9

The Dimension of a Variety

The most important invariant of a linear subspace of affine space is its dimension. For affine varieties, we have seen numerous examples which have a clearly defined dimension, at least from a naive point of view. In this chapter, we will carefully define the dimension of any affine or projective variety and show how to compute it. We will also show that this notion accords well with what we would expect intuitively. In keeping with our general philosophy, we consider the computational side of dimension theory right from the outset.

§1 The Variety of a Monomial Ideal

We begin our study of dimension by considering monomial ideals. In particular, we want to compute the dimension of the variety defined by such an ideal. Suppose, for example, we have the ideal $I = \langle x^2y, x^3\rangle$ in $k[x, y]$. Letting H_x denote the line in k^2 defined by $x = 0$ (so $H_x = \mathbf{V}(x)$) and H_y the line $y = 0$, we have

$$\begin{aligned} \mathbf{V}(I) &= \mathbf{V}(x^2y) \cap \mathbf{V}(x^3) \\ &= (H_x \cup H_y) \cap H_x \\ &= (H_x \cap H_x) \cup (H_y \cap H_x) \\ &= H_x. \end{aligned} \tag{1}$$

Thus, $\mathbf{V}(I)$ is the y-axis H_x. Since H_x has dimension 1 as a vector subspace of k^2, it is reasonable to say that it also has dimension 1 as a variety.

As a second example, consider the ideal

$$I = \langle y^2z^3, x^5z^4, x^2yz^2\rangle \subset k[x, y, z].$$

Let H_x be the plane defined by $x = 0$ and define H_y and H_z similarly. Also, let H_{xy} be the line $x = y = 0$. Then we have

$$\begin{aligned} \mathbf{V}(I) &= \mathbf{V}(y^2z^3) \cap \mathbf{V}(x^5z^4) \cap \mathbf{V}(x^2yz^2) \\ &= (H_y \cup H_z) \cap (H_x \cup H_z) \cap (H_x \cup H_y \cup H_z) \\ &= H_z \cup H_{xy}. \end{aligned}$$

To verify this, note that the plane H_z belongs to each of the three terms in the second line and, hence, to their intersection. Thus, $\mathbf{V}(I)$ will consist of the plane H_z together, perhaps, with some other subset not contained in H_z. Collecting terms not contained in H_z, we have $H_y \cap H_x \cap (H_x \cup H_y)$, which equals H_{xy}. Thus, $\mathbf{V}(I)$ is the union of the (x, y)-plane H_z and the z-axis H_{xy}. We will say that the dimension of a union of finitely many vector subspaces of k^n is the biggest of the dimensions of the subspaces, and so the dimension of $\mathbf{V}(I)$ is 2 in this example.

The variety of any monomial ideal may be assigned a dimension in much the same fashion. But first we need to describe what a variety of a general monomial ideal looks like. In k^n, a vector subspace defined by setting some subset of the variables $x_1, \dots, x_n$ equal to zero is called a *coordinate subspace.*

Proposition 1. *The variety of a monomial ideal in $k[x_1, \dots, x_n]$ is a finite union of coordinate subspaces of k^n.*

Proof. First, note that if $x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r}$ is a monomial in $k[x_1, \dots, x_n]$ with $\alpha_j \geq 1$ for $1 \leq j \leq r$, then

$$\mathbf{V}(x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r}) = H_{x_{i_1}} \cup \dots \cup H_{x_{i_r}},$$

where $H_{x_k} = \mathbf{V}(x_k)$. Thus, the variety defined by a monomial is a union of coordinate hyperplanes. Note also that there are only n such hyperplanes.

Since a monomial ideal is generated by a finite collection of monomials, the variety corresponding to a monomial ideal is a finite intersection of unions of coordinate hyperplanes. By the distributive property of intersections over unions, any finite intersection of unions of coordinate hyperplanes can be rewritten as a finite union of intersections of coordinate hyperplanes [see (1) for an example of this]. But the intersection of any collection of coordinate hyperplanes is a coordinate subspace. $\square$

When we write the variety of a monomial ideal I as a union of finitely many coordinate subspaces, we can omit a subspace if it is contained in another in the union. Thus, we can write $\mathbf{V}(I)$ as a union of coordinate subspaces.

$$\mathbf{V}(I) = V_1 \cup \dots \cup V_p,$$

where $V_i \not\subset V_j$ for $i \neq j$. In fact, such a decomposition is unique, as you will show in Exercise 8.

Let us make the following provisional definition. We will always assume that k is infinite.

Definition 2. *Let V be a variety which is the union of a finite number of linear subspaces of k^n. Then the dimension of V, denoted* $\dim V$*, is the largest of the dimensions of the subspaces.*

Thus, the dimension of the union of two planes and a line is 2, and the dimension of a union of three lines is 1. To compute the dimension of the variety corresponding to a monomial ideal, we merely find the maximum of the dimensions of the coordinate subspaces contained in $\mathbf{V}(I)$.

Although this is easy to do for any given example, it is worth systematizing the computation. Let $I = \langle m_1, \ldots, m_t \rangle$ be a proper ideal generated by the monomials m_j. In trying to compute dim $\mathbf{V}(I)$, we need to pick out the component of

$$\mathbf{V}(I) = \bigcap_{j=1}^{t} \mathbf{V}(m_j)$$

of largest dimension. If we can find a collection of variables $x_{i_1}, \ldots, x_{i_r}$ such that at least one of these variables appears in each m_j, then the coordinate subspace defined by the equations $x_{i_1} = \cdots = x_{i_r} = 0$ is contained in $\mathbf{V}(I)$. This means we should look for variables which occur in as many of the different m_j as possible. More precisely, for $1 \leq j \leq t$, let

$$M_j = \{k \in \{1, \ldots, n\} : x_k \text{ divides the monomial } m_j\}$$

be the set of subscripts of variables occurring with positive exponent in m_j. (Note that M_j is nonempty by our assumption that $I \neq k[x_1, \ldots, x_n]$.) Then let

$$\mathcal{M} = \{J \subset \{1, \ldots, n\} : J \cap M_j \neq \emptyset \text{ for all } 1 \leq j \leq t\}$$

consist of all subsets of $\{1, \ldots, n\}$ which have nonempty intersection with *every* set M_j. (Note that $\mathcal{M}$ is not empty because $\{1, \ldots, n\} \in \mathcal{M}$.) If we let $|J|$denote the number of elements in a set J, then we have the following.

Proposition 3. *With the notation above,*

$$\dim \mathbf{V}(I) = n - \min(|J| : J \in \mathcal{M}).$$

Proof. Let $J = \{i_1, \ldots, i_r\}$ be an element of $\mathcal{M}$ such that $|J| = r$ is minimal in $\mathcal{M}$. Since each monomial m_j contains some power of some x_{i_k}, $1 \leq k \leq r$, the coordinate subspace $W = \mathbf{V}(x_{i_1}, \ldots, x_{i_r})$ is contained in $\mathbf{V}(I)$. The dimension of W is $n - r = n - |J|$, and hence, by Definition 2, the dimension of $\mathbf{V}(I)$ is at least $n - |J|$.

If $\mathbf{V}(I)$ had dimension larger than $n - r$, then for some $s < r$ there would be a coordinate subspace $W' = V(x_{k_1}, \ldots, x_{k_s})$ contained in $\mathbf{V}(I)$. Each monomial m_j would vanish on W' and, in particular, it would vanish at the point $p \in W'$ whose k_i-th coordinate is 0 for $1 \leq i \leq s$ and whose other coordinates are 1. Hence, at least one of the x_{k_i} must divide m_j, and it would follow that $J' = \{k_1, \ldots, k_s\} \in \mathcal{M}$. Since $|J'| = s < r$, this would contradict the minimality of r. Thus, the dimension of $\mathbf{V}(I)$ must be as claimed. □

Let us check this on the second example given above. To match the notation of the proposition, we relabel the variables x, y, z as x_1, x_2, x_3, respectively. Then

$$I = \langle x_2^2 x_3^3, x_1^5 x_3^4, x_1^2 x_2 x_3^2 \rangle = \langle m_1, m_2, m_3 \rangle,$$

where

$$m_1 = x_2^2 x_3^3, \quad m_2 = x_1^5 x_3^4, \quad m_3 = x_1^2 x_2 x_3^2.$$

Using the notation of the discussion preceding Proposition 3,

$$M_1 = \{2, 3\}, \quad M_2 = \{1, 3\}, \quad M_3 = \{1, 2, 3\},$$

so that

$$\mathcal{M} = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3\}\}.$$

Then $\min(|J| : J \in \mathcal{M}) = 1$, which implies that

$$\dim \mathbf{V}(I) = 3 - \min_{J \in \mathcal{M}} |J| = 3 - 1 = 2.$$

Generalizing this example, note that if some variable, say x_i, appears in every monomial in a set of generators for a proper monomial ideal I, then it will be true that $\dim \mathbf{V}(I) = n - 1$ since $J = \{i\} \in \mathcal{M}$. For a converse, see Exercise 4.

It is also interesting to compare a monomial ideal I to its radical $\sqrt{I}$. In the exercises, you will show that $\sqrt{I}$ is a monomial ideal when I is. We also know from Chapter 4 that $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$ for any ideal I. It follows from Definition 2 that $\mathbf{V}(I)$ and $\mathbf{V}(\sqrt{I})$ have the same dimension (since we defined dimension in terms of the underlying variety). In Exercise 10 you will check that this is consistent with the formula given in Proposition 3.

EXERCISES FOR §1

1. For each of the following monomial ideals I, write $\mathbf{V}(I)$ as a union of coordinate subspaces.
   a. $I = \langle x^5, x^4yz, x^3z \rangle \subset k[x, y, z]$.
   b. $I = \langle wx^2y, xyz^3, wz^5 \rangle \subset k[w, x, y, z]$.
   c. $I = \langle x_1x_2, x_3 \cdots x_n \rangle \subset k[x_1, \ldots, x_n]$.
2. Find $\dim \mathbf{V}(I)$ for each of the following monomial ideals.
   a. $I = \langle xy, yz, xz \rangle \subset k[x, y, z]$.
   b. $I = \langle wx^2z, w^3y, wxyz, x^5z^6 \rangle \subset k[w, x, y, z]$.
   c. $I = \langle u^2vwyz, wx^3y^3, uxy^7z, y^3z, uwx^3y^3z^2 \rangle \subset k[u, v, w, x, y, z]$.
3. Show that $W \subset k^n$ is a coordinate subspace if and only if W can be spanned by a subset of the basis vectors $\{\mathbf{e}_i : 1 \leq i \leq n\}$, where $\mathbf{e}_i$ is the vector consisting of all zeros except for a 1 in the i-th place.
4. Suppose that $I \subset k[x_1, \ldots, x_n]$ is a monomial ideal such that $\dim \mathbf{V}(I) = n - 1$.
   a. Show that the monomials in any generating set for I have a nonconstant common factor.
   b. Write $\mathbf{V}(I) = V_1 \cup \cdots \cup V_p$, where V_i is a coordinate subspace and $V_i \not\subset V_j$ for $i \neq j$. Suppose, in addition, that exactly one of the V_i has dimension $n - 1$. What is the maximum that p (the number of components) can be? Give an example in which this maximum is achieved.
5. Let I be a monomial ideal in $k[x_1, \ldots, x_n]$ such that $\dim \mathbf{V}(I) = 0$.
   a. What is $\mathbf{V}(I)$ in this case?
   b. Show that $\dim \mathbf{V}(I) = 0$ if and only if for each $1 \leq i \leq n$, $x_i^{\ell_i} \in I$ for some $\ell_i \geq 1$. Hint: In Proposition 3, when will it be true that $\mathcal{M}$ contains only $J = \{1, \ldots, n\}$?

6. Let $\langle m_1, \ldots, m_r \rangle \subset k[x_1, \ldots, x_n]$ be a monomial ideal generated by $r \leq n$ monomials. Show that $\dim \mathbf{V}(m_1, \ldots, m_r) \geq n - r$.
7. Show that a coordinate subspace is an irreducible variety when the field k is infinite.
8. In this exercise, we will relate the decomposition of the variety of a monomial ideal I as a union of coordinate subspaces given in Proposition 1 with the decomposition of $\mathbf{V}(I)$ into irreducible components. We will assume that the field k is infinite.
   a. If $\mathbf{V}(I) = V_1 \cup \cdots \cup V_k$, where the V_j are coordinate subspaces such that $V_i \not\subset V_j$ if $i \neq j$, then show that this union is the minimal decomposition of $\mathbf{V}(I)$ into irreducible varieties given in Theorem 4 of Chapter 4, §6.
   b. Deduce that the V_i in part (a) are unique up to the order in which they are written.
9. Let $I = \langle m_i, \ldots, m_s \rangle$ be a monomial ideal in $k[x_1, \ldots, x_n]$. For each $1 \leq j \leq s$, let $M_j = \{k : x_k \text{ divides } m_j\}$ as in the text, and consider the monomial

$$m'_j = \prod_{k \in M_j} x_k.$$

   Note that m'_j contains exactly the same variables as m_j, but all to the first power.
   a. Show that $m'_j \in \sqrt{I}$ for each $1 \leq j \leq s$.
   b. Show that $\sqrt{I} = \langle m'_1, \ldots, m'_s \rangle$. Hint: Use Lemmas 2 and 3 of Chapter 2, §4.
10. Let I be a monomial ideal. Using Exercise 9, show the equality $\dim \mathbf{V}(I) = \dim \mathbf{V}(\sqrt{I})$ follows from the dimension formula given in Proposition 3.

§2 The Complement of a Monomial Ideal

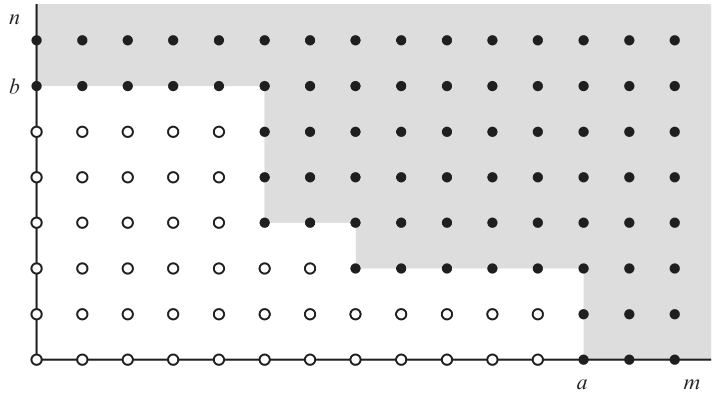
One of Hilbert's key insights in his famous paper *Über die Theorie der algebraischen Formen* [see HILBERT (1890)] was that the dimension of the variety associated to a monomial ideal could be characterized by the growth of the number of monomials not in the ideal as the total degree increases. We have alluded to this phenomenon in several places in Chapter 5 (notably in Exercise 12 of §3).

In this section, we will make a careful study of the monomials not contained in a monomial ideal $I \subset k[x_1, \ldots, x_n]$. Since there may be infinitely many such monomials, our goal will be to find a formula for the number of monomials $x^\alpha \notin I$ which have total degree less than some bound. The results proved here will play a crucial role in §3 when we define the dimension of an arbitrary variety.

Example 1. Consider a proper monomial ideal I in $k[x, y]$. Since I is proper (that is, $I \neq k[x, y]$), $\mathbf{V}(I)$ is either
a. The origin $\{(0, 0)\}$,
b. the x-axis,
c. the y-axis, or
d. the union of the x-axis and the y-axis.

In case (a), by Exercise 5 of §1, we must have $x^a \in I$ and $y^b \in I$ for some integers $a, b > 0$. Here, the number of monomials not in I will be finite, equal to some constant

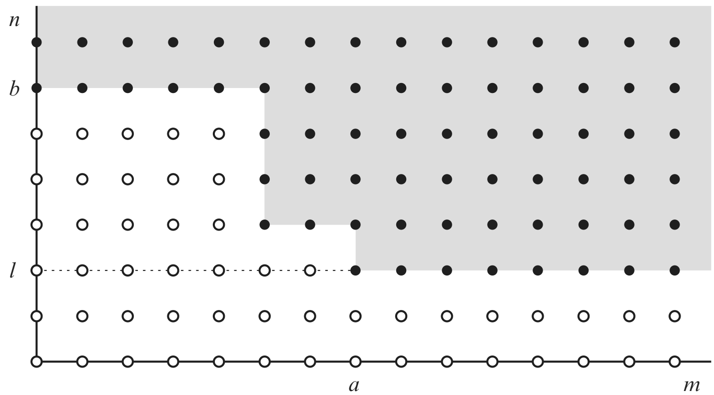
$C_0 \leq a \cdot b$. If we assume that a and b are as small as possible, we get the following picture when we look at exponents:



$(m,n) \longleftrightarrow x^m y^n$

The monomials in I are indicated by solid dots, while those not in I are open circles.

In case (b), since $\mathbf{V}(I)$ is the x-axis, no power x^k of x can belong to I. On the other hand, since the y-axis does not belong to $\mathbf{V}(I)$, we must have $y^b \in I$ for some minimal integer $b > 0$. The picture would be as follows:



$(m,n) \longleftrightarrow x^m y^n$

As the picture indicates, we let l denote the minimum exponent of y that occurs among *all* monomials in I. Note that $l \leq b$, and we also have $l > 0$ since no positive power of x lies in I. Then the monomials in the complement of I are precisely the monomials

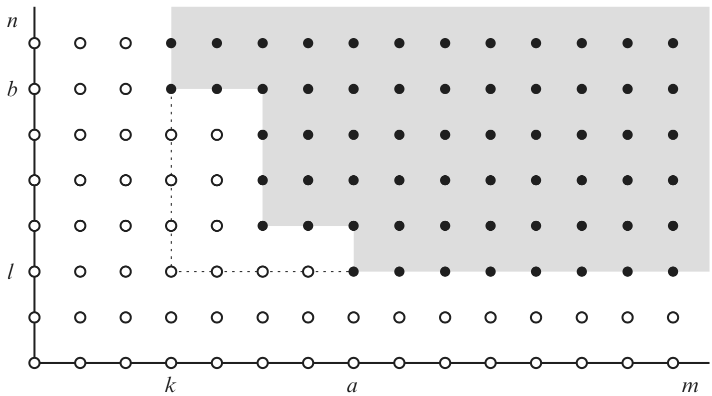
$$\{x^i y^j : i \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq l - 1\},$$

corresponding to the exponents on l copies of the horizontal axis in $\mathbb{Z}^2_{\geq 0}$, together with a finite number of other monomials. These additional monomials can be characterized as those monomials $m \notin I$ with the property that $x^r m \in I$ for some $r > 0$. In the above picture, they correspond to the open circles on or above the dotted line.

Thus, the monomials in the complement of I consist of l "lines" of monomials together with a finite set of monomials. This description allows us to "count" the number of monomials not in I. More precisely, in Exercise 1, you will show that if $s > l$, the l "lines" contain precisely $l(s+1) - (1+2+\cdots+l-1)$ monomials of total degree $\leq s$. In particular, if s is large enough (more precisely, we must have $s > a+b$, where a is indicated in the above picture), the number of monomials not in I of total degree $\leq s$ equals $ls + C_0$, where C_0 is some constant depending only on I.

In case (c), the situation is similar to (b), except that the "lines" of monomials are parallel to the vertical axis in the plane $\mathbb{Z}^2_{\geq 0}$ of exponents. In particular, we get a similar formula for the number of monomials not in I of total degree $\leq s$ once s is sufficiently large.

In case (d), let k be the minimum exponent of x that occurs among all monomials of I, and similarly let l be the minimum exponent of y. Note that k and l are positive since xy must divide every monomial in I. Then we have the following picture when we look at exponents:



$(m,n) \longleftrightarrow x^m y^n$

The monomials in the complement of I consist of the k "lines" of monomials

$$\{x^i y^j : 0 \leq i \leq k-1,\ j \in \mathbb{Z}_{\geq 0}\}$$

parallel to the vertical axis, the l "lines" of monomials

$$\{x^i y^j : i \in \mathbb{Z}_{\geq 0},\ 0 \leq j \leq l-1\}$$

parallel to the horizontal axis, together with a finite number of other monomials (indicated by open circles inside or on the boundary of the region indicated by the dotted lines).

Thus, the monomials not in I consist of $l + k$ "lines" of monomials together with a finite set of monomials. For s large enough (in fact, for $s > a + b$, where a and b are as in the above picture) the number of monomials not in I of total degree $\leq s$ will be $(l + k)s + C_0$, where C_0 is a constant. See Exercise 1 for the details of this claim.

The pattern that appears in Example 1, namely, that the monomials in the complement of a monomial ideal $I \subset k[x, y]$ consist of a number of infinite families parallel to the "coordinate subspaces" in $\mathbb{Z}^2_{\geq 0}$, together with a finite collection of monomials, generalizes to arbitrary monomial ideals. In §3, this will be the key to understanding how to define and compute the dimension of an arbitrary variety.

To discuss the general situation, we will introduce some new notation. For each monomial ideal I, we let

$$C(I) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : x^\alpha \notin I\}$$

be the set of exponents of monomials *not* in I. This will be our principal object of study. We also set

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), \\ e_2 &= (0, 1, \dots, 0), \\ &\vdots \\ e_n &= (0, 0, \dots, 1). \end{aligned}$$

Further, we define the *coordinate subspace of* $\mathbb{Z}^n_{\geq 0}$ determined by $e_{i_1}, \dots, e_{i_r}$, where $i_1 < \cdots < i_r$, to be the set

$$[e_{i_1}, \dots, e_{i_r}] = \{a_1 e_{i_1} + \cdots + a_r e_{i_r} : a_j \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq j \leq r\}.$$

We say that $[e_{i_1}, \dots, e_{i_r}]$ is an r-dimensional coordinate subspace. Finally, a subset of $\mathbb{Z}^n_{\geq 0}$ is a *translate* of a coordinate subspace $[e_{i_1}, \dots, e_{i_r}]$ if it is of the form

$$\alpha + [e_{i_1}, \dots, e_{i_r}] = \{\alpha + \beta : \beta \in [e_{i_1}, \dots, e_{i_r}]\},$$

where $\alpha = \sum_{i \notin \{i_1, \dots, i_r\}} a_i e_i$ for $a_i \geq 0$. This restriction on α means that we are translating by a vector perpendicular to $[e_{i_1}, \dots, e_{i_r}]$. For example, the set $\{(1, l) : l \in \mathbb{Z}_{\geq 0}\} = e_1 + [e_2]$ is a translate of the subspace $[e_2]$ in the plane $\mathbb{Z}^2_{\geq 0}$ of exponents.

With these definitions in hand, our discussion of monomial ideals in $k[x, y]$ from Example 1 can be summarized as follows.

a. If $\mathbf{V}(I)$ is the origin, then $C(I)$ consists of a finite number of points.
b. If $\mathbf{V}(I)$ is the x-axis, then $C(I)$ consists of a finite number of translates of $[e_1]$ and, possibly, a finite number of points not on these translates.
c. If $\mathbf{V}(I)$ is the y-axis, then $C(I)$ consists of a finite number of translates of $[e_2]$ and, possibly, a finite number of points not on these translates.
d. If $\mathbf{V}(I)$ is the union of the x-axis and the y-axis, then $C(I)$ consists of a finite number of translates of $[e_1]$, a finite number of translates of $[e_2]$, and, possibly, a finite number of points not on either set of translates.

In the exercises, you will carry out a similar analysis for monomial ideals in the polynomial ring in three variables.

Now let us turn to the general case. We first observe that there is a direct correspondence between the coordinate subspaces in $\mathbf{V}(I)$ and the coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$ contained in $C(I)$.

Proposition 2. *Let $I \subset k[x_1, \ldots, x_n]$ be a proper monomial ideal.*
(i) *The coordinate subspace $\mathbf{V}(x_i : i \notin \{i_1, \ldots, i_r\})$ is contained in $\mathbf{V}(I)$ if and only if $[e_{i_1}, \ldots, e_{i_r}] \subset C(I)$.*
(ii) *The dimension of $\mathbf{V}(I)$ is the dimension of the largest coordinate subspace in $C(I)$.*

Proof. (i)$\Rightarrow$: First note that $W = \mathbf{V}(x_i : i \notin \{i_1, \ldots, i_r\})$ contains the point p whose i_j-th coordinate is 1 for $1 \leq j \leq r$ and whose other coordinates are 0. For any $\alpha \in [e_{i_1}, \ldots, e_{i_r}]$, the monomial x^α can be written in the form $x^\alpha = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}}$. Then $x^\alpha = 1$ at p, so that $x^\alpha \notin I$ since $p \in W \subset \mathbf{V}(I)$ by hypothesis. This shows that $\alpha \in C(I)$.

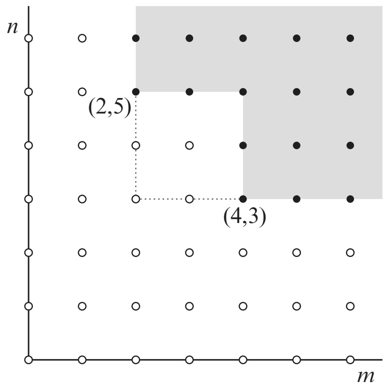
$\Leftarrow$: Suppose that $[e_{i_1}, \ldots e_{i_r}] \subset C(I)$. Then, since I is proper, every monomial in I contains at least one variable other than $x_{i_1}, \ldots, x_{i_r}$.This means that every monomial in I vanishes on any point $(a_1, \ldots, a_n) \in k^n$ for which $a_i = 0$ when $i \notin \{i_1, \ldots, i_r\}$. So every monomial in I vanishes on the coordinate subspace $\mathbf{V}(x_i : i \notin \{i_1, \ldots, i_r\})$, and, hence, the latter is contained in $\mathbf{V}(I)$.

(ii) Note that the coordinate subspace $\mathbf{V}(x_i : i \notin \{i_1, \ldots, i_r\})$ has dimension r. It follows from part (i) that the dimensions of the coordinate subspaces of k^n contained in $\mathbf{V}(I)$ and the coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$ contained in $C(I)$ are the same. By Definition 2 of §1, $\dim \mathbf{V}(I)$ is the maximum of the dimensions of the coordinate subspaces of k^n contained in $\mathbf{V}(I)$, so the statement follows. $\square$

We can now characterize the complement of a monomial ideal.

Theorem 3. *If $I \subset k[x_1, \ldots, x_n]$ is a proper monomial ideal, then the set $C(I) \subset \mathbb{Z}^n_{\geq 0}$ of exponents of monomials not lying in I can be written as a finite (but not necessarily disjoint) union of translates of coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$.*

Before proving the theorem, consider, for example, the ideal $I = \langle x^4y^3, x^2y^5 \rangle$.



$(m,n) \longleftrightarrow x^m y^n$

Here, it is easy to see that $C(I)$ is the finite union

$$C(I) = [e_1] \cup (e_2 + [e_1]) \cup (2e_2 + [e_1]) \cup [e_2] \cup (e_1 + [e_2])$$
$$\cup \{(3,4)\} \cup \{(3,3)\} \cup \{(2,4)\} \cup \{(2,3)\}.$$

We regard the last four sets in this union as being translates of the 0-dimensional coordinate subspace, which is the origin in $\mathbb{Z}^2_{\geq 0}$.

Proof of Theorem 3. If I is the zero ideal, the theorem is trivially true, so we can assume that $I \neq 0$. The proof is by induction on the number of variables n. If $n = 1$, then $I = \langle x^k \rangle$ for some integer $k > 0$. The only monomials not in I are $1, x, \ldots, x^{k-1}$, and hence $C(I) = \{0, 1, \ldots, k-1\} \subset \mathbb{Z}_{\geq 0}$. Thus, the complement consists of k points, all of which are translates of the origin.

So assume that the result holds for $n - 1$ variables and that we have a monomial ideal $I \subset k[x_1, \ldots, x_n]$. For each integer $j \geq 0$, let I_j be the ideal in $k[x_1, \ldots, x_{n-1}]$ generated by monomials m with the property that $m \cdot x_n^j \in I$. Then $C(I_j)$ consists of exponents $\alpha \in \mathbb{Z}^{n-1}_{\geq 0}$ such that $x^\alpha x_n^j \in I$. Geometrically, this says that $C(I_j) \subset \mathbb{Z}^{n-1}_{\geq 0}$ corresponds to the intersection of $C(I)$ and the hyperplane $(0, \ldots, 0, j) + [e_1, \ldots, e_{n-1}]$ in $\mathbb{Z}^n_{\geq 0}$.

Because I is an ideal, we have $I_j \subset I_{j'}$ when $j < j'$. By the ascending chain condition for ideals, there is an integer j_0 such that $I_j = I_{j_0}$ for all $j \geq j_0$. For any integer j, we let $C(I_j) \times \{j\}$ denote the set $\{(\alpha, j) \in \mathbb{Z}^n_{\geq 0} : \alpha \in C(I_j) \subset \mathbb{Z}^{n-1}_{\geq 0}\}$. Then we claim the monomials $C(I)$ not lying in I can be written as

$$C(I) = (C(I_{j_0}) \times \mathbb{Z}_{\geq 0}) \cup \bigcup_{j=0}^{j_0 - 1} (C(I_j) \times \{j\}). \tag{1}$$

To prove this claim, first note that $C(I_j) \times \{j\} \subset C(I)$ by the definition of $C(I_j)$. To show that $C(I_{j_0}) \times \mathbb{Z}_{\geq 0} \subset C(I)$, observe that $I_j = I_{j_0}$ when $j \geq j_0$, so that $C(I_{j_0}) \times \{j\} \subset C(I)$ for these j's. When $j < j_0$, we have $x^\alpha x_n^j \notin I$ whenever $x^\alpha x_n^{j_0} \notin I$ since I is an ideal, which shows that $C(I_{j_0}) \times \{j\} \subset C(I)$ for $j < j_0$. We conclude that $C(I)$ contains the right-hand side of (1).

