Galois Fields and Hardware Design

Construction of Galois Fields, Basic Properties, Uniqueness, Containment, Closure, Polynomial Functions over Galois Fields

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Agenda

- Introduction to Field Construction
- Constructing $\mathbb{F}_{2^k}$ and its elements
- Addition, multiplication and inverses over GFs
- Conjugates and their minimal polynomials
- GF containment and algebraic closure
- Hardware design over GFs
- Then we will verify hardware over GFs using Gröbner bases

Definition

An integral domain R is a set with two operations $(+, \cdot)$ such that:

1. The elements of R form an abelian group under $+$ with additive identity 0.
2. The multiplication is associative and commutative, with multiplicative identity 1.
3. The distributive law holds: $a(b+c) = ab + ac$.
4. The cancellation law holds: if $ab = ac$ and $a \neq 0$, then $b = c$.

Examples: $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_p, \mathbb{F}[x], \mathbb{F}[x, y]$. Finite rings $\mathbb{Z}_n, n \neq p$ are not integral domains.

Euclidean Domains

Definition

A Euclidean domain $\mathbb{D}$ is an integral domain where:

1. associated with each non-zero element $a \in \mathbb{D}$ is a non-negative integer $f(a)$ s.t. $f(a) \leq f(ab)$ if $b \neq 0$; and
2. $\forall a, b\ (b \neq 0), \exists (q, r)$ s.t. $a = qb + r$, where either $r = 0$ or $f(r) < f(b)$.

- Can apply the Euclid's algorithm to compute $g = GCD(g_1, \ldots, g_t)$
- Then $g = \sum_i u_i g_i$

Euclidean Domains

- $\mathbb{D} = \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_p$
- The ring $\mathbb{F}[x]$ is a Euclidean domain where $\mathbb{F}$ is any field
- The ring $R = \mathbb{F}[x, y]$ is NOT a Euclidean domain where $\mathbb{F}$ is any field
  - For $x, y \in R, GCD(x, y) = 1$, but cannot write $1 = f_1(x, y) \cdot x + f_2(x, y)y$
- $\mathbb{Z}_{2^k}$ is neither and integral domain not a Euclidean domain

Definition

Let $\mathbb{D}$ be a Euclidean domain, and $p \in \mathbb{D}$ be a prime element. Then $\mathbb{D} \pmod{p}$ is a field.

- That is why $\mathbb{Z} \pmod{p}$ is a field
- In $\mathbb{R}[x], x^2 + 1$ is a prime — actually called an irreducible polynomial
- So $\mathbb{R}[x] \pmod{x^2 + 1}$ is a field and is the field of complex numbers $\mathbb{C}$
- $\mathbb{R}[x] \pmod{p} = \{f(x) \mid \forall g(x) \in \mathbb{R}[x], f(x) = g(x) \pmod{p}\}$

$\mathbb{R}[x] \pmod{x^2+1} = \mathbb{C}$

- Let $f, g \in \mathbb{R}[x] \pmod{x^2+1}$
- $f =$ remainder of division by x^2+1, it is linear
- Let $f = ax + b,\ \ g = cx + d$

$$\begin{aligned} f \cdot g &= (ax+b)(cx+d) \pmod{x^2+1} \\ &= acx^2 + (ad+bc)x + bd \pmod{x^2+1} \\ &= (ad+bc)x + (bd - ac) \text{ after reducing by } \ x^2 = -1 \end{aligned}$$

- Replace x with $i = \sqrt{-1}$, and we get $\mathbb{C}$
- $\mathbb{C}$ is a 2 (=degree(x^2+1)) dimensional extension of $\mathbb{R}$
- Intuitively, that is why $\mathbb{C} \supset \mathbb{R}$ (containment and closure)

Recall from my previous slides:

From Rings to Fields

Rings $\supset$ Integral Domains $\supset$ Unique Factorization Domains $\supset$ Euclidean Domains $\supset$ Fields

Now you know the reason for this containment

Construct Galois Extension Fields

- $\mathbb{F}_p[x]$ is a Euclidean domain, let $P(x)$ be irreducible over $\mathbb{F}_p$, and let degree of $P(x) = k$
- $\mathbb{F}_p[x] \pmod{P(x)} = \mathbb{F}_{p^k}$, a finite field of p^k elements
- Denote GFs as $\mathbb{F}_q$, $q = p^k$ for prime p and $k \geq 1$
- $\mathbb{F}_{p^k}$ is a k-dimensional **extension** of $\mathbb{F}_p$, so $\mathbb{F}_p \subset \mathbb{F}_{p^k}$
- Our interest $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P(x)}$ where $P(x) \in \mathbb{F}_2[x]$ is a degree-k irreducible polynomial

- Irreducible polynomials of any degree k always exist over $\mathbb{F}_2$, so $\mathbb{F}_{2^k}$ can be constructed for arbitrary $k \geq 1$

Table: Some irreducible polynomials in $\mathbb{F}_2[x]$.

