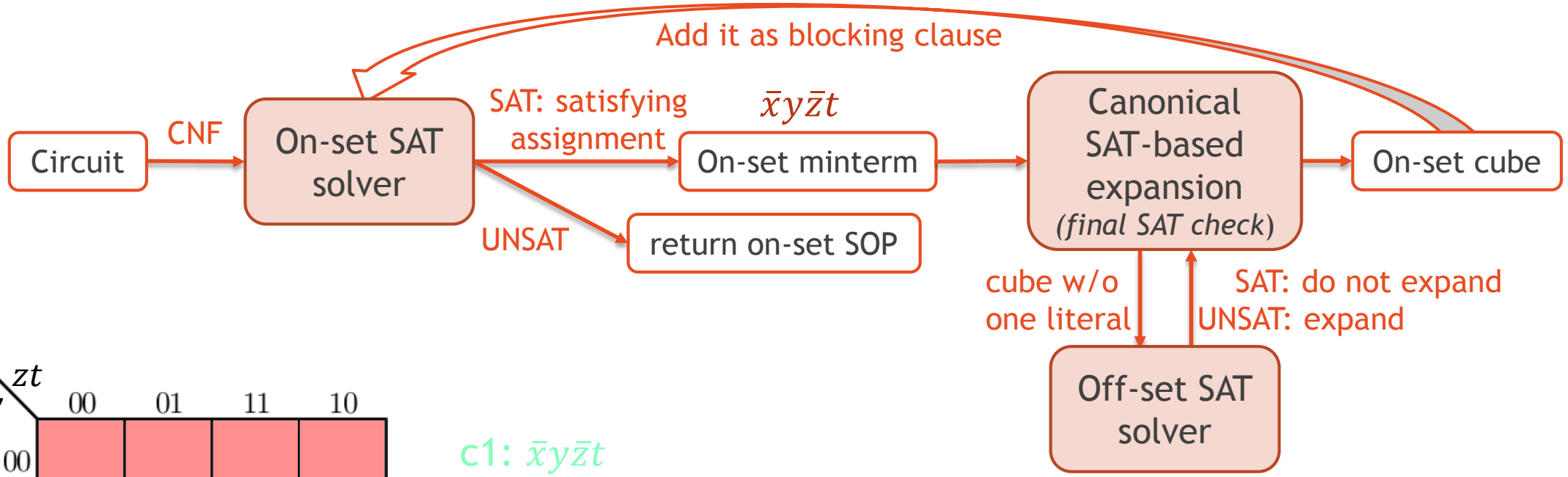
Expansion of Minterms to Cubes

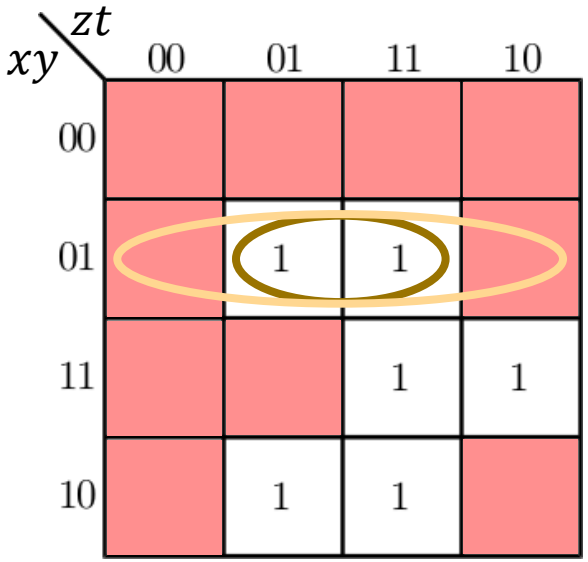




c1: $\bar{x}y\bar{z}t$
c2: $-y\bar{z}t$
c3: $\bar{x}-\bar{z}t$
c4: $\bar{x}y-t$

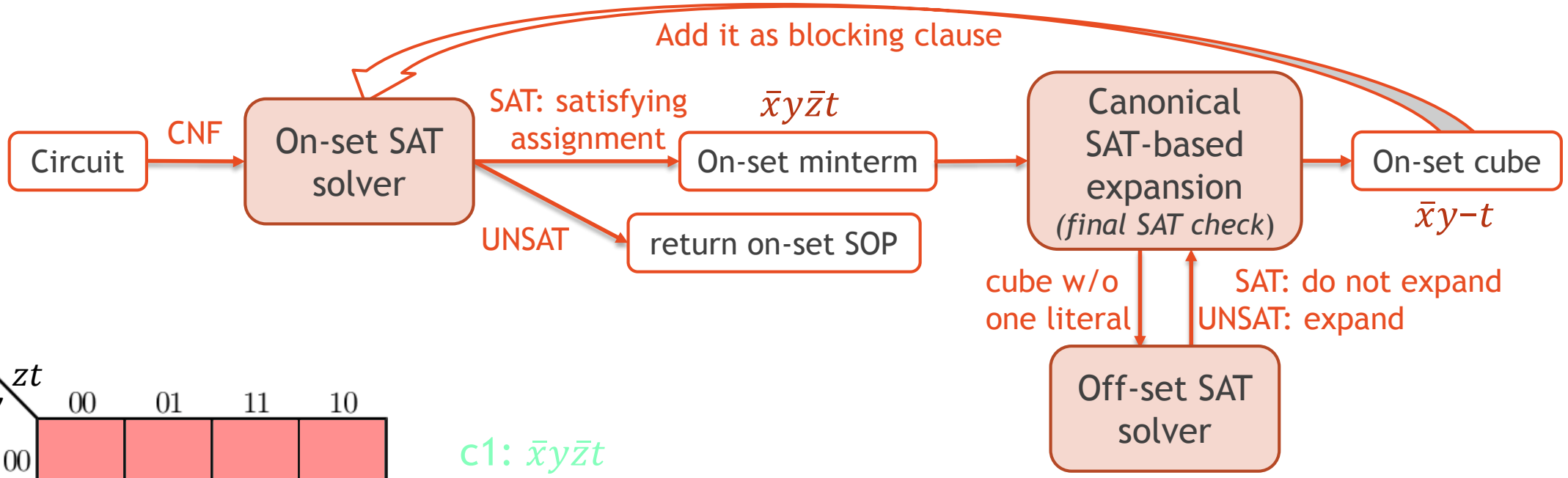
Expansion of Minterms to Cubes

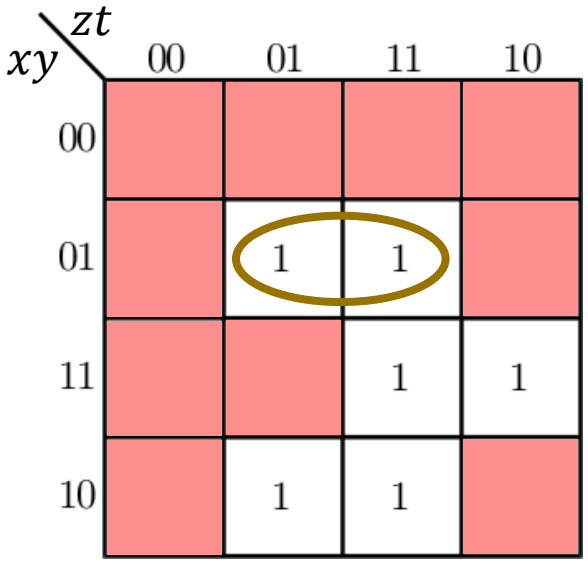




c1: $\bar{x}y\bar{z}t$
c2: $-y\bar{z}t$
c3: $\bar{x}-\bar{z}t$
c4: $\bar{x}y-t$
c5: $\bar{x}y$--