To prove the opposite inclusion, take $\alpha = (\alpha_1, \ldots, \alpha_n) \in C(I)$. Then we have $\alpha \in C(I_{\alpha_n}) \times \{\alpha_n\}$ by definition. If $\alpha_n < j_0$, then α obviously lies in the right-hand side of (1). On the other hand, if $\alpha_n \geq j_0$, then $I_{\alpha_n} = I_{j_0}$ shows that $\alpha \in C(I_{j_0}) \times \mathbb{Z}_{\geq 0}$, and our claim is proved.

If we apply our inductive assumption, we can write $C(I_0), \ldots, C(I_{j_0})$ as finite unions of translates of coordinate subspaces of $\mathbb{Z}^{n-1}_{\geq 0}$. Substituting these finite unions into the right-hand side of (1), we immediately see that $C(I)$ is also a finite union of translates of coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$. □

Our next goal is to find a formula for the number of monomials of total degree $\leq s$ in the complement of a monomial ideal $I \subset k[x_1, \ldots, x_n]$. Here is one of the key facts we will need.

Lemma 4. *The number of monomials of total degree $\leq s$ in $k[x_1, \ldots, x_m]$ is the binomial coefficient $\binom{m+s}{s}$.*

Proof. See Exercise 11 of Chapter 5, §3. □

In what follows, we will refer to $|\alpha| = \alpha_1 + \cdots + \alpha_n$ as the *total degree* of $\alpha \in \mathbb{Z}^n_{\geq 0}$. This is also the total degree of the monomial x^α. Using this terminology, Lemma 4 easily implies that the number of points of total degree $\leq s$ in an m-dimensional coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ is $\binom{m+s}{s}$ (see Exercise 5). Observe that when m is fixed, the expression

$$\binom{m+s}{s} = \binom{m+s}{m} = \frac{1}{m!}(s+m)(s+m-1)\cdots(s+1)$$

is a polynomial of degree m in s. Note that the coefficient of s^m is $1/m!$.

What about the number of monomials of total degree $\leq s$ in a *translate* of an m-dimensional coordinate subspace in $\mathbb{Z}^n_{\geq 0}$? Consider, for instance, the translate $a_{m+1}e_{m+1} + \cdots + a_n e_n + [e_1, \ldots, e_m]$ of the coordinate subspace $[e_1, \ldots, e_m]$. Then, since $a_{m+1}, \ldots, a_n$ are fixed, the number of points in the translate with total degree $\leq s$ is just equal to the number of points in $[e_1, \ldots, e_m]$ of total degree $\leq s - (a_{m+1} + \cdots + a_n)$ provided, of course, that $s > a_{m+1} + \cdots + a_n$. More generally, we have the following.

Lemma 5. *Let $\alpha + [e_{i_1}, \ldots, e_{i_m}]$ be a translate of the coordinate subspace $[e_{i_1}, \ldots, e_{i_m}] \subset \mathbb{Z}^n_{\geq 0}$, where as usual $\alpha = \sum_{i \notin \{i_1, \ldots, i_m\}} a_i e_i$.*

(i) *The number of points in $\alpha + [e_{i_1}, \ldots, e_{i_m}]$ of total degree $\leq s$ is equal to*

$$\binom{m+s-|\alpha|}{s-|\alpha|},$$

provided that $s > |\alpha|$.

(ii) *For $s > |\alpha|$, this number of points is a polynomial function of s of degree m, and the coefficient of s^m is $1/m!$.*

Proof. (i) If $s > |\alpha|$, then each point β in $\alpha + [e_{i_1}, \ldots, e_{i_m}]$ of total degree $\leq s$ has the form $\beta = \alpha + \gamma$, where $\gamma \in [e_{i_1}, \ldots, e_{i_m}]$ and $|\gamma| \leq s - |\alpha|$. The formula given in (i) follows using Lemma 4 to count the number of possible γ.

(ii) See Exercise 6. □

We are now ready to prove a connection between the dimension of $\mathbf{V}(I)$ for a monomial ideal and the degree of the polynomial function which counts the number of points of total degree $\leq s$ in $C(I)$.

Theorem 6. *If $I \subset k[x_1, \ldots, x_n]$ is a monomial ideal with $\dim \mathbf{V}(I) = d$, then for all s sufficiently large, the number of monomials not in I of total degree $\leq s$ is a polynomial of degree d in s. Further, the coefficient of s^d in this polynomial is positive.*

Proof. We need to determine the number of points in $C(I)$ of total degree $\leq s$. By Theorem 3, we know that $C(I)$ can be written as a finite union

$$C(I) = T_1 \cup T_2 \cup \cdots \cup T_t,$$

where each T_i is a translate of a coordinate subspace in $\mathbb{Z}^n_{\geq 0}$. We can assume that $T_i \neq T_j$ for $i \neq j$.

The *dimension* of T_i is the dimension of the associated coordinate subspace. Since I is an ideal, it follows easily that a coordinate subspace $[e_{i_1}, \ldots, e_{i_r}]$ lies in $C(I)$ if and only if some translate does. By hypothesis, $\mathbf{V}(I)$ has dimension d, so that by Proposition 2, each T_i has dimension $\leq d$, with equality occurring for at least one T_i.

We will sketch the remaining steps in the proof, leaving the verification of several details to the reader as exercises. To count the number of points of total degree $\leq s$ in $C(I)$, we must be careful, since $C(I)$ is a union of coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$ that may not be disjoint [for instance, see part (d) of Example 1]. If we use the superscript s to denote the subset consisting of elements of total degree $\leq s$, then it follows that

$$C(I)^s = T_1^s \cup T_2^s \cup \cdots \cup T_t^s.$$

The number of elements in $C(I)^s$ will be denoted $|C(I)^s|$.

In Exercise 7, you will develop a general counting principle (called the Inclusion-Exclusion Principle) that allows us to count the elements in a finite union of finite sets. If the sets in the union have common elements, we cannot simply add to find the total number of elements because that would count some elements in the union more than once. The Inclusion-Exclusion Principle gives "correction terms" that eliminate this multiple counting. Those correction terms are the numbers of elements in double intersections, triple intersections, etc., of the sets in question.

If we apply the Inclusion-Exclusion Principle to the above union for $C(I)^s$, we easily obtain

$$(2) \qquad |C(I)^s| = \sum_i |T_i^s| - \sum_{i<j} |T_i^s \cap T_j^s| + \sum_{i<j<k} |T_i^s \cap T_j^s \cap T_k^s| - \cdots .$$

By Lemma 5, we know that for s sufficiently large, the number of points in T_i^s is a polynomial of degree $m_i = \dim(T_i) \leq d$ in s, and the coefficient of s^{m_i} is $1/m_i!$. From this it follows that $|C(I)^s|$ is a polynomial of degree at most d in s when s is sufficiently large.

We also see that the first sum in (2) is a polynomial of degree d in s when s is sufficiently large. The degree is exactly d because some of the T_i have dimension d and the coefficients of the leading terms are positive and hence can't cancel. If we can show that the remaining sums in (2) correspond to polynomials of smaller degree, it will follow that $|C(I)^s|$ is given by a polynomial of degree d in s. This will also show that the coefficient of s^d is positive.

You will prove in Exercise 8 that the intersection of two distinct translates of coordinate subspaces of dimensions m and r in $\mathbb{Z}^n_{\geq 0}$ is either empty or a translate of a coordinate subspace of dimension $< \max(m, r)$. Let us see how this applies to a nonzero term $|T_i^s \cap T_j^s|$ in the second sum of (2). Since $T_i \neq T_j$, Exercise 8 implies that $T = T_i \cap T_j$ is the translate of a coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ of dimension $< d$, so that by Lemma 5, the number of points in $T^s = T^s_j \cap T^s_j$ is a polynomial in s of degree $< d$. Adding these up for all $i < j$, we see that the second sum in (2) is a polynomial of degree $< d$ in s for s sufficiently large. The other sums in (2) are handled similarly, and it follows that $|C(I)^s|$ is a polynomial of the desired form when s is sufficiently large. $\square$

Let us see how this theorem works in the example $I = \langle x^4y^3, x^2y^5\rangle$ discussed following Theorem 3. Here, we have already seen that $C(I) = C_0 \cup C_1$, where

$$\begin{aligned} C_I &= [e_1] \cup (e_2 + [e_1]) \cup (2e_2 + [e_1]) \cup [e_2] \cup (e_1 + [e_2]), \\ C_0 &= \{(3,4),(3,3),(2,4),(2,3)\}. \end{aligned}$$

To count the number of points of total degree $\leq s$ in C_1, we count the number in each translate and subtract the number which are counted more than once. (In this case, there are no triple intersections to worry about. Do you see why?) The number of points of total degree $\leq s$ in $[e_2]$ is $\binom{1+s}{s} = \binom{1+s}{1} = s+1$ and the number in $e_1 + [e_2]$ is $\binom{1+s-1}{s-1} = s$. Similarly, the numbers in $[e_1], e_2 + [e_1]$, and $2e_2{+}[e_1]$ are $s+1, s$, and $s-1$, respectively. Of the possible intersections of pairs of these, only six are nonempty and each consists of a single point. You can check that $(1,2),(1,1),(1,0),(0,2),(0,1),(0,0)$ are the six points belonging to more than one translate. Thus, for large s, the number of points of total degree less $\leq s$ in C_1 is

$$|C_1^s| = (s+1) + s + (s+1) + s + (s-1) - 6 = 5s - 5.$$

Since there are four points in C_0, the number of points of total degree $\leq s$ in $C(I)$ is

$$|C_1^s| + |C_0^s| = (5s-5) + 4 = 5s - 1,$$

provided that s is sufficiently large. (In Exercise 9 you will show that in this case, s is "sufficiently large" as soon as $s \geq 7$.)

Theorem 6 shows that the dimension of the affine variety defined by a monomial ideal is equal to the degree of the polynomial in s which counts the number of points in $C(I)$ of total degree $\leq s$ for s large. This gives a purely algebraic definition of dimension. In §3, we will extend these ideas to general ideals.

The polynomials that occur in Theorem 6 have the property that they take integer values when the variable s is a sufficiently large integer. For later purposes, it will be useful to characterize this class of polynomials. The first thing to note is that polynomials with this property need *not* have integer coefficients. For example, the polynomial $\frac{1}{2}s(s-1)$ takes integer values whenever s is an integer, but does not have integer coefficients. The reason is that either s or $s-1$ must be even, hence, divisible by 2. Similarly, the polynomial $\frac{1}{3\cdot 2}s(s-1)(s-2)$ takes integer values for any integer s: no matter what s is, one of the three consecutive integers $s-2, s-1, s$ must be divisible by 3 and at least one of them divisible by 2. It is easy to generalize this argument and show that

$$\begin{aligned}\binom{s}{d} &= \frac{s(s-1)\cdots(s-(d-1))}{d!}\\ &= \frac{1}{d\cdot(d-1)\cdots 2\cdot 1}s(s-1)\cdots(s-(d-1))\end{aligned}$$

takes integer values for any integer s (see Exercise 10). Further, in Exercises 11 and 12, you will show that any polynomial of degree d which takes integer values for sufficiently large integers s can be written uniquely as an integer linear combination of the polynomials

$$\binom{s}{0} = 1, \binom{s}{1} = s, \binom{s}{2} = \frac{s(s-1)}{2}, \quad \ldots,\\ \binom{s}{d} = \frac{s(s-1)\cdots(s-(d-1))}{d!}.$$

Using this fact, we obtain the following sharpening of Theorem 6.

Proposition 7. *If $I \subset k[x_1, \ldots, x_n]$ is a monomial ideal with $\dim \mathbf{V}(I) = d$, then for all s sufficiently large, the number of points in $C(I)$ of total degree $\leq s$ is a polynomial of degree d in s which can be written in the form*

$$\sum_{i=0}^{d} a_i \binom{s}{d-i},$$

where $a_i \in \mathbb{Z}$ for $0 \leq i \leq d$ and $a_0 > 0$.

In the final part of this section, we will study the projective variety associated with a monomial ideal. This makes sense because every monomial ideal is homogeneous (see Exercise 13). Thus, a monomial ideal $I \subset k[x_1, \ldots, x_n]$ determines a projective variety $\mathbf{V}_p(I) \subset \mathbb{P}^{n-1}(k)$, where we use the subscript p to remind us that we are in projective space. In Exercise 14, you will show that $\mathbf{V}_p(I)$ is a finite union of projective linear subspaces which have dimension one less than the *dimension* of their affine counterparts. As in the affine case, we define the *dimension* of a finite union of projective linear subspaces to be the maximum of the dimensions of the subspaces. Then Theorem 6 shows that the dimension of the projective variety $\mathbf{V}_p(I)$ of a monomial ideal I is one less than the degree of the polynomial in s counting the number of monomials not in I of total degree $\leq s$.

In this case it turns out to be more convenient to consider the polynomial in s counting the number of monomials whose total degree is *equal* to s. The reason resides in the following proposition.

Proposition 8. *Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal and let $\mathbf{V}_p(I)$ be the projective variety in $\mathbb{P}^{n-1}(k)$ defined by I. If $\dim \mathbf{V}_p(I) = d-1$, then for all s sufficiently large, the number of monomials not in I of total degree s is given by a polynomial of* *the form*

$$\sum_{i=0}^{d-1} b_i \binom{s}{d-1-i}$$

of degree $d-1$ *in* s*, where* $b_i \subset \mathbb{Z}$ *for* $0 \le i \le d-1$ *and* $b_0 > 0$.

Proof. As an affine variety, $\mathbf{V}(I) \subset k^n$ has dimension d, so that by Theorem 6, the number of monomials not in I of total degree $\le s$ is a polynomial $p(s)$ of degree d for s sufficiently large. We also know that the coefficient of s^d is positive. It follows that the number of monomials of total degree *equal* to s is given by

$$p(s) - p(s-1)$$

for s large enough. In Exercise 15, you will show that this polynomial has degree $d-1$ and that the coefficient of s^{d-1} is positive. Since it also takes integer values when s is a sufficiently large integer, it follows from the remarks preceding Proposition 7 that $p(s) - p(s-1)$ has the desired form. □

In particular, this proposition says that for the projective variety defined by a monomial ideal, the dimension and the degree of the polynomial in the statement are equal. In §3, we will extend these results to the case of arbitrary homogeneous ideals $I \subset k[x_1, \ldots, x_n]$.

EXERCISES FOR §2

1. In this exercise, we will verify some of the claims made in Example 1. Remember that $I \subset k[x, y]$ is a proper monomial ideal.
   a. In case (b) of Example 1, show that if $s > l$, then the l "lines" of monomials contain $l(s+1) - (1 + 2 + \cdots + l - 1)$ monomials of total degree $\le s$.
   b. In case (b), conclude that the number of monomials not in I of total degree $\le s$ is given by $ls + C_0$ for s sufficiently large. Explain how to compute C_0 and show that $s > a + b$ guarantees that s is sufficiently large. Illustrate your answer with a picture that shows what can go wrong if s is too small.
   c. In case (d) of Example 1, show that the constant C_0 in the polynomial function giving the number of points in $C(I)$ of total degree $\le s$ is equal to the finite number of monomials not contained in the "lines" of monomials, minus $l \cdot k$ for the monomials belonging to both families of lines, minus $1 + 2 + \cdots + (l-1)$, minus $1 + \cdots + (k-1)$.
2. Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal. Suppose that in $\mathbb{Z}^n_{\ge 0}$, the translate $\alpha + [e_{i_1}, \ldots, e_{i_r}]$ is contained in $C(I)$. If $\alpha = \sum_{i \notin \{i_1, \ldots, i_r\}} a_i e_i$, show that $C(I)$ contains all translates $\beta + [e_{i_1}, \ldots, e_{i_r}]$ for all β of the form $\beta = \sum_{i \notin \{i_1, \ldots, i_r\}} b_i e_i$, where $0 \le b_i \le a_i$ for all i. In particular, $[e_{i_1}, \ldots, e_{i_r}] \subset C(I)$. Hint: I is an ideal.
3. In this exercise, you will find monomial ideals $I \subset k[x, y, z]$ with a given $C(I) \subset \mathbb{Z}^3_{\ge 0}$.
   a. Suppose that $C(I)$ consists of one translate of $[e_1, e_2]$ and two translates of $[e_2, e_3]$. Use Exercise 2 to show that $C(I) = [e_1, e_2] \cup [e_2, e_3] \cup (e_1 + [e_2, e_3])$.
   b. Find a monomial ideal I so that $C(I)$ is as described in part a. Hint: Study all monomials of small degree to see whether or not they lie in I.

c. Suppose now that $C(I)$ consists of one translate of $[e_1, e_2]$, two translates of $[e_2, e_3]$, and one additional translate (not contained in the others) of the line $[e_2]$. Use Exercise 2 to give a precise description of $C(I)$.
d. Find a monomial ideal I so that $C(I)$ is as in part (c).

4. Let I be a monomial ideal in $k[x, y, z]$. In this exercise, we will study $C(I) \subset \mathbb{Z}^3_{\geq 0}$.
a. Show that $\mathbf{V}(I)$ must be one of the following possibilities: the origin; one, two, or three coordinate lines; one, two, or three coordinate planes; or the union of a coordinate plane and a perpendicular coordinate axis.
b. Show that if $\mathbf{V}(I)$ contains only the origin, then $C(I)$ has a finite number of points.
c. Show that if $\mathbf{V}(I)$ is a union of one, two, or three coordinate lines, then $C(I)$ consists of a finite number of translates of $[e_1]$, $[e_2]$, and/or $[e_3]$, together with a finite number of points not on these translates.
d. Show that if $\mathbf{V}(I)$ is a union of one, two or three coordinate planes, then $C(I)$ consists of a finite number of translates of $[e_1, e_2]$, $[e_1, e_3]$, and/or $[e_2, e_3]$ plus, possibly, a finite number of translates of $[e_1]$, $[e_2]$, and/or $[e_3]$ (where a translate of $[e_i]$ cannot occur unless $[e_i, e_j] \subset C(I)$ for some $j \neq i$) plus, possibly, a finite number of points not on these translates.
e. Finally, show that if $\mathbf{V}(I)$ is the union of a coordinate plane and the perpendicular coordinate axis, then $C(I)$ consists of a finite nonzero number of translates of a single coordinate plane $[e_i, e_j]$, plus a finite nonzero number of translates of $[e_k]$, $k \neq i, j$, plus, possibly, a finite number of translates of $[e_i]$ and/or $[e_j]$, plus a finite number of points not on any of these translates.

5. Show that the number of points in any m-dimensional coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ of total degree $\leq s$ is given by $\binom{m+s}{s}$.

6. Prove part (ii) of Lemma 5.

7. In this exercise, you will develop a counting principle, called the Inclusion-Exclusion Principle. The idea is to give a general method for counting the number of elements in a union of finite sets. We will use the notation $|A|$ for the number of elements in the finite set A.
a. Show that for any two finite sets A and B.
$$|A \cup B| = |A| + |B| - |A \cap B|.$$
b. Show that for any three finite sets A, B, C,
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$
c. Using induction on the number of sets, show that the number of elements in a union of n finite sets $A_1 \cup \cdots \cup A_n$ is equal to the sum of the $|A_i|$, minus the sum of all double intersections $|A_i \cap A_j|$, $i < j$, plus the sum of all the threefold intersections $|A_i \cap A_j \cap A_k|$, $i < j < k$, minus the sum of the fourfold intersections, etc. This can be written as the following formula:
$$|A_1 \cup \cdots \cup A_n| = \sum_{r=1}^{n} (-1)^{r-1} \left(\sum_{1 \leq i_1 < \cdots < i_r \leq n} |A_{i_1} \cap \cdots \cap A_{i_r}| \right).$$

8. In this exercise, you will show that the intersection of two translates of different coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$ is a translate of a lower dimensional coordinate subspace.
a. Let $A = \alpha + [e_{i_1} \ldots, e_{i_m}]$, where $\alpha = \sum_{i \notin \{i_1,\ldots,i_m\}} a_i e_i$, and let $B = \beta + [e_{j_1}, \ldots, e_{J_r}]$, where $\beta = \sum_{i \notin \{j_i,\ldots,j_r\}} b_i e_i$ If $A \neq B$ and $A \cap B \neq \emptyset$, then show that
$$[e_{i_1}, \ldots, e_{i_m}] \neq [e_{j_1}, \ldots, e_{j_r}]$$

and that $A \cap B$ is a translate of

$$[e_{i_1}, \ldots, e_{i_m}] \cap [e_{j_1}, \ldots, e_{j_r}].$$

b. Deduce that $\dim A \cap B < \max(m, r)$.

9. Show that if $s \geq 7$, then the number of elements in $C(I)$ of total degree $\leq s$ for the monomial ideal I in the example following Theorem 6 is given by the polynomial $5s - 1$.
10. Show that the polynomial

$$p(s) = \binom{s}{d} = \frac{s(s-1)\cdots(s-(d-1))}{d!}$$

takes integer values for all integers s. Note that p is a polynomial of degree d in s.

11. In this exercise, we will show that every polynomial $p(s)$ of degree $\leq d$ which takes integer values for every $s \in \mathbb{Z}_{\geq 0}$ can be written as a unique linear combination with integer coefficients of the polynomials $\binom{s}{0}, \binom{s}{1}, \binom{s}{2}, \ldots, \binom{s}{d}$.

a. Show that the polynomials

$$\binom{s}{0}, \binom{s}{1}, \binom{s}{2}, \ldots, \binom{s}{d}$$

are linearly independent in the sense that

$$a_0\binom{s}{0} + a_1\binom{s}{1} + \cdots + a_d\binom{s}{d} = 0$$

for all s implies that $a_0 = a_1 = \cdots = a_d = 0$.

b. Show that any two polynomials $p(s)$ and $q(s)$ of degree $\leq d$ which take the same values at the $d + 1$ points $s = 0, 1, \ldots, d$ must be identical. Hint: How many roots does the polynomial $p(s) - q(s)$ have?

c. Suppose we want to construct a polynomial $p(s)$ that satisfies

$$\begin{aligned} p(0) &= c_0, \\ p(1) &= c_1, \\ &\ \vdots \\ p(d) &= c_d, \end{aligned} \tag{3}$$

where the c_i are given values in $\mathbb{Z}$. Show that if we set

$$\begin{aligned} \Delta_0 &= c_0, \\ \Delta_1 &= c_1 - c_0, \\ \Delta_2 &= c_2 - 2c_1 + c_0, \\ &\ \vdots \\ \Delta_d &= \sum_{n=0}^{d} (-1)^n \binom{d}{n} c_{d-n}, \end{aligned}$$

then the polynomial

$$p(s) = \Delta_0\binom{s}{0} + \Delta_1\binom{s}{1} + \cdots + \Delta_d\binom{s}{d} \tag{4}$$

satisfies the equations in (3). Hint: Argue by induction on d. [The polynomial in (4) is called a Newton–Gregory interpolating polynomial.]

d. Explain why the polynomial in (4) takes integer values for all integer s. Hint: Recall that the c_i in (3) are integers. See also Exercise 10.

e. Deduce from parts (a)–(d) that every polynomial of degree d which takes integer values for all integer $s \geq 0$ can be written as a unique integer linear combination of $\binom{s}{0}, \ldots, \binom{s}{d}$.

12. Suppose that $p(s)$ is a polynomial of degree d which takes integer values when s is a sufficiently large integer, say $s \geq a$. We want to prove that $p(s)$ is an integer linear combination of the polynomials $\binom{s}{0}, \ldots, \binom{s}{d}$ studied in Exercises 10 and 11. We can assume that a is a positive integer.

a. Show that the polynomial $p(s+a)$ can be expressed in terms of $\binom{s}{0}, \ldots, \binom{s}{d}$ and conclude that $p(s)$ is an integer linear combination of $\binom{s-a}{0}, \ldots, \binom{s-a}{d}$.

b. Use Exercise 10 to show that $p(s)$ takes integer values for all $s \in \mathbb{Z}$ and conclude that $p(s)$ is an integer linear combination of $\binom{s}{0}, \ldots, \binom{s}{d}$.

13. Show that every monomial ideal is a homogeneous ideal.

14. Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal.

a. In k^n, let $\mathbf{V}(x_{i_1}, \ldots, x_{i_r})$ be a coordinate subspace of dimension $n-r$ contained in $\mathbf{V}(I)$. Prove that $\mathbf{V}_p(x_{i_1}, \ldots, x_{i_r}) \subset \mathbf{V}_p(I)$ in $\mathbb{P}^{n-1}(k)$. Also show that $\mathbf{V}_p(x_{i_1}, \ldots, x_{i_r})$ looks like a copy of $\mathbb{P}^{n-r-1}$ sitting inside $\mathbb{P}^{n-1}$. Thus, we say that $\mathbf{V}_p(x_{i_1}, \ldots, x_{i_r})$ is a projective linear subspace of dimension $n-r-1$.

b. Prove the claim made in the text that $\mathbf{V}_p(I)$ is a finite union of projective linear subspaces of dimension one less than their affine counterparts.

15. Verify the statement in the proof of Proposition 8 that if $p(s)$ is a polynomial of degree d in s with a positive coefficient of s^d, then $p(s) - p(s-1)$ is a polynomial of degree $d-1$ with a positive coefficient of s^{d-1}.

§3 The Hilbert Function and the Dimension of a Variety

In this section, we will define the Hilbert function of an ideal I and use it to define the dimension of a variety V. We will give the basic definitions in both the affine and projective cases. The basic idea will be to use the experience gained in the last section and define dimension in terms of the number of monomials not contained in the ideal I. In the affine case, we will use the number of monomials not in I of total degree $\leq s$, whereas in the projective case, we consider those of total degree *equal* to s.

However, we need to note from the outset that the results from §2 do not apply directly because when I is not a monomial ideal, different monomials not in I can be dependent on one another. For instance, if $I = \langle x^2 - y^2 \rangle$, neither the monomial x^2 nor y^2 belongs to I, but their difference does. So we should not regard x^2 and y^2 as two monomials not in I. Rather, to generalize §2, we will need to consider the number of monomials of total degree $\leq s$ which are "linearly independent modulo" I.

In Chapter 5, we defined the quotient of a ring modulo an ideal. There is an analogous operation on vector spaces which we will use to make the above ideas precise. Given a vector space V and a subspace $W \subset V$, it is not difficult to show that the relation on V defined by $v \sim v'$ if $v - v' \in W$ is an equivalence relation (see Exercise 1). The set of equivalence classes of $\sim$ is denoted V/W, so that

$$V/W = \{[v] : v \in V\}.$$

In the exercises, you will check that the operations $[v] + [v'] = [v + v']$ and $a[v] = [av]$, where $a \in k$ and $v, v' \in V$ are well-defined and make V/W into a k-vector space, called the *quotient space* of V modulo W.

When V is finite-dimensional, we can compute the dimension of V/W as follows.

Proposition 1. *Let W be a subspace of a finite-dimensional vector space V. Then W and V/W are also finite-dimensional vector spaces, and*

$$\dim V = \dim W + \dim V/W.$$

Proof. If V is finite-dimensional, it is a standard fact from linear algebra that W is also finite-dimensional. Let $v_1, \ldots, v_m$ be a basis of W, so that $\dim W = m$. In V, the vectors $v_1, \ldots, v_m$ are linearly independent and, hence, can be extended to a basis $v_1, \ldots, v_m, v_{m+1}, \ldots v_{m+n}$ of V. Thus, $\dim V = m + n$. We claim that $[v_{m+1}], \ldots, [v_{m+n}]$ form a basis of V/W.

To see that they span, take $[v] \in V/W$. If we write $v = \sum_{i=1}^{m+n} a_i v_i$, then $v \sim a_{m+1}v_{m+1} + \cdots + a_{m+n}v_{m+n}$ since their difference is $a_1v_1 + \cdots + a_m v_m \in W$. It follows that in V/W, we have

$$[v] = [a_{m+1}v_{m+1} + \cdots + a_{m+n}v_{m+n}] = a_{m+1}[v_{m+1}] + \cdots + a_{m+n}[v_{m+n}].$$

The proof that $[v_{m+1}], \ldots, [v_{m+n}]$ are linearly independent is left to the reader (see Exercise 2). This proves the claim, and the proposition follows immediately. □

The Dimension of an Affine Variety

Considered as a vector space over k, the polynomial ring $k[x_1, \ldots, x_n]$ has infinite dimension, and the same is true for any nonzero ideal (see Exercise 3). To get something finite-dimensional, we will restrict ourselves to polynomials of total degree $\leq s$. Hence, we let

$$k[x_1, \ldots, x_n]_{\leq s}$$

denote the set of polynomials of total degree $\leq s$ in $k[x_1, \ldots, x_n]$. By Lemma 4 of §2, it follows that $k[x_1, \ldots, x_n]_{\leq s}$ is a vector space of dimension $\binom{n+s}{s}$. Then, given an ideal $I \subset k[x_1, \ldots, x_n]$, we let

$$I_{\leq s} = I \cap k[x_1, \ldots, x_n]_{\leq s}$$

denote the set of polynomials in I of total degree $\leq s$. Note that $I_{\leq s}$ is a vector subpace of $k[x_1, \ldots, x_n]_{\leq s}$. We are now ready to define the affine Hilbert function of I.

