8

Projective Algebraic Geometry

So far all of the varieties we have studied have been subsets of affine space k^n. In this chapter, we will enlarge k^n by adding certain "points at ∞" to create n-dimensional projective space $\mathbb{P}^n(k)$. We will then define projective varieties in $\mathbb{P}^n(k)$ and study the projective version of the algebra-geometry correspondence. The relation between affine and projective varieties will be considered in §4; in §5, we will study elimination theory from a projective point of view. By working in projective space, we will get a much better understanding of the Extension Theorem in Chapter 3. The chapter will end with a discussion of the geometry of quadric hypersurfaces and an introduction to Bezout's Theorem.

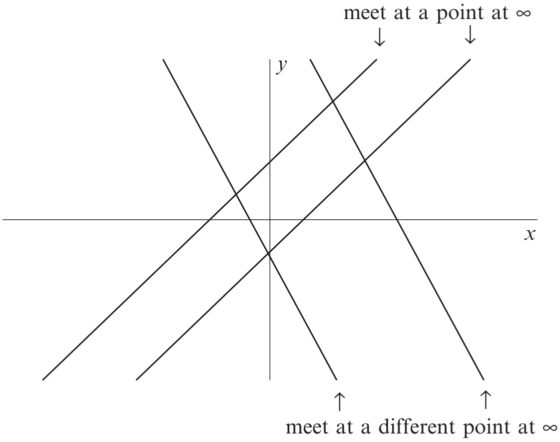
§1 The Projective Plane

This section will study the projective plane $\mathbb{P}^2(\mathbb{R})$ over the real numbers $\mathbb{R}$. We will see that, in a certain sense, the plane $\mathbb{R}^2$ is missing some "points at ∞," and by adding them to $\mathbb{R}^2$, we will get the projective plane $\mathbb{P}^2(\mathbb{R})$. Then we will introduce *homogeneous coordinates* to give a more systematic treatment of $\mathbb{P}^2(\mathbb{R})$.

Our starting point is the observation that two lines in $\mathbb{R}^2$ intersect in a point, *except* when they are parallel. We can take care of this exception if we view parallel lines as meeting at some sort of point at ∞. As indicated by the picture at the top of the following page, there should be different points at ∞, depending on the direction of the lines. To approach this more formally, we introduce an equivalence relation on lines in the plane by setting $L_1 \sim L_2$ if L_1 and L_2 are parallel. Then an equivalence class $[L]$ consists of all lines parallel to a given line L. The above discussion suggests that we should introduce one point at ∞ for each equivalence class $[L]$. We make the following provisional definition.

Definition 1. *The* **projective plane over** $\mathbb{R}$, *denoted* $\mathbb{P}^2(\mathbb{R})$, *is the set*

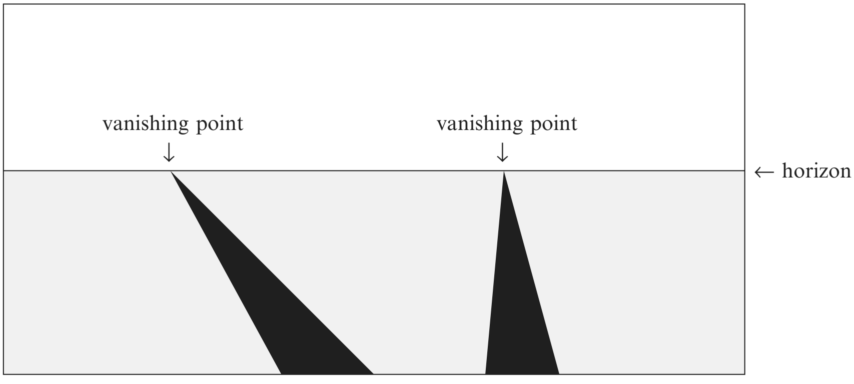
$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{\textit{one point at } \infty \textit{ for each equivalence class of parallel lines}\}.$$



We will let $[L]_\infty$ denote the common point at ∞ of all lines parallel to L. Then we call the set $\overline{L} = L \cup [L]_\infty \subset \mathbb{P}^2(\mathbb{R})$ the *projective line* corresponding to L. Note that two projective lines always meet at exactly one point: if they are not parallel, they meet at a point in $\mathbb{R}^2$; if they are parallel, they meet at their common point at ∞.