Expansion of Minterms to Cubes

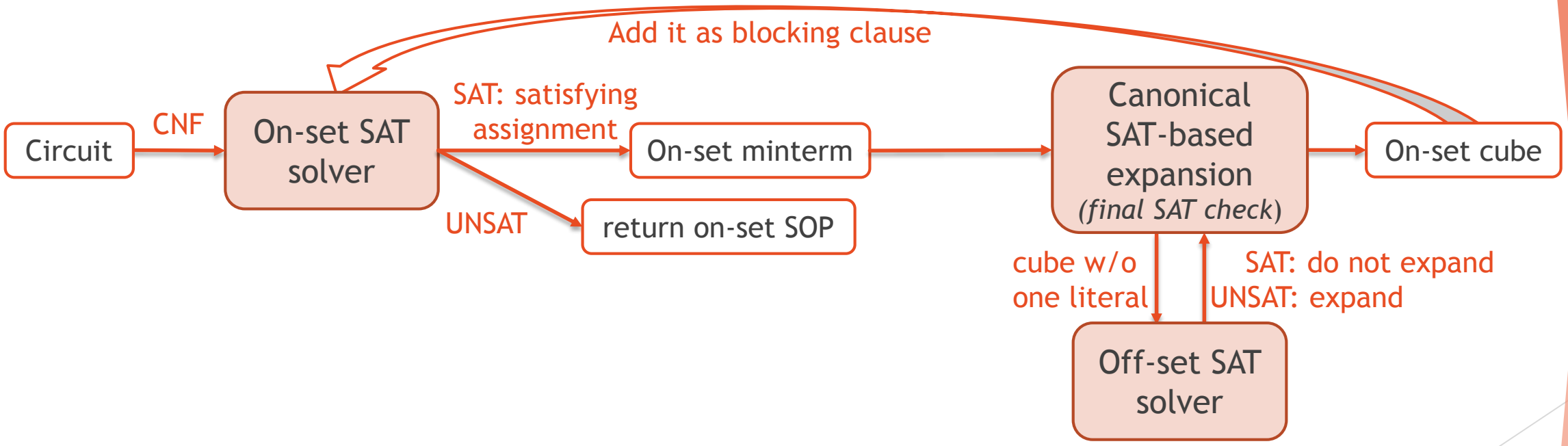




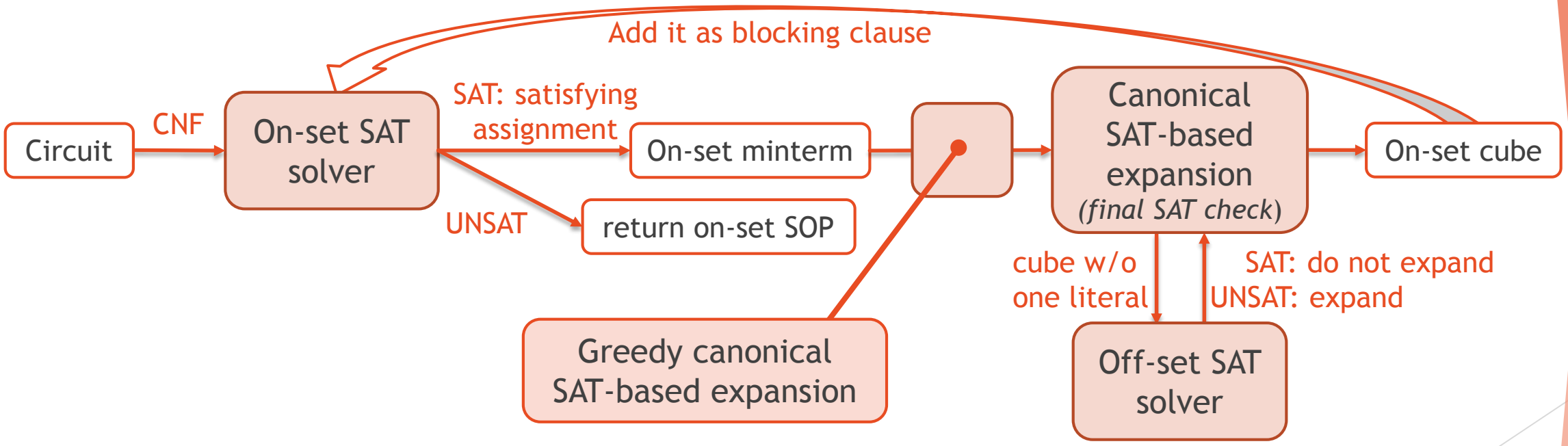
c1: $\bar{x}y\bar{z}t$
c2: $-y\bar{z}t$
c3: $\bar{x}{-}\bar{z}t$
c4: $\bar{x}y{-}t$
c5: $\bar{x}y{-}{-}$

- Canonical expansion: literals are always removed in the same order
- Prime cubes: no literal can be removed without changing the value of the function