Definition 2. *Let I be an ideal in $k[x_1, \ldots, x_n]$. The* **affine Hilbert function** *of I is the function on the nonnegative integers s defined by*

$$\begin{aligned} {}^aHF_I(s) &= \dim k[x_1, \ldots, x_n]_{\leq s}/I_{\leq s} \\ &= \dim k[x_1, \ldots, x_n]_{\leq s} - \dim I_{\leq s} \end{aligned}$$

(where the second equality is by Proposition 1).

With this terminology, the results of §2 for monomial ideals can be restated as follows.

Proposition 3. *Let I be a proper monomial ideal in $k[x_1, \ldots, x_n]$.*

(i) *For all $s \geq 0$, ${}^aHF_I(s)$ is the number of monomials not in I of total degree $\leq s$.*

(ii) *For all s sufficiently large, the affine Hilbert function of I is given by a polynomial function*

$$ {}^aHF_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i}, $$

where $b_i \in \mathbb{Z}$ and b_0 is positive.

(iii) *The degree of the polynomial in part* (ii) *is the maximum of the dimensions of the coordinate subspaces contained in* $\mathbf{V}(I)$.

Proof. To prove (i), first note that $\{x^\alpha : |\alpha| \leq s\}$ is a basis of $k[x_1, \ldots, x_n]_{\leq s}$ as a vector space over k. Further, Lemma 3 of Chapter 2, §4 shows that $\{x^\alpha : |\alpha| \leq s, x^\alpha \in I\}$ is a basis of $I_{\leq s}$. Consequently, the monomials in $\{x^\alpha : |\alpha| \leq s, x^\alpha \notin I\}$ are exactly what we add to a basis of $I_{\leq s}$ to get a basis of $k[x_1, \ldots, x_n]_{\leq s}$. It follows from the proof of Proposition 1 that $\{[x^\alpha] : |\alpha| \leq s, x^\alpha \notin I\}$ is a basis of the quotient space $k[x_1, \ldots, x_n]_{\leq s}/I_{\leq s}$, which completes the proof of (i).

Parts (ii) and (iii) follow easily from (i) and Proposition 7 of §2. □

We are now ready to link the ideals of §2 to arbitrary ideals in $k[x_1, \ldots, x_n]$. The key ingredient is the following observation due to Macaulay. As in Chapter 8, §4, we say that a monomial order $>$ on $k[x_1, \ldots, x_n]$ is a *graded order* if $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$.

Proposition 4. *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let $>$ be a graded order on $k[x_1, \ldots, x_n]$. Then the monomial ideal $\langle \mathrm{LT}(I) \rangle$ has the same affine Hilbert function as I.*

Proof. Fix s and consider the leading monomials $\mathrm{LM}(f)$ of all elements $f \in I_{\leq s}$. There are only finitely many such monomials, so that

$$ \{\mathrm{LM}(f) : f \in I_{\leq s}\} = \{\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_m)\} \tag{1} $$

for some polynomials $f_1, \ldots, f_m \in I_{\leq s}$. By rearranging and deleting duplicates, we can assume that $\mathrm{LM}(f_1) > \mathrm{LM}(f_2) > \cdots > \mathrm{LM}(f_m)$. We claim that $f_1, \ldots, f_m$ are a basis of $I_{\leq s}$ as a vector space over k.

To prove this, consider a nontrivial linear combination $a_1 f_1 + \cdots + a_m f_m$ and choose the smallest i such that $a_i \neq 0$. Given how we ordered the leading monomials, there is nothing to cancel $a_i \mathrm{LT}(f_i)$, so the linear combination is nonzero. Hence, $f_1, \ldots, f_m$ are linearly independent. Next, let $W = [f_1, \ldots, f_m] \subset I_{\leq s}$ be the subspace spanned by $f_1, \ldots, f_m$. If $W \neq I_{\leq s}$, pick $f \in I_{\leq s} - W$ with $\mathrm{LM}(f)$ minimal. By (1), $\mathrm{LM}(f) = \mathrm{LM}(f_i)$ for some i, and hence, $\mathrm{LT}(f) = \lambda \mathrm{LT}(f_i)$ for some $\lambda \in k$. Then $f - \lambda f_i \in I_{\leq s}$ has a smaller leading monomial, so that $f - \lambda f_i \in W$ by the minimality of $\mathrm{LM}(f)$. This implies $f \in W$, which is a contradiction. It follows that $W = [f_1, \ldots, f_m] = I_{\leq s}$, and we conclude that $f_1, \ldots, f_m$ are a basis.

The monomial ideal $\langle \mathrm{LT}(I) \rangle$ is generated by the leading terms (or leading monomials) of elements of I. Thus, $\mathrm{LM}(f_i) \in \langle \mathrm{LT}(I) \rangle_{\leq s}$ since $f_i \in I_{\leq s}$. We claim that $\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_m)$ are a vector space basis of $\langle \mathrm{LT}(I) \rangle_{\leq s}$. Arguing as above, it is easy to see that they are linearly independent. It remains to show that they span, i.e., that $[\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_m)] = \langle \mathrm{LT}(I) \rangle_{\leq s}$. By Lemma 3 of Chapter 2, §4, it suffices to show that

$$(2) \qquad \{\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_m)\} = \{\mathrm{LM}(f) : f \in I, \mathrm{LM}(f) \text{ has total degree} \leq s\}.$$

To relate this to (1), note that $>$ is a graded order, which implies that for *any* nonzero polynomial $f \in k[x_1, \ldots, x_n]$, $\mathrm{LM}(f)$ has the same total degree as f. In particular, if $\mathrm{LM}(f)$ has total degree $\leq s$, then so does f, which means that (2) follows immediately from (1).

Thus, $I_{\leq s}$ and $\langle \mathrm{LT}(I) \rangle_{\leq s}$ have the same dimension (since they both have bases consisting of m elements), and then the dimension formula of Proposition 1 implies that

$$\begin{aligned} {}^aHF_I(s) &= \dim k[x_1, \ldots, x_n]_{\leq s}/I_{\leq s} \\ &= \dim k[x_1, \ldots, x_n]_{\leq s}/\langle \mathrm{LT}(I) \rangle_{\leq s} = {}^aHF_{\langle \mathrm{LT}(I) \rangle}(s). \end{aligned}$$

This proves the proposition. □

If we combine Propositions 3 and 4, it follows immediately that if I is *any* ideal in $k[x_1, \ldots, x_n]$ and s is sufficiently large, the affine Hilbert function of I can be written

$${}^aHF_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i},$$

where the b_i are integers and b_0 is positive. This leads to the following definition.

Definition 5. *The polynomial which equals ${}^aHF_I(s)$ for sufficiently large s is called the* **affine Hilbert polynomial** *of I and is denoted ${}^aHP_I(s)$.*

As an example, consider the ideal $I = \langle x^3y^2 + 3x^2y^2 + y^3 + 1 \rangle \subset k[x, y]$. If we use grlex order, then $\langle \mathrm{LT}(I) \rangle = \langle x^3y^2 \rangle$, and using the methods of §2, one can show that the number of monomials not in $\langle \mathrm{LT}(I) \rangle$ of total degree $\leq s$ equals $5s - 5$ when $s \geq 3$. From Propositions 3 and 4, we obtain

$${}^aHF_I(s) = {}^aHF_{\langle \mathrm{LT}(I) \rangle}(s) = 5s - 5$$

when $s \geq 3$. It follows that the affine Hilbert polynomial of I is

$${}^aHP_I(s) = 5s - 5.$$

By definition, the affine Hilbert function of an ideal I coincides with the affine Hilbert polynomial of I when s is sufficiently large. The smallest integer s_0 such that ${}^aHP_I(s) = {}^aHF_I(s)$ for all $s \geq s_0$ is called the *index of regularity* of I. Determining the index of regularity is of considerable interest and importance in many computations with ideals, but we will not pursue this topic in detail here.

We next compare the degrees of the affine Hilbert polynomials for I and $\sqrt{I}$.

Proposition 6. *If $I \subset k[x_1, \ldots, x_n]$ is an ideal, then the affine Hilbert polynomials of I and $\sqrt{I}$ have the same degree.*

Proof. For a monomial ideal I, we know that the degree of the affine Hilbert polynomial is the dimension of the largest coordinate subspace of k^n contained in $\mathbf{V}(I)$. Since $\sqrt{I}$ is monomial by Exercise 9 of §1 and $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$, it follows immediately that ${}^aHP_I$ and ${}^aHP_{\sqrt{I}}$ have the same degree.

Now let I be an arbitrary ideal in $k[x_1, \ldots, x_n]$ and pick any graded order $>$ in $k[x_1, \ldots, x_n]$. We claim that

$$\langle \text{LT}(I) \rangle \subset \langle \text{LT}(\sqrt{I}) \rangle \subset \sqrt{\langle \text{LT}(I) \rangle}. \tag{3}$$

The first containment is immediate from $I \subset \sqrt{I}$. To establish the second, let x^α be a monomial in $\text{LT}(\sqrt{I})$. This means that there is a polynomial $f \in \sqrt{I}$ such that $\text{LT}(f) = x^\alpha$. We know $f^r \in I$ for some $r \geq 0$, and it follows that $x^{r\alpha} = \text{LT}(f^r) \in \langle \text{LT}(I) \rangle$. Thus, $x^\alpha \in \sqrt{\langle \text{LT}(I) \rangle}$.

In Exercise 8, we will prove that if $I_1 \subset I_2$ are any ideals of $k[x_1, \ldots, x_n]$, then $\deg {}^aHP_{I_2} \leq \deg {}^aHP_{I_1}$. If we apply this fact to (3), we obtain the inequalities

$$\deg {}^aHP_{\sqrt{\langle \text{LT}(I) \rangle}} \leq \deg {}^aHP_{\langle \text{LT}(\sqrt{I}) \rangle} \leq \deg {}^aHP_{\langle \text{LT}(I) \rangle}.$$

By the result for monomial ideals, the two outer terms here are equal and we conclude that ${}^aHP_{\langle \text{LT}(I) \rangle}$ and ${}^aHP_{\langle \text{LT}(\sqrt{I}) \rangle}$ have the same degree. By Proposition 4, the same is true for ${}^aHP_I$ and ${}^aHP_{\sqrt{I}}$, and the proposition is proved. □

This proposition is evidence of something that is not at all obvious, namely, that the degree of the affine Hilbert polynomial has *geometric* meaning in addition to its *algebraic* significance in indicating how far $I_{\leq s}$ is from being all of $k[x_1, \ldots, x_n]_{\leq s}$. Recall that $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$ for all ideals. Thus, the degree of the affine Hilbert polynomial is the same for a large collection of ideals defining the same variety. Moreover, we know from §2 that the degree of the affine Hilbert polynomial is the same as our intuitive notion of the dimension of the variety of a monomial ideal. So it should be no surprise that in the general case, we define dimension in terms of the degree of the affine Hilbert function. We will always assume that the field k is infinite.

Definition 7. *The* **dimension** *of an affine variety $V \subset k^n$, denoted* dim V*, is the degree of the affine Hilbert polynomial of the corresponding ideal $I = \mathbf{I}(V) \subset k[x_1, \ldots, x_n]$.*

As an example, consider the twisted cubic $V = \mathbf{V}(y - x^2, z - x^3) \subset \mathbb{R}^3$. In Chapter 1, we showed that $I = \mathbf{I}(V) = \langle y - x^2, z - x^3 \rangle \subset \mathbb{R}[x, y, z]$. Using grlex order, a Groebner basis for I is $\{y^3 - z^2, x^2 - y, xy - z, xz - y^2\}$, so that $\langle \text{LT}(I) \rangle =$

$\langle y^3, x^2, xy, xz\rangle$. Then

$$\begin{aligned}\dim V &= \deg {}^a\!HP_I\\ &= \deg {}^a\!HP_{\langle \mathrm{LT}(I)\rangle}\\ &= \text{maximum dimension of a coordinate subspace in } \mathbf{V}(\langle \mathrm{LT}(I)\rangle)\end{aligned}$$

by Propositions 3 and 4. Since

$$\mathbf{V}(\langle \mathrm{LT}(I)\rangle) = \mathbf{V}(y^3, x^2, xy, xz) = \mathbf{V}(x, y) \subset \mathbb{R}^3,$$

we conclude that $\dim V = 1$. This agrees with our intuition that the twisted cubic should be 1-dimensional since it is a curve in $\mathbb{R}^3$.

For another example, let us compute the dimension of the variety of a monomial ideal. In Exercise 10, you will show that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$ when I is a monomial ideal and k is infinite. Then Proposition 6 implies that

$$\dim \mathbf{V}(I) = \deg {}^a\!HP_{\mathbf{I}(\mathbf{V}(I))} = \deg {}^a\!HP_{\sqrt{I}} = \deg {}^a\!HP_I,$$

and it follows from part (iii) of Proposition 3 that $\dim \mathbf{V}(I)$ is the maximum dimension of a coordinate subspace contained in $\mathbf{V}(I)$. This agrees with the provisional definition of dimension given in §2. In Exercise 10, you will see that this can fail when k is a finite field.

An interesting exceptional case is the empty variety. Note that $1 \in \mathbf{I}(V)$ if and only if $k[x_1, \ldots, x_n]_{\leq s} = \mathbf{I}(V)_{\leq s}$ for all s. Hence,

$$V = \emptyset \iff {}^a\!HP_{\mathbf{I}(V)} = 0.$$

Since the zero polynomial does not have a degree, we do not assign a dimension to the empty variety.

One drawback of Definition 7 is that to find the dimension of a variety V, we need to know $\mathbf{I}(V)$, which, in general, is difficult to compute. It would be much nicer if $\dim V$ were the degree of ${}^a\!HP_I$, where I is an arbitrary ideal defining V. Unfortunately, this is not true in general. For example, if $I = \langle x^2 + y^2\rangle \subset \mathbb{R}[x, y]$, it is easy to check that ${}^a\!HP_I(s)$ has degree 1. Yet $V = \mathbf{V}(I) = \{(0, 0)\} \subset \mathbb{R}^2$ is easily seen to have dimension 0. Thus, $\dim \mathbf{V}(I) \neq \deg {}^a\!HP_I$ in this case (see Exercise 11 for the details).

When the field k is algebraically closed, these difficulties go away. More precisely, we have the following theorem that tells us how to compute the dimension in terms of any defining ideal.

Theorem 8 (The Dimension Theorem). *Let $V = \mathbf{V}(I)$ be an affine variety, where $I \subset k[x_1, \ldots, x_n]$ is an ideal. If k is algebraically closed, then*

$$\dim V = \deg {}^a\!HP_I.$$

Furthermore, if $>$ is a graded order on $k[x_1, \ldots, x_n]$, then

$$\begin{aligned}\dim V &= \deg {}^a\!HP_{\langle \mathrm{LT}(I)\rangle}\\ &= \textit{maximum dimension of a coordinate subspace in } \mathbf{V}(\langle \mathrm{LT}(I)\rangle).\end{aligned}$$

Finally, the last two equalities hold over any field k when $I = \mathbf{I}(V)$.

Proof. Since k is algebraically closed, the Nullstellensatz implies that $\mathbf{I}(V) = \mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$. Then

$$\dim V = \deg {}^aHP_{\mathbf{I}(V)} = \deg {}^aHP_{\sqrt{I}} = \deg {}^aHP_I,$$

where the last equality is by Proposition 6. The second part of the theorem now follows immediately using Propositions 3 and 4.

□

In other words, over an algebraically closed field, to compute the dimension of a variety $V = \mathbf{V}(I)$, one can proceed as follows:

- Compute a Groebner basis for I using a graded order such as grlex or grevlex.
- Compute the maximal dimension d of a coordinate subspace contained in $\mathbf{V}(\langle \mathrm{LT}(I)\rangle)$. Note that Proposition 3 of §1 gives an algorithm for doing this.

Then $\dim V = d$ follows from Theorem 8.

The Dimension of a Projective Variety

Our discussion of the dimension of a projective variety $V \subset \mathbb{P}^n(k)$ will parallel what we did in the affine case and, in particular, many of the arguments are the same. We start by defining the Hilbert function and the Hilbert polynomial for an arbitrary homogeneous ideal $I \subset k[x_0, \ldots, x_n]$. As above, we assume that k is infinite.

As we saw in §2, the projective case uses total degree equal to s rather than $\leq s$. Since polynomials of total degree s do not form a vector space (see Exercise 13), we will work with *homogeneous* polynomials of total degree s. Let

$$k[x_0, \ldots, x_n]_s$$

denote the set of homogeneous polynomials of total degree s in $k[x_0, \ldots, x_n]$, together with the zero polynomial. In Exercise 13, you will show that $k[x_0, \ldots, x_n]_s$ is a vector space of dimension $\binom{n+s}{s}$. If $I \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal, we let

$$I_s = I \cap k[x_0, \ldots, x_n]_s$$

denote the set of homogeneous polynomials in I of total degree s (and the zero polynomial). Note that I_s is a vector subspace of $k[x_0, \ldots, x_n]_s$. Then the *Hilbert function* of I is defined by

$$HF_I(s) = \dim k[x_0, \ldots, x_n]_s / I_s.$$

Strictly speaking, we should call this the projective Hilbert function, but the above terminology is customary in algebraic geometry.

When I is a monomial ideal, the argument of Proposition 3 adapts easily to show that $HF_I(s)$ is the number of monomials not in I of total degree s. It follows from Proposition 8 of §2 that for s sufficiently large, we can express the Hilbert function of a monomial ideal in the form

$$(4) \qquad HF_I(S) = \sum_{i=0}^{d} b_i \binom{s}{d-i},$$

where $b_i \in \mathbb{Z}$ and b_0 is positive. We also know that d is the largest dimension of a projective coordinate subspace contained in $\mathbf{V}(I) \subset \mathbb{P}^n(k)$.

As in the affine case, we can use a monomial order to link the Hilbert function of a homogeneous ideal to the Hilbert function of a monomial ideal.

Proposition 9. *Let $I \subset k[x_0, \dots, x_n]$ be a homogeneous ideal and let $>$ be a monomial order on $k[x_0, \dots, x_n]$. Then the monomial ideal $\langle \mathrm{LT}(I) \rangle$ has the same Hilbert function as I.*

Proof. The argument is similar to the proof of Proposition 4. However, since we do *not* require that $>$ be a graded order, some changes are needed.

For a fixed s, we can find $f_1, \dots, f_m \in I_s$ such that

$$\{\mathrm{LM}(f) : f \in I_s\} = \{\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_m)\} \tag{5}$$

and we can assume that $\mathrm{LM}(f_1) > \mathrm{LM}(f_2) > \cdots > \mathrm{LM}(f_m)$. As in the proof of Proposition 4, $f_1, \dots, f_m$ form a basis of I_s as a vector space over k.

Now consider $\langle \mathrm{LT}(I) \rangle_s$. We know $\mathrm{LM}(f_i) \in \langle \mathrm{LT}(I) \rangle_s$ since $f_i \in I_s$ and we need to show that $\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_m)$ form a vector space basis of $\langle \mathrm{LT}(I) \rangle_s$. The leading terms are distinct, so as above, they are linearly independent. It remains to prove that they span. By Lemma 3 of Chapter 2, §4, it suffices to show that

$$\{\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_m)\} = \{\mathrm{LM}(f) : f \in I, \mathrm{LM}(f) \text{ has total degree } s\}. \tag{6}$$

To relate this to (5), suppose that $\mathrm{LM}(f)$ has total degree s for some $f \in I$. If we write f as a sum of homogeneous polynomials $f = \sum_i h_i$, where h_i has total degree i, it follows that $\mathrm{LM}(f) = \mathrm{LM}(h_s)$. Since I is a homogeneous ideal, we have $h_s \in I$. Thus, $\mathrm{LM}(f) = \mathrm{LM}(h_s)$ where $h_s \in I_s$, and, consequently, (6) follows from (5). From here, the argument is identical to what we did in Proposition 4, and we are done. □

If we combine Proposition 9 with the description of the Hilbert function for a monomial ideal given by (4), we see that for any homogeneous ideal $I \subset k[x_0, \dots, x_n]$, the Hilbert function can be written

$$HF_I(S) = \sum_{i=0}^{d} b_i \binom{s}{d-i}$$

for s sufficiently large. The polynomial on the right of this equation is called the *Hilbert polynomial* of I and is denoted $HP_I(s)$.

We then define the dimension of a projective variety in terms of the Hilbert polynomial as follows.

Definition 10. *The* **dimension** *of a projective variety $V \subset \mathbb{P}^n(k)$, denoted* $\dim V$, *is the degree of the Hilbert polynomial of the corresponding homogeneous ideal $I = \mathbf{I}(V) \subset k[x_0, \dots, x_n]$. (Note that I is homogeneous by Proposition 4 of Chapter 8, §3.)*

Over an algebraically closed field, we can compute the dimension as follows.

Theorem 11 (The Dimension Theorem). *Let $V = \mathbf{V}(I) \subset \mathbb{P}^n(k)$ be a projective variety, where $I \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal. If V is nonempty and k is algebraically closed, then*

$$\dim V = \deg HP_I.$$

Furthermore, for any monomial order on $k[x_0, \ldots, x_n]$, we have

$$\begin{aligned} \dim V &= \deg HP_{\langle \mathrm{LT}(I)\rangle} \\ &= \textit{maximum dimension of a projective coordinate subspace in } \mathbf{V}(\langle \mathrm{LT}(I)\rangle). \end{aligned}$$

Finally, the last two equalities hold over any field k when $I = \mathbf{I}(V)$.

Proof. The first step is to show that I and $\sqrt{I}$ have Hilbert polynomials of the same degree. The proof is similar to what we did in Proposition 6 and is left as an exercise.

By the projective Nullstellensatz, we know that $\mathbf{I}(V) = \mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$, and, from here, the proof is identical to what we did in the affine case (see Theorem 8). □

For our final result, we compare the dimension of affine and projective varieties.

Theorem 12.

(i) *Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then, for $s \geq 1$, we have*

$$HF_I(s) = {}^aHF_I(s) - {}^aHF_I(s-1).$$

There is a similar relation between Hilbert polynomials. Consequently, if $V \subset \mathbb{P}^n(k)$ is a nonempty projective variety and $C_V \subset k^{n+1}$ is its affine cone (see Chapter 8, §3), then

$$\dim C_V = \dim V + 1.$$

(ii) *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let $I^h \subset k[x_0, \ldots, x_n]$ be its homogenization with respect to x_0 (see §4 of Chapter 8). Then for $s \geq 0$, we have*

$${}^aHF_I(s) = HF_{I^h}(s).$$

There is a similar relation between Hilbert polynomials. Consequently, if $V \subset k^n$ is an affine variety and $\overline{V} \subset \mathbb{P}^n(k)$ is its projective closure (see Chapter 8, §4), then

$$\dim V = \dim \overline{V}.$$

Proof. We will use the subscripts a and p to indicate the affine and projective cases respectively. The first part of (i) follows easily by reducing to the case of a monomial ideal and using the results of §2. We leave the details as an exercise. For the second part of (i), note that the affine cone C_V is simply the *affine* variety in k^{n+1} defined by $\mathbf{I}_p(V)$. Further, it is easy to see that $\mathbf{I}_a(C_V) = \mathbf{I}_p(V)$ (see Exercise 19). Thus, the dimensions of V and C_V are the degrees of $HP_{\mathbf{I}_p(V)}$ and ${}^aHP_{\mathbf{I}_p(V)}$, respectively. Then $\dim C_V = \dim V + 1$ follows from Exercise 15 of §2 and the relation just proved between the Hilbert polynomials.

To prove the first part of (ii), consider the maps

$$\phi : k[x_1, \ldots, x_n]_{\leq s} \longrightarrow k[x_0, \ldots, x_n]_s,$$
$$\psi : k[x_0, \ldots, x_n]_s \longrightarrow k[x_1, \ldots, x_n]_{\leq s}$$

defined by the formulas

$$\phi(f) = x_0^s f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right) \quad \text{for} f \in k[x_1, \ldots, x_n]_{\leq s},$$
$$(7) \qquad \psi(F) = F(1, x_1, \ldots, x_n) \quad \text{for } F \in k[x_0, \ldots, x_n]_s.$$

We leave it as an exercise to check that these are linear maps that are inverses of each other, and hence, $k[x_1, \ldots, x_n]_{\leq s}$ and $k[x_0, \ldots, x_n]_s$ are isomorphic vector spaces. You should also check that if $f \in k[x_1, \ldots, x_n]_{\leq s}$ has total degree $d \leq s$, then

$$\phi(f) = x_0^{s-d} f^h,$$

where f^h is the homogenization of f as defined in Proposition 7 of Chapter 8, §2.

Under these linear maps, you will check in the exercises that

$$\phi(I_{\leq s}) \subset I_s^h,$$
$$(8) \qquad \psi(I_s^h) \subset I_{\leq s},$$

and it follows easily that the above inclusions are equalities. Thus, $I_{\leq s}$ and I_s^h are also isomorphic vector spaces.

This shows that $k[x_1, \ldots, x_n]_{\leq s}$ and $k[x_0, \ldots, x_n]_s$ have the same dimension, and the same holds for $I_{\leq s}$ and I_s^h. By the dimension formula of Proposition 1, we conclude that

$$^aHP_I(s) = \dim k[x_1, \ldots, x_n]_{\leq s}/I_{\leq s}$$
$$(9) \qquad = \dim k[x_0, \ldots, x_n]_s/I_s^h = HP_{I^h}(s),$$

which is what we wanted to prove.

For the second part of (ii), suppose $V \subset k^n$. Let $I = \mathbf{I}_a(V) \subset k[x_1, \ldots, x_n]$ and let $I^h \subset k[x_0, \ldots, x_n]$ be the homogenization of I with respect to x_0. Then $\overline{V}$ is defined to be $\mathbf{V}_p(I^h) \subset \mathbb{P}^n(k)$. Furthermore, we know from Exercise 8 of Chapter 8, §4 that $I^h = \mathbf{I}_p(\overline{V})$. Then

$$\dim V = \deg {}^aHP_I = \deg HP_{I^h} = \dim \overline{V}$$

follows immediately from the first part of (ii), and the theorem is proved. □

Some computer algebra systems can compute Hilbert polynomials. REDUCE has a command to find the affine Hilbert polynomial of an ideal, whereas Macaulay 2 and CoCoA will compute the projective Hilbert polynomial of a homogeneous ideal.

EXERCISES FOR §3

1. In this exercise, you will verify that if V is a vector space and W is a subspace of V, then V/W is a vector space.

a. Show that the relation on V defined by $v \sim v'$ if $v - v' \in W$ is an equivalence relation.
b. Show that the addition and scalar multiplication operations on the equivalence classes defined in the text are well-defined. That is, if $v, v', w, w' \in V$ are such that $[v] = [v']$ and $[w] = [w']$, then show that $[v + w] = [v' + w']$ and $[av] = [av']$ for all $a \in k$.
c. Verify that V/W is a vector space under the operations given in part b.

2. Let V be a finite-dimensional vector space and let W be a vector subspace of V. If $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}\}$ is a basis of V such that $\{v_1, \ldots, v_m\}$ is a basis for W, then show that $[v_{m+1}], \ldots, [v_{m+n}]$ are linearly independent in V/W.
3. Show that a nonzero ideal $I \subset k[x_1, \ldots, x_n]$ is infinite-dimensional as a vector space over k. Hint: Pick $f \neq 0$ in I and consider $x^\alpha f$.
4. The proofs of Propositions 4 and 9 involve finding vector space bases of $k[x_1, \ldots, x_n]_{\leq s}$ and $k[x_1, \ldots, x_n]_s$ where the elements in the bases have distinct leading terms. We showed that such bases exist, but our proof was nonconstructive. In this exercise, we will illustrate a method for actually finding such a basis. We will only discuss the homogeneous case, but the method applies equally well to the affine case.

   The basic idea is to start with any basis of I, and order the elements according to their leading terms. If two of the basis elements have the same leading monomial, we can replace one of them with a k-linear combination that has a smaller leading monomial. Continuing in this way, we will get the desired basis.