Degree	Irreducible Polynomials
1	$x; x+1$
2	x^2+x+1
3	$x^3+x+1; x^3+x^2+1$
4	$x^4+x+1; x^4+x^3+1; x^4+x^3+x^2+x+1$

- $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P(x)}$, let α be a root of $P(x)$, i.e. $P(\alpha) = 0$
- $P(x)$ has no roots in $\mathbb{F}_2$ (irreducible); root lies in its algebraic extension $\mathbb{F}_{2^k}$
- Any element $A \in \mathbb{F}_{2^k}$:
  $A = \sum_{i=0}^{k-1}(a_i \cdot \alpha^i) = a_0 + a_1 \cdot \alpha + \cdots + a_{k-1} \cdot \alpha^{k-1}$ where $a_i \in \mathbb{F}_2$
- The "degree" of $A < k$
- Think of $A = \{a_{k-1}, \ldots, a_0\}$ as a bit-vector

Example of $\mathbb{F}_{16}$

- $\mathbb{F}_{2^4}$ as $\mathbb{F}_2[x] \pmod{P(x)}$, where $P(x) = x^4 + x^3 + 1$, $P(\alpha) = 0$
- Any element $A \in \mathbb{F}_{16} = a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0$ (degree < 4)

Table: Bit-vector, Exponential and Polynomial representation of elements in $\mathbb{F}_{2^4} = \mathbb{F}_2[x] \pmod{x^4 + x^3 + 1}$

$a_3a_2a_1a_0$	Expo	Poly	$a_3a_2a_1a_0$	Expo	Poly
0000	0	0	1000	α^3	α^3
0001	1	1	1001	α^4	α^3+1
0010	α	α	1010	α^{10}	$\alpha^3+\alpha$
0011	α^{12}	$\alpha+1$	1011	α^5	$\alpha^3+\alpha+1$
0100	α^2	α^2	1100	α^{14}	$\alpha^3+\alpha^2$
0101	α^9	α^2+1	1101	α^{11}	$\alpha^3+\alpha^2+1$
0110	α^{13}	$\alpha^2+\alpha$	1110	α^8	$\alpha^3+\alpha^2+\alpha$
0111	α^7	$\alpha^2+\alpha+1$	1111	α^6	$\alpha^3+\alpha^2+\alpha+1$

Add, Mult in $\mathbb{F}_{2^k}$

Definition

The characteristic of a finite field $\mathbb{F}_q$ with unity element 1 is the smallest integer n such that $1 + \cdots + 1$ (n times) $= 0$.

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- What is the characteristic of $\mathbb{F}_{2^k}$? Of $\mathbb{F}_{p^k}$?

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- What is the characteristic of $\mathbb{F}_{2^k}$? Of $\mathbb{F}_{p^k}$?
- Characteristic $= 2$ and p, respectively, of course!

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- In $\mathbb{F}_{2^k}$ coefficients reduced modulo 2

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$$
\begin{aligned}
\alpha^5 + \alpha^{11} &= \alpha^3 + \alpha + 1 + \alpha^3 + \alpha^2 + 1 \\
&= 2 \cdot \alpha^3 + \alpha^2 + \alpha + 2 \\
&= \alpha^2 + \alpha \quad (\text{as characteristic of } \mathbb{F}_{2^k} = 2) \\
&= \alpha^{13}
\end{aligned}
$$