Expansion of Minterms to Cubes

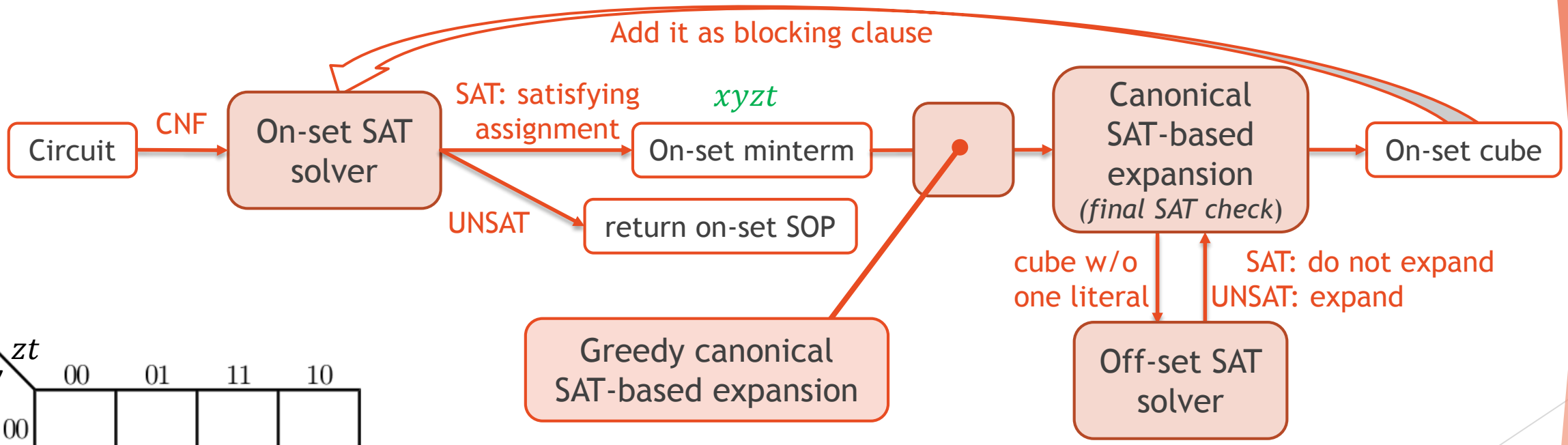


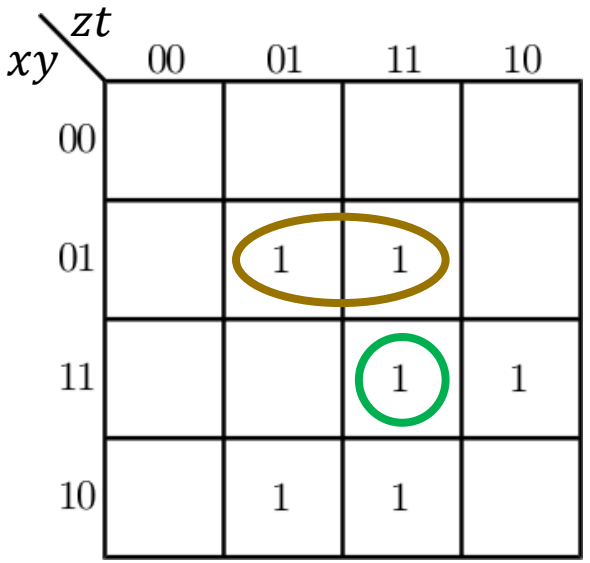
Expansion of Minterms to Cubes



- Ensure that more than one uncovered minterm is covered

Expansion of Minterms to Cubes



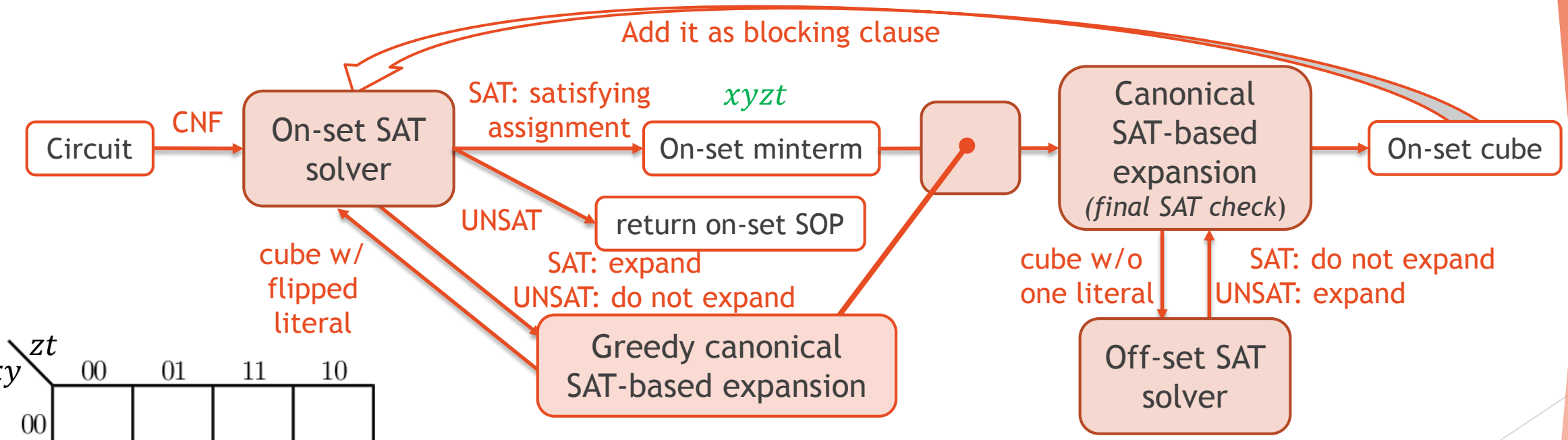


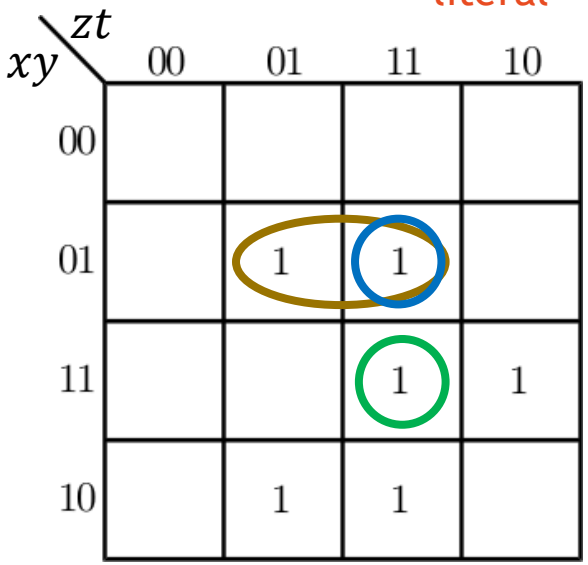
blocked: $\bar{x}y{-}t$

$xyzt$

- Ensure that more than one uncovered minterm is covered

Expansion of Minterms to Cubes





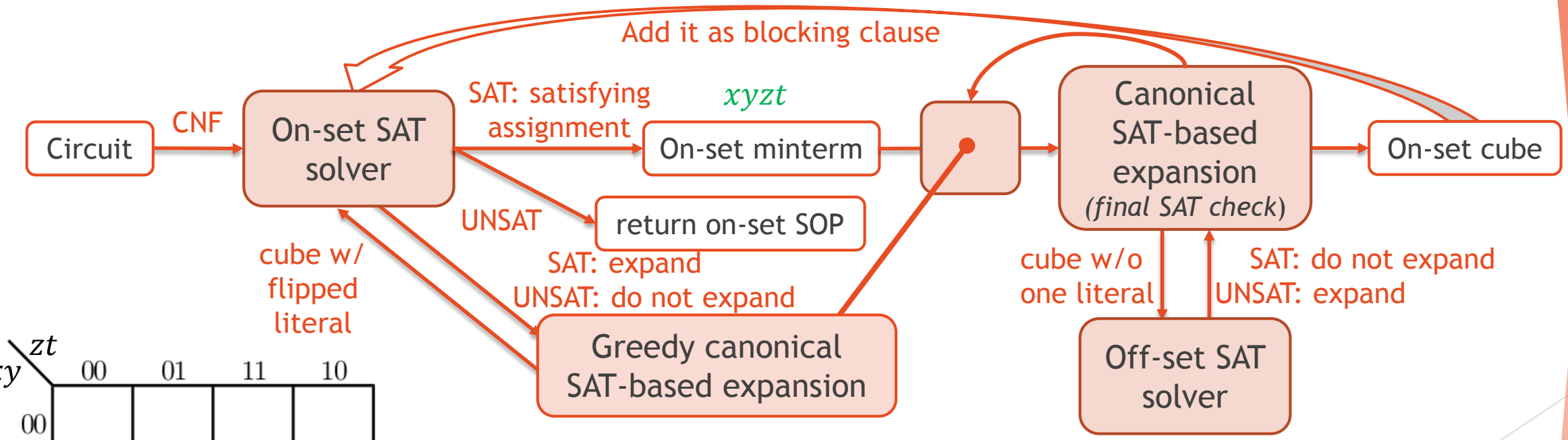
blocked: $\bar{x}y{-}t$

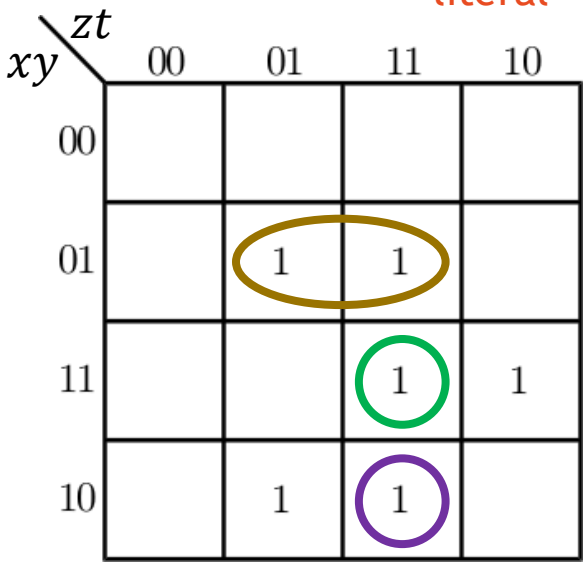
$xyzt$

$\bar{x}yzt$

- Ensure that more than one uncovered minterm is covered