   To see how this works in practice, let I be a homogeneous ideal in $k[x, y, z]$, and suppose that $\{x^3 - xy^2, x^3 + x^2y - z^3, x^2y - y^3\}$ is a basis for I_3. We will use grlex order with $x > y > z$.

   a. Show that if we subtract the first polynomial from the second, leaving the third polynomial unchanged, then we get a new basis for I_3.
   b. The second and third polynomials in this new basis now have the same leading monomial. Show that if we change the third polynomial by subtracting the second polynomial from it and multiplying the result $by - 1$, we end up with a basis $\{x^3 - xy^2, x^2y + xy^2 - z^3, xy^2 + y^3 - z^3\}$ for I_3 in which all three leading monomials are distinct.
5. Let $I = \langle x^3 - xyz, y^4 - xyz^2, xy - z^2\rangle$. Using grlex order with $x > y > z$ find bases of I_3 and I_4 where the elements in the bases have distinct leading monomials. Hint: Use the method of Exercise 4.
6. Use the methods of §2 to compute the affine Hilbert polynomials for each of the following ideals.
   a. $I = \langle x^3y, xy^2\rangle \subset k[x, y]$.
   b. $I = \langle x^3y^2 + 3x^2y^2 + y^3 + 1\rangle \subset k[x, y]$.
   c. $I = \langle x^3yz^5, xy^3z^2\rangle \subset k[x, y, z]$.
   d. $I = \langle x^3 - yz^2, y^4 - x^2yz\rangle \subset k[x, y, z]$.
7. Find the index of regularity [that is, the smallest s_0 such that ${}^aHF_I(s) = {}^aHP_I(s)$ for all $s \geq s_0$] for each of the ideals in Exercise 6.
8. In this exercise, we will show that if $I_1 \subset I_2$ are ideals in $k[x_1, \ldots, x_n]$, then
$$\deg {}^aHP_{I_2} \leq \deg {}^aHP_{I_1}.$$
   a. Show that $I_1 \subset I_2$ implies $C(\langle \text{LT}(I_2)\rangle) \subset C(\langle \text{LT}(I_1)\rangle)$ in $\mathbb{Z}^n_{\geq 0}$.
   b. Show that for $s \geq 0$, the affine Hilbert *functions* satisfy the inequality
$${}^aHF_{I_2}(s) \leq {}^aHF_{I_1}(s)$$
   c. From part (b), deduce the desired statement about the degrees of the affine Hilbert *polynomials.* Hint: Argue by contradiction and consider the values of the polynomials as $s \to \infty$.

d. If $I_1 \subset I_2$ are homogeneous ideals in $k[x_0, \ldots, x_n]$, prove an analogous inequality for the degrees of the Hilbert polynomials of I_1 and I_2.

9. Use Definition 7 to show that a point $p = (a_1, \ldots, a_n) \in k^n$ gives a variety of dimension zero. Hint: Use Exercise 7 of Chapter 4, §5 to describe $\mathbf{I}(\{p\})$.
10. Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal, and assume that k is an infinite field. In this exercise, we will study $\mathbf{I}(\mathbf{V}(I))$.
    a. Show that $\mathbf{I}(\mathbf{V}(x_{i_1}, \ldots, x_{i_r})) = \langle x_{i_1}, \ldots, x_{i_r}\rangle$. Hint: Use Proposition 5 of Chapter 1, §1.
    b. Show that an intersection of monomial ideals is a monomial ideal. Hint: Use Lemma 3 of Chapter 2, §4.
    c. Show that $\mathbf{I}(\mathbf{V}(I))$ is a monomial ideal. Hint: Use parts(a) and (b) together with Theorem 15 of Chapter 4, §3.
    d. The final step is to show that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$. We know that $\sqrt{I} \subset \mathbf{I}(\mathbf{V}(I))$, and since $\mathbf{I}(\mathbf{V}(I))$ is a monomial ideal, you need only prove that $x^{\alpha} \in \mathbf{I}(\mathbf{V}(I))$ implies that $x^{r\alpha} \in I$ for some $r > 0$. Hint: If $I = \langle m_1, \ldots, m_t\rangle$ and $x^{r\alpha} \notin I$ for $r > 0$, show that for every j, there is x_{i_j} such that x_{i_j} divides m_j but not x^{α}. Use $x_{i_1}, \ldots, x_{i_t}$ to obtain a contradiction.
    e. Let $\mathbb{F}_2$ be a field with of two elements and let $I = \langle x\rangle \subset \mathbb{F}_2[x, y]$. Show that $\mathbf{I}(\mathbf{V}(I)) = \langle x, y^2 - y\rangle$. This is bigger than $\sqrt{I}$ and is not a monomial ideal.
11. Let $I = \langle x^2 + y^2\rangle \subset \mathbb{R}[x, y]$.
    a. Show carefully that $\deg {}^{a}HP_I = 1$.
    b. Use Exercise 9 to show that $\dim \mathbf{V}(I) = 0$.
12. Compute the dimension of the affine varieties defined by the following ideals. You may assume that k is algebraically closed.
    a. $I = \langle xz, xy - 1\rangle \subset k[x, y, z]$.
    b. $I = \langle zw - y^2, xy - z^3\rangle \subset k[x, y, z, w]$.
13. Consider the polynomial ring $k[x_0, \ldots, x_n]$.
    a. Given an example to show that the set of polynomials of total degree s is not closed under addition and, hence, does not form a vector space.
    b. Show that the set of homogeneous polynomials of total degree s (together with the zero polynomial) is a vector space over k.
    c. Use Lemma 5 of §2 to show that this vector space has dimension $\binom{n+s}{s}$. Hint: Consider the number of polynomials of total degree $\leq s$ and $\leq s - 1$.
    d. Give a second proof of the dimension formula of part (c) using the isomorphism of Exercise 20 below.
14. If I is a homogeneous ideal, show that the Hilbert polynomials HP_I and $HP_{\sqrt{I}}$ have the same degree. Hint: The quickest way is to use Theorem 12.
15. We will study when the Hilbert polynomial is zero.
    a. If $I \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal, prove that $\langle x_0, \ldots, x_n\rangle^r \subset I$ for some $r \geq 0$ if and only if the Hilbert polynomial of I is the zero polynomial.
    b. Conclude that if $V \subset \mathbb{P}^n(k)$ is a variety, then $V = \emptyset$ if and only if its Hilbert polynomial is the zero polynomial. Thus, the empty variety in $\mathbb{P}^n(k)$ does not have a dimension.
16. Compute the dimension of the following projective varieties. Assume that k is algebraically closed.
    a. $I = \langle x^2 - y^2, x^3 - x^2y + y^3\rangle \subset k[x, y, z]$.
    b. $I = \langle y^2 - xz, x^2y - z^2w, x^3 - yzw\rangle \subset k[x, y, z, w]$.
17. In this exercise, we will see that in general, there is no relation between the number of variables n, the number r of polynomials in a basis of I, and the dimension of $V = \mathbf{V}(I)$. Let $V \subset \mathbb{P}^3(k)$ be the curve given by the projective parametrization $x = t^3u^2$, $y = t^4u$, $z = t^5$, $w = u^5$. Since this is a curve in 3-dimensional space, our intuition would lead us to believe that V should be defined by two equations. Assume that k is algebraically closed.

a. Use Theorem 12 of Chapter 8, §5 to find an ideal $I \subset k[x, y, z, w]$ such that $V = \mathbf{V}(I)$ in $\mathbb{P}^3(k)$. If you use grevlex for a certain ordering of the variables, you will get a basis of I containing three elements.
b. Show that I_2 is 1-dimensional and I_3 is 6-dimensional.
c. Show that I cannot be generated by two elements. Hint: Suppose that $I = \langle A, B\rangle$, where A and B are homogeneous. By considering I_2, show that A or B must be a multiple of $y^2 - xz$, and then derive a contradiction by looking at I_3.

A much more difficult question would be to prove that there are no two homogeneous polynomials A, B such that $V = \mathbf{V}(A, B)$.

18. This exercise is concerned with the proof of part (i) of Theorem 12.
a. Use the methods of §2 to show that $HF_I(s) = {}^aHF_I(s) - {}^aHF_I(s-1)$ whenever I is a monomial ideal.
b. Prove that $HF_I(s) = {}^aHF_I(s) - {}^aHF_I(s-1)$ for an arbitrary homogeneous ideal I.
19. If $V \subset \mathbb{P}^n(k)$ is a nonempty projective variety and $C_V \subset k^{n+1}$ is its affine cone, then prove that $\mathbf{I}_p(V) = \mathbf{I}_a(C_V)$ in $k[x_0, \ldots, x_n]$.
20. This exercise is concerned with the proof of part (ii) of Theorem 12.
a. Show that the maps ϕ and ψ defined in (7) are linear maps and verify that they are inverses of each other.
b. Prove (8) and conclude that $\phi : I_{\leq s} \to I_s^h$ is an isomorphism whose inverse is ψ.

§4 Elementary Properties of Dimension

Using the definition of the dimension of a variety from §3, we can now state several basic properties of dimension. As in §3, we assume that the field k is infinite.

Proposition 1. *Let V_1 and V_2 be projective or affine varieties. If $V_1 \subset V_2$, then* $\dim V_1 \leq \dim V_2$.

Proof. We leave the proof to the reader as Exercise 1. □

We next will study the relation between the dimension of a variety and the number of defining equations. We begin with the case where V is defined by a single equation.

Proposition 2. *Let k be an algebraically closed field and let $f \in k[x_0, \ldots, x_n]$ be a nonconstant homogeneous polynomial. Then the dimension of the projective variety in $\mathbb{P}^n(k)$ defined by f is*

$$\dim \mathbf{V}(f) = n - 1.$$

Proof. Fix a monomial order $>$ on $k[x_0, \ldots, x_n]$. Since k is algebraically closed, Theorem 11 of §3 says the dimension of $\mathbf{V}(f)$ is the maximum dimension of a projective coordinate subspace contained in $\mathbf{V}(\langle \mathrm{LT}(I)\rangle)$, where $I = \langle f\rangle$. One can check that $\langle \mathrm{LT}(I)\rangle = \langle \mathrm{LT}(f)\rangle$, and since $\mathrm{LT}(f)$ is a nonconstant monomial, the projective variety $\mathbf{V}(\mathrm{LT}(f))$ is a union of subspaces of $\mathbb{P}^n(k)$ of dimension $n-1$. It follows that $\dim \mathbf{V}(I) = n - 1$. □

Thus, when k is algebraically closed, a hypersurface $\mathbf{V}(f)$ in $\mathbb{P}^n$ always has dimension $n-1$. We leave it as an exercise for the reader to prove the analogous statement for affine hypersurfaces.

It is important to note that these results are *not valid* if k is not algebraically closed. For instance, let $I = \langle x^2 + y^2 \rangle$ in $\mathbb{R}[x, y]$. In §3, we saw that $\mathbf{V}(f) = \{(0, 0)\} \subset \mathbb{R}^2$ has dimension 0, yet Proposition 2 would predict that the dimension was 1. In fact, over a nonalgebraically closed field, the variety in k^n or $\mathbb{P}^n$ defined by a single polynomial can have any dimension between 0 and $n-1$.

The following theorem establishes the analogue of Proposition 2 when the ambient space $\mathbb{P}^n(k)$ is replaced by an arbitrary variety V. Note that if I is an ideal and f is a polynomial, then $\mathbf{V}(I + \langle f \rangle) = \mathbf{V}(I) \cap \mathbf{V}(f)$.

Theorem 3. *Let k be an algebraically closed field and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. If $\dim \mathbf{V}(I) > 0$ and f is any nonconstant homogeneous polynomial, then*

$$\dim \mathbf{V}(I) \geq \dim \mathbf{V}(I + \langle f \rangle) \geq \dim \mathbf{V}(I) - 1$$

Proof. To compute the dimension of $\mathbf{V}(I + \langle f \rangle)$, we will need to compare the Hilbert polynomials HP_I and $HP_{I+\langle f \rangle}$. We first note that since $I \subset I + \langle f \rangle$, Exercise 8 of §3 implies that

$$\deg HP_I \geq \deg HP_{I+\langle f \rangle},$$

from which we conclude that $\dim \mathbf{V}(I) \geq \dim \mathbf{V}(I + \langle f \rangle)$ by Theorem 11 of §3.

To obtain the other inequality, suppose that f has total degree $r > 0$. Fix a total degree $s \geq r$ and consider the map

$$\pi : k[x_0, \ldots, x_n]_s / I_s \longrightarrow k[x_0, \ldots, x_n]_s / (I + \langle f \rangle)_s$$

which sends $[g] \in k[x_0, \ldots, x_n]_s / I_s$ to $\pi([g]) = [g] \in k[x_0, \ldots, x_n]_s / (I + \langle f \rangle)_s$. In Exercise 4, you will check that π is a well-defined linear map. It is easy to see that π is onto, and to investigate its kernel, we will use the map

$$\alpha_f : k[x_0, \ldots, x_n]_{s-r} / I_{s-r} \longrightarrow k[x_0, \ldots, x_n]_s / I_s$$

defined by sending $[h] \in k[x_0, \ldots, x_n]_{s-r} / I_{s-r}$ to $\alpha_f([h]) = [fh] \in k[x_0, \ldots, x_n]_s / I_s$. In Exercise 5, you will show that α_f is also a well-defined linear map.

We claim that the kernel of π is exactly the image of α_f, i.e., that

$$(1) \quad \alpha_f(k[x_0, \ldots, x_n]_{s-r} / I_{s-r}) = \{[g] : \pi([g]) = [0] \text{ in } k[x_0, \ldots, x_n]_s / (I + \langle f \rangle)_s\}.$$

To prove this, note that if $h \in k[x_0, \ldots, x_n]_{s-r}$, then $fh \in (I + \langle f \rangle)_s$ and, hence, $\pi([fh]) = [0]$ in $k[x_0, \ldots, x_n]_s / (I + \langle f \rangle)_s$. Conversely, if $g \in k[x_0, \ldots, x_n]_s$ and $\pi([g]) = [0]$, then $g \in (I + \langle f \rangle)_s$. This means $g = g' + fh$ for some $g' \in I$. If we write $g' = \sum_i g'_i$ and $h = \sum_i h_i$ as sums of homogeneous polynomials, where g'_i and h_i have total degree i, it follows that $g = g'_s + fh_{s-r}$ since g and f are homogeneous. Since I is a homogeneous ideal, we have $g'_s \in I_s$, and it follows that $[g] = [fh_{s-r}] = \alpha_f([h_{s-r}])$ in $k[x_0, \ldots, x_n]_s / I_s$. This shows that $[g]$ is in the image of α_f and completes the proof of (1).

Since π is onto and we know its kernel by (1), the dimension theorem for linear mappings shows that

$$\dim k[x_0,\ldots,x_n]_s/I_s = \dim \alpha_f(k[x_0,\ldots,x_n]_{s-r}/I_{s-r}) + \dim(k[x_0,\ldots,x_n]_s/I + \langle f\rangle)_s.$$

Now certainly,

$$\text{(2)} \qquad \dim \alpha_f(k[x_0,\ldots,x_n]_{s-r}/I_{s-r}) \leq \dim k[x_0,\ldots,x_n]_{s-r}/I_{s-r},$$

with equality if and only if α_f is one-to-one. Hence,

$$\dim k[x_0,\ldots,x_n]_s/(I+\langle f\rangle)_s \geq \dim k[x_0,\ldots,x_n]_s/I_s - \dim k[x_0,\ldots,x_n]_{s-r}/I_{s-r}.$$

In terms of Hilbert functions, this tells us that

$$HF_{I+\langle f\rangle}(s) \geq HF_I(s) - HF_I(s-r)$$

whenever $s \geq r$. Thus, if s is sufficiently large, we obtain the inequality

$$\text{(3)} \qquad HP_{I+\langle f\rangle}(s) \geq HP_I(s) - HP_I(s-r)$$

for the Hilbert polynomials.

Suppose that HP_I has degree d. Then it is easy to see that the polynomial on the right-hand side of (3) had degree $d-1$ (the argument is the same as used in Exercise 15 of §2). Thus, (3) shows that $HP_{I+\langle f\rangle}(s)$ is $\geq$ a polynomial of degree $d-1$ for s sufficiently large, which implies deg $HP_{I+\langle f\rangle}(s) \geq d-1$ [see, for example, part (c) of Exercise 8 of §3]. Since k is algebraically closed, we conclude that $\dim \mathbf{V}(I+\langle f\rangle) \geq \dim \mathbf{V}(I) - 1$ by Theorem 8 of §3. □

By carefully analyzing the proof of Theorem 3, we can give a condition that ensures that $\dim \mathbf{V}(I+\langle f\rangle) = \dim \mathbf{V}(I) - 1$.

Corollary 4. *Let k be an algebraically closed field and let $I \subset k[x_0,\ldots,x_n]$ be a homogeneous ideal. Let f be a nonconstant homogeneous polynomial whose class in the quotient ring $k[x_0,\ldots,x_n]/I$ is not a zero divisor. Then*

$$\dim \mathbf{V}(I+\langle f\rangle) = \dim \mathbf{V}(I) - 1$$

when $\dim \mathbf{V}(I) > 0$, *and* $\mathbf{V}(I+\langle f\rangle) = \emptyset$ *when* $\dim \mathbf{V}(I) = 0$.

Proof. As we observed in the proof of Theorem 3, the inequality (2) is an equality if the multiplication map α_f is one-to-one. We claim that the latter is true if $[f] \in k[x_0,\ldots,x_n]/I$ is not a zero divisor. Namely, suppose that $[h] \in k[x_0,\ldots,x_n]_{s-r}/I_{s-r}$ is nonzero. This implies that $h \notin I_{s-r}$ and, hence, $h \notin I$ since $I_{s-r} = I \cap k[x_0,\ldots,x_n]_{s-r}$. Thus, $[h] \in k[x_0,\ldots,x_n]/I$ is nonzero, so that $[f][h] = [fh]$ is nonzero in $k[x_0,\ldots,x_n]/I$ by our assumption on f. Thus, $fh \notin I$ and, hence, $\alpha_f([h]) = [fh]$ is nonzero in $k[x_0,\ldots,x_n]_s/I_s$. This shows that α_f is one-to-one.

Since (2) is an equality, the proof of Theorem 3 shows that we also get the equality

$$\dim(k[x_0,\ldots,x_n]_s/(I+\langle f\rangle)_s = \dim k[x_0,\ldots,x_n]_s/I_s - \dim k[x_0,\ldots,x_n]_{s-r}/I_{s-r}$$

when $s \geq r$. In terms of Hilbert polynomials, this says $HP_{I+\langle f\rangle}(s) = HP_I(s) - HP_I(s-r)$, and it follows immediately that $\dim \mathbf{V}(I+\langle f\rangle) = \dim \mathbf{V}(I) - 1$. □

We remark that Theorem 3 can fail for affine varieties, even when k is algebraically closed. For example, consider the ideal $I = \langle xz, yz\rangle \subset \mathbb{C}[x, y, z]$. One easily sees that in $\mathbb{C}^3$, we have $\mathbf{V}(I) = \mathbf{V}(z) \cup V(x, y)$, so that $\mathbf{V}(I)$ is the union of the (x, y)-plane and the z-axis. In particular, $\mathbf{V}(I)$ has dimension 2 (do you see why?). Now, let $f = z-1 \in \mathbb{C}[x, y, z]$. Then $\mathbf{V}(f)$ is the plane $z = 1$ and it follows that $\mathbf{V}(I + \langle f\rangle) = \mathbf{V}(I) \cap \mathbf{V}(f)$ consists of the single point $(0, 0, 1)$ (you will check this carefully in Exercise 7). By Exercise 9 of §3, we know that a point has dimension 0. Yet Theorem 3 would predict that $\mathbf{V}(I + \langle f\rangle)$ had dimension at least 1.

What goes "wrong" here is that the planes $z = 0$ and $z = 1$ are parallel and, hence, do not meet in affine space. We are missing a component of dimension 1 at infinity. This is an example of the way dimension theory works more satisfactorily for homogeneous ideals and projective varieties. It is possible to formulate a version of Theorem 3 that is valid for affine varieties, but we will not pursue that question here.

Our next result extends Theorem 3 to the case of several polynomials $f_1, \ldots, f_r$.

Proposition 5. *Let k be an algebraically closed field and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. Let $f_1, \ldots, f_r$ be nonconstant homogeneous polynomials in $k[x_0, \ldots, x_n]$ such that $r \leq \dim \mathbf{V}(I)$. Then*

$$\dim \mathbf{V}(I + \langle f_1, \ldots, f_r\rangle) \geq \dim \mathbf{V}(I) - r.$$

Proof. The result follows immediately from Theorem 3 by induction on r. □

In the exercises, we will ask you to derive a condition on the polynomials $f_1, \ldots, f_r$ which guarantees that the dimension of $\mathbf{V}(f_1, \ldots, f_r)$ is exactly equal to $n - r$.

Our next result concerns varieties of dimension 0.

Proposition 6. *Let V be a nonempty affine or projective variety. Then V consists of finitely many points if and only if* $\dim V = 0$.

Proof. We will give the proof only in the affine case. Let $>$ be a graded order on $k[x_1, \ldots, x_n]$. If V is finite, then let a_j, for $j = 1, \ldots, m_i$, be the distinct elements of k appearing as i-th coordinates of points of V. Then

$$f = \prod_{j=1}^{m_i} (x_i - a_j) \in \mathbf{I}(V)$$

and we conclude that $\text{LT}(f) = x_i^{m_i} \in \langle \text{LT}(\mathbf{I}(V))\rangle$. This implies that $\mathbf{V}(\langle \text{LT}(\mathbf{I}(V))\rangle) = \{0\}$ and then Theorem 8 of §3 implies that $\dim V = 0$.

Now suppose that $\dim V = 0$. Then the affine Hilbert polynomial of $\mathbf{I}(V)$ is a constant C, so that

$$\dim k[x_1, \ldots, x_n]_{\leq s}/\mathbf{I}(V)_{\leq s} = C$$

for s sufficiently large. If we also have $s \geq C$, then the classes $[1], [x_i], [x_i^2], \ldots, [x_i^s] \in k[x_1, \ldots, x_n]_{\leq s}/\mathbf{I}(V)_{\leq s}$ are $s + 1$ vectors in a vector space of dimension $C \leq s$ and, hence, they must be linearly dependent. But a nontrivial linear relation

$$[0] = \sum_{j=0}^{s} a_j [x_i^j] = \left[\sum_{j=0}^{s} a_j x_i^j \right]$$

means that $\Sigma_{j=0}^{s} a_j x_i^j$ is a nonzero polynomial in $\mathbf{I}(V)_{\le s}$. This polynomial vanishes on V, which implies that there are only finitely many distinct i-th coordinates among the points of V. Since this is true for all $1 \le i \le n$, it follows that V must be finite. □

If, in addition, k is algebraically closed, then we see that the six conditions of Theorem 6 of Chapter 5, §3 are equivalent to $\dim V = 0$. In particular, given any defining ideal I of V, we get a simple criterion for detecting when a variety has dimension 0.

Now that we understand varieties of dimension 0, let us record some interesting properties of positive dimensional varieties.

Proposition 7. *Let k be algebraically closed.*

(i) *Let $V \subset \mathbb{P}^n(k)$ be a projective variety of dimension > 0. Then $V \cap \mathbf{V}(f) \ne \emptyset$ for every nonconstant homogeneous polynomial $f \in k[x_0, \ldots, x_n]$. Thus, a positive dimensional projective variety meets every hypersurface in $\mathbb{P}^n(k)$.*

(ii) *Let $W \subset k^n$ be an affine variety of dimension > 0. If $\overline{W}$ is the projective closure of W in $\mathbb{P}^n(k)$, then $W \ne \overline{W}$. Thus, a positive dimensional affine variety always has points at infinity.*

Proof. (i) Let $V = \mathbf{V}(I)$. Since $\dim V > 0$, Theorem 3 shows that $\dim V \cap \mathbf{V}(f) \ge \dim V - 1 \ge 0$. Let us check carefully that this guarantees $V \cap \mathbf{V}(f) \ne \emptyset$.

If $V \cap \mathbf{V}(f) = \emptyset$, then the projective Nullstellensatz implies that $\langle x_0, \ldots, x_n \rangle^r \subset I + \langle f \rangle$ for some $r \ge 0$. By Exercise 15 of §3, it follows that $HP_{I+\langle f \rangle}$ is the zero polynomial. Yet if you examine the proof of Theorem 3, the inequality given for $HP_{I+\langle f \rangle}$ shows that this polynomial cannot be zero when $\dim V > 0$. We leave the details as an exercise.

(ii) The points at infinity of W are $\overline{W} \cap \mathbf{V}(x_0)$, where $\mathbf{V}(x_0)$ is the hyperplane at infinity. By Theorem 12 of §3, we have $\dim \overline{W} = \dim W > 0$, and then (i) implies that $\overline{W} \cap \mathbf{V}(x_0) \ne \emptyset$. □

We next study the dimension of the union of two varieties.

Proposition 8. *If V and W are varieties either both in k^n or both in $\mathbb{P}^n(k)$, then*

$$\dim(V \cup W) = \max(\dim V, \dim W).$$

Proof. The proofs for the affine and projective cases are nearly identical, so we will give only the affine proof.

Let $I = \mathbf{I}(V)$ and $J = \mathbf{I}(W)$, so that $\dim V = \deg {}^aHP_I$ and $\dim W = \deg {}^aHP_J$. By Theorem 15 of Chapter 4, §3, $\mathbf{I}(V \cup W) = \mathbf{I}(V) \cap \mathbf{I}(W) = I \cap J$. It is more convenient to work with the product ideal IJ and we note that

$$IJ \subset I \cap J \subset \sqrt{IJ}$$

(see Exercise 15). By Exercise 8 of §3, we conclude that

$$\deg {}^aHP_{\sqrt{IJ}} \leq \deg {}^aHP_{I\cap J} \leq \deg {}^aHP_{IJ}.$$

Proposition 6 of §3 says that the outer terms are equal. We conclude that $\dim(V \cup W) = \deg {}^aHP_{IJ}$.

Now fix a graded order $>$ on $k[x_1, \ldots, x_n]$. By Propositions 3 and 4 of §3, it follows that $\dim V$, $\dim W$, and $\dim(V \cup W)$ are given by the maximal dimension of a coordinate subspace contained in $\mathbf{V}(\langle \mathrm{LT}(I)\rangle)$,$\mathbf{V}(\langle \mathrm{LT}(J)\rangle)$ and $\mathbf{V}(\langle \mathrm{LT}(IJ)\rangle)$ respectively. In Exercise 16, you will prove that

$$\langle \mathrm{LT}(IJ)\rangle \supset \langle \mathrm{LT}(I)\rangle \cdot \langle \mathrm{LT}(J)\rangle.$$

This implies

$$\mathbf{V}(\langle \mathrm{LT}(IJ)\rangle) \subset \mathbf{V}(\langle \mathrm{LT}(I)\rangle) \cup \mathbf{V}(\langle \mathrm{LT}(J)\rangle).$$

Since k is infinite, every coordinate subspace is irreducible (see Exercise 7 of §1), and as a result, a coordinate subspace is contained in $\mathbf{V}(\langle \mathrm{LT}(IJ)\rangle)$ lies in either $\mathbf{V}(\langle \mathrm{LT}(I)\rangle)$ or $\mathbf{V}(\langle \mathrm{LT}(J)\rangle)$. This implies $\dim(V \cup W) \leq \max(\dim V, \dim W)$. The opposite inequality follows from Proposition 1, and the proposition is proved. $\square$

This proposition has the following useful corollary.

Corollary 9. *The dimension of a variety is the largest of the dimensions of its irreducible components.*

Proof. If $V = V_1 \cup \cdots \cup V_r$ is the decomposition of V into irreducible components, then Proposition 8 and an induction on r shows that

$$\dim V = \max\{\dim V_1, \ldots, \dim V_r\},$$

as claimed. $\square$

This corollary allows us to reduce to the case of an irreducible variety when computing dimensions. The following result shows that for irreducible varieties, the notion of dimension is especially well-behaved.

Proposition 10. *Let k be an algebraically closed field and let $V \subset \mathbb{P}^n(k)$ be an irreducible variety.*

(i) *If $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial which does not vanish identically on V, then* $\dim(V \cap \mathbf{V}(f)) = \dim V - 1$ *when* $\dim V > 0$, *and* $V \cap \mathbf{V}(f) = \emptyset$ *when* $\dim V = 0$.

(ii) *If $W \subset V$ is a variety such that $W \neq V$, then* $\dim W < \dim V$.

Proof. (i) By Proposition 4 of Chapter 5, §1, we know that $\mathbf{I}(V)$ is a prime ideal and $k[V] \cong k[x_0, \ldots, x_n]/\mathbf{I}(V)$ is an integral domain. Since $f \notin \mathbf{I}(V)$, the class of f is nonzero in $k[x_0, \ldots, x_n]/\mathbf{I}(V)$ and, hence, is not a zero divisor. The desired conclusion then follows from Corollary 4.

(ii) If W is a proper subvariety of V, then we can find $f \in \mathbf{I}(W) - \mathbf{I}(V)$. Thus, $W \subset V \cap \mathbf{V}(f)$, and it follows from (i) and Proposition 1 that

$$\dim W \leq \dim(V \cap \mathbf{V}(f)) = \dim V - 1 < \dim V.$$

This completes the proof of the proposition. □

Part (i) of Proposition 10 asserts that when V is irreducible and f does not vanish on V, then some component of $V \cap \mathbf{V}(f)$ has dimension $\dim V - 1$. With some more work, it can be shown that *every* component of $V \cap \mathbf{V}(f)$ has dimension $\dim V - 1$. See, for example, Theorem 3.8 in Chapter IV of KENDIG (1977) or Theorem 5 of Chapter 1, §6 of SHAFAREVICH (1974).

In the next section, we will see that there is a way to understand the meaning of the dimension of an *irreducible* variety V in terms of the coordinate ring $k[V]$ and the field of rational functions $k(V)$ of V that we introduced in Chapter 5.