At first sight, one might expect that a line in the plane should have two points at ∞, corresponding to the two ways we can travel along the line. However, the reason why we want only one is contained in the previous paragraph: if there were two points at ∞, then parallel lines would have two points of intersection, not one. So, for example, if we parametrize the line $x = y$ via $(x, y) = (t, t)$, then we can approach its point at ∞ using either $t \to \infty$ or $t \to -\infty$.

A common way to visualize points at ∞ is to make a perspective drawing. Pretend that the earth is flat and consider a painting that shows two roads extending infinitely far in different directions:



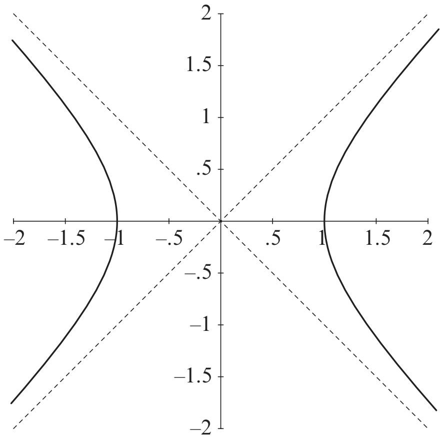
For each road, the two sides (which are parallel, but appear to be converging) meet at the same point on the horizon, which in the theory of perspective is called a *vanishing* *point*. Furthermore, any line parallel to one of the roads meets at the same vanishing point, which shows that the vanishing point represents the point at ∞ of these lines. The same reasoning applies to any point on the horizon, so that the horizon in the picture represents points at ∞. (Note that the horizon does not contain all of them—it is missing the point at ∞ of lines parallel to the horizon.)

The above picture reveals another interesting property of the projective plane: the points at ∞ form a special projective line, which is called the *line at* ∞. It follows that $\mathbb{P}^2(\mathbb{R})$ has the projective lines $\overline{L} = L \cup [L]_\infty$, where L is a line in $\mathbb{R}^2$, together with the line at ∞. In the exercises, you will prove that two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ determine a unique point and two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line. Note the symmetry in these statements: when we interchange "point" and "projective line" in one, we get the other. This is an instance of the *principle of duality*, which is one of the fundamental concepts of projective geometry.

For an example of how points at ∞ can occur in other contexts, consider the parametrization of the hyperbola $x^2 - y^2 = 1$ given by the equations

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2}, \\ y &= \frac{2t}{1-t^2}. \end{aligned}$$

When $t \neq \pm 1$, it is easy to check that this parametrization covers all of the hyperbola except $(-1, 0)$. But what happens when $t = \pm 1$? Here is a picture of the hyperbola:



If we let $t \to 1^-$, then the corresponding point (x, y) travels along the first quadrant portion of the hyperbola, getting closer and closer to the asymptote $x = y$. Similarly, if $t \to 1^+$, we approach $x = y$ along the third quadrant portion of the hyperbola. Hence, it becomes clear that $t = 1$ should correspond to the point at ∞ of the asymptote $x = y$. Similarly, one can check that $t = -1$ corresponds to the point at ∞ of $x = -y$. (In the exercises, we will give a different way to see what happens when $t = \pm 1$.)

Thus far, our discussion of the projective plane has introduced some nice ideas, but it is not entirely satisfactory. For example, it is not really clear why the line at ∞ should be called a projective line. A more serious objection is that we have no unified way of naming points in $\mathbb{P}^2(\mathbb{R})$. Points in $\mathbb{R}^2$ are specified by coordinates, but points at ∞ are specified by lines. To avoid this asymmetry, we will introduce *homogeneous coordinates* on $\mathbb{P}^2(\mathbb{R})$.

To get homogeneous coordinates, we will need a new definition of projective space. The first step is to define an equivalence relation on nonzero points of $\mathbb{R}^3$ by setting

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$$

if there is a nonzero real number λ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. One can easily check that $\sim$ is an equivalence relation on $\mathbb{R}^3 - \{0\}$ (where as usual 0 refers to the origin $(0, 0, 0)$ in $\mathbb{R}^3$). Then we can redefine projective space as follows.