Expansion of Minterms to Cubes





blocked: $\bar{x}y{-}t$

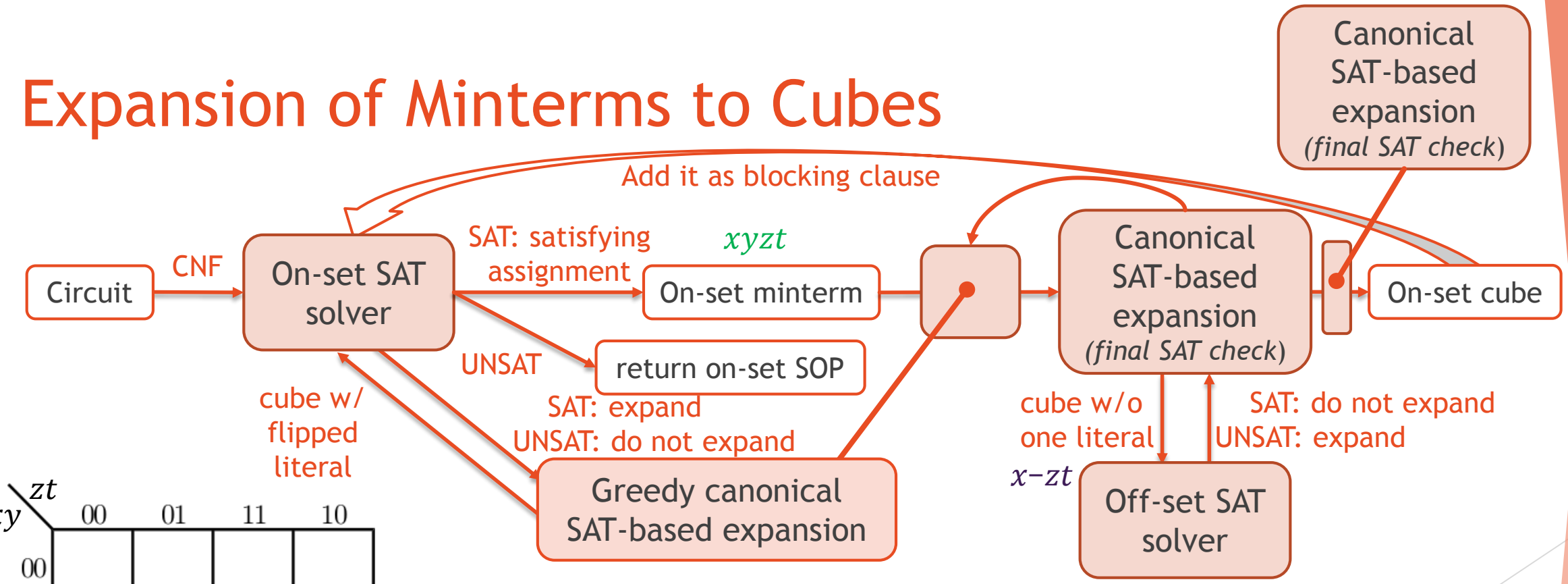
$xyzt$

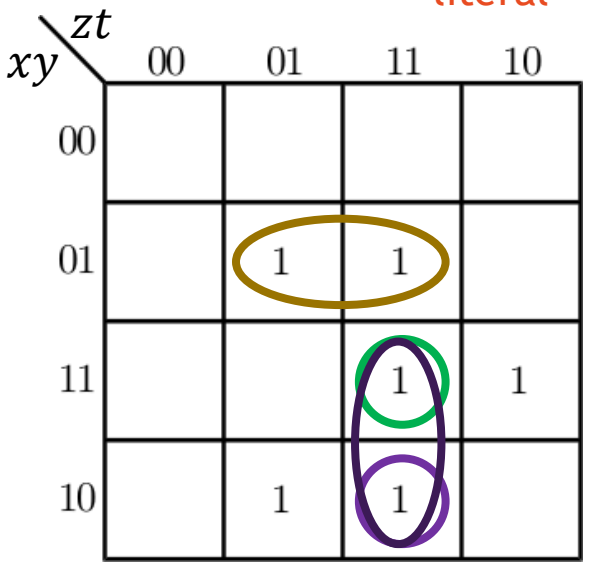
$\bar{x}yzt$

$x\bar{y}zt \rightarrow x{-}zt$

- Ensure that more than one uncovered minterm is covered

Expansion of Minterms to Cubes





blocked: $\bar{x}y-t$

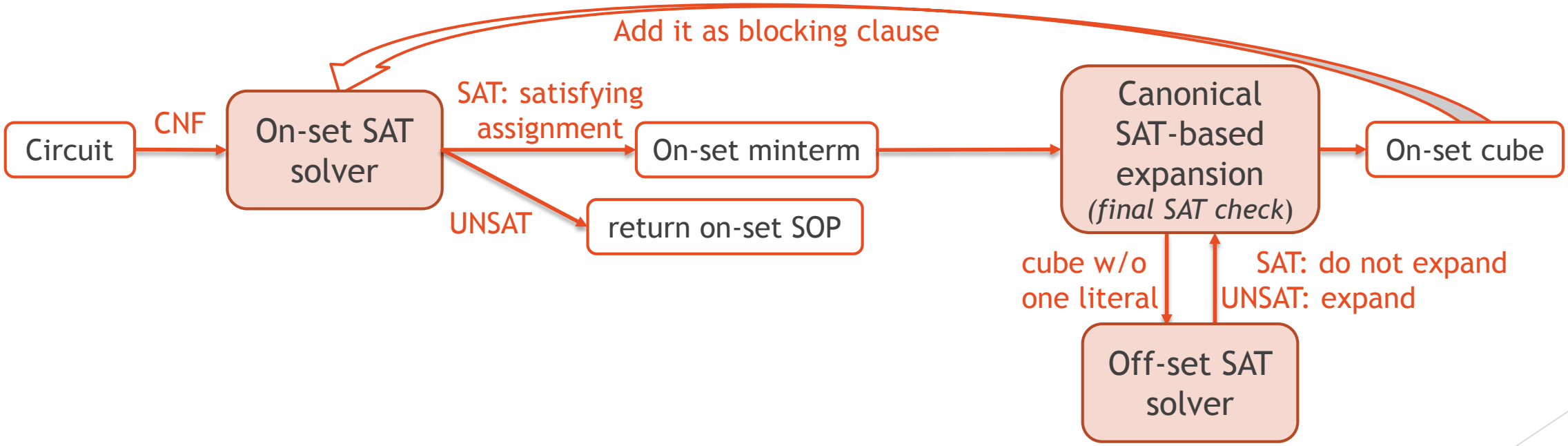
$xyzt$

$\bar{x}yzt$

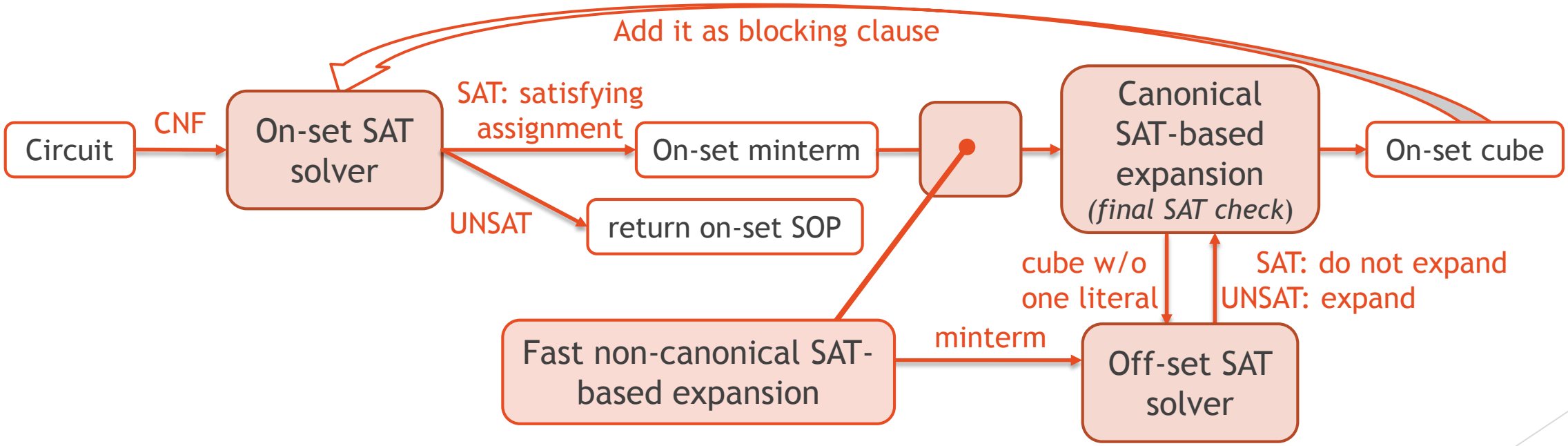
$x\bar{y}zt \rightarrow x-zt$

- Ensure that more than one uncovered minterm is covered
- Canonical expansion: literals are always removed in the same order
- At the end: iterate the final SAT check on all literals