EXERCISES FOR §4

1. Prove Proposition 1. Hint: Use Exercise 8 of the previous section.
2. Let k be an algebraically closed field. If $f \in k[x_1, \ldots, x_n]$ is a nonconstant polynomial, show that the affine hypersurface $\mathbf{V}(f) \subset k^n$ has dimension $n - 1$.
3. In $\mathbb{R}^4$, give examples of four different affine varieties, each defined by a single equation, that have dimensions 0, 1, 2, 3, respectively.
4. In this exercise, we study the mapping

$$\pi : k[x_0, \ldots, x_n]_s/I_s \longrightarrow k[x_0, \ldots, x_n]_s/(I + \langle f \rangle)_s$$

defined by $\pi([g]) = [g]$ for all $g \in k[x_0, \ldots, x_n]_s$.

   a. Show that π is well-defined. That is, show that the image of the class $[g]$ does not depend on which representative g in the class that we choose. We call π the *natural projection* from $k[x_0, \ldots, x_n]_s/I_s$ to $k[x_0, \ldots, x_n]_s/(I + \langle f \rangle)_s$.
   b. Show that π is a linear mapping of vector spaces.
   c. Show that the natural projection π is onto.
5. Show that if f is a homogeneous polynomial of degree r and I is a homogeneous ideal, then the map

$$\alpha_f : k[x_0, \ldots, x_n]_{s-r}/I_{s-r} \longrightarrow k[x_0, \ldots, x_n]_s/I_s$$

defined by $\alpha_f([h]) = [f \cdot h]$ is a well-defined linear mapping. That is, show that $\alpha_f([h])$ does not depend on the representative h for the class and that α preserves the vector space operations.
6. Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of total degree $r > 0$.
   a. Find a formula for the Hilbert polynomial of $\langle f \rangle$. Your formula should depend only on n and r (and, of course, s). In particular, all such polynomials f have the same Hilbert polynomial. Hint: Examine the proofs of Theorem 3 and Corollary 4 in the case when $I = \{0\}$.

b. More generally, suppose that $V = \mathbf{V}(I)$ and that the class of f is not a zero divisor in $k[x_0, \ldots, x_n]/I$. Then show that the Hilbert polynomial of $I + \langle f \rangle$ depends only on I and r.

If we vary f, we can regard the varieties $\mathbf{V}(f) \subset \mathbb{P}^n(k)$ as an algebraic family of hypersurfaces. Similarly, varying f gives the family of varieties $V \cap \mathbf{V}(f)$. By parts (a) and (b), the Hilbert polynomials are constant as we vary f. In general, once a technical condition called "flatness" is satisfied, Hilbert polynomials are constant on any algebraic families of varieties.

7. Let $I = \langle xz, yz \rangle$. Show that $\mathbf{V}(I + \langle z - 1 \rangle) = \{(0, 0, 1)\}$.
8. Let $R = k[x_0, \ldots, x_n]$. A sequence $f_1, \ldots, f_r$ of $r \leq n + 1$ nonconstant homogeneous polynomials is called an R-sequence if the class $[f_{j+1}]$ is not a zero divisor in $R/\langle f_1, \ldots, f_j \rangle$ for each $1 \leq j < r$.
   a. Show for example that for $r \leq n$, $x_0, \ldots, x_r$ is an R-sequence.
   b. Show that if k is algebraically closed and $f_1, ..., f_r$ is an R-sequence, then
   $$\dim \mathbf{V}(f_1, \ldots, f_r) = n - r.$$
   Hint: Use Corollary 4 and induction on r. Work with the ideals $I_j = \langle f_1, \ldots, f_j \rangle$ for $1 \leq j \leq r$.
9. Let $R = k[x_0, \ldots, x_n]$ be the polynomial ring. A homogeneous ideal I is said to be a *complete intersection* if it can be generated by an R-sequence. A projective variety V is called a *complete intersection* if $\mathbf{I}(V)$ is a complete intersection.
   a. Show that every irreducible linear subspace of $\mathbb{P}^n(k)$ is a complete intersection.
   b. Show that hypersurfaces are complete intersections when k is algebraically closed.
   c. Show that projective closure of the union of the (x, y)- and (z, w)-planes in k^4 is not a complete intersection.
   d. Let V be the affine twisted cubic $\mathbf{V}(y - x^2, z - x^3)$ in k^3. Is the projective closure of V a complete intersection?

   Hint for parts (c) and (d): Use the technique of Exercise 17 of §3.
10. Suppose that $I \subset k[x_1, \ldots, x_n]$ is an ideal. In this exercise, we will prove that the affine Hilbert polynomial is constant if and only if the quotient ring $k[x_1, \ldots, x_n]/I$ is finite-dimensional as a vector space over k. Furthermore, when this happens, we will show that the constant is the dimension of $k[x_1, \ldots, x_n]/I$ as a vector space over k.
    a. Let $\alpha_s : k[x_1, \ldots, x_n]_{\leq s}/I_{\leq s} \to k[x_1, \ldots, x_n]/I$ be the map defined by $\alpha_s([f]) = [f]$. Show that α_s is well-defined and one-to-one.
    b. If $k[x_1, \ldots, x_n]/I$ is finite-dimensional, show that α_s is an isomorphism for s sufficiently large and conclude that the affine Hilbert polynomial is constant (and equals the dimension of $k[x_1, \ldots, x_n]/I$). Hint: Pick a basis $[f_1], \ldots, [f_m]$ of $k[x_1, \ldots, x_n]/I$ and let s be bigger than the total degrees of $f_1, \ldots, f_m$.
    c. Now suppose the affine Hilbert polynomial is constant. Show that if $s \leq t$, the image of α_t contains the image of α_s. If s and t are large enough, conclude that the images are equal. Use this to show that α_s is an isomorphism for s sufficiently large and conclude that $k[x_1, \ldots, x_n]/I$ is finite-dimensional.
11. Let $V \subset k^n$ be finite. In this exercise, we will prove that $k[x_1, \ldots, x_n]/\mathbf{I}(V)$ is finite-dimensional and that its dimension is $|V|$, the number of points in V. If we combine this with the previous exercise, we see that the affine Hilbert polynomial of $\mathbf{I}(V)$ is the constant $|V|$. Suppose that $V = \{p_1, \ldots, p_m\}$, where $m = |V|$.
    a. Define a map $\phi : k[x_1, \ldots, x_n]/\mathbf{I}(V) \to k^m$ by $\phi([f]) = (f(p_1), \ldots, f(p_m))$. Show that ϕ is a well-defined linear map and show that it is one-to-one.
    b. Fix i and let $W_i = \{p_j : j \neq i\}$. Show that $1 \in \mathbf{I}(W_i) + \mathbf{I}(\{p_i\})$. Hint: Show that $\mathbf{I}(\{p_i\})$ is a maximal ideal.

c. By part (b), we can find $f_i \in \mathbf{I}(W_i)$ and $g_i \in \mathbf{I}(\{p_i\})$ such that $f_i + g_i = 1$. Show that $\phi(f_i)$ is the vector in k^m which has a 1 in the i-th coordinate and 0's elsewhere.
d. Conclude that ϕ is an isomorphism and that $\dim k[x_1, \ldots, x_n]/\mathbf{I}(V) = |V|$.

12. Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. In this exercise we will study the geometric significance of the coefficient b_0 of the Hilbert polynomial

$$HP_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i}.$$

We will call b_0 the *degree* of I. The *degree* of a projective variety is then defined to be the degree of $\mathbf{I}(V)$ and, as we will see, the degree is in a sense a generalization of the total degree of the defining equation for a hypersurface. Note also that we can regard Exercises 10 and 11 as studying the degrees of ideals and varieties with constant affine Hilbert polynomial.

a. Show that the degree of the ideal $\langle f \rangle$ is the same as the total degree of f. Also, if k is algebraically closed, show that the degree of the hypersurface $\mathbf{V}(f)$ is the same as the total degree of f_{red}, the reduction of f defined in Chapter 4, §2. Hint: Use Exercise 6.
b. Show that if I is a complete intersection (Exercise 9) generated by the elements of an R-sequence $f_1, \ldots, f_r$, then the degree of I is the product

$$\deg f_1 \cdot \deg f_2 \cdots \deg f_r,$$

of the total degrees of the f_i. Hint: Look carefully at the proof of Theorem 3. The hint for Exercise 8 may be useful.
c. Determine the degree of the projective closure of the standard twisted cubic.

13. Verify carefully the claim made in the proof of Proposition 7 that $HP_{I+\langle f \rangle}$ cannot be the zero polynomial when $\dim V > 0$. Hint: Look at the inequality (3) from the proof of Theorem 3.
14. This exercise will explore what happens if we weaken the hypotheses of Proposition 7.
a. Let $V = \mathbf{V}(x) \subset k^2$. Show that $V \cap \mathbf{V}(x-1) = \emptyset$ and explain why this does not contradict part (a) of the proposition.
b. Let $W = \mathbf{V}(x^2 + y^2 - 1) \subset \mathbb{R}^2$. Show that $W = \overline{W}$ in $\mathbb{P}^2(\mathbb{R})$ and explain why this does not contradict part (b) of the proposition.
15. If $I, J \subset k[x_1, \ldots, x_n]$ are ideals, prove that $IJ \subset I \cap J \subset \sqrt{IJ}$.
16. Show that if I and J are any ideals and $>$ is any monomial ordering, then

$$\langle \mathrm{LT}(I) \rangle \cdot \langle \mathrm{LT}(J) \rangle \subset \langle \mathrm{LT}(IJ) \rangle.$$

17. Using Proposition 10, we can get an alternative definition of the dimension of an irreducible variety. We will assume that the field k is algebraically closed and that $V \subset \mathbb{P}^n(k)$ is irreducible.
a. If $\dim V > 0$, prove that there is an irreducible variety $W \subset V$ such that $\dim W = \dim V - 1$. Hint: Use Proposition 10 and look at the irreducible components of $V \cap \mathbf{V}(f)$.
b. If $\dim V = m$, prove that one can find a chain of $m + 1$ irreducible varieties

$$V_0 \subset V_1 \subset \cdots \subset V_m = V$$

such that $V_i \neq V_{i+1}$ for $0 \le i \le m - 1$.
c. Show that it is impossible to find a similar chain of length greater than $m+1$ and conclude that the dimension of an irreducible variety is one less than the length of the longest strictly increasing chain of irreducible varieties contained in V.
18. Prove an affine version of part (ii) of Proposition 10.

§5 Dimension and Algebraic Independence

In §3, we defined the dimension of an affine variety as the degree of the affine Hilbert polynomial. This was useful for proving the properties of dimension in §4, but Hilbert polynomials do not give the full story. In algebraic geometry, there are many ways to formulate the concept of dimension and we will explore two of the more interesting approaches in this section and the next.

If $V \subset k^n$ is an affine variety, recall from Chapter 5 that the *coordinate ring* $k[V]$ consists of all polynomial functions on V. This is related to the ideal $\mathbf{I}(V)$ by the natural ring isomorphism $k[V] \cong k[x_1, \ldots, x_n]/\mathbf{I}(V)$ (which is the identity on k) discussed in Theorem 7 of Chapter 5, §2. To see what $k[V]$ has to do with dimension, note that for any $s \geq 0$, there is a well-defined linear map

$$(1) \qquad k[x_1, \ldots, x_n]_{\leq s}/\mathbf{I}(V)_{\leq s} \longrightarrow k[x_1, \ldots, x_n]/\mathbf{I}(V) \cong k[V]$$

which is one-to-one (see Exercise 10 of §4). Thus, we can regard $k[x_1, \ldots, x_n]_{\leq s}/\mathbf{I}(V)_{\leq s}$ as a finite-dimensional "piece" of $k[V]$ that approximates $k[V]$ more and more closely as s gets larger. Since the degree of ${}^aHP_{\mathbf{I}(V)}$ measures how fast these finite-dimensional approximations are growing, we see that $\dim V$ tells us something about the "size" of $k[V]$.

This discussion suggests that we should be able to formulate the dimension of V directly in terms of the ring $k[V]$. To do this, we will use the notion of algebraically independent elements.

Definition 1. *We say that elements $\phi_1, \ldots, \phi_r \in k[V]$ are* **algebraically independent over** *k if there is no nonzero polynomial p of r variables with coefficients in k such that $p(\phi_1, \ldots, \phi_r) = 0$ in $k[V]$.*

Note that if $\phi_1, \ldots, \phi_r \in k[V]$ are algebraically independent over k, then the ϕ_i's are distinct and nonzero. It is also easy to see that any subset of $\{\phi_1, \ldots, \phi_r\}$ is also algebraically independent over k (see Exercise 1 for the details).

The simplest example of algebraically independent elements occurs when $V = k^n$. If k is an infinite field, we have $\mathbf{I}(V) = \{0\}$ and, hence, $k[V] = k[x_1, \ldots, x_n]$. Here, the elements $x_1, \ldots, x_n$ are algebraically independent over k since $p(x_1, \ldots, x_n) = 0$ means that p is the zero polynomial.

For another example, let V be the twisted cubic in $\mathbb{R}^3$, so that $\mathbf{I}(V) = \langle y - x^2, z - x^3 \rangle$. Let us show that $[x] \in \mathbb{R}[V]$ is algebraically independent over $\mathbb{R}$. Suppose p is a polynomial with coefficients in $\mathbb{R}$ such that $p([x]) = [0]$ in $\mathbb{R}[V]$. By the way we defined the ring operations in $\mathbb{R}[V]$, this means $[p(x)] = [0]$, so that $p(x) \in \mathbf{I}(V)$. But it is easy to show that $\mathbb{R}[x] \cap \langle y - x^2, z - x^3 \rangle = \{0\}$, which proves that p is the zero polynomial. On the other hand, we leave it to the reader to verify that $[x], [y] \in \mathbb{R}[V]$ are not algebraically independent over $\mathbb{R}$ since $[y] - [x]^2 = [0]$ in $\mathbb{R}[V]$.

We can relate the dimension of V to the number of algebraically independent elements in the coordinate ring $k[V]$ as follows.

Theorem 2. *Let $V \subset k^n$ be an affine variety. Then the dimension of V equals the maximal number of elements of $k[V]$ which are algebraically independent over k.*

Proof. We will first show that if $d = \dim V$, then we can find d elements of $k[V]$ which are algebraically independent over k. To do this, let $I = \mathbf{I}(V)$ and consider the ideal of leading terms $\langle \mathrm{LT}(I) \rangle$ for some graded order on $k[x_1, \dots, x_n]$. By Theorem 8 of §3, we know that d is the maximum dimension of a coordinate subspace contained in $\mathbf{V}(\langle \mathrm{LT}(I) \rangle)$. A coordinate subspace $W \subset \mathbf{V}(\langle \mathrm{LT}(I) \rangle)$ of dimension d is defined by the vanishing of $n - d$ coordinates, so that we can write $W = \mathbf{V}(x_j : j \notin \{i_1, \dots, i_d\})$ for some $1 \le i_1 < \cdots < i_d \le n$. We will show that $[x_{i_1}], \dots, [x_{id}] \in k[V]$ are algebraically independent over k.

If we let $p \in k^n$ be the point whose i_j-th coordinate is 1 for $1 \le j \le d$ and whose other coordinates are 0, then $p \in W \subset \mathbf{V}(\langle \mathrm{LT}(I) \rangle)$. Then every monomial in $\langle \mathrm{LT}(I) \rangle$ vanishes at p and, hence, no monomial in $\langle \mathrm{LT}(I) \rangle$ can involve only $x_{i_1}, \dots, x_{i_d}$ (this is closely related to the proof of Proposition 2 of §2). Since $\langle \mathrm{LT}(I) \rangle$ is a monomial ideal, this implies that $\langle \mathrm{LT}(I) \rangle \cap k[x_{i_1}, \dots, x_{i_d}] = \{0\}$. Then

$$I \cap k[x_{i_1}, \dots, x_{i_d}] = \{0\} \tag{2}$$

since a nonzero element $f \in I \cap k[x_{i_1}, \dots, x_{i_d}]$ would give the nonzero element $\mathrm{LT}(f) \in \langle \mathrm{LT}(I) \rangle \cap k[x_{i_1}, \dots, x_{i_d}]$.

We can now prove that $[x_{i_1}], \dots, [x_{i_d}] \in k[V]$ are algebraically independent over k. Let p be a polynomial with coefficients in k such that $p([x_{i_1}], \dots, [x_{i_d}]) = [0]$. Then $[p(x_{i_1}, \dots, x_{i_d})] = [0]$ in $k[V]$, which shows that $p(x_{i_1}, \dots, x_{i_d}) \in I$. By (2), it follows that $p(x_{i_1}, \dots, x_{i_d}) = 0$, and since $x_{i_1}, \dots, x_{i_d}$ are variables, we see that p is the zero polynomial. Since $d = \dim V$, we have found the desired number of algebraically independent elements.

The final step in the proof is to show that if r elements of $k[V]$ are algebraically independent over k, then $r \le \dim V$. So assume that $[f_1], \dots, [f_r] \in k[V]$ are algebraically independent. Let N be the largest of the total degrees of $f_1, \dots, f_r$ and let $y_1, \dots, y_r$ be new variables. If $p \in k[y_1, \dots, y_r]$ is a polynomial of total degree $\le s$, then it is easy to check that the polynomial $p(f_1, \dots, f_r) \in k[x_1, \dots, x_n]$ has total degree $\le Ns$ (see Exercise 2). Then consider the map

$$\alpha : k[y_1, \dots, y_r]_{\le s} \longrightarrow k[x_1, \dots, x_n]_{\le Ns}/I_{\le Ns} \tag{3}$$

which sends $p(y_1, \dots, y_r) \in k[y_1, \dots, y_r]_{\le s}$ to the coset $[p(f_1, \dots, f_r)] \in k[x_1, \dots, x_n]_{\le Ns}/I_{\le Ns}$. We leave it as an exercise to show that α is a well-defined linear map.

We claim that α is one-to-one. To see why, suppose that $p \in k[y_1, \dots, y_r]_{\le s}$ and $[p(f_1, \dots, f_r)] = [0]$ in $k[x_1, \dots, x_n]_{\le Ns}/I_{\le Ns}$. Using the map (1), it follows that

$$[p(f_1, \dots, f_r)] = p([f_1], \dots, [f_r]) = [0] \quad \text{in } k[x_1, \dots, x_n]/I \cong k[V].$$

Since $[f_1], \dots, [f_r]$ are algebraically independent and p has coefficients in k, it follows that p must be the zero polynomial. Hence, α is one-to-one.

Comparing dimensions in (3), we see that

$$(4) \qquad {}^aHF_I(Ns) = \dim k[x_1, \ldots, x_n]_{\leq Ns}/(I_{\leq Ns}) \geq \dim k[y_1, \ldots, y_r]_{\leq s}.$$

Since $y_1, \ldots, y_r$ are variables, Lemma 4 of §2 shows that the dimension of $k[y_1, \ldots, y_r]_{\leq s}$ is $\binom{r+s}{s}$, which is a polynomial of degree r in s. In terms of the affine Hilbert polynomial, this implies

$${}^aHP_1(Ns) \geq \binom{r+s}{s} = \text{a polynomial of degree } r \text{ in } s$$

for s sufficiently large. It follows that ${}^aHP_I(Ns)$ and, hence, ${}^aHP_I(s)$ must have degree at least r. Thus, $r \leq \dim V$, which completes the proof of the theorem. □

As an application, we can show that isomorphic varieties have the same dimension.

Corollary 3. *Let V and V' be affine varieties which are isomorphic (as defined in Chapter 5, §4). Then* $\dim V = \dim V'$.

Proof. By Theorem 9 of Chapter 5, §4 we know V and V' are isomorphic if and only if there is a ring isomorphism $\alpha : k[V] \to k[V']$ which is the identity on k. Then elements $\phi_1, \ldots, \phi_r \in k[V]$ are algebraically independent over k if and only if $\alpha(\phi_1), \ldots, \alpha(\phi_r) \in k[V']$ are. We leave the easy proof of this assertion as an exercise. From here, the corollary follows immediately from Theorem 2. □

In the proof of Theorem 2, note that the $d = \dim V$ algebraically independent elements we found in $k[V]$ came from the coordinates. We can use this to give another formulation of dimension.

Corollary 4. *Let $V \subset k^n$ be an affine variety. Then the dimension of V is equal to the largest integer r for which there exist r variables $x_{i_1}, \ldots, x_{i_r}$ such that $\mathbf{I}(V) \cap k[x_{i_1}, \ldots, x_{i_r}] = \{0\}$ [that is, such that $\mathbf{I}(V)$ does not contain any polynomial in these variables which is not identically zero].*

Proof. First, from (2), it follows that we can find $d = \dim V$ such variables. Suppose that we could find $d+1$ variables, $x_{j_1}, \ldots, x_{j_{d+1}}$ such that $I \cap k[x_{j_1}, \ldots, x_{j_{d+1}}] = \{0\}$. Then the argument following (2) would imply that $[x_{j_1}], \ldots, [x_{j_{d+1}}] \in k[V]$ were algebraically independent over k. Since $d = \dim V$, this is impossible by Theorem 2. □

In the exercises, you will show that if k is algebraically closed, then Corollary 4 remains true if we replace $\mathbf{I}(V)$ with *any* defining ideal I of V. Since we know how to compute $I \cap k[x_{i_1}, \ldots, x_{i_r}]$ by elimination theory, Corollary 4 then gives us an alternative method (though not an efficient one) for computing the dimension of a variety.

We can also interpret Corollary 4 in terms of projections. If we choose r variables $x_{i_1}, \ldots, x_{i_r}$, then we get the projection map $\pi : k^n \to k^r$ defined by $\pi(a_1, \ldots, a_n) = (a_{i_1}, \ldots, a_{i_r})$. Also, let $\tilde{I} = \mathbf{I}(V) \cap k[x_{i_1}, \ldots, x_{i_r}]$ be the appropriate elimination ideal. If k is algebraically closed, then the Closure Theorem from §2 of Chapter 3 shows that

$\mathbf{V}(\tilde{I}) \cap k^r$ is the smallest variety containing the projection $\pi(V)$. It follows that

$$\tilde{I} = \{0\} \Longleftrightarrow \mathbf{V}(\tilde{I}) = k^r$$
$$\Longleftrightarrow \text{the smallest variety containing } \pi(V) \text{ is } k^r.$$

In general, a subset of k^r is said to be *Zariski dense* if the smallest variety containing it is k^r. Thus, Corollary 4 shows that the dimension of V is the largest dimension of a coordinate subspace for which the projection of V is Zariski dense in the subspace.

We can regard the above map π as a linear map from k^n to itself which leaves the i_j-th coordinate unchanged for $1 \leq j \leq r$ and sends the other coordinates to 0. It is then easy to show that $\pi \circ \pi = \pi$ and that the image of π is $k^r \subset k^n$ (see Exercise 8). More generally, a linear map $\pi : k^n \to k^n$ is called a *projection* if $\pi \circ \pi = \pi$. If π has rank r, then the image of π is an r-dimensional subspace H of k^n, and we say that π is a *projection onto* H.

Now let π be a projection onto a subspace $H \subset k^n$. Under π, any variety $V \subset k^n$ gives a subset $\pi(V) \subset H$. Then we can interpret the dimension of V in terms of its projections $\pi(V)$ as follows.

Proposition 5. *Let k be an algebraically closed field and let $V \subset k^n$ be an affine variety. Then the dimension of V is the largest dimension of a subspace $H \subset k^n$ for which a projection of V onto H is Zariski dense.*

Proof. If V has dimension d, then by the above remarks, we can find a projection of V onto a d-dimensional coordinate subspace which has Zariski dense image.

Now let $\pi : k^n \to k^n$ be an arbitrary projection onto an r-dimensional subspace H of k^n. We need to show that $r \leq \dim V$ whenever $\pi(V)$ is Zariski dense in H. From linear algebra, we can find a basis of k^n so that in the new coordinate system, $\pi(a_1, \ldots, a_n) = (a_1, \ldots, a_r)$ [see, for example, section 3.4 of FINKBEINER (1978)]. Since changing coordinates does not affect the dimension (this follows from Corollary 3 since a coordinate change gives isomorphic varieties), we are reduced to the case of a projection onto a coordinate subspace, and then the proposition follows from the above remarks. □

Let π be a projection of k^n onto a subspace H of dimension r. By the Closure Theorem from Chapter 3, §2 we know that if $\pi(V)$ is Zariski dense in H, then we can find a proper variety $W \subset H$ such that $H - W \subset (V)$. Thus, $\pi(V)$ "fills up" most of the r-dimensional subspace H, and hence, it makes sense that this should force V to have dimension at least r. So Proposition 5 gives a very geometric way of thinking about the dimension of a variety.

For the final part of the section, we will assume that V is an *irreducible* variety. By Proposition 4 of Chapter 5, §1, we know that $\mathbf{I}(V)$ is a prime ideal and that $k[V]$ is an integral domain. As in §5 of Chapter 5, we can then form the field of fractions of $k[V]$, which is the *field of rational functions* on V and is denoted $k(V)$. For elements of $k(V)$, the definition of algebraic independence over k is the same as that given for elements of $k[V]$ in Definition 1. We can relate the dimension of V to $k(V)$ as follows.

Theorem 6. *Let $V \subset k^n$ be an irreducible affine variety. Then the dimension of V equals the maximal number of elements of $k(V)$ which are algebraically independent over k.*

Proof. Let $d = \dim V$. Since $k[V] \subset k(V)$, any d elements of $k[V]$ which are algebraically independent over k will have the same property when regarded as elements of $k(V)$. So it remains to show that if $\phi_1, \ldots, \phi_r \in k(V)$ are algebraically independent, then $r \leq \dim V$. Each ϕ_i is a quotient of elements of $k[V]$, and if we pick a common denominator f, then we can write $\phi_i = [f_i]/[f]$ for $1 \leq i \leq r$. Note also that $[f] \neq [0]$ in $k[V]$. We need to modify the proof of Theorem 2 to take the denominator f into account.

Let N be the largest of the total degrees of $f, f_1, \ldots, f_r$, and let $y_1, \ldots, y_r$ be new variables. If $p \in k[y_1, \ldots, y_r]$ is a polynomial of total degree $\leq s$, then we leave it as an exercise to show that

$$f^s p(f_1/f, \ldots, f_r/f)$$

is a polynomial in $k[x_1, \ldots, x_n]$ of total degree $\leq Ns$ (see Exercise 10). Then consider the map

$$(5) \qquad \beta : k[y_1, \ldots, y_r]_{\leq s} \longrightarrow k[x_1, \ldots, x_n]_{\leq Ns}/I_{\leq Ns}$$

sending a polynomial $p(y_1, \ldots, y_r) \in k[y_1, \ldots, y_r]_{\leq s}$ to $[f^s p(f_1/f, \ldots, f_r/f)] \in k[x_1, \ldots, x_n]_{\leq Ns}/I_{\leq Ns}$. We leave it as an exercise to show that β is a well-defined linear map.

To show that β is one-to-one, suppose that $p \in k[y_1, \ldots, y_r]_{\leq s}$ and that $[f^s p(f_1/f, \ldots, f_r/f)] = [0]$ in $k[x_1, \ldots, x_n]_{\leq Ns}/I_{\leq Ns}$. Using the map (1), it follows that

$$[f^s p(f_1/f, \ldots, f_r/f)] = [0] \quad \text{in } k[x_1, \ldots, x_n]/I \cong k[V].$$

However, if we work in $k(V)$, then we can write this as

$$[f]^s p([f_1]/[f], \ldots, [f_r]/[f]) = [f]^s p(\phi_1, \ldots, \phi_r) = [0] \quad \text{in } k(V).$$

Since $k(V)$ is a field and $[f] \neq [0]$, it follows that $p(\phi_1, \ldots, \phi_r) = [0]$. Then p must be the zero polynomial since $\phi_1, \ldots, \phi_r$ are algebraically independent and p has coefficients in k. Thus, β is one-to-one.

Once we know that β is one-to-one in (5), we get the the inequality (4), and from here, the proof of Theorem 2 shows that $\dim V \geq r$. This proves the theorem. □

As a corollary of this theorem, we can prove that birationally equivalent varieties have the same dimension.

Corollary 7. *Let V and V' be irreducible affine varieties which are birationally equivalent (as defined in Chapter 5, §5). Then* $\dim V = \dim V'$.

Proof. In Theorem 10 of Chapter 5, §5, we showed that two irreducible affine varieties V and V' are birationally equivalent if and only if there is an isomorphism $k(V) \cong k(V')$

of their function fields which is the identity on k. The remainder of the proof is identical to what we did in Corollary 3. □

In field theory, there is a concept of *transcendence degree* which is closely related to what we have been studying. In general, when we have a field K containing k, we have the following definition.

Definition 8. *Let K be a field containing k. Then we say that K has* **transcendence degree** d **over** k *provided that d is the largest number of elements of K which are algebraically independent over k.*

If we combine this definition with Theorem 6, then for any irreducible affine variety V, we have

$$\dim V = \text{the transcendence degree of } k(V) \text{ over } k.$$

Many books on algebraic geometry use this as the definition of the dimension of an irreducible variety. The dimension of an arbitrary variety is then defined to be the maximum of the dimensions of its irreducible components.

For an example of transcendence degree, suppose that k is infinite, so that $k(V) = k(x_1, \ldots, x_n)$ when $V = k^n$. Since k^n has dimension n, we conclude that the field $k(x_1, \ldots, x_n)$ has transcendence degree n over k. It is clear that the transcendence degree is at least n, but it is less obvious that no $n + 1$ elements of $k(x_1, \ldots, x_n)$ can be algebraically independent over k. So our study of dimension yields some insights into the structure of fields.