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\end{aligned}$$

Addition in $\mathbb{F}_{2^k}$ is Bit-vector XOR operation

Add, Mult in $\mathbb{F}_{2^k}$

$$\begin{aligned}
\alpha^4 \cdot \alpha^{10} &= (\alpha^3+1)(\alpha^3+\alpha) \\
&= \alpha^6 + \alpha^4 + \alpha^3 + \alpha \\
&= \alpha^4 \cdot \alpha^2 + (\alpha^4 + \alpha^3) + \alpha \\
&= (\alpha^3+1) \cdot \alpha^2 + (1) + \alpha \quad (\text{as} \;\; \alpha^4 = \alpha^3 + 1) \\
&= \alpha^5 + \alpha^2 + \alpha + 1 \\
&= \alpha^4 \cdot \alpha + \alpha^2 + \alpha + 1 \\
&= (\alpha^3+1) \cdot \alpha + \alpha^2 + \alpha + 1 \\
&= \alpha^4 + \alpha^2 + 1 \\
&= \alpha^3 + \alpha^2
\end{aligned}$$

Reduce everything $\pmod{P(x) = x^4 + x^3 + 1}$, and $-1 = +1$ in $\mathbb{F}_{2^k}$

Every non-zero element has an inverse

- How to find the inverse of α?
- HW for you: think Euclidean algorithm!
- What is the inverse of α in our $\mathbb{F}_{16}$ example?

Lemma

Let A be any non-zero element in $\mathbb{F}_q$, then $A^{q-1} = 1$.

Theorem

[Generalized Fermat's Little Theorem] Given a finite field $\mathbb{F}_q$, each element $A \in \mathbb{F}_q$ satisfies: $A^q \equiv A$ or $A^q - A \equiv 0$

Example

Given $\mathbb{F}_{2^2} = \{0, 1, \alpha, \alpha + 1\}$ with $P(x) = x^2 + x + 1$, where $P(\alpha) = 0$.

$$0^{2^2} = 0; \quad 1^{2^2} = 1; \quad \alpha^{2^2} = \alpha \pmod{\alpha^2 + \alpha + 1}$$

and

$$(\alpha + 1)^{2^2} = \alpha + 1 \pmod{\alpha^2 + \alpha + 1}$$

- An irreducible poly $P(x)$ is primitive if its root α can generate all non-zero elements of the field.
- $\mathbb{F}_q = \{0, 1 = \alpha^{q-1}, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q-2}\}$
- $x^4 + x^3 + 1$ is primitive but $x^4 + x^3 + x^2 + x + 1$ is not

$$
\begin{aligned}
\alpha^4 &= \alpha^3 + \alpha^2 + \alpha + 1 \\
\alpha^5 &= \alpha^4 \cdot \alpha \\
&= (\alpha^3 + \alpha^2 + \alpha + 1)(\alpha) \\
&= (\alpha^4) + \alpha^3 + \alpha^2 + \alpha \\
&= (\alpha^3 + \alpha^2 + \alpha + 1) + (\alpha^3 + \alpha^2 + \alpha) \\
&= 1
\end{aligned}
$$

Conjugates of α

Theorem

Let $f(x) \in \mathbb{F}_2[x]$ be an arbitrary polynomial, and let β be an element in $\mathbb{F}_{2^k}$ for any $k > 1$. If β is a root of $f(x)$, then for any $l \geq 0, \beta^{2^l}$ is also a root of $f(x)$. Elements β^{2^l} are conjugates of each other.

Example

Let $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{P(x) = x^4 + x^3 + 1}$. Let $P(\alpha) = 0$. Let us find conjugates of α as α^{2^l}.

$$
\begin{aligned}
l &= 1 : \alpha^2 \\
l &= 2 : \alpha^4 = \alpha^3 + 1 \\
l &= 3 : \alpha^8 = \alpha^3 + \alpha^2 + \alpha \\
l &= 4 : \alpha^{16} = \alpha \quad \text{(conjugates start to repeat)}
\end{aligned}
$$

So $\alpha, \alpha^2, \alpha^3 + 1, \alpha^3 + \alpha^2 + \alpha$ are conjugates of each other.

Get the irreducible polynomial back from conjugates

Example

Over $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{x^4 + x^3 + 1}$, conjugate elements:

- $\alpha, \alpha^2, \alpha^4, \alpha^8$
- $\alpha^3, \alpha^6, \alpha^{12}, \alpha^{24}$
- $\alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{56}$
- α^5, α^{10}

Minimal Polynomial of an element β

Let e be the smallest integer such that $\beta^{2^e} = \beta$. Construct the polynomial $f(x) = \prod_{i=0}^{e-1}(x + \beta^{2^i})$. Then $f(x)$ is an irreducible polynomial, and it is also called the irreducible polynomial of β.