Expansion of Minterms to Cubes

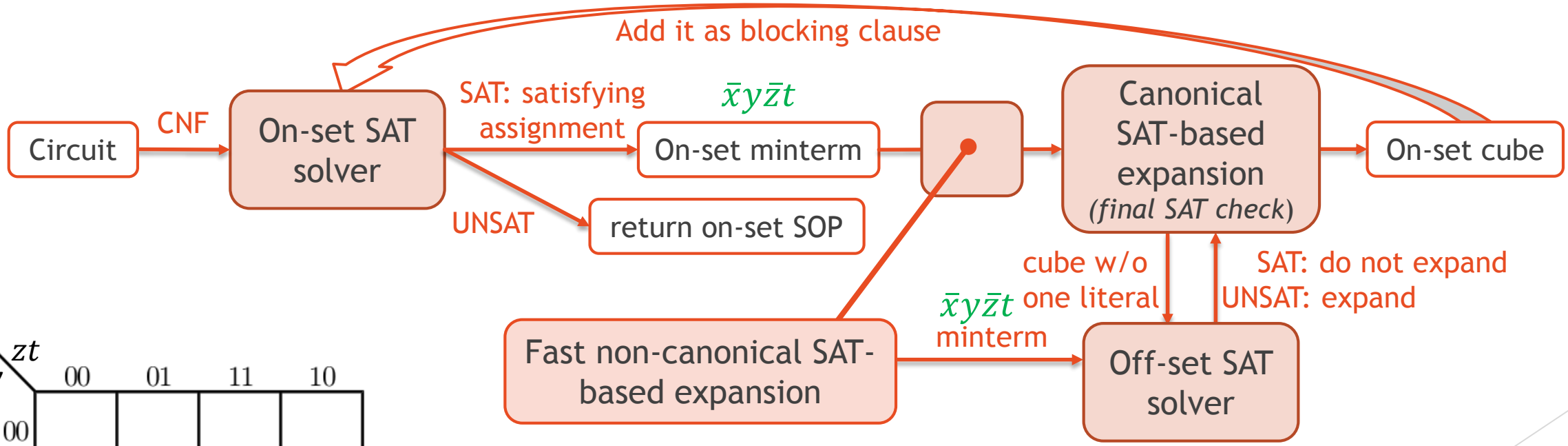


Expansion of Minterms to Cubes



- Assume the values of the minterm

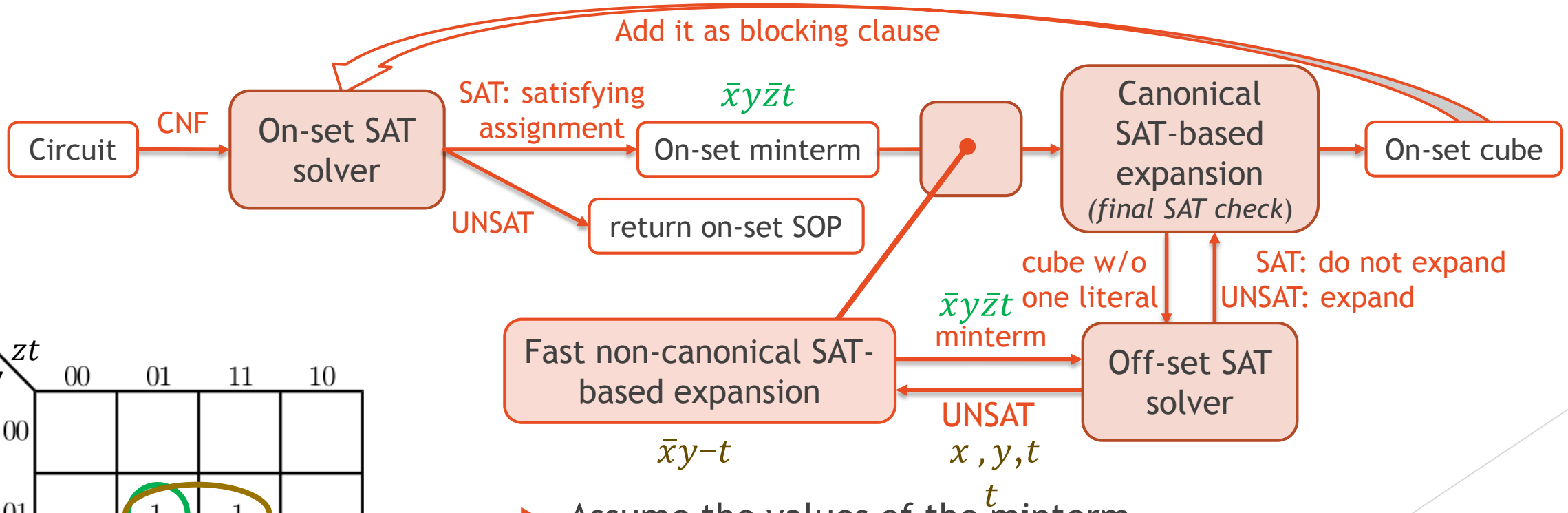
Expansion of Minterms to Cubes

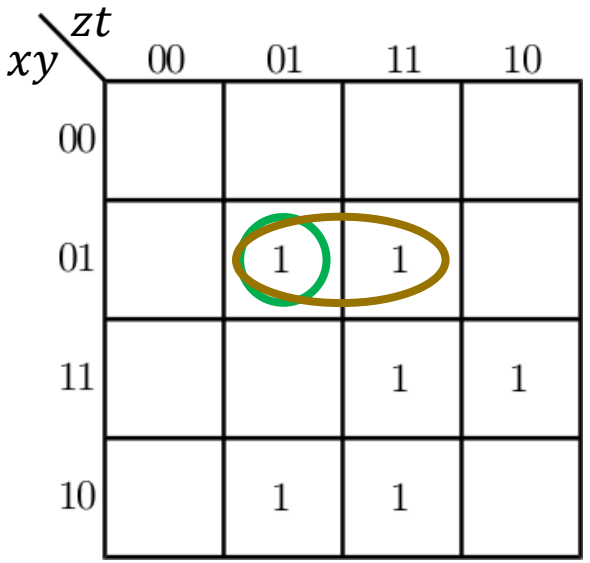


xy \ zt	00	01	11	10
00				
01		1	1	
11			1	1
10		1	1	

- Assume the values of the minterm

Expansion of Minterms to Cubes





- Assume the values of the minterm
- Get the set of literals used to prove unsatisfiability (procedure “analyse final” in MiniSAT)
- Extend the literals that are not returned in the set