To fully understand transcendence degree, one needs to study more about algebraic and transcendental field extensions. A good reference is Chapters VII and X of LANG (1965).

EXERCISES FOR §5

1. Let $\phi_1, \ldots, \phi_r \in k[V]$ be algebraically independent over k.
   a. Prove that the ϕ_i are distinct and nonzero.
   b. Prove that any nonempty subset of $\{\phi_1, \ldots, \phi_r\}$ consists of algebraically independent elements over k.
   c. Let $y_1, \ldots, y_r$ be variables and consider the map $\alpha : k[y_1, \ldots., y_r] \to k[V]$ defined by $\alpha(p) = p(\phi_1, \ldots, \phi_r)$. Show that α is a one-to-one ring homomorphism.
2. This exercise is concerned with the proof of Theorem 2.
   a. If $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ have total degree $\leq N$ and $p \in k[x_1, \ldots, x_n]$ has total degree $\leq s$, show that $p(f_1, \ldots, f_r)$ has total degree $\leq Ns$.
   b. Show that the map α defined in the proof of Theorem 2 is a well-defined linear map.
3. Complete the proof of Corollary 3.
4. Let k be an algebraically closed field and let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Show that the dimension of $\mathbf{V}(I)$ is equal to the largest integer r for which there exist r variables $x_{i_1}, \ldots, x_{i_r}$ such that $I \cap k[x_{i_1}, \ldots, x_{i_r}] = \{0\}$. Hint: Use I rather than $\mathbf{I}(V)$ in the proof of Theorem 2. Be sure to explain why $\dim V = \deg {}^a HP_I$.

5. Let $I = \langle xy - 1\rangle \subset k[x, y]$. What is the projection of $\mathbf{V}(I)$ to the x-axis and to the y-axis? Note that $\mathbf{V}(I)$ projects densely to both axes, but in neither case is the projection the whole axis.
6. Let k be infinite and let $I = \langle xy, xz\rangle \subset k[x, y, z]$.
   a. Show that $I \cap k[x] = 0$, but that $I \cap k[x, y]$ and $I \cap k[x, z]$ are not equal to 0.
   b. Show that $I \cap k[y, z] = 0$, but that $I \cap k[x, y, z] \neq 0$.
   c. What do you conclude about the dimension of $\mathbf{V}(I)$?
7. Here is a more complicated example of the phenomenon exhibited in Exercise 6. Again, assume that k is infinite and let $I = \langle zx - x^2, zy - xy\rangle \subset k[x, y, z]$.
   a. Show that $I \cap k[z] = 0$. Is either $I \cap k[x, z]$ or $I \cap k[y, z]$ equal to 0?
   b. Show that $I \cap k[x, y] = 0$, but that $I \cap k[x, y, z] \neq 0$.
   c. What does part (b) say about $\dim \mathbf{V}(I)$?
8. Given $1 \leq i_1 < \cdots < i_r \leq n$, define a linear map $\pi : k^n \to k^n$ by letting $\pi(a_1, \ldots, a_n)$ be the vector whose i_jth coordinate is a_{i_j} for $1 \leq j \leq r$ and whose other coordinates are 0. Show that $\pi \circ \pi = \pi$ and determine the image of π.
9. In this exercise, we will show that there can be more than one projection onto a given subspace $H \subset k^n$.
   a. Show that the matrices
   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
   both define projections from $\mathbb{R}^2$ onto the x-axis. Draw a picture that illustrates what happens to a typical point of $\mathbb{R}^2$ under each projection.
   b. Show that there is a one-to-one correspondence between projections of $\mathbb{R}^2$ onto the x-axis and nonhorizontal lines in $\mathbb{R}^2$ through the origin.
   c. More generally, fix an r-dimensional subspace $H \subset k^n$. Show that there is a one-to-one correspondence between projections of k^n onto H and $(n - r)$-dimensional subspaces $H' \subset k^n$ which satisfy $H \cap H' = \{0\}$. Hint: Consider the kernel of the projection.
10. This exercise is concerned with the proof of Theorem 6.
    a. If $f, f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ have total degree $\leq N$ and $p \in k[x_1, \ldots, x_n]$ has total degree $\leq s$, show that $f^s p(f_1/f, \ldots, f_r/f)$ is a polynomial in $k[x_1, \ldots, x_n]$.
    b. Show that the polynomial of part (a) has total degree $\leq Ns$.
    c. Show that the map β defined in the proof of Theorem 6 is a well-defined linear map.
11. Complete the proof of Corollary 7.
12. Suppose that $\phi : V \to W$ is a polynomial map between affine varieties (see Chapter 5, §1). We proved in §4 of Chapter 5 that ϕ induces a ring homomorphism $\phi^* : k[W] \to k[V]$ which is the identity on k. From ϕ, we get the subset $\phi(V) \subset W$. We say that ϕ is *dominating* if the smallest variety of W containing $\phi(V)$ is W itself. Thus, ϕ is dominating if its image is Zariski dense in W.
    a. Show that ϕ is dominating if and only if the homomorphism $\phi^* : k[W] \to k[V]$ is one-to-one. Hint: Show that $W' \subset W$ is a proper subvariety if and only if there is nonzero element $[f] \in k[W]$ such that $W' \subset W \cap \mathbf{V}(f)$.
    b. If ϕ is dominating, show that $\dim V \geq \dim W$. Hint: Use Theorem 2 and part (a).
13. This exercise will study the relation between parametrizations and dimension. Assume that k is an infinite field.
    a. Suppose that $F : k^m \to V$ is a polynomial parametrization of a variety V (as defined in Chapter 3, §3). Thus, m is the number of parameters and V is the smallest variety containing $F(k^m)$. Prove that $m \geq \dim V$.
    b. Give an example of a polynomial parametrization $F : k^m \to V$ where $m > \dim V$.

c. Now suppose that $F : k^m - W \to V$ is a rational parametrization of V (as defined in Chapter 3, §3). We know that V is irreducible by Proposition 6 of Chapter 4, §5. Show that we can define a field homomorphism $F^* : k(V) \to k(t_1, \ldots, t_m)$ which is one-to-one. Hint: See the proof of Theorem 10 of Chapter 5, §5.
d. If $F : k^m - W \to V$ is a rational parametrization, show that $m \geq \dim V$.

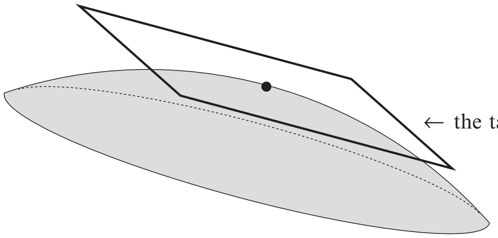
14. In this exercise, we will show how to define the field of rational functions on an irreducible projective variety $V \subset \mathbb{P}^n(k)$. If we take a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$, then f does *not* give a well-defined function on V. To see why, let $p \in V$ have homogeneous coordinates $(a_0, \ldots, a_n)$. Then $(\lambda a_0, \ldots, \lambda a_n)$ are also homogeneous coordinates for p, and
$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n),$$
where d is the total degree of f.
a. Explain why the above equation makes it impossible for us to define $f(p)$ as a single-valued function on V.
b. If $g \in k[x_0, \ldots, x_n]$ also has total degree d and $g \notin \mathbf{I}(V)$, then show that $\phi = f/g$ is a well-defined function on the nonempty set $V - V \cap \mathbf{V}(g) \subset V$.
c. We say that $\phi = f/g$ and $\phi' = f'/g'$ are equivalent on V, written $\phi \sim \phi'$, provided that there is a proper variety $W \subset V$ such that $\phi = \phi'$ on $V - W$. Prove that $\sim$ is an equivalence relation. An equivalence class for $\sim$ is called a *rational function* on V, and the set of all equivalence classes is denoted $k(V)$. Hint: Your proof will use the fact that V is irreducible.
d. Show that addition and multiplication of equivalence classes is well-defined and makes $k(V)$ into a field. We call $k(V)$ the *field of rational functions* of the projective variety V.
e. If U_i is the affine part of $\mathbb{P}^n(k)$ where $x_i = 1$, then we get an irreducible affine variety $V \cap U_i \subset U_i \cong k^n$. If $V \cap U_i \neq \emptyset$, show that $k(V)$ is isomorphic to the field $k(V \cap U_i)$ of rational functions on the affine variety $V \cap U_i$. Hint: You can assume $i = 0$. What do you get when you set $x_0 = 1$ in the quotient f/g considered in part (b)?

15. Suppose that $V \subset \mathbb{P}^n(k)$ is irreducible and let $k(V)$ be its rational function field as defined in Exercise 14.
a. Prove that $\dim V$ is the transcendence degree of $k(V)$ over k. Hint: Reduce to the affine case.
b. We say that two irreducible projective varieties V and V' (lying possibly in different projective spaces) are *birationally equivalent* if any of their affine portions $V \cap U_i$ and $V' \cap U_j$ are birationally equivalent in the sense of Chapter 5, §5. Prove that V and V' are birationally equivalent if and only if there is a field isomorphism $k(V) \cong k(V')$ which is the identity on k. Hint: Use the previous exercise and Theorem 10 of Chapter 5, §5.
c. Prove that birationally equivalent projective varieties have the same dimension.

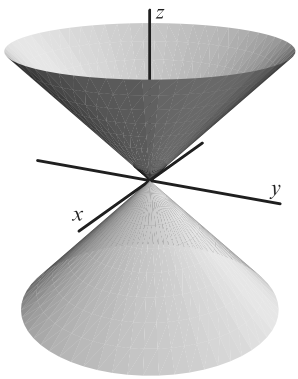
§6 Dimension and Nonsingularity

This section will explore how dimension is related to the geometric properties of a variety V. The discussion will be rather different from §5, where the algebraic properties of $k[V]$ and $k(V)$ played a dominant role. We will introduce some rather sophisticated concepts, and some of the theorems will be proved only in special cases. For convenience, we will assume that V is always an affine variety.

When we look at a surface $V \subset \mathbb{R}^3$, one intuitive reason for saying that it is 2-dimensional is that at a point p on V, a small portion of the surface looks like a small portion of the plane. This is reflected by the way the tangent plane approximates V at p:



Of course, we have to be careful because the surface may have points where there does not seem to be a tangent plane. For example, consider the cone $\mathbf{V}(x^2 + y^2 - z^2)$. There seems to be a nice tangent plane everywhere except at the origin:



In this section, we will introduce the concept of a *nonsingular* point p of a variety V, and we will give a careful definition of the *tangent space* $T_p(V)$ of V at p. Our discussion will generalize what we did for curves in §4 of Chapter 3. The tangent space gives useful information about how the variety V behaves near the point p. This is the so-called "local viewpoint." Although we have not discussed this topic previously, it plays an important role in algebraic geometry. In general, properties which reflect the behavior of a variety near a given point are called *local properties.*

We begin with a discussion of the tangent space. For a curve V defined by an equation $f(x, y) = 0$ in $\mathbb{R}^2$, we saw in Chapter 3 that the line tangent to the curve at a point $(a, b) \in V$ is defined by the equation

$$\frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b) = 0,$$

provided that the two partial derivatives do not vanish (see Exercise 4 of Chapter 3, §4). We can generalize this to an arbitrary variety as follows.

Definition 1. *Let $V \subset k^n$ be an affine variety and let $p = (p_1, \ldots, p_n) \in V$ be a point.*

(i) *If $f \in k[x_1, \ldots, x_n]$ is a polynomial, the* **linear part** *of f at p, denoted $d_p(f)$, is defined to be the polynomial*

$$d_p(f) = \frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n).$$

Note that $d_p(f)$ has total degree ≤ 1.

(ii) *The* **tangent space** *of V at p, denoted $T_p(V)$, is the variety*

$$T_p(V) = \mathbf{V}(d_p(f) : f \in \mathbf{I}(V)).$$

If we are working over $\mathbb{R}$, then the partial derivative $\frac{\partial f}{\partial x_i}$ has the usual meaning. For other fields, we use the *formal partial derivative*, which is defined by

$$\frac{\partial}{\partial x_i}\left(\sum_{\alpha_1,\ldots,\alpha_n} c_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1} \ldots x_i^{\alpha_i} \ldots x_n^{\alpha_n}\right) = \sum_{\alpha_1,\ldots,\alpha_n} c_{\alpha_1\ldots\alpha_n} \alpha_i x_1^{\alpha_1} \ldots x_i^{\alpha_i - 1} \ldots x_n^{\alpha_n}.$$

In Exercise 1, you will show that the usual rules of differentiation apply to $\frac{\partial}{\partial x_i}$. We first prove some simple properties of $T_p(V)$.

Proposition 2. *Let $p \in V \subset k^n$.*

(i) *If $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$ then $T_p(V) = \mathbf{V}(d_p(f_1), \ldots, d_p(f_s))$.*

(ii) *$T_p(V)$ is the translate of a linear subspace of k^n.*

Proof. (i) By the product rule, it is easy to show that

$$d_p(hf) = h(p) \cdot d_p(f) + d_p(h) \cdot f(p)$$

(see Exercise 2). This implies $d_p(hf) = h(p) \cdot d_p(f)$ when $f(p) = 0$, and it follows that if $g = \Sigma_{i=1}^{s} h_i f_i \in \mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$, then

$$d_p(g) = \sum_{i=1}^{s} d_p(h_i f_i) = \sum_{i=1}^{s} h_i(p) \cdot d_p(f_i) \in \langle d_p(f_1), \ldots, d_p(f_s) \rangle.$$

This shows that $T_p(V)$ is defined by the vanishing of the $d_p(f_i)$.

(ii) Introduce new coordinates on k^n by setting $X_i = x_i - p_i$ for $1 \leq i \leq n$. This coordinate system is obtained by translating p to the origin. By part (i), we know that $T_p(V)$ is given by $d_p(f_1) = \cdots = d_p(f_s) = 0$. Since each $d_p(f_i)$ is linear in $X_1, \ldots, X_n$, it follows that $T_p(V)$ is a linear subspace with respect to the X_i. In terms of the original coordinates, this means that $T_p(V)$ is the translate of a subspace of k^n. □

We can get an intuitive idea of what the tangent space means by thinking about *Taylor's formula* for a polynomial of several variables. For a polynomial of one variable, one has the standard formula

$$f(x) = f(a) + f'(a)(x - a) + \text{terms involving higher powers of } x - a.$$

For $f \in k[x_1, \dots, x_n]$, you will show in Exercise 3 that if $p = (p_1, \dots, p_n)$, then

$$f = f(p) + \frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n)$$
$$+ \text{ terms of total degree } \geq 2 \text{ in } x_1 - p_1, \dots, x_n - p_n.$$

This is part of Taylor's formula for f at p. When $p \in V$ and $f \in \mathbf{I}(V)$, we have $f(p) = 0$, so that

$$f = d_p(f) + \text{terms of total degree} \geq 2 \text{ in } x_1 - p_1, \dots, x_n - p_n.$$

Thus $d_p(f)$ is the best linear approximation of f near p. Now suppose that $\mathbf{I}(V) = \langle f_1, \dots, f_s \rangle$. Then V is defined by the vanishing of the f_i, so that the best linear approximation to V near p should be defined by the vanishing of the $d_p(f_i)$. By Proposition 2, this is exactly the tangent space $T_p(V)$.

We can also think about $T_p(V)$ in terms of lines that meet V with "higher multiplicity" at p. In Chapter 3, this was how we defined the tangent line for curves in the plane. In the higher dimensional case, suppose that we have $p \in V$ and let L be a line through p. We can parametrize L by $F(t) = p + tv$, where $v \in k^n$ is a vector parallel to L. If $f \in k[x_1, \dots, x_n]$, then $f \circ F(t)$ is a polynomial in the variable t, and note that $f \circ F(0) = f(p)$. Thus, 0 is a root of this polynomial whenever $f \in \mathbf{I}(V)$. We can use the multiplicity of this root to decide when L is contained in $T_p(V)$.

Proposition 3. *If L is a line through p parametrized by $F(t) = p + tv$, then $L \subset T_p(V)$ if and only if 0 is a root of multiplicity ≥ 2 of $f \circ F(t)$ for all $f \in \mathbf{I}(V)$.*

Proof. If we write the parametrization of L as $x_i = p_i + tv_i$ for $1 \leq i \leq n$, where $p = (p_1, \dots, p_n)$ and $v = (v_1, \dots, v_n)$, then, for any $f \in \mathbf{I}(V)$, we have

$$g(t) = f \circ F(t) = f(p_1 + v_1 t, \dots, p_n + tv_n).$$

As we noted above, $g(0) = 0$ because $p \in V$, so that $t = 0$ is a root of $g(t)$. In Exercise 5 of Chapter 3, §4, we showed that $t = 0$ is a root of multiplicity ≥ 2 if and only if we also have $g'(0) = 0$. Using the chain rule for functions of several variables, we obtain

$$\frac{dg}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = \frac{\partial f}{\partial x_1}v_1 + \cdots + \frac{\partial f}{\partial x_n}v_n.$$

If follows that that $g'(0) = 0$ if and only if

$$0 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)v_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)((p_i + v_i) - p_i).$$

The expression on the right in this equation is $d_p(f)$ evaluated at the point $p + v \in L$, and it follows that $p + v \in T_p(V)$ if and only if $g'(0) = 0$ for all $f \in \mathbf{I}(V)$. Since $p \in L$, we know that $L \subset T_p(V)$ is equivalent to $p + v \in T_p(V)$, and the proposition is proved. □

It is time to look at some examples.

Example 4. Let $V \subset \mathbb{C}^n$ be the hypersurface defined by $f = 0$, where $f \in k[x_1, \ldots, x_n]$ is a nonconstant polynomial. By Proposition 9 of Chapter 4, §2, we have

$$\mathbf{I}(V) = \mathbf{I}(\mathbf{V}(f)) = \sqrt{\langle f \rangle} = \langle f_{red} \rangle,$$

where $f_{red} = f_1 \cdots f_r$ is the product of the distinct irreducible factors of f. We will assume that $f = f_{red}$. This implies that

$$V = \mathbf{V}(f) = \mathbf{V}(f_1 \cdots f_r) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_r)$$

is the decomposition of V into irreducible components (see Exercise 9 of Chapter 4, §6). In particular, every component of V has dimension $n - 1$ by the affine version of Proposition 2 of §4.

Since $\mathbf{I}(V) = \langle f \rangle$, it follows from Proposition 2 that for any $p \in V$, $T_p(V)$ is the linear space defined by the single equation

$$\frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n) = 0.$$

This implies that

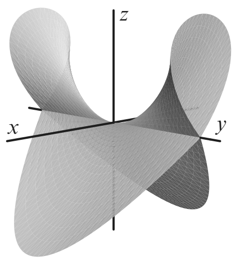
$$\text{(1)} \qquad \dim T_p(V) = \begin{cases} n-1 & \text{at least one } \frac{\partial f}{\partial x_i}(p) \neq 0 \\ n & \text{all } \frac{\partial f}{\partial x_i}(p) = 0. \end{cases}$$

You should check how this generalizes Proposition 2 of Chapter 3, §4.

For a specific example, consider $V = \mathbf{V}(x^2 - y^2z^2 + z^3)$. In Exercise 4, you will show that $f = x^2 - y^2z^2 + z^3 \in \mathbb{C}[x, y, z]$ is irreducible, so that $\mathbf{I}(V) = \langle f \rangle$. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2yz^2, \quad \frac{\partial f}{\partial z} = -2y^2z + 3z^2.$$

We leave it as an exercise to show that on V, the partials vanish simultaneously only on the y-axis, which lies in V. Thus, the tangent spaces $T_p(V)$ are all 2-dimensional, except along the y-axis, where they are all of $\mathbb{C}^3$. Over $\mathbb{R}$, we get the following picture of V (which appeared earlier in §2 of Chapter 1):



When we give the definition of nonsingular point later in this section, we will see that the points of V on the y-axis are the singular points, whereas other points of V are nonsingular.

Example 5. Now consider the curve $C \subset \mathbb{C}^3$ obtained by intersecting the surface V of Example 4 with the plane $x+y+z=0$. Thus, $C = V(x+y+z, x^2 - y^2z^2 + z^3)$. Using the techniques of §3, you can verify that $\dim C = 1$.

In the exercises, you will also show that $(f_1, f_2) = (x+y+z, x^2 - y^2z^2 + z^3)$ is a prime ideal, so that C is an irreducible curve. Since a prime ideal is radical, the Nullstellensatz implies that $I(C) = (f_1, f_2)$. Thus, for $p = (a, b, c) \in C$, it follows that $T_p(C)$ is defined by the linear equations

$$\begin{aligned} d_p(f_1) &= 1 \cdot (x-a) + 1 \cdot (y-b) + 1 \cdot (z-c) = 0, \\ d_p(f_2) &= 2a \cdot (x-a) + (-2bc^2) \cdot (y-b) + (-2b^2c + 3c^2) \cdot (z-c) = 0. \end{aligned}$$

This is a system of linear equations in $x-a$, $y-b$, $z-c$, and the matrix of coefficients is

$$J_p(f_1, f_2) = \begin{pmatrix} 1 & 1 & 1 \\ 2a & -2bc^2 & -2b^2c + 3c^2 \end{pmatrix}.$$

Let rank $(J_p(f_1, f_2))$ denote the rank of this matrix. Since $T_p(C)$ is a translate of the kernel of $J_p(f_1, f_2)$, it follows that

$$\dim T_p(C) = 3 - \operatorname{rank}(J_p(f_1, f_2)).$$

In the exercises, you will show that $T_p(C)$ is 1-dimensional at all points of C except for the origin, where $T_0(C)$ is the 2-dimensional plane $x+y+z=0$.

In these examples, we were careful to always compute $\mathbf{I}(V)$. It would be much nicer if we could use any set of defining equations of V. Unfortunately, this does not always work: if $V = \mathbf{V}(f_1, \ldots, f_s)$, then $T_p(V)$ need *not* be defined by $d_p(f_1) = \cdots = d_p(f_s) = 0$. For example, let V be the y-axis in k^2. Then V is defined by $x^2 = 0$, but you can easily check that $T_p(V) \neq \mathbf{V}(d_p(x^2))$ for all $p \in V$. However, in Theorem 9, we will find a nice condition on $f_1, \ldots, f_s$ which, when satisfied, will allow us to compute $T_p(V)$ using the $d_p(f_i)$'s.

Examples 4 and 5 indicate that the nicest points on V are the ones where $T_p(V)$ has the same dimension as V. But this principle does not apply when V has irreducible components of different dimensions. For example, let $V = V(xz, yz) \subset \mathbb{R}^3$. This is the union of the (x, y)-plane and the z-axis, and it is easy to check that

$$\dim T_p(V) = \begin{cases} 2 & p \text{ is on the } (x, y)\text{-plane minus the origin} \\ 1 & p \text{ is on the } z\text{-axis minus the origin} \\ 3 & p \text{ is the origin.} \end{cases}$$

Excluding the origin, points on the z-axis have a 1-dimensional tangent space, which seems intuitively correct. Yet at such a point, we have $\dim T_p(V) < \dim V$. The problem, of course, is that we are on a component of the wrong dimension.

To avoid this difficulty, we need to define the dimension of a variety at a point.

Definition 6. *Let V be an affine variety. For $p \in V$, the* **dimension of** V **at** p, *denoted* $\dim_p V$, *is the maximum dimension of an irreducible component of V containing p.*

By Corollary 9 of §4, we know that $\dim V$ is the maximum of $\dim_p V$ as p varies over all points of V. If V is a hypersurface in $\mathbb{C}^n$, it is easy to compute $\dim_p V$, for in Example 4, we showed that every irreducible component of V has dimension $n - 1$. It follows that $\dim_p V = n - 1$ for all $p \in V$. On the other hand, if $V \subset k^n$ is an arbitrary variety, the theory developed in §§3 and 4 enables us to compute $\dim V$, but unless we know how to decompose V into irreducible components, more subtle tools are needed to compute $\dim_p V$. This will be discussed in §7 when we study the properties of the tangent cone.

We can now define what it means for a point $p \in V$ to be nonsingular.

Definition 7. *Let p be a point on an affine variety V. Then p is* **nonsingular** *(or* **smooth***) provided* $\dim T_p(V) = \dim_p V$. *Otherwise, p is a* **singular** *point of V.*

If we look back at our previous examples, it is easy to identify which points are nonsingular and which are singular. In Example 5, the curve C is irreducible, so that $\dim_p C = 1$ for all $p \in C$ and, hence, the singular points are where $\dim T_p(C) \neq 1$ (only one in this case). For the hypersurfaces $V = \mathbf{V}(f)$ considered in Example 4, we know that $\dim_p V = n - 1$ for all $p \in V$, and it follows from (1) so that p is singular if and only if all of the partial derivatives of f vanish at p. This means that the singular points of V form the variety

$$\Sigma = \mathbf{V}\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right). \tag{2}$$

In general, the singular points of a variety V have the following properties.

Theorem 8. *Let $V \subset k^n$ be an affine variety and let*

$$\Sigma = \{p \in V : p \textit{ is a singular point of } V\}.$$

We call Σ the **singular locus** *of V. Then:*

(i) *Σ is an affine variety contained in V.*
(ii) *If $p \in \Sigma$, then* $\dim T_p(V) > \dim_p V$.
(iii) *Σ contains no irreducible component of V.*
(iv) *If V_i and V_j are distinct irreducible components of V, then $V_i \cap V_j \subset \Sigma$.*

Proof. A complete proof of this theorem is beyond the scope of the book. Instead, we will assume that V is a hypersurface in $\mathbb{C}^n$ and show that the theorem holds in this case. As we discuss each part of the theorem, we will give references for the general case.

Let $V = \mathbf{V}(f) \subset \mathbb{C}^n$ be a hypersurface such that $\mathbf{I}(V) = \langle f \rangle$. We noted earlier that $\dim_p V = n - 1$ and that Σ consists of those points of V where all of the partial derivatives of f vanish simultaneously. Then (2) shows that Σ is an affine variety, which proves (i) for hypersurfaces. A proof in the general case is given in the Corollary to Theorem 6 in Chapter II, §2 of SHAFAREVICH (1974).

Part (ii) of the theorem says that at a singular point of V, the tangent space is too big. When V is a hypersurface in $\mathbb{C}^n$, we know from (1) that if p is a singular point, then $\dim T_p(V) = n > n - 1 = \dim_p V$. This proves (ii) for hypersurfaces, and the general case follows from Theorem 3 in Chapter II, §1 of SHAFAREVICH (1974).

Part (iii) says that on each irreducible component of V, the singular locus consists of a proper subvariety. Hence, most points of a variety are nonsingular. To prove this for a hypersurface, let $V = \mathbf{V}(f) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_r)$ be the decomposition of V into irreducible components, as discussed in Example 4. Suppose that Σ contains one of the components, say $\mathbf{V}(f_1)$. Then every $\frac{\partial f}{\partial x_i}$ vanishes on $\mathbf{V}(f_1)$. If we write $f = f_1 g$, where $g = f_2 \cdots f_r$, then

$$\frac{\partial f}{\partial x_i} = f_1 \frac{\partial g}{\partial x_i} + \frac{\partial f_1}{\partial x_i} g$$

by the product rule. Since f_1 certainly vanishes on $\mathbf{V}(f_1)$, it follows that $\frac{\partial f_1}{\partial x_i} g$ also vanishes on $\mathbf{V}(f_1)$. By assumption, f_1 is irreducible, so that

$$\frac{\partial f_1}{\partial x_i} g \in \mathbf{I}(\mathbf{V}(f_1)) = \langle f_1 \rangle.$$

This says that f_1 divides $\frac{\partial f_1}{\partial x_i} g$ and, hence, f_1 divides $\frac{\partial f_1}{\partial x_i}$ or g. The latter is impossible since g is a product of irreducible polynomials distinct from f_1 (meaning that none of them is a constant multiple of f_1). Thus, f_1 must divide $\frac{\partial f_1}{\partial x_i}$ for all i. Since $\frac{\partial f_1}{\partial x_i}$ has smaller total degree than f_1, we must have $\frac{\partial f_1}{\partial x_i} = 0$ for all i, and it follows that f_1 is constant (see Exercise 9). This contradiction proves that Σ contains no component of V.

When V is an arbitrary irreducible variety, a proof that Σ is a proper subvariety can be found in the corollary to Theorems 4.1 and 4.3 in Chapter IV of KENDIG (1977). See also the discussion preceding the definition of singular point in Chapter II, §1 of SHAFAREVICH (1974). If V has two or more irreducible components, the claim follows from the irreducible case and part (iv) below. See Exercise 11 for the details.