Get the irreducible polynomial back from conjugates

Minimal polynomial of any element β is: $f(x) = \prod_{i=0}^{e-1}(x + \beta^{2^i})$

Example

Over $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{x^4 + x^3 + 1}$, conjugate elements and their minimal polynomials are:

- $\alpha, \alpha^2, \alpha^4, \alpha^8 :\ f_1(x) = (x+\alpha)(x+\alpha^2)(x+\alpha^4)(x+\alpha^8) = x^4 + x^3 + 1$
- $\alpha^3, \alpha^6, \alpha^{12}, \alpha^{24} :\ f_2(x) = x^4 + x^3 + x^2 + 1$
- $\alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{56} :\ f_3(x) = x^4 + x + 1$
- $\alpha^5, \alpha^{10} :\ f_4(x) = x^2 + x + 1$

Some observations....

Note that $f_4 = x^2 + x + 1$ is the polynomial used to construct $\mathbb{F}_4$. Also notice that associated with every element in $\mathbb{F}_{2^k}$ is a minimal polynomial and its roots (conjugates), that demonstrate the containment of fields and also the uniqueness of the fields upto the labeling of the elements.

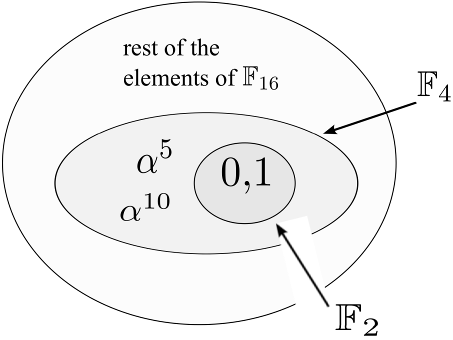


Figure: Containment of fields: $\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{16}$

Additive & Multiplicative closure: $\alpha^5 + \alpha^{10} = 1, \quad \alpha^5 \cdot \alpha^{10} = 1.$

Containment and Closure

Theorem

$\mathbb{F}_{2^n} \subset \mathbb{F}_{2^m}$ *if* n *divides* m.

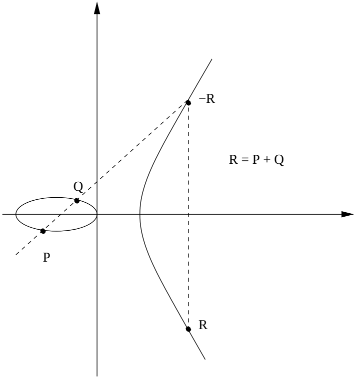
Example: $\mathbb{F}_2 \subset F_{2^2} \subset \mathbb{F}_{2^4} \subset \mathbb{F}_{2^8} \subset \dots$
$\mathbb{F}_2 \subset \mathbb{F}_{2^3} \subset \mathbb{F}_{2^6} \subset \dots$
$\mathbb{F}_2 \subset \mathbb{F}_{2^5} \subset \mathbb{F}_{2^{10}} \subset \dots$
$\mathbb{F}_2 \subset \mathbb{F}_{2^7} \subset \mathbb{F}_{2^{14}} \subset \dots$ and so on

Algebraic Closure of $\mathbb{F}_q$

The algebraic closure of $\mathbb{F}_{2^k}$ is the union of ALL such fields $\mathbb{F}_{2^n}$ where $k \mid n$.

Elliptic Curve Cryptography

$$y^2 + xy = x^3 + ax^2 + b \text{ over } \mathrm{GF}(2^k)$$



Compute Slope: $\dfrac{y_2 - y_1}{x_2 - x_1}$

Computation of inverses over $\mathbb{F}_{2^k}$ is expensive

Point addition using Projective Co-ordinates

- Curve: $Y^2 + XYZ = X^3Z + aX^2Z^2 + bZ^4$ over $\mathbb{F}_{2^k}$
- Let $(X_3,\ Y_3,\ Z_3) = (X_1,\ Y_1,\ Z_1) + (X_2,\ Y_2,\ 1)$

$$
\begin{aligned}
A &= Y_2 \cdot Z_1^2 + Y_1 & E &= A \cdot C \\
B &= X_2 \cdot Z_1 + X_1 & X_3 &= A^2 + D + E \\
C &= Z_1 \cdot B & F &= X_3 + X_2 \cdot Z_3 \\
D &= B^2 \cdot (C + aZ_1^2) & G &= X_3 + Y_2 \cdot Z_3 \\
Z_3 &= C^2 & Y_3 &= E \cdot F + Z_3 \cdot G
\end{aligned}
$$