Outline

- SAT-based SOP Generation
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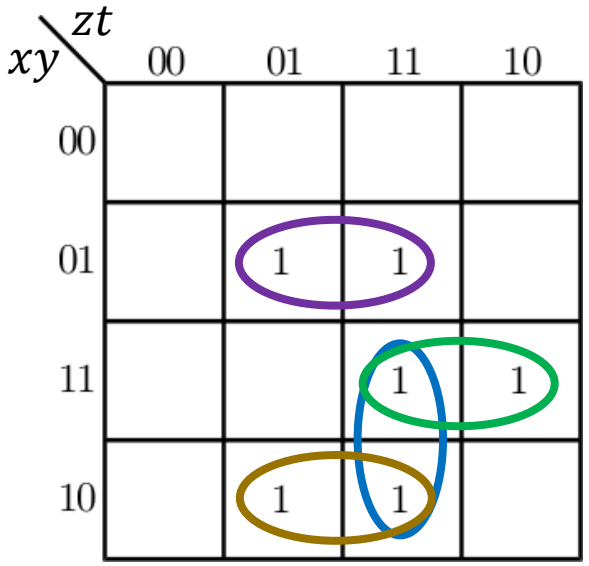
Removing Redundant Cubes

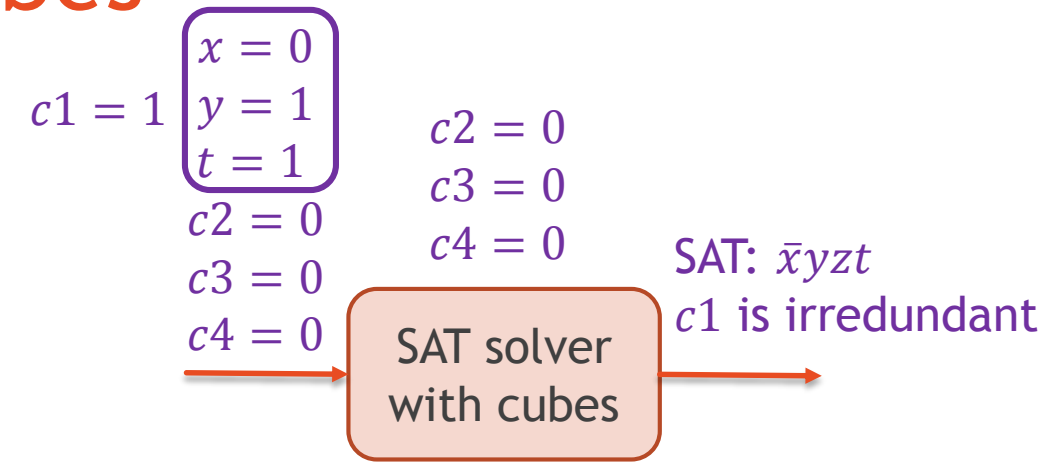


$c1 = \bar{x}yt$

$c2 = xzt$

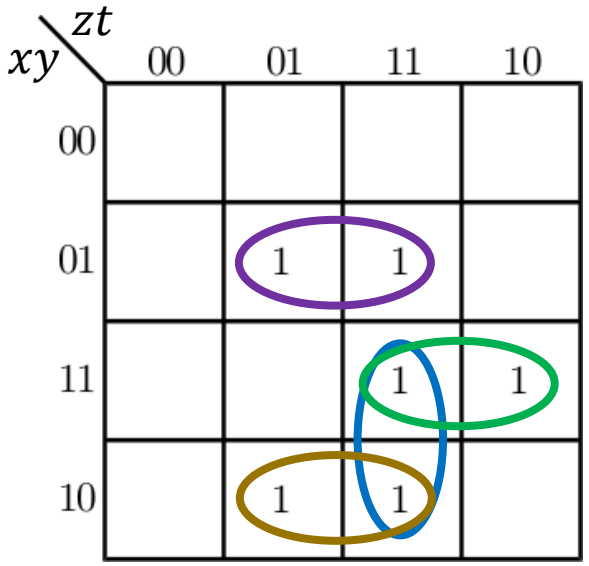
$c3 = xyz$

$c4 = x\bar{y}t$



- Initialize a SAT solver with all cubes
- Find an assignment for which $ci = 1$ and the other irredundant cubes evaluate to 0
  - SAT: ci is irredundant
  - UNSAT: ci is redundant, remove it from the SOP
- Canonical: cubes are removed always in the same order

Removing Redundant Cubes

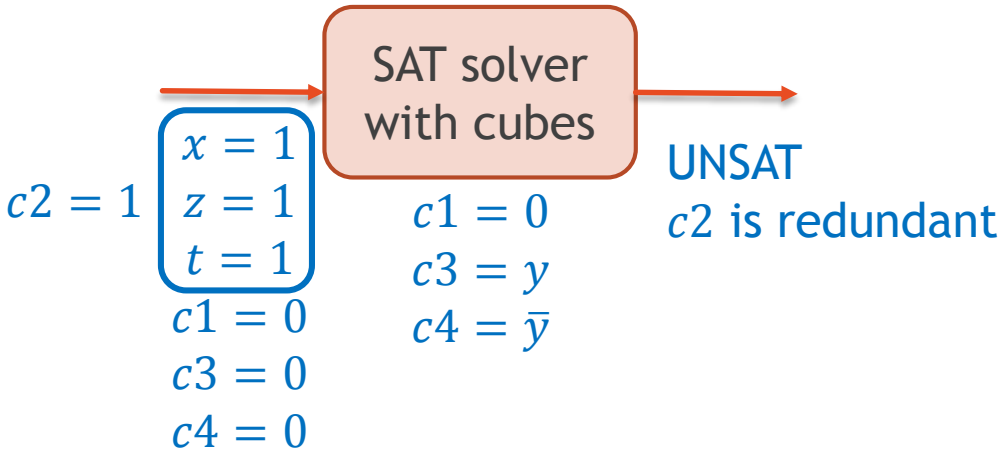


$c1 = \bar{x}yt$

$c2 = xzt$

$c3 = xyz$

$c4 = x\bar{y}t$



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Methods for Runtime Improvement

- Generate simultaneously on-set and off-set SOPs

On-set SOP

x_1	x_2	x_3	f
-	1	1	1
1	1	-	1

$f = x_2x_3 + x_1x_2$

Off-set SOP

x_1	x_2	x_3	f
0	0	-	0
0	-	0	0
1	0	-	0

$\bar{f} = \bar{x}_1\,\bar{x}_2 + \bar{x}_1\,\bar{x}_3 + x_1\bar{x}_2$

Avoid generating first the larger SOP

Future work: generate each set in parallel

- Prioritize outputs with large SOP
  - Sort outputs by size of their input support
  - Generate SOPs in decreasing order

S_3: 9 inputs

S_2: 7 inputs

S_1: 5 inputs

S_0: 3 inputs

Early termination when the large SOP exceed resource limits

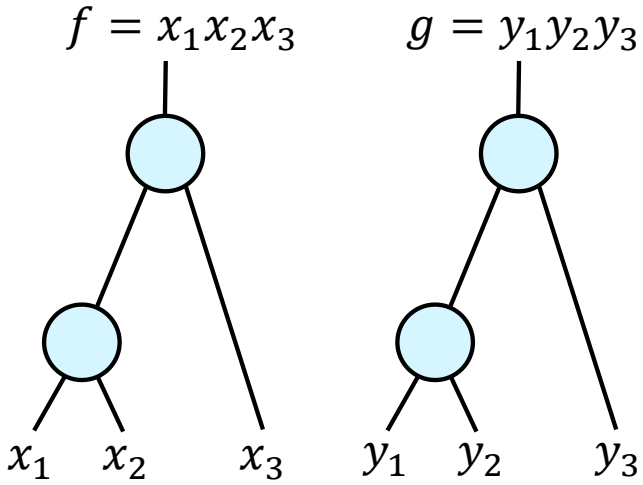
Future work: generate SOP of each output in parallel