Finally, part (iv) of the theorem says that a nonsingular point of a variety lies on a unique irreducible component. In the hypersurface case, suppose that $V = \mathbf{V}(f) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_r)$ and that $p \in \mathbf{V}(f_i) \cap \mathbf{V}(f_i)$ for $i \neq j$. Then we can write $f = gh$, where f_i divides g and f_j divides h. Hence, $g(p) = h(p) = 0$, and then an easy argument using the product rule shows that $\frac{\partial f}{\partial x_i}(p) = 0$ for all i. This proves that $\mathbf{V}(f_i) \cap \mathbf{V}(f_j) \subset \Sigma$, so that (iv) is true for hypersurfaces. When V is an arbitrary variety, see Theorem 6 in Chapter II, §2 of SHAFAREVICH (1974). □

In some cases, it is also possible to show that a point of a variety V is nonsingular without having to compute $\mathbf{I}(V)$. To formulate a precise result, we will need some notation. Given $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, let $J(f_1, \ldots, f_r)$ be the $r \times n$ matrix of partial derivatives

$$J(f_1, \ldots, f_r) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_r}{\partial x_1} & \cdots & \dfrac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

Given $p \in k^n$, evaluating this matrix at p gives an $r \times n$ matrix of numbers denoted $J_p(f_1, \ldots, f_r)$. Then we have the following result.

Theorem 9. *Let $V = \mathbf{V}(f_1, \ldots, f_r) \subset \mathbb{C}^n$ be an arbitrary variety and suppose that $p \in V$ is a point where $J_p(f_1, \ldots, f_r)$ has rank r. Then p is a nonsingular point of V and lies on a unique irreducible component of V of dimension $n - r$.*

Proof. As with Theorem 8, we will only prove this for a hypersurface $V = \mathbf{V}(f) \subset \mathbb{C}^n$, which is the case $r = 1$ of the theorem. Here, note that f is now *any* defining equation of V, and, in particular, it could happen that $\mathbf{I}(V) \neq \langle f \rangle$. But we still know that f vanishes on V, and it follows from the definition of tangent space that

$$(3) \qquad T_p(V) \subset \mathbf{V}(d_p(f)).$$

Since $r = 1$, $J_p(f)$ is the row vector whose entries are $\frac{\partial f}{\partial x_i}(p)$, and our assumption that $J_p(f)$ has rank 1 implies that at least one of the partials is nonzero at p. Thus, $d_p(f)$ is a nonzero linear function of $x_i - p_i$, and it follows from (3) that $\dim T_p(V) \leq n - 1$. If we compare this to (1), we see that p is a nonsingular point of V, and by part (iv) of Theorem 8, it lies on a unique irreducible component of V. Since the component has the predicted dimension $n - r = n - 1$, we are done. For the general case, see Theorem (1.16) of MUMFORD (1976). □

Theorem 9 is important for several reasons. First of all, it is very useful for determining the nonsingular points and dimension of a variety. For instance, it is now possible to redo Examples 4 and 5 without having to compute $\mathbf{I}(V)$ and $\mathbf{I}(C)$. Another aspect of Theorem 9 is that it relates nicely to our intuition that the dimension should drop by one for each equation defining V. This is what happens in the theorem, and in fact we can sharpen our intuition as follows. Namely, the dimension should drop by one for each defining equation, provided the defining equations are sufficiently independent [which means that $\mathrm{rank}(J_p(f_1, \ldots, f_r)) = r$]. In Exercise 16, we see a more precise way to state this. Furthermore, note that our intuition applies to the nonsingular part of V.

Theorem 9 is also related to some important ideas from advanced courses in the calculus of several variables. In particular, the *Implicit Function Theorem* has the same hypothesis concerning $J_p(f_1, \ldots, f_r)$ as Theorem 9. When $V = \mathbf{V}(f_1, \ldots, f_r)$ satisfies this hypothesis, the complex variable version of the Implicit Function Theorem asserts that near p, the variety V looks like the graph of a nice function, and we get a vivid picture of why V has dimension $n - r$ at p. To understand the full meaning of Theorem 9, one needs to study the notion of a *manifold.* A nice discussion of this topic and its relation to nonsingularity and dimension can be found in KENDIG (1977).

EXERCISES FOR §6

1. We will discuss the properties of the formal derivative defined in the text.
   a. Show that $\frac{\partial}{\partial x_i}$ is k-linear and satisfies the product rule.
   b. Show that $\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} f\right)$ for all i and j.
   c. If $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, compute $\frac{\partial}{\partial x_i}(f_1^{\alpha_1} \cdots f_r^{\alpha_r})$.
   d. Formulate and prove a version of the chain rule for computing the partial derivatives of a polynomial of the form $F(f_1, \ldots, f_r)$. Hint: Use part (c).
2. Prove that $d_p(hf) = h(p) \cdot d_p(f) + d_p(h) \cdot f(p)$.
3. Let $p = (p_1, \ldots, p_n) \in k^n$ and let $f \in k[x_1, \ldots, x_n]$.
   a. Show that f can be written as a polynomial in $x_i - p_i$. Hint: $x_i^m = ((x_i - p_i) + p_i)^m$.
   b. Suppose that when we write f as a polynomial in $x_i - p_i$, every term has total degree at least 2. Show that $\frac{\partial f}{\partial x_i}(p) = 0$ for all i.
   c. If we write f as a polynomial in $x_i - p_i$, show that the constant term is $f(p)$ and the linear term is $d_p(f)$. Hint: Use part (b).
4. As in Example 4, let $f = x^2 - y^2z^2 + z^3 \in \mathbb{C}[x, y, z]$ and let $V = \mathbf{V}(f) \subset \mathbb{C}^3$.
   a. Show carefully that f is irreducible in $\mathbb{C}[x, y, z]$.
   b. Show that V contains the y-axis.
   c. Let $p \in V$. Show that the partial derivatives of f all vanish at p if and only if p lies on the y-axis.
5. Let A be an $m \times n$ matrix, where $n \geq m$. If $r \leq m$, we say that a matrix B is an $r \times r$ *submatrix* of A provided that B is the matrix obtained by first selecting r columns of A, and then selecting r rows from those columns.
   a. Pick a 3×4 matrix of numbers and write down all of its 3×3 and 2×2 submatrices.
   b. Show that A has rank $< r$ if and only if all $t \times t$ submatrices of A have determinant zero for all $r \leq t \leq m$. Hint: The rank of a matrix is the maximum number of linearly independent columns. If A has rank s, it follows that you can find an $m \times s$ submatrix of rank s. Now use the fact that the rank is also the maximum number of linearly independent rows. What is the criterion for an $r \times r$ matrix to have rank $< r$?
6. As in Example 5, let $C = \mathbf{V}(x + y + z, x^2 - y^2z^2 + z^3) \subset \mathbb{C}^3$ and let I be the ideal $I = \langle x + y + z, x^2 - y^2z^2 + z^3\rangle \subset \mathbb{C}[x, y, z]$.
   a. Show that I is a prime ideal. Hint: Introduce new coordinates $X = x + y + z$, $Y = y$, and $Z = z$. Show that $I = \langle X, F(Y, Z)\rangle$ for some polynomial in Y, Z. Prove that $\mathbb{C}[X, Y, Z]/I \cong \mathbb{C}[Y, Z]/\langle F\rangle$ and show that $F \in \mathbb{C}[Y, Z]$ is irreducible.
   b. Conclude that C is an irreducible variety and that $\mathbf{I}(C) = I$.
   c. Compute the dimension of C.
   d. Determine all points $(a, b, c) \in C$ such that the 2×3 matrix

   $$J_p(f_1, f_2) = \begin{pmatrix} 1 & 1 & 1 \\ 2a & -2bc^2 & -2b^2c + 3c^2 \end{pmatrix}$$

   has rank < 2. Hint: Use Exercise 5.
7. Let $f = x^2 \in k[x, y]$. In k^2, show that $T_p(\mathbf{V}(f)) \neq \mathbf{V}(d_p(f))$ for all $p \in V$.
8. Let $V = \mathbf{V}(xy, xz) \subset k^3$ and assume that k is an infinite field.
   a. Compute $\mathbf{I}(V)$.
   b. Verify the formula for $\dim T_p(V)$ given in the text.
9. Suppose that $f \in k[x_1, \ldots, x_n]$ is a polynomial such that $\frac{\partial}{\partial x_i} f = 0$ for all i. If k has characteristic 0 (which means that k contains a field isomorphic to $\mathbb{Q}$), then show that f must be the constant.

10. The result of Exercise 9 may be false if k does not have characteristic 0.
   a. Let $f = x^2 + y^2 \in \mathbb{F}_2[x, y]$, where $\mathbb{F}_2$ is a field with two elements. What are the partial derivatives of f?
   b. To analyze the case when k does not have characteristic 0, we need to define the *characteristic* of k. Given any field k, show that there is a ring homomorphism $\phi : \mathbb{Z} \to k$ which sends $n > 0$ in $\mathbb{Z}$ to $1 \in k$ added to itself n times. If ϕ is one-to-one, argue that k contains a copy of $\mathbb{Q}$ and hence has characteristic 0.
   c. If k does not have characteristic 0, it follows that the map ϕ of part (b) cannot be one-to-one. Show that the kernel must be the ideal $\langle p \rangle \subset \mathbb{Z}$ for some prime number p. We say that k has *characteristic* p in this case. Hint: Use the Isomorphism Theorem from Exercise 16 of Chapter 5, §2 and remember that k is an integral domain.
   d. If k has characteristic p, show that $(a + b)^p = a^p + b^p$ for every $a, b \in k$.
   e. Suppose that k has characteristic p and let $f \in k[x_1, \ldots, x_n]$. Show that all partial derivatives of f vanish identically if and only if every exponent of every monomial appearing in f is divisible by p.
   f. Suppose that k is algebraically closed and has characteristic p. If $f \in k[x_1, \ldots, x_n]$ is irreducible, then show that some partial derivative of f must be nonzero. This shows that Theorem 8 is true for hypersurfaces over *any* algebraically closed field. Hint: If all partial derivatives vanish, use parts (d) and (e) to write f as *a* p-th power. Why do you need k to be algebraically closed?
11. Let $V = V_1 \cup \cdots \cup V_r$ be a decomposition of a variety into its irreducible components.
   a. Suppose that $p \in V$ lies in a unique irreducible component V_i. Show that $T_p(V) = T_p(V_i)$. This reflects the local nature of the tangent space. Hint: One inclusion follows easily from $V_i \subset V$. For the other inclusion, pick a function $f \in \mathbf{I}(W) - \mathbf{I}(V_i)$, where W is the union of the other irreducible components. Then $g \in \mathbf{I}(V_i)$ implies $fg \in \mathbf{I}(V)$.
   b. With the same hypothesis as part (a), show that p is nonsingular in V if and only if it is nonsingular in V_i.
   c. Let Σ be the singular locus of V and let Σ_i be the singular locus of V_i. Prove that

$$\Sigma = \bigcap_{i \neq j} (V_i \cap V_j) \cup \bigcup_i \Sigma_i.$$

   Hint: Use part (b) and part (iv) of Theorem 8.
   d. If each Σ_i is a proper subset of V_i, then show that Σ contains no irreducible components of V. This shows that part (iii) of Theorem 8 follows from the irreducible case.
12. Find all singular points of the following curves in k^2. Assume that k is algebraically closed.
   a. $y^2 = x^3 - 3$.
   b. $y^2 = x^3 - 6x^2 + 9x$.
   c. $x^2y^2 + x^2 + y^2 + 2xy(x + y + 1) = 0$.
   d. $x^2 = x^4 + y^4$.
   e. $xy = x^6 + y^6$.
   f. $x^2y + xy^2 = x^4 + y^4$.
   g. $x^3 = y^2 + x^4 + y^4$.
13. Find all singular points of the following surfaces in k^3. Assume that k is algebraically closed.
   a. $xy^2 = z^2$.
   b. $x^2 + y^2 = z^2$.
   c. $x^2y + x^3 + y^3 = 0$.
   d. $x^3 - zxy + y^3 = 0$.
14. Show that $\mathbf{V}(y - x^2 + z^2, 4x - y^2 + w^3) \subset \mathbb{C}^4$ is a nonempty smooth surface.

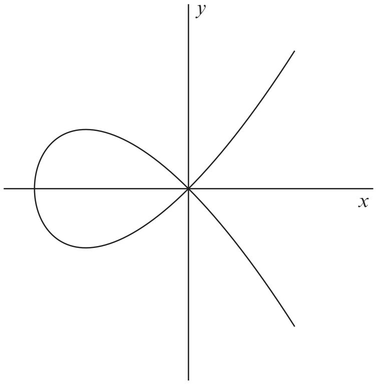
15. Let $V \subset k^n$ be a hypersurface with $\mathbf{I}(V) = \langle f \rangle$. Show that if V is not a hyperplane and $p \in V$ is nonsingular, then either the variety $V \cap T_p(V)$ has a singular point at p or the restriction of f to $T_p(V)$ has an irreducible factor of multiplicity ≥ 2. Hint: Pick coordinates so that $p = 0$ and $T_p(V)$ is defined by $x_1 = 0$. Thus, we can regard $T_p(V)$ as a copy of k^{n-1}, then $V \cap T_p(V)$ is a hypersurface in k^{n-1}. Then the restriction of f to $T_p(V)$ is the polynomial $f(0, x_2, \ldots, x_n)$. See also Example 4.
16. Let $V \subset \mathbb{C}^n$ be irreducible and let $p \in V$ be a nonsingular point. Suppose that V has dimension d.
    a. Show that we can find polynomials $f_1, \ldots, f_{n-d} \in \mathbf{I}(V)$ such that $T_p(V) = V(d_p(f_1), \ldots, d_p(f_{n-d}))$.
    b. If $f_1, \ldots, f_{n-d}$ are as in part (a) show that $J_p(f_1, \ldots, f_{n-d})$ has rank $n - d$ and conclude that V is an irreducible component of $V(f_1, \ldots, f_{n-d})$. This shows that although V itself may not be defined by $n - d$ equations, it is a component of a variety that is. Hint: Use Theorem 9.
17. Suppose that $V \subset \mathbb{C}^n$ is irreducible of dimension d and suppose that $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$.
    a. Show that $p \in V$ is nonsingular if and only if $J_p(f_1, \ldots, f_s)$ has rank $n - d$. Hint: Use Proposition 2.
    b. By Theorem 8, we know that V has nonsingular points. Use this and part (a) to prove that $d \geq n - s$. How does this relate to Proposition 5 of §4?
    c. Let $\mathcal{D}$ be the set of determinants of all $(n - d) \times (n - d)$ submatrices of $J(f_1, \ldots, f_s)$. Prove that the singular locus of V is $\Sigma = V \cap \mathbf{V}(g : g \in \mathcal{D})$. Hint: Use part (a) and Exercise 5. Also, what does part (ii) of Theorem 8 tell you about the rank of $J_p(f_1, \ldots, f_s)$?

§7 The Tangent Cone

In this final section of the book, we will study the *tangent cone* of a variety V at a point p. When p is nonsingular, we know that, near p, V is nicely approximated by its tangent space $T_p(V)$. This clearly fails when p is singular, for as we saw in Theorem 8 of §6, the tangent space has the wrong dimension (it is too big). To approximate V near a singular point, we need something different.

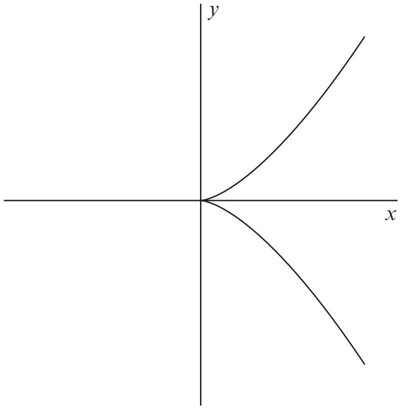
We begin with an example.

Example 1. Consider the curve $y^2 = x^2(x + 1)$, which has the following picture in the plane $\mathbb{R}^2$:



We see that the origin is a singular point. Near this point, the curve is approximated by the lines $x = \pm y$. These lines are defined by $x^2 - y^2 = 0$, and if we write the defining equation of the curve as $f(x, y) = x^2 - y^2 + x^3 = 0$, we see that $x^2 - y^2$ is the nonzero homogeneous component of f of smallest total degree.

Similarly, consider the curve $y^2 - x^3 = 0$:



The origin is again a singular point, and the nonzero homogeneous component of $y^2 - x^3$ of smallest total degree is y^2. Here, $\mathbf{V}(y^2)$ is the x-axis and gives a nice approximation of the curve near $(0, 0)$.

In both of the above curves, we approximated the curve near the singular point using the smallest nonzero homogeneous component of the defining equation. To generalize this idea, suppose that $p = (p_1, \ldots, p_n) \in k^n$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ let

$$(x - p)^\alpha = (x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^{\alpha_n}$$

and note that $(x - p)^\alpha$ has total degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Now, given any polynomial $f \in k[x_1, \ldots, x_n]$ of total degree d, we can write f as a polynomial in $x_i - p_i$, so that f is a k-linear combination of $(x - p)^\alpha$ for $|\alpha| \leq d$. If we group according to total degree, we can write

$$(1) \qquad f = f_{p,0} + f_{p,1} + \cdots + f_{p,d},$$

where $f_{p.j}$ is a k-linear combination of $(x - p)^\alpha$ for $|\alpha| = j$. Note that $f_{p,0} = f(p)$ and $f_{p,1} = d_p(f)$ (as defined in Definition 1 of the previous section). In the exercises, you will discuss Taylor's formula, which shows how to express $f_{p,j}$ in terms of the partial derivatives of f at p. In many situations, it is convenient to translate p to the origin so that we can use homogeneous components. We can now define the tangent cone.

Definition 2. *Let $V \subset k^n$ be an affine variety and let $p = (p_1, \ldots, p_n) \in V$.*

(i) *If $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial, then $f_{p.min}$ is defined to be $f_{p,j}$, where j is the smallest integer such that $f_{p,j} \neq 0$ in* (1).

(ii) *The* **tangent cone** *of V at p, denoted $C_p(V)$, is the variety*

$$C_p(V) = \mathbf{V}(f_{p,min} : f \in \mathbf{I}(V)).$$

The tangent cone gets its name from the following proposition.

Proposition 3. *Let $p \in V \subset k^n$. Then $C_p(V)$ is the translate of the affine cone of a variety in $\mathbb{P}^{n-1}(k)$.*

Proof. Introduce new coordinates on k^n by letting $X_i = x_i - p_i$. Relative to this coordinate system, we can assume that p is the origin 0. Then $f_{0,min}$ is a homogeneous polynomial in $X_1, \ldots, X_n$, and as f varies over $\mathbf{I}(V)$, the $f_{0,min}$ generate a homogeneous ideal $J \subset k[X_1, \ldots, X_n]$. Then $C_p(V) = \mathbf{V}_a(J) \subset k^n$ by definition. Since J is homogeneous, we also get a projective variety $W = \mathbf{V}_p(J) \subset \mathbb{P}^{n-1}(k)$, and as we saw in Chapter 8, this means that $C_p(V)$ is an affine cone $C_W \subset k^n$ of W. This proves the proposition. $\square$

The tangent cone of a hypersurface $V \subset k^n$ is especially easy to compute. In Exercise 2 you will show that if $\mathbf{I}(V) = \langle f \rangle$, then $C_p(V)$ is defined by the single equation $f_{p,min} = 0$. This is exactly what we did in Example 1. However, when $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$ has more generators, it need *not* follow that $C_p(V) = \mathbf{V}((f_1)_{p,min}, \ldots, (f_s)_{p,min})$. For example, suppose that V is defined by $xy = xz + z(y^2 - z^2) = 0$. In Exercise 3, you will show that $\mathbf{I}(V) = \langle xy, xz + z(y^2 - z^2) \rangle$. To see that $C_0(V) \neq \mathbf{V}(xy, xz)$, note that $f = yz(y^2 - z^2) = y(xz + z(y^2 - z^2)) - z(xy) \in \mathbf{I}(V)$. Then $f_{0,min} = yz(y^2 - z^2)$ vanishes on $C_0(V)$, yet does not vanish on all of $\mathbf{V}(xy, xz)$.

We can overcome this difficulty by using an appropriate Groebner basis. The result is stated most efficiently when the point p is the origin.

Proposition 4. *Assume that the origin 0 is a point of $V \subset k^n$. Let x_0 be a new variable and pick a monomial order on $k[x_0, x_1, \ldots, x_n]$ such that among monomials of the same total degree, any monomial involving x_0 is greater than any monomial involving only $x_1, \ldots, x_n$ (lex and grlex with $x_0 > \cdots > x_n$ satisfy this condition).*

(i) *Let $\mathbf{I}(V)^h \subset k[x_0, x_1, \ldots, x_n]$ be the homogenization of $\mathbf{I}(V)$ and let $G_1, \ldots, G_s$ be a Groebner basis of $\mathbf{I}(V)^h$ with respect to the above monomial order. Then*

$$C_0(V) = \mathbf{V}((g_1)_{0,min}, \ldots, (g_s)_{0,min}),$$

where $g_i = G_i(1, x_1, \ldots, x_n)$ is the dehomogenization of G_i.

(ii) *Suppose that k is algebraically closed, and let I be any ideal such that $V = \mathbf{V}(I)$. If $G_1, \ldots, G_s$ are a Groebner basis of I^h, then*

$$C_0(V) = \mathbf{V}((g_1)_{0,min}, \ldots, (g_s)_{0,min}),$$

where $g_i = G_i(1, x_1, \ldots, x_n)$ is the dehomogenization of G_i.

Proof. In this proof, we will write f_j and f_{min} rather than $f_{0,j}$ and $f_{0,min}$.

(i) Let $I = \mathbf{I}(V)$. It suffices to show that $f_{min} \in \langle (g_1)_{min}, \ldots, (g_s)_{min}\rangle$ for all $f \in I$. If this fails to hold, then we can find $f \in I$ with $f_{min} \notin \langle (g_1)_{min}, \ldots, (g_s)_{min}\rangle$ such that $\text{LT}(f_{min})$ is minimal [note that we can regard f_{min} as a polynomial in $k[x_0, x_1, \ldots, x_n]$, so that $\text{LT}(f_{min})$ is defined]. If we write f as a sum of homogeneous components

$$f = f_{min} + \cdots + f_d,$$

where d is the total degree of f, then

$$f^h = f_{min}x_0^a + \cdots + f_d \in I^h$$

for some a. By the way we chose the monomial order on $k[x_0, x_1, \ldots, x_n]$, it follows that $\text{LT}(f^h) = \text{LT}(f_{min})x_0^a$. Since $G_1, \ldots, G_s$ form a Groebner basis, we know that some $\text{LT}(G_i)$ divides $\text{LT}(f_{min})x_0^a$.

If g_i is the dehomogenization of G_i, it is easy to see that $g_i \in I$. We leave it as an exercise to show that

$$\text{LT}(G_i) = \text{LT}((g_i)_{min})x_0^b$$

for some b. This implies that $\text{LT}(f_{min}) = cx^\alpha \text{LT}((g_i)_{min})$ for some nonzero $c \in k$ and some monomial x^α in $x_1, \ldots, x_n$. Now consider $\tilde{f} = f - cx^\alpha g_i \in I$. Since $f_{min} \notin \langle (g_1)_{min}, \ldots, (g_s)_{min}\rangle$, we know that $f_{min} - cx^\alpha(g_i)_{min} \neq 0$, and it follows easily that

$$\tilde{f}_{min} = f_{min} - cx^\alpha(g_i)_{min}.$$

Then $\text{LT}(\tilde{f}_{min}) < \text{LT}(f_{min})$ since the leading terms of f_{min} and $cx^\alpha(g_i)_{min}$ are equal. This contradicts the minimality of $\text{LT}(f_{min})$, and (i) is proved. In the exercises, you will show that $g_1, \ldots, g_n$ are a basis of I, though not necessarily a Groebner basis.

(ii) Let W denote the variety $\mathbf{V}(f_{min} : f \in I)$. If we apply the argument of part (i) to the ideal I, we see immediately that

$$W = \mathbf{V}((g_1)_{min}, \ldots, (g_s)_{min}).$$

It remains to show that W is the tangent cone at the origin. Since $I \subset \mathbf{I}(V)$, the inclusion $C_0(V) \subset W$ is obvious by the definition of tangent cone. Going the other way, suppose that $g \in \mathbf{I}(V)$. We need to show that g_{min} vanishes on W. By the Nullstellensatz, we know that $g^m \in I$ for some m and, hence, $(g^m)_{min} = 0$ on W. In the exercises, you will check that $(g^m)_{min} = (g_{min})^m$ and it follows that g_{min} vanishes on W. This completes the proof of the proposition.

□

In practice, this proposition is usually used over an algebraically closed field, for here, part (ii) says that we can compute the tangent cone using *any* set of defining equations of the variety.

For an example of how to use Proposition 4, suppose $V = \mathbf{V}(xy, xz+z(y^2-z^2))$. If we set $I = \langle xy, xz+z(y^2-z^2)\rangle$, the first step is to determine $I^h \subset k[w, x, y, z]$, where w is the homogenizing variable. Using grlex order on $k[x, y, z]$, a Groebner basis for I is $\{xy, xz + z(y^2 - z^2), x^2z - xz^3\}$. By the theory developed in §4 of Chapter 8,

$\{xy, xzw + z(y^2 - z^2), x^2zw - xz^3\}$ is a basis of I^h. In fact, it is a Groebner basis for grlex order, with the variables ordered $x > y > z > w$ (see Exercise 5). However, this monomial order does not satisfy the hypothesis of Proposition 4, but if we use grlex with $w > x > y > z$, then a Groebner basis is

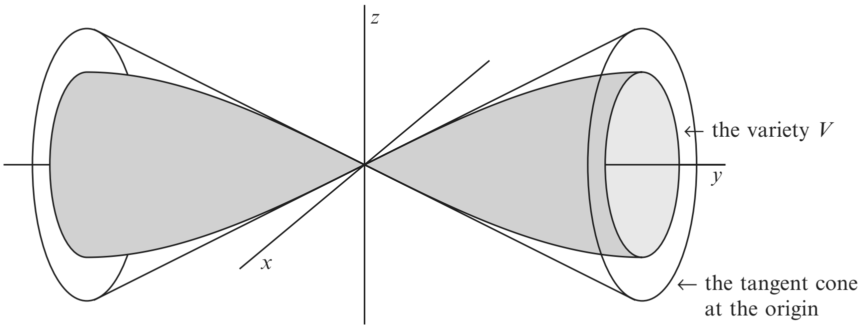
$$\{xy, xzw + z(y^2 - z^2), yz(y^2 - z^2)\}.$$

Proposition 4 shows that if we dehomogenize and take minimal homogeneous components, then the tangent cone at the origin is given by

$$C_0(V) = \mathbf{V}(xy, xz, yz(y^2 - z^2)).$$

In the exercises, you will show that this is a union of five lines through the origin in k^3.

We will next study how the tangent cone approximates the variety V near the point p. Recall from Proposition 3 that $C_p(V)$ is the translate of an affine cone, which means that $C_p(V)$ is made up of lines through p. So to understand the tangent cone, we need to describe which lines through p lie in $C_p(V)$. We will do this using secant lines. More precisely, let L be a line in k^n through p. Then L is a *secant line* of V if it meets V in a point distinct from p. Here is the crucial idea: if we take secant lines determined by points of V getting closer and closer to p, then the "limit" of the secant lines should lie on the tangent cone. You can see this in the following picture.

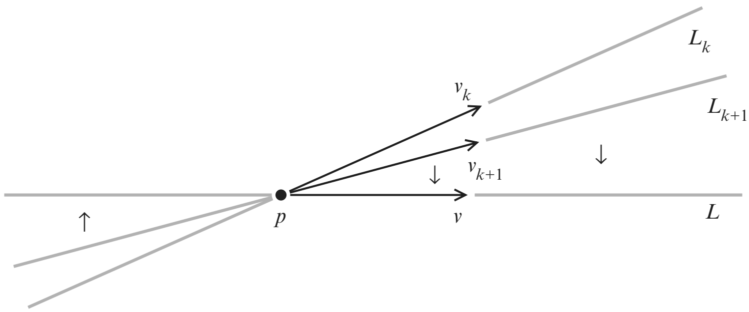


To make this idea precise, we will work over the complex numbers $\mathbb{C}$. Here, it is possible to define what it means for a sequence of points $q_k \in \mathbb{C}^n$ to converge to $q \in \mathbb{C}^n$. For instance, if we think of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, this means that the coordinates of q_k converge to the coordinates of q. We will assume that the reader has had some experience with sequences of this sort.

We will treat lines through their parametrizations. So suppose we have parametrized L via $p + tv$, where $v \in \mathbb{C}^n$ is a nonzero vector parallel to L and $t \in \mathbb{C}$. Then we define a limit of lines as follows.

Definition 5. *We say that a line $L \subset \mathbb{C}^n$ through a point $p \in \mathbb{C}^n$ is a* **limit of lines** *$\{L_k\}_{k=1}^{\infty}$ through p if given a parametrization $p + tv$ of L, there exist parametrizations $p + tv_k$ of L_k such that* $\lim_{k\to\infty} v_k = v$ *in $\mathbb{C}^n$.*

This notion of convergence corresponds to the following picture:



Now we can state a precise version of how the tangent cone approximates a complex variety near a point.

Theorem 6. *Let $V \subset \mathbb{C}^n$ be an affine variety. Then a line L through $p \in V$ lies in the tangent cone $C_p(V)$ if and only if there exists a sequence $\{q_k\}_{k=1}^{\infty}$ of points in $\mathbf{V} - \{p\}$ converging to p such that if L_k is the secant line containing p and q_k, then the lines L_k converge to the given line L.*

Proof. By translating p to the origin, we may assume that $p = 0$. Let $\{q_k\}$ be a sequence of points on V converging to the origin and suppose the lines L_k through 0 and q_k converge (in the sense of Definition 5) to some line L through the origin. We want to show that $L \subset C_0(V)$.