No inverses, just addition and multiplication

Multiplication in GF(2^4)

Input:
$A = (a_3 a_2 a_1 a_0)$
$B = (b_3 b_2 b_1 b_0)$
$A = a_0 + a_1 \cdot \alpha + a_2 \cdot \alpha^2 + a_3 \cdot \alpha^3$
$B = b_0 + b_1 \cdot \alpha + b_2 \cdot \alpha^2 + b_3 \cdot \alpha^3$

Irreducible Polynomial:
$P = (11001)$
$P(x) = x^4 + x^3 + 1, \quad P(\alpha) = 0$

Result:
Output $G = A \times B \pmod{P(x)}$

Multiplication over GF(2^4)

$$
\begin{array}{ccccccc}
 & & & a_3 & a_2 & a_1 & a_0 \\
\times & & & b_3 & b_2 & b_1 & b_0 \\
\hline
 & & & a_3 \cdot b_0 & a_2 \cdot b_0 & a_1 \cdot b_0 & a_0 \cdot b_0 \\
 & & a_3 \cdot b_1 & a_2 \cdot b_1 & a_1 \cdot b_1 & a_0 \cdot b_1 & \\
 & a_3 \cdot b_2 & a_2 \cdot b_2 & a_1 \cdot b_2 & a_0 \cdot b_2 & & \\
a_3 \cdot b_3 & a_2 \cdot b_3 & a_1 \cdot b_3 & a_0 \cdot b_3 & & & \\
\hline
s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0
\end{array}
$$

In polynomial expression:
$S = s_0 + s_1 \cdot \alpha + s_2 \cdot \alpha^2 + s_3 \cdot \alpha^3 + s_4 \cdot \alpha^4 + s_5 \cdot \alpha^5 + s_6 \cdot \alpha^6$

S should be further reduced $\pmod{P(x)}$

Multiplication over GF(2^4)

$$
\begin{array}{ccc|cccc|cl}
s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 & & \\
\hline
 & & & s_4 & 0 & 0 & s_4 & \Leftarrow & s_4 \cdot \alpha^4 \pmod{P(\alpha)} \\
 & & & s_5 & 0 & s_5 & s_5 & \Leftarrow & s_5 \cdot \alpha^5 \pmod{P(\alpha)} \\
 & & + & s_6 & s_6 & s_6 & s_6 & \Leftarrow & s_6 \cdot \alpha^6 \pmod{P(\alpha)} \\
\hline
 & & & g_3 & g_2 & g_1 & g_0 & &
\end{array}
$$

$$
\begin{aligned}
s_4 \cdot \alpha^4 \pmod{\alpha^4 + \alpha^3 + 1} &= s_4(\alpha^3 + 1) = s_4 \cdot \alpha^3 + s_4 \\
s_5 \cdot \alpha^5 \pmod{\alpha^4 + \alpha^3 + 1} &= s_5(\alpha^3 + \alpha + 1) = s_5 \cdot \alpha^3 + s_5 \cdot \alpha + s_5 \\
s_6 \cdot \alpha^6 \pmod{\alpha^4 + \alpha^3 + 1} &= s_6(\alpha^3 + \alpha^2 + \alpha + 1) \\
&= s_6 \cdot \alpha^3 + s_6 \cdot \alpha^2 + s_6 \cdot \alpha + s_6
\end{aligned}
$$

$$G = g_0 + g_1 \cdot \alpha + g_2 \cdot \alpha^2 + g_3 \cdot \alpha^3$$

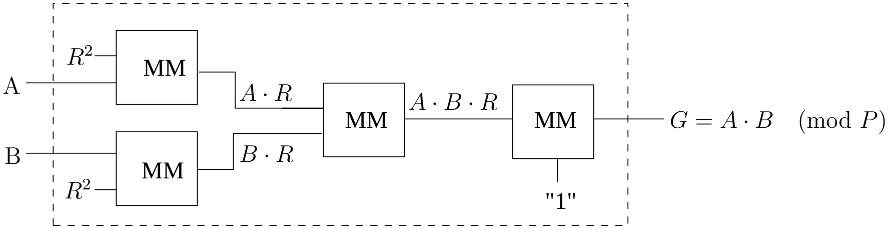


Figure: *Montgomery* multiplier over GF(2^k)