Methods for Runtime Improvement

- Detect isomorphic outputs

  Benefit from structure sharing

  - Isomorphic functions: implement an identical function using different inputs



Isomorphic class 1
Func: f, g
Repr: $f = x_1x_2x_3$

  - Generate SOP only for one output per class (the representative)
  - Duplicate the generated SOP for the other outputs from the class

- Share one CNF among all outputs

  Benefit from logic sharing

  - Generate one CNF for the complete circuit
  - For each output, initialize the SAT solver with the CNF part of the output

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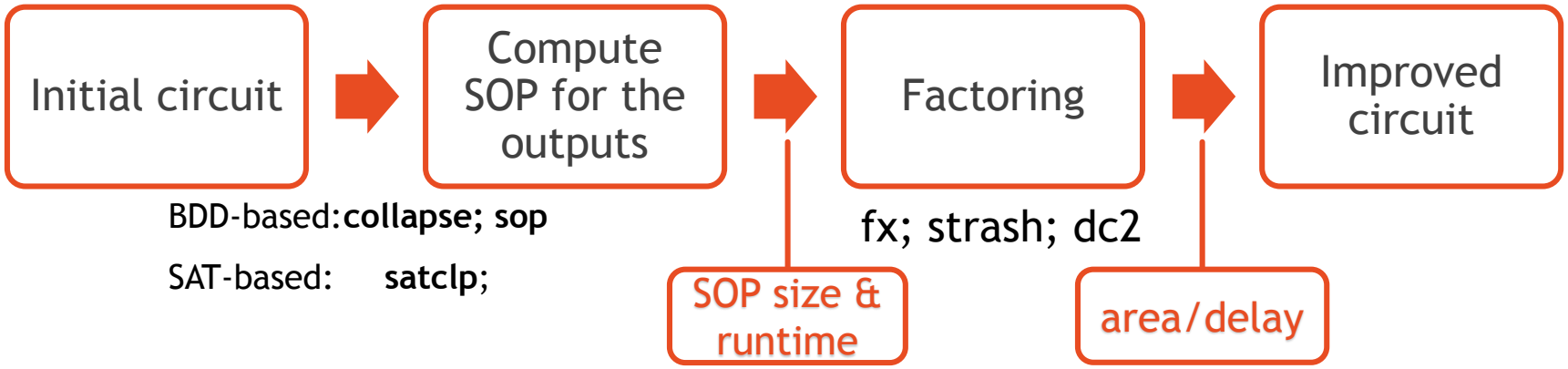
Experimental Setup

- Implemented the SAT-based SOP generation in ABC
- Compared flows for SOP generation

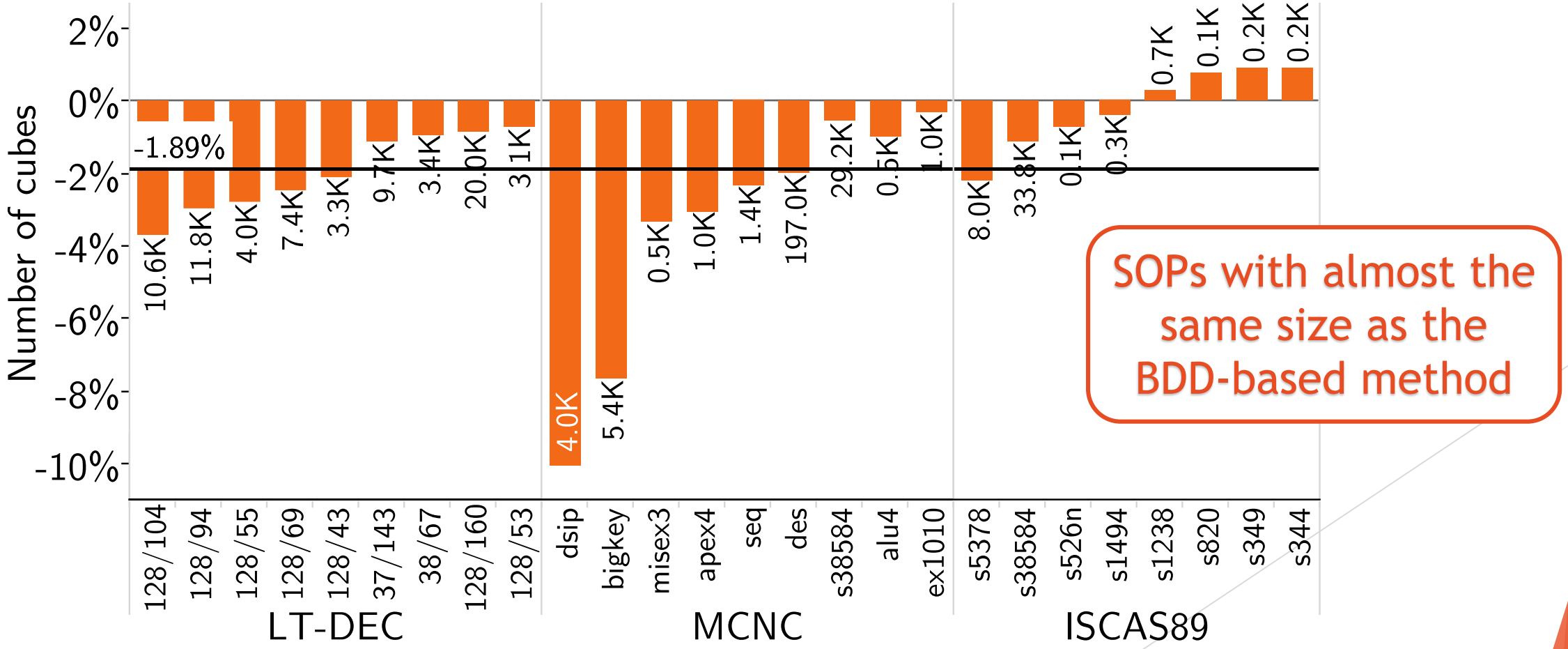


- For the BDD-based SOPs, generate 5 SOPs using different variable orders
- For the SAT-based SOPs, generate 12 SOPs using different options
  - Non-canonical or canonical SOPs
  - Order variables by fanout number
  - Reverse variable order
  - Shared CNF by primary outputs