By the definition of L_k converging to L, we can find parametrizations tv_k of L_k (remember that $p = 0$) such that the v_k converge to v as $k \to \infty$. Since $q_k \in L_k$, we can write $q_k = t_k v_k$ for some complex number t_k. Note that $t_k \neq 0$ since $q_k \neq p$. We claim that the t_k converge to 0. This follows because as $k \to \infty$, we have $v_k \to v \neq 0$ and $t_k v_k = q_k \to 0$. (A more detailed argument will be given in Exercise 8.)

Now suppose that f is any polynomial that vanishes on V. As in the proof of Proposition 4, we write f_{min} and f_j rather than $f_{0,min}$ and $f_{0,j}$. If f has total degree d, then we can write $f = f_l + f_{l+1} + \cdots + f_d$, where $f_l = f_{min}$. Since $q_k = t_k u_k \in V$, we have

$$0 = f(t_k v_k) = f_l(t_k v_k) + \cdots + f_d(t_k v_k). \tag{2}$$

Each f_i is homogeneous of degree i, so that $f_i(t_k v_k) = t_k^i f_i(v_k)$. Thus,

$$0 = t_k^l f_l(v_k) + \cdots + t_k^d f_d(v_k). \tag{3}$$

Since $t_k \neq 0$, we can divide through by t_k^l to obtain

$$0 = f_l(v_k) + t_k f_{l+1}(v_k) + \cdots + t_k^{d-l} f_d(v_k). \tag{4}$$

Letting $k \to \infty$, the right-hand side in (4) tends to $f_l(v)$ since $v_k \to v$ and $t_k \to 0$. We conclude that $f_l(v) = 0$, and since $f_l(tv) = t^l f_l(v) = 0$ for all t, it follows that $L \subset C_0(V)$. This shows that $C_0(V)$ contains all limits of secant lines determined by sequences of points in V converging to 0.

To prove the converse, we will first study the set

$$(5) \qquad \mathcal{V} = \{(v, t) \in \mathbb{C}^n \times \mathbb{C} : tv \in V, t \neq 0\} \subset \mathbb{C}^{n+1}.$$

If $(v, t) \in \mathcal{V}$, note that the L determined by 0 and $tv \in V$ is a secant line. Thus, we want to know what happens to $\mathcal{V}$ as $t \to 0$. For this purpose, we will study the Zariski closure $\overline{\mathcal{V}}$ of $\mathcal{V}$, which is the smallest variety in $\mathbb{C}^{n+1}$ containing $\mathcal{V}$. We claim that

$$(6) \qquad \overline{\mathcal{V}} = \mathcal{V} \cup (C_0(V) \times \{0\}).$$

From §4 of Chapter 4, we know that $\overline{\mathcal{V}} = \mathbf{V}(\mathbf{I}(\mathcal{V}))$. So we need to calculate the functions that vanish on $\mathcal{V}$. If $f \in \mathbf{I}(V)$, write $f = f_l + \cdots + f_d$ where $f_l = f_{min}$, and set

$$\tilde{f} = f_l + t f_{l+1} + \cdots + t^{d-l} f_d \in \mathbb{C}[t, x_1, \ldots, x_n].$$

We will show that

$$(7) \qquad \mathbf{I}(\mathcal{V}) = \langle \tilde{f} : f \in \mathbf{I}(V) \rangle.$$

One direction of the proof is easy, for $f \in \mathbf{I}(V)$ and $(v, t) \in \mathcal{V}$ imply $f(tv) = 0$, and then equations (2), (3), and (4) show that $\tilde{f}(v, t) = 0$. Conversely, suppose that $g \in \mathbb{C}[t, x_1, \ldots, x_n]$ vanishes on $\mathcal{V}$. Write $g = \Sigma_i g_i t^i$, where $g_i \in \mathbb{C}[x_1, \ldots, x_n]$, and let $g_i = \Sigma_j g_{ij}$ be the decomposition of g_i into the sum of its homogeneous components. If $(v, t) \in \mathcal{V}$, then for every $\lambda \in \mathbb{C} - \{0\}$, we have $(\lambda v, \lambda^{-1} t) \in \mathcal{V}$ since $(\lambda^{-1} t) \cdot (\lambda v) = tv \in V$. Thus,

$$0 = g(\lambda v, \lambda^{-1} t) = \sum_{i,j} g_{ij}(\lambda v)(\lambda^{-1} t)^i = \sum_{i,j} \lambda^j g_{ij}(v) \lambda^{-i} t^i = \sum_{i,j} \lambda^{j-i} g_{ij}(v) t^i$$

for all $\lambda \neq 0$. Letting $m = j - i$, we can organize this sum according to powers of λ:

$$0 = \sum_m \left(\sum_i g_{i,m+i}(v) t^i \right) \lambda^m.$$

Since this holds for all $\lambda \neq 0$, it follows that $\Sigma_i g_{i,m+i}(v) t^i = 0$ for all m and, hence, $\Sigma_i g_{i,m+i} t^i \in \mathbf{I}(\mathcal{V})$. Let $f_m = \Sigma_i g_{i,m+i} \in \mathbb{C}[x_1, \ldots, x_n]$. Since $(v, 1) \in \mathcal{V}$ for all $v \in V$, it follows that $f_m \in \mathbf{I}(V)$. If we let i_0 be the smallest i such that $g_{i,m+i} \neq 0$, then

$$\tilde{f}_m = g_{i_0, m+i_0} + g_{i_0+1, m+i_0+1} t + \cdots,$$

so that $\Sigma_i g_{i,m+i} t^i = t^{i_0} \tilde{f}_m$. From this, it follows immediately that $g \in \langle \tilde{f} : f \in \mathbf{I}(V) \rangle$, and (7) is proved.

From (7), we have $\overline{\mathcal{V}} = \mathbf{V}(\tilde{f} : f \in \mathbf{I}(V))$. To compute this variety, let $(v, t) \in \mathbb{C}^{n+1}$, and first suppose that $t \neq 0$. Using (2), (3), and (4), it follows easily that $\tilde{f}(v, t) = 0$ if and only if $f(tv) = 0$. Thus,

$$\overline{\mathcal{V}} \cap \{(v,t) : t \neq 0\} = \mathcal{V}.$$

Now suppose $t = 0$. If $f = f_{min} + \cdots + f_d$, it follows from the definition of $\tilde{f}$ that $\tilde{f}(v,0) = 0$ if and only if $f_{min}(v) = 0$. Hence,

$$\overline{\mathcal{V}} \cap \{(v,t) : t = 0\} = C_0(V) \times \{0\},$$

and (6) is proved.

To complete the proof of Theorem 6, we will need the following fact about Zariski closure.

Proposition 7. *Let $Z \subset W \subset \mathbb{C}^n$ be affine varieties and assume that W is the Zariski closure of $W - Z$. If $z \in Z$ is any point, then there is a sequence of points $\{w_k \in W - Z\}_{k=1}^{\infty}$ which converges to z.*

Proof. The proof of this is beyond the scope of the book. In Theorem (2.33) of MUMFORD (1976), this result is proved for irreducible varieties in $\mathbb{P}^n(\mathbb{C})$. Exercise 9 will show how to deduce Proposition 7 from Mumford's theorem. □

To apply this proposition to our situation, let $Z = C_0(V) \times \{0\} \subset W = \overline{\mathcal{V}}$. By (6), we see that $W - Z = \overline{\mathcal{V}} - C_0(V) \times \{0\} = \mathcal{V}$ and, hence, $W = \overline{\mathcal{V}}$ is the Zariski closure of $W - Z$. Then the proposition implies that any point in $Z = C_0(V) \times \{0\}$ is a limit of points in $W - Z = \mathcal{V}$.

We can now finish the proof of Theorem 6. Suppose a line L parametrized by tv is contained in $C_0(V)$. Then $v \in C_0(V)$, which implies that $(v,0) \in C_0(V) \times \{0\}$. By the above paragraph, we can find points $(v_k, t_k) \in \mathcal{V}$ which converge to $(v,0)$. If we let L_k be the line parametrized by tv_k, then $v_k \to v$ shows that $L_k \to L$. Furthermore, since $q_k = t_k v_k \in V$ and $t_k \neq 0$, we see that L_k is the secant line determined by $q_k \in V$. Finally, as $k \to \infty$, we have $q_k = t_k \cdot v_k \to 0 \cdot v = 0$, which shows that L is a limit of secant lines of points $q_k \in V$ converging to 0. This completes the proof of the theorem. □

If we are working over an infinite field k, we may not be able to define what it means for secant lines to converge to a line in the tangent cone. So it is not clear what the analogue of Theorem 6 should be. But if $p = 0$ is in V over k, we can still form the set $\mathcal{V}$ as in (5), and every secant line still gives a point $(v,t) \in \mathcal{V}$ with $t \neq 0$. A purely algebraic way to discuss limits of secant lines as $t \to 0$ would be to take the smallest variety containing $\mathcal{V}$ and see what happens when $t = 0$. This means looking at $\overline{\mathcal{V}} \cap (k^n \times \{0\})$, which by (6) is exactly $C_0(V) \times \{0\}$. You should check that the proof of (6) is valid over k, so that the decomposition

$$\overline{\mathcal{V}} = \mathcal{V} \cup (C_0(V) \times \{0\})$$

can be regarded as the extension of Theorem 6 to the infinite field k. In Exercise 10, we will explore some other interesting aspects of the variety $\overline{\mathcal{V}}$.

Another way in which the tangent cone approximates the variety is in terms of dimension. Recall from §6 that $\dim_p V$ is the maximum dimension of an irreducible component of V containing p.

Theorem 8. *Let p be a point on an affine variety $V \subset k^n$. Then* $\dim_p V = \dim C_p(V)$.

Proof. This is a standard result in advanced courses in commutative algebra [see, for example, Theorem 13.9 in MATSUMURA (1986)]. As in §6, we will only prove this for the case of a hypersurface in $\mathbb{C}^n$. If $V = \mathbf{V}(f)$, we know that $C_p(V) = \mathbf{V}(f_{p,min})$ by Exercise 2. Thus, both V and $C_p(V)$ are hypersurfaces, and, hence, both have dimension $n-1$ at all points. This shows that $\dim_p V = \dim C_p(V)$. □

This is a nice result because it enables us to compute $\dim_p V$ without having to decompose V into its irreducible components.

The final topic of this section will be the relation between the tangent cone and the tangent space. In the exercises, you will show that for any point p of a variety V, we have

$$C_p(V) \subset T_p(V).$$

In terms of dimensions, this implies that

$$\dim C_p(V) \leq \dim T_p(V).$$

Then the following corollary of Theorem 8 tells us when these coincide.

Corollary 9. *Assume that k is algebraically closed and let p be a point of a variety $V \subset k^n$. Then the following are equivalent:*

(i) *p is a nonsingular point of V.*
(ii) $\dim C_p(V) = \dim T_p(V)$.
(iii) $C_p(V) = T_p(V)$.

Proof. Since $\dim C_p(V) = \dim_p V$ by Theorem 8, the equivalence of (i) and (ii) is immediate from the definition of a nonsingular point. The implication (iii) ⇒ (ii) is trivial, so that it remains to prove (ii) ⇒ (iii).

Since k is algebraically closed, we know that k is infinite, which implies that the linear space $T_p(V)$ is an irreducible variety in k^n. [When $T_p(V)$ is a coordinate subspace, this follows from Exercise 7 of §1. See Exercise 12 below for the general case.] Thus, if $C_p(V)$ has the same dimension $T_p(V)$, the equality $C_p(V) = T_p(V)$ follows immediately from the affine version of Proposition 10 of §4 (see Exercise 18 of §4). □

If we combine Theorem 6 and Corollary 9, it follows that at a nonsingular point p of a variety $V \subset \mathbb{C}^n$, the tangent space at p is the union of all limits of secant lines determined by sequences of points in V converging to p. This is a powerful generalization of the idea from elementary calculus that the tangent line to a curve is a limit of secant lines.

EXERCISES FOR §7

1. Suppose that k is a field of characteristic 0. Given $p \in k^n$ and $f \in k[x_1, \ldots, x_n]$, we know that f can be written in the form $f = \Sigma_\alpha c_\alpha (x - p)^\alpha$, where $c_\alpha \in k$ and $(x - p)^\alpha$ is as in the text. Given α, define
$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$
where $\frac{\partial^{\alpha_i}}{\partial^{\alpha_i} x_i}$ means differentiation α_i times with respect to x_i. Finally, set
$$\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!.$$
   a. Show that
$$\frac{\partial^\alpha (x - p)^\beta}{\partial x^\alpha}(p) = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$
   Hint: There are two cases to consider: when $\beta_i < \alpha_i$, for some i and when $\beta_i \geq \alpha_i$ for all i.
   b. If $f = \Sigma_\alpha c_\alpha (x - p)^\alpha$, then show that
$$c_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p),$$
   and conclude that
$$f = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p)(x - p)^\alpha.$$
   This is *Taylor's formula* for f at p. Hint: Be sure to explain where you use the characteristic 0 assumption.
   c. Write out the formula of part (b) explicitly when $f \in k[x, y]$ has total degree 3.
   d. What formula do we get for $f_{p,j}$ in terms of the partial derivatives of f?
   e. Give an example to show that over a finite field, it may be impossible to express f in terms of its partial derivatives. Hint: See Exercise 10 of §6.
2. Let $V \subset k^n$ be a hypersurface.
   a. If $\mathbf{I}(V) = \langle f \rangle$, prove that $C_P(V) = \mathbf{V}(f_{p,min})$.
   b. If k is algebraically closed and $V = \mathbf{V}(f)$, prove that the conclusion of part (a) is still true. Hint: See the proof of part (ii) of Proposition 4.
3. In this exercise, we will show that the ideal $I = \langle xy, xz + z(y^2 - z^2) \rangle \subset k[x, y, z]$ is a radical ideal when k has characteristic 0.
   a. Show that
$$\langle x, z(y^2 - z^2) \rangle = \langle x, z \rangle \cap \langle x, y - z \rangle \cap \langle x, y + z \rangle.$$
   Furthermore, show that the three ideals on the right-hand side of the equation are prime. Hint: Work in $k[x, y, z]/\langle x \rangle \cong k[y, z]$ and use the fact that $k[y, z]$ has unique factorization. Also explain why this result fails if k is the field $\mathbb{F}_2$ consisting of two elements.
   b. Show that
$$\langle y, xz - z^3 \rangle = \langle y, z \rangle \cap \langle y, x - z^2 \rangle,$$
   and show that the two ideals on the right-hand side of the equation are prime.

c. Prove that $I = \langle x, z(y^2-z^2)\rangle \cap \langle y, xz-z^3\rangle$. Hint: One way is to use the ideal intersection algorithm from Chapter 4, §3. There is also an elementary argument.
d. By parts (a), (b) and (c) we see that I is an intersection of five prime ideals. Show that I is a radical ideal. Also, use this decomposition of I to describe $V = \mathbf{V}(I) \subset k^3$.
e. If k is algebraically closed, what is $\mathbf{I}(V)$?

4. This exercise is concerned with the proof of Proposition 4. Fix a monomial order $>$ on $k[x_0, \ldots, x_n]$ with the properties described in the statement of the proposition.
a. If $g \in k[x_1, \ldots, x_n]$ is the dehomogenization of $G \in k[x_0, \ldots, x_n]$, prove that $\mathrm{LT}(G) = \mathrm{LT}(g_{min})x_0^b$ for some b.
b. If $G_1, \ldots, G_s$ is a basis of I^h, prove that the dehomogenizations $g_1, \ldots, g_s$ form a basis of I. In Exercise 5, you will show that if the G_i's are a Groebner basis for $>$, the g_i's may fail to be a Groebner basis for I with respect to the monomial induced order on $k[x_1, \ldots, x_n]$.
c. If $f, g \in k[x_1, \ldots, x_n]$, show that $(f \cdot g)_{min} = f_{min} \cdot g_{min}$. Conclude that $(f^m)_{min} = (f_{min})^m$.

5. We will continue our study of the variety $V = \mathbf{V}(xy, xz + z(y^2 - z^2))$ begun in the text.
a. If we use grlex with $w > x > y > z$, show that a Groebner basis for $I^h \subset k[w, x, y, z]$ is $\{xy, xzw + z(y^2 - z^2), yz(y^2 - z^2)\}$.
b. If we dehomogenize the Groebner basis of part (a), we get a basis of I. Show that this basis is *not* a Groebner basis of I for grlex with $x > y > z$.
c. Use Proposition 4 to show that the tangent cone $C_0(V)$ is a union of five lines through the origin in k^3 and compare your answer to part (e) of Exercise 3.

6. Compute the dimensions of the tangent cone and the tangent space at the origin of the varieties defined by the following ideals:
a. $\langle xz, xy\rangle \subset k[x, y, z]$.
b. $\langle x - y^2, x - z^3\rangle \subset k[x, y, z]$.

7. In §3 of Chapter 3, we used elimination theory to show that the tangent surface of the twisted cubic $\mathbf{V}(y - x^2, z - x^3) \subset \mathbb{R}^3$ is defined by the equation

$$x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)z^2 = 0.$$

a. Show that the singular locus of the tangent surface S is exactly the twisted cubic. Hint: Two different ideals may define the same variety. For an example of how to deal with this, see equation (14) in Chapter 3, §4.
b. Compute the tangent space and tangent cone of the surface S at the origin.

8. Suppose that in $\mathbb{C}^n$ we have two sequences of vectors v_k and $t_k v_k$, where $t_k \in \mathbb{C}$, such that $v_k \to v \neq 0$ and $t_k v_k \to 0$. We claim that $t_k \to 0$ in $\mathbb{C}$. To prove this, define the length of a complex number $t = x + iy$ to be $|t| = \sqrt{x^2 + y^2}$ and define the length of $v = (z_1, \ldots, z_n) \in \mathbb{C}^n$ to be $|v| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Recall that $v_k \to v$ means that for every $\epsilon > 0$, there is N such that $|v_k - v| < \epsilon$ for $k \geq N$.
a. If we write $v = (z_1, \ldots, z_n)$ and $v_k = (z_{k1}, \ldots, z_{kn})$, then show that $v_k \to v$ implies $z_{kj} \to z_j$ for all j. Hint: Observe that $|z_j| \leq |v|$.
b. Pick a nonzero component z_j of v. Show that $z_{kj} \to z_j \neq 0$ and $t_k z_{kj} \to 0$. Then divide by z_j and conclude that $t_k \to 0$.

9. Theorem (2.33) of MUMFORD (1976) states that if $W \subset \mathbb{P}^n(\mathbb{C})$ is an irreducible projective variety and $Z \subset W$ is a projective variety not equal to W, then any point in Z is a limit of points in $W - Z$. Our goal is to apply this to prove Proposition 7.
a. Let $Z \subset W \subset \mathbb{C}^n$ be affine varieties such that W is the Zariski closure of $W - Z$. Show that Z contains no irreducible component of W.

b. Show that it suffices to prove Proposition 7 in the case when W is irreducible. Hint: If p lies in Z, then it lies in some component W_1 of W. What does part (a) tell you about $W_1 \cap Z \subset W_1$?
c. Let $Z \subset W \subset \mathbb{C}^n$, where W is irreducible and $Z \neq W$, and let $\overline{Z}$ and $\overline{W}$ be their projective closures in $\mathbb{P}^n(\mathbb{C})$. Show that the irreducible case of Proposition 7 follows from Mumford's Theorem (2.33). Hint: Use $\overline{Z} \cup (\overline{W} - W) \subset \overline{W}$.
d. Show that the converse of the proposition is true in the following sense. Let $p \in \mathbb{C}^n$. If $p \notin V - W$ and p is a limit of points in $V - W$, then show that $p \in W$. Hint: Show that $p \in V$ and recall that polynomials are continuous.

10. Let $V \subset k^n$ be a variety containing the origin and let $\mathcal{V} \subset k^{n+1}$ be the set described in (5). Given $\lambda \in k$, consider the "slice" $(k^n \times \{\lambda\}) \cap \overline{\mathcal{V}}$. Assume that k is infinite.
    a. When $\lambda \neq 0$, show that this slice equals $V_\lambda \times \{\lambda\}$, where $V_\lambda = \{v \in k^n : \lambda v \in V\}$. Also show that V_λ is an affine variety.
    b. Show that $V_1 = V$, and, more generally, for $\lambda \neq 0$, show that V_λ is isomorphic to V. Hint: Consider the polynomial map defined by sending $(x_1, \ldots, x_n)$ to $(\lambda x_1, \ldots, \lambda x_n)$.
    c. Suppose that $k = \mathbb{R}$ or $\mathbb{C}$ and that $\lambda \neq 0$ is close to the origin. Explain why V_λ gives a picture of V where we have expanded the scale by a factor of $1/\lambda$. Conclude that as $\lambda \to 0$, V_λ shows what V looks like as we "zoom in" at the origin.
    d. Use (6) to show that $V_0 = C_0(V)$. Explain what this means in terms of the "zooming in" described in part (c).
11. If $p \in V \subset k^n$, show that $C_p(V) \subset T_p(V)$.
12. If k is an infinite field and $V \subset k^n$ is a subspace (in the sense of linear algebra), then prove that V is irreducible. Hint: In Exercise 7 of §1, you showed that this was true when V was a coordinate subspace. Now pick an appropriate basis of k^n.
13. Let $W \subset \mathbb{P}^{n-1}(\mathbb{C})$ be a nonempty projective variety and let $C_W \subset \mathbb{C}^n$ be its affine cone.
    a. Prove that the tangent cone of C_W at the origin is C_W.
    b. Prove that the origin is a smooth point of C_W if and only if W is a projective linear subspace of $\mathbb{P}^{n-1}(\mathbb{C})$. Hint: Use Corollary 9.

In Exercises 14-17, we will study the "blow-up" of a variety V at a point $p \in V$. The blowing-up process gives us a map of varieties $\pi : \widetilde{V} \to V$ such that away from p, the two varieties look the same, but at p, $\widetilde{V}$ can be much larger than V, depending on what the tangent cone $C_p(V)$ looks like.

14. Let k be an arbitrary field. In §5 of Chapter 8, we studied varieties in $\mathbb{P}^{n-1} \times k^n$, where $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(k)$. Let $y_1, \ldots, y_n$ be homogeneous coordinates in $\mathbb{P}^{n-1}$ and let $x_1, \ldots, x_n$ be coordinates in k^n. Then the $(y_1, \ldots, y_n)$-homogeneous polynomials $x_i y_j - x_j y_i$ (this is the terminology of Chapter 8, §5) define a variety $\Gamma \subset \mathbb{P}^{n-1} \times k^n$. This variety has some interesting properties.
    a. If $(p, q) \in \mathbb{P}^{n-1} \times k^n$, then interpreting p as homogeneous coordinates and q as ordinary coordinates, show that $(p, q) \in \Gamma$ if and only if $q = tp$ for some $t \in k$ (which might be zero).
    b. If $q \neq 0$ is in k^n, show that $(\mathbb{P}^{n-1} \times \{q\}) \cap \Gamma$ consists of a single point [which can be thought of as (q, q), where the first q is the point of $\mathbb{P}^{n-1}$ with homogeneous coordinates given by $q \in k^n - \{0\}$]. On the other hand, when $q = 0$, show that $(\mathbb{P}^{n-1} \times \{q\}) \cap \Gamma = \mathbb{P}^{n-1} \times \{0\}$.
    c. If $\pi : \Gamma \to k^n$ is the projection map, conclude that $\pi^{-1}(q)$ consists of a single point, except when $q = 0$, in which case $\pi^{-1}(0)$ is a copy of $\mathbb{P}^{n-1}$. Hence, we can regard Γ

as the variety obtained by removing the origin from k^n and replacing it by a copy of $\mathbb{P}^{n-1}$.

d. To see what the $\mathbb{P}^{n-1} \times \{0\} \subset \Gamma$ means, consider a line L through the origin parametrized by tv. Show that the points $(v, tv) \in \mathbb{P}^{n-1} \times k^n$ lie in Γ and, hence, describe a curve $L \subset \Gamma$. Investigate where this curve meets $\mathbb{P}^{n-1} \times \{0\}$ and conclude that distinct lines through the origin in k^n give distinct points in $\pi^{-1}(0)$. Thus, the difference between Γ and k^n is that Γ separates tangent directions at the origin. We call $\pi : \Gamma \to k^n$ the *blow-up* of k^n at the origin.

15. This exercise is a continuation of Exercise 14. Let $V \subset k^n$ be a variety containing the origin and assume that the origin is not an irreducible component of V. Our goal here is to define the blow-up of V at the origin. Let $\Gamma \subset \mathbb{P}^{n-1} \times k^n$ be as in the previous exercise. Then $\widetilde{V} \subset \Gamma$ is defined to be the smallest variety in $\mathbb{P}^{n-1} \times k^n$ containing $(\mathbb{P}^{n-1} \times (V - \{0\})) \cap \Gamma$. If $\pi : \Gamma \to k^n$ is as in Exercise 14, then prove that $\pi(\widetilde{V}) \subset V$. Hint: First show that $\widetilde{V} \subset \mathbb{P}^{n-1} \times V$.

    This exercise shows we have a map $\pi : \widetilde{V} \to V$, which is called the *blow-up* of V at the origin. By Exercise 14, we know that $\pi^{-1}(q)$ consists of a single point for $q \neq 0$ in V. In Exercise 16, you will describe $\pi^{-1}(0)$ in terms of the tangent cone of V at the origin.

16. Let $V \subset k^n$ be a variety containing the origin and assume that the origin is not an irreducible component of V. We know that tangent cone $C_0(V)$ is the affine cone C_W over some projective variety $W \subset \mathbb{P}^{n-1}$. We call W the *projectivized tangent cone* of V at 0. The goal of this exercise is to show that if $\pi : \widetilde{V} \to V$ is the blow-up of V at 0 as defined in Exercise 15, then $\pi^{-1}(0) = W \times \{0\}$.

a. Show that our assumption that $\{0\}$ is not an irreducible component of V implies that k is infinite and that V is the Zariski closure of $V - \{0\}$.

b. Let $g \in k[y_1, \ldots, y_n, x_1, \ldots, x_n]$. Then show that $g \in \mathbf{I}(\widetilde{V})$ if and only if $g(q, tq) = 0$ for all $q \in V - \{0\}$ and all $t \in k - \{0\}$. Hint: Use part (a) of Exercise 14.

c. Then show that $g \in \mathbf{I}(\widetilde{V})$ if and only if $g(q, tq) = 0$ for all $q \in V$ and all $t \in k$. Hint: Use parts (a) and (b).

d. Explain why $\mathbf{I}(\widetilde{V})$ is generated by $(y_1, \ldots, y_n)$-homogeneous polynomials.

e. Assume that $g = \Sigma_\alpha g_\alpha(y_1, \ldots, y_n)x^\alpha \in \mathbf{I}(\widetilde{V})$. By part (d), we may assume that the g_α are all homogeneous of the same total degree d. Let

$$f(x_1, \ldots, x_n) = \sum_\alpha g_\alpha(x_1, \ldots, x_n)x^\alpha.$$

Then show that $f \in \mathbf{I}(V)$. Hint: First show that $g(x_1, \ldots, x_n, tx_1, \ldots, tx_n) = f(x_1, \ldots, x_n)t^d$, and then use part (c).

f. Prove that $W \times \{0\} \subset \widetilde{V} \cap (\mathbb{P}^{n-1} \times \{0\})$. Hint: It suffices to show that $g(v, 0) = 0$ for $g \in \mathbf{I}(\widetilde{V})$ and $v \in C_0(V)$. In the notation of part (e) note that $g(v, 0) = g_0(v)$. If $g_0 \neq 0$, show that $g_0 = f_{min}$, where f is the polynomial defined in part (e).

g. Prove that $V \cap (\mathbb{P}^{n-1} \times \{0\}) \subset W \times \{0\}$. Hint: If $f = f_l + \cdots + f_d \in \mathbf{I}(V)$, where $f_l = f_{min}$, let g be the remainder of $t^l f$ on division by $tx_1 - y_1, \ldots, tx_n - y_n$. Show that t does not appear in g and that $g \in \mathbf{I}(\widetilde{V})$. Then compute $g(v, 0)$ using the techniques of parts (e) and (f).

A line in the tangent cone can be regarded as a way of approaching the origin through points of V. So we can think of the projectivized tangent cone W as describing all possible ways of approaching the origin within V. Then $\pi^{-1}(0) = W \times \{0\}$ means that each of these different ways gives a distinct point in the blow-up. Note how this generalizes Exercise 14.

17. Assume that k is an algebraically closed field and suppose that $V = \mathbf{V}(f_1, \ldots, f_s) \subset k^n$ contains the origin.
    a. By analyzing what you did in part (g) of Exercise 16, explain how to find defining equations for the blow-up $\tilde{V}$.
    b. Compute the blow-up at the origin of $\mathbf{V}(y^2 - x^2 - x^3)$ and describe how your answer relates to the first picture in Example 1.
    c. Compute the blow-up at the origin of $\mathbf{V}(y^2 - x^3)$.

    Note that in parts (b) and (c), the blow-up is a smooth curve. In general, blowing-up is an important tool in what is called *desingularizing* a variety with singular points.