Montgomery Multiply: $F = A \cdot B \cdot R^{-1}$, $R = \alpha^k$

- Barrett architectures do not require precomputed R^{-1}
- We can verify 163-bit circuits, and also catch bugs!
- Conventional techniques fail beyond 16-bit circuits

Verification: The Mathematical Problem

Let us take verification of GF multipliers as an example:

- Given specification polynomial: $f : Z = A \cdot B \pmod{P(x)}$ over $\mathbb{F}_{2^k}$, for given k, and given $P(x)$, s.t. $P(\alpha) = 0$
- Given circuit implementation C
  - Primary inputs: $A = \{a_0, \ldots, a_{k-1}\}, B = \{b_0, \ldots, b_{k-1}\}$
  - Primary Output $Z = \{z_0, \ldots, z_{k-1}\}$
  - $A = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{k-1}\alpha^{k-1}$
  - $B = b_0 + b_1\alpha + \cdots + b_{k-1}\alpha^{k-1},\ Z = z_0 + z_1\alpha + \cdots + z_{k-1}\alpha^{k-1}$
- Does the circuit C correctly compute specification f?

Mathematically:

- Construct a miter between the spec f and implementation C
- Model the circuit (gates) as polynomials $\{f_1, \ldots, f_s\} \in \mathbb{F}_{2^k}[x_1, \ldots, x_d]$
- Apply Weak Nullstellensatz

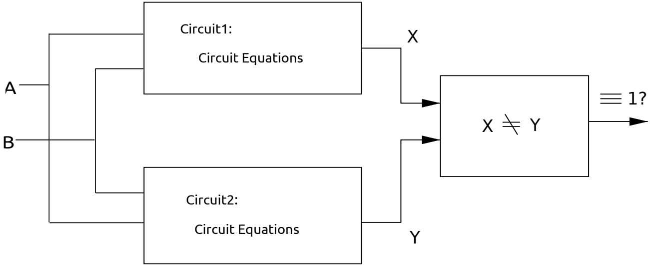


Figure: The equivalence checking setup: miter.

Spec can be a polynomial f, or a circuit implementation C

Model the miter gate as: $t(X - Y) = 1$, where t is a free variable

Verify a polynomial spec against circuit C

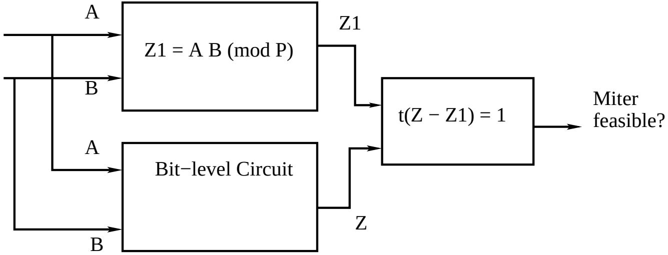


Figure: The equivalence checking setup: miter.

- When $Z = Z_1$, $t(Z - Z_1) = 1$ has no solution: infeasible miter
- When $Z \neq Z_1$: let $t^{-1} = (Z - Z_1)$. Then $t \cdot (t^{-1}) = 1$ always has a solution!
- Apply Nullstellensatz over $\mathbb{F}_{2^k}$

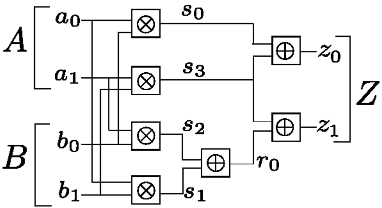


Figure: A 2-bit Multiplier

- Write $A = a_0 + a_1\alpha$ as a polynomial $f_A : A + a_0 + a_1\alpha$
- Polynomials modeling the entire circuit: ideal $J = \langle f_1, \ldots, f_{10}\rangle$