Experimental Setup

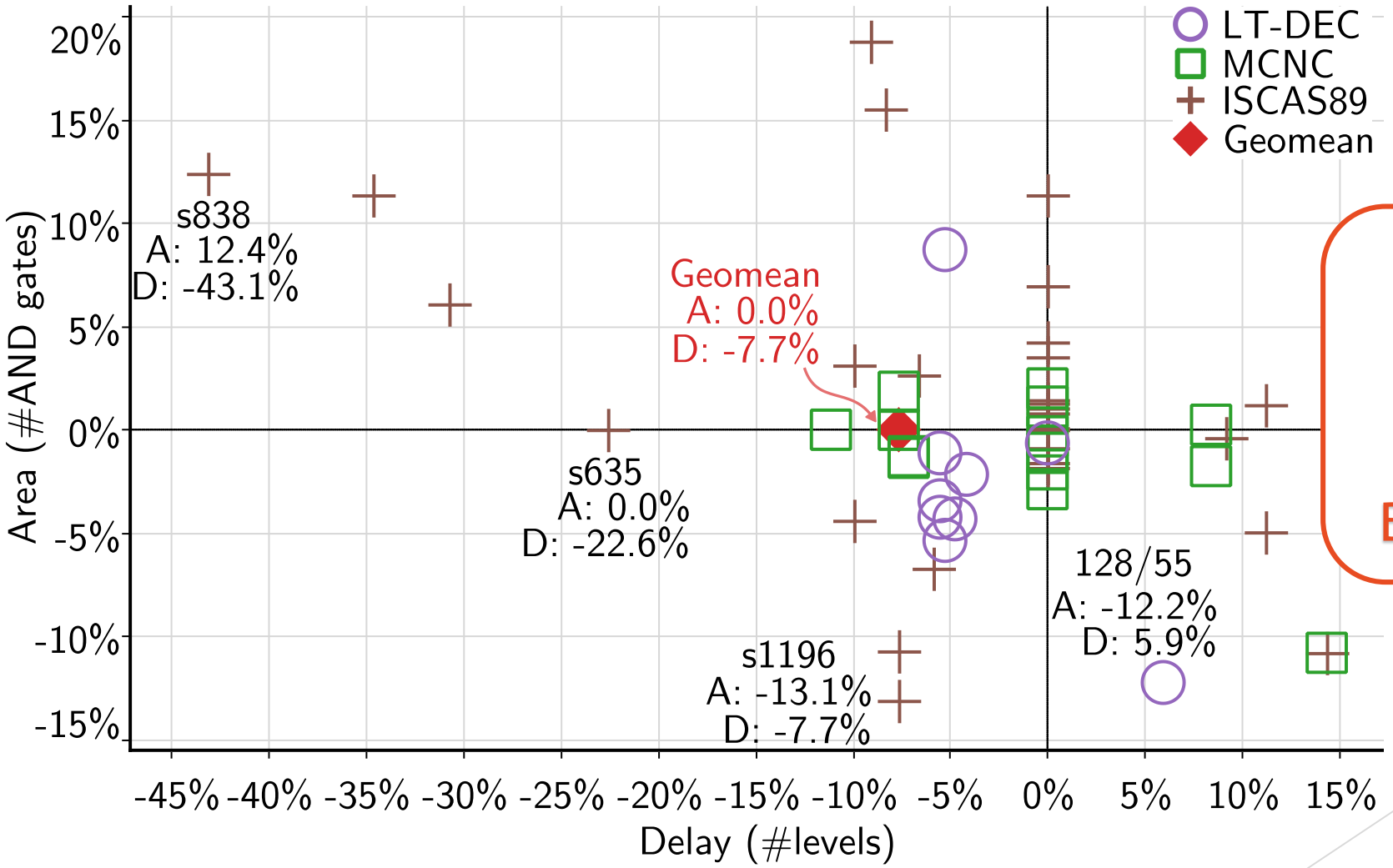
- Benchmarks
  - Set of 20 large MCNC benchmarks
  - Set of 32 ISCAS'89 benchmarks
  - Set of 9 logic tables from instruction decoder unit (LT-DEC)

Comparison of SOP Size



SOPs with almost the same size as the BDD-based method

Comparison of Area-Delay Product



Pareto optimal multi-level implementations compared to the BDD-based method

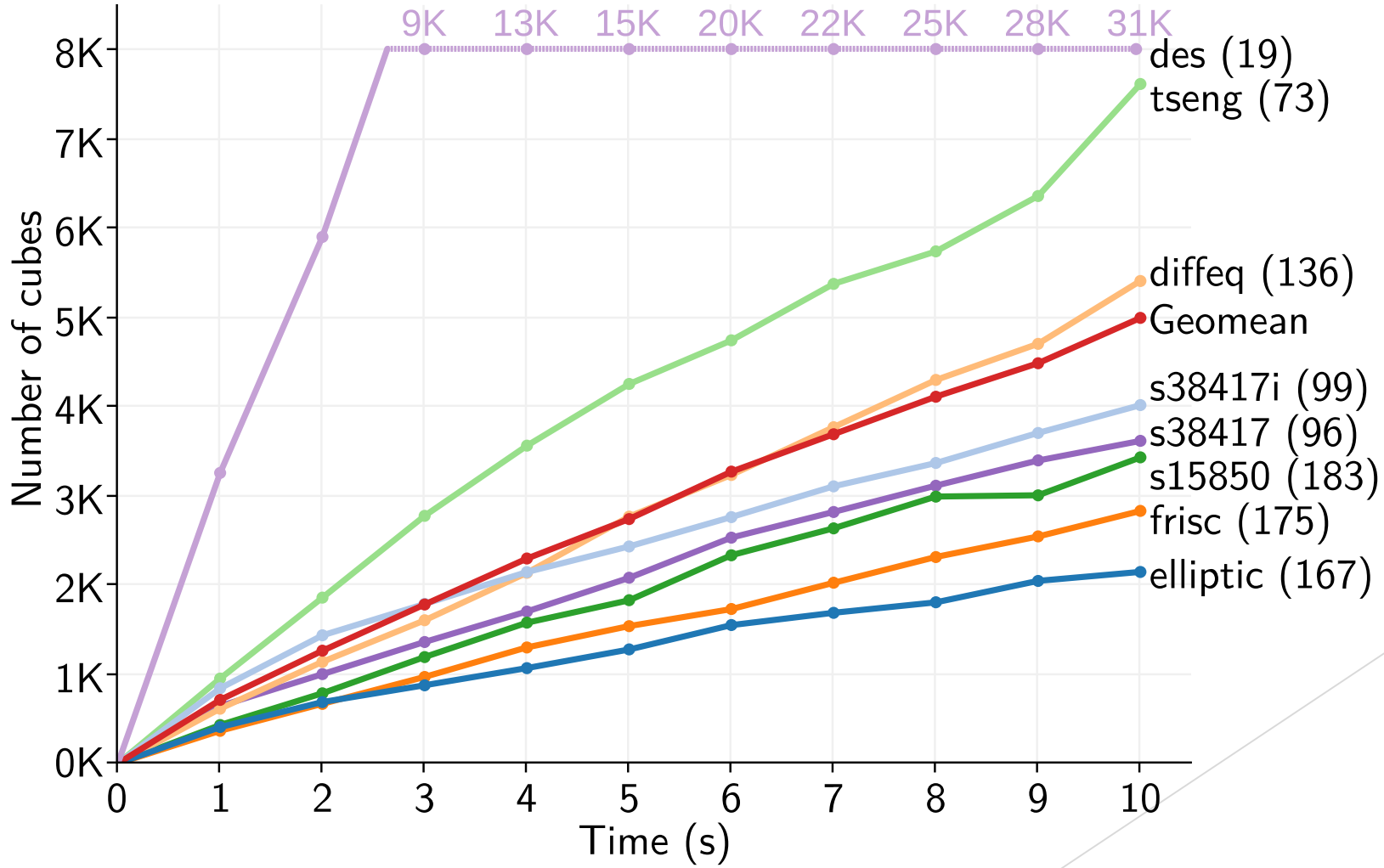
Comparison of Runtime

- Public benchmarks
  - On average, 8x slower than the BDD-based method
  - Faster for circuits with many isomorphic outputs
    - S35932 – 10 isomorphic classes for 2048 combinatorial outputs -> SAT-based: 0.06 sec; BDD-based: 1.83 sec
- Industrial benchmarks
  - Faster due to isomorphism: generated SOPs for only ~30% of the outputs
  - Hundreds of PIs/POs -> constructing the global BDD in one manager is problematic

Comparison of Runtime

- Runtime profile (for the LT-DEC benchmarks)
  - Generating minterms 8%
  - Expanding minterms to cubes 84%
  - Removing redundant cubes 2%
  - Other operations 6%

Partial SOPs



Outline

- SAT-based SOP Generation
  - Generation of Minterms
  - Expansion of Minterms to Cubes
  - Removing Redundant Cubes
- Methods for Runtime Improvement
- Experimental Results
- Conclusion

Conclusion

- We presented a complete SAT-based algorithm for **progressive generation** of **irredundant (non-)canonical SOPs**
- Quality of results
  - Decreased SOP size up to 10% compared to the BDD-based method
  - Area-delay product post factoring can be improved up to 36%
- Runtime: 8x slower on average, but faster for isomorphic circuits
- Future work
  - Improving the runtime: using parallelism, faster cube expansion
  - Dedicated SAT-based multi-output SOP computation