$f_1 : z_0 + z_1\alpha + Z$; $f_2 : b_0 + b_1\alpha + B$; $f_3 : a_0 + a_1\alpha + A$; $f_4 :$ $s_0 + a_0 \cdot b_0$; $f_5 : s_1 + a_0 \cdot b_1$; $f_6 : s_2 + a_1 \cdot b_0$; $f_7 : s_3 + a_1 \cdot b_1$; $f_8 :$ $r_0 + s_1 + s_2$; $f_9 : z_0 + s_0 + s_3$; $f_{10} : z_1 + r_0 + s_3$

Continue with multiplier verification

- So far, ideal $J = \langle f_1, \ldots, f_{10} \rangle$ models the implementation
- Let polynomial $f : Z - A \cdot B$ denote the spec
- Miter polynomial $f_m : t \cdot (Z - Z_1) - 1$
- Update the ideal representation of the miter: $J = J + \langle f, f_m \rangle$
- Finally: ideal $J = \langle f_1, \ldots, f_{10},\ f,\ f_m \rangle$ represents the miter circuit
- $J \subseteq \mathbb{F}_{2^k}[A, B, Z, Z_1, a_0, a_1, b_0, b_1, r_0, s_0, \ldots, s_3, t]$
- Verification problem: is the variety $V_{\mathbb{F}_4}(J) = \emptyset$?
- How will we solve this problem?

Theorem (Weak Nullstellensatz over $\mathbb{F}_{2^k}$)

Let ideal $J = \langle f_1, \ldots, f_s \rangle \subset \mathbb{F}_{2^k}[x_1, \ldots, x_n]$ *be an ideal. Let* $J_0 = \langle x_1^{2^k} - x_1, \ldots, x_n^{2^k} - x_n \rangle$ *be the ideal of all vanishing polynomials. Then:*

$$V_{\mathbb{F}_{2^k}}(J) = \emptyset \iff V_{\overline{\mathbb{F}_{2^k}}}(J + J_0) = \emptyset \iff reducedGB(J + J_0) = \{1\}$$

Proof:

$$\begin{aligned} V_{\mathbb{F}_{2^k}}(J) &= V_{\overline{\mathbb{F}_{2^k}}}(J) \cap \mathbb{F}_{2^k} \\ &= V_{\overline{\mathbb{F}_{2^k}}}(J) \cap V_{\mathbb{F}_{2^k}}(J_0) = V_{\overline{\mathbb{F}_{2^k}}}(J) \cap V_{\overline{\mathbb{F}_{2^k}}}(J_0) \\ &= V_{\overline{\mathbb{F}_{2^k}}}(J + J_0) \end{aligned}$$

Remember: $V_{\mathbb{F}_q}(J_0) = V_{\overline{\mathbb{F}_q}}(J_0)$. The variety of J_0 does not change over the field or the closure!

Apply Weak Nullstellesatz to the Miter

- Note: Word-level polynomials $f_A : A + a_0 + a_1\alpha \in \mathbb{F}_{2^k}$
- Gate level polynomials $f_4 : s_0 + a_0 \cdot b_0 \in \mathbb{F}_2$
- Since $\mathbb{F}_2 \subset \mathbb{F}_{2^k}$, we can treat ALL polynomials of the miter, collectively, over the larger field $\mathbb{F}_{2^k}$, so $J \subseteq \mathbb{F}_{2^k}[A, B, Z, Z_1, a_0, a_1, \ldots, z_0, z_1]$
- Consider word-level vanishing polynomials: $A^{2^2} - A$
- What about bit-level vanishing polynomials: $a_0^2 - a_0$
- So, $J_0 = \langle W^{2^k} - W, B^2 - B \rangle$, where W are all the word-level variables, and B are all the bit-level variables
- Now compute $G = GB(J + J_0)$. If $G = \{1\}$, the circuit is correct. Otherwise there is definitely a BUG within the field $\mathbb{F}_{2^k}$

Polynomial Functions over $\mathbb{F}_q$

- Any combinational circuit with k-bit inputs and k-bit output
  - Implements a function $f : \mathbb{B}^k \rightarrow \mathbb{B}^k$
  - Can be viewed as a function $f : \mathbb{F}_{2^k} \rightarrow \mathbb{F}_{2^k}$ or $f : \mathbb{Z}_{2^k} \rightarrow \mathbb{Z}_{2^k}$
  - Need symbolic representations: view them as polynomial functions
- Treat the circuit $f : \mathbb{B}^k \rightarrow \mathbb{B}^k$ as a polynomial function
- Please see the last section in my book chapter