Definition 2. $\mathbb{P}^2(\mathbb{R})$ *is the set of equivalence classes of* $\sim$ *on* $\mathbb{R}^3 - \{0\}$. *Thus, we can write*

$$\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 - \{0\})/ \sim .$$

If a triple $(x, y, z) \in \mathbb{R}^3 - \{0\}$ *corresponds to a point* $p \in \mathbb{P}^2(\mathbb{R})$, *we say that* (x, y, z) *are* **homogeneous coordinates** *of* p.

At this point, it is not clear that Definitions 1 and 2 give the same object, although we will see shortly that this is the case.

Homogeneous coordinates are different from the usual notion of coordinates in that they are not unique. For example, $(1, 1, 1)$, $(2, 2, 2)$, (π, π, π) and $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ are all homogeneous coordinates of the same point in projective space. But the nonuniqueness of the coordinates is not so bad since they are all multiples of one another.

As an illustration of how we can use homogeneous coordinates, let us define the notion of a projective line.

Definition 3. *Given real numbers A, B, C, not all zero, the set*

$$\{p \in \mathbb{P}^2(\mathbb{R}) : p \text{ has homogeneous coordinates } (x, y, z) \\ \text{with } Ax + By + Cz = 0\}$$

is called a **projective line** *of* $\mathbb{P}^2(\mathbb{R})$.

An important observation is that if $Ax + By + Cz = 0$ holds for one set (x, y, z) of homogeneous coordinates of $p \in \mathbb{P}^2(\mathbb{R})$, then it holds for *all* homogeneous coordinates of p. This is because the others can be written $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$, so that $A \cdot \lambda x + B \cdot \lambda y + C \cdot \lambda z = \lambda(Ax + By + Cz) = 0$. Later in this chapter, we will use the same idea to define varieties in projective space.

To relate our two definitions of projective plane, we will use the map

$$\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}) \tag{1}$$

defined by sending $(x, y) \in \mathbb{R}^2$ to the point $p \in \mathbb{P}^2(\mathbb{R})$ whose homogeneous coordinates are $(x, y, 1)$. This map has the following properties.

Proposition 4. *The map* (1) *is one-to-one and the complement of its image is the projective line H_∞ defined by $z = 0$.*

Proof. First, suppose that (x, y) and (x', y') map to the same point p in $\mathbb{P}^2(\mathbb{R})$. Then $(x, y, 1)$ and $(x', y', 1)$ are homogeneous coordinates of p, so that $(x, y, 1) = \lambda(x', y', 1)$ for some λ. Looking at the third coordinate, we see that $\lambda = 1$ and it follows that $(x, y) = (x', y')$.

Next, let (x, y, z) be homogeneous coordinates of a point $p \in \mathbb{P}^2(\mathbb{R})$. If $z = 0$, then p is on the projective line H_∞. On the other hand, if $z \neq 0$, then we can multiply by $1/z$ to see that $(x/z, y/z, 1)$ gives homogeneous coordinates for p. This shows that p is in the image of map (1). We leave it as an exercise to show that the image of the map is disjoint from H_∞, and the proposition is proved. □

We will call H_∞ the *line at* ∞. It is customary (though somewhat sloppy) to identify $\mathbb{R}^2$ with its image in $\mathbb{P}^2(\mathbb{R})$, so that we can write projective space as the disjoint union

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup H_\infty.$$

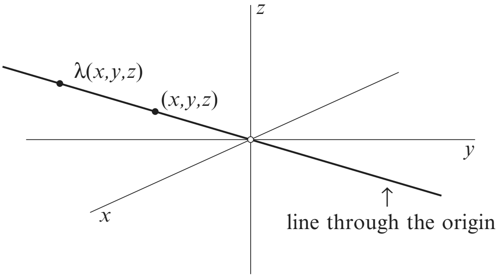
This is beginning to look familiar. It remains to show that H_∞ consists of points at ∞ in our earlier sense. Thus, we need to study how lines in $\mathbb{R}^2$ (which we will call *affine lines*) relate to projective lines. The following table tells the story:

$$\begin{array}{ccccc} \text{affine line} & & \text{projective line} & & \text{point at } \infty \\ L : y = mx + b & \to & \overline{L} : y = mx + bz & \to & (1, m, 0) \\ L : x = c & \to & \overline{L} : x = cz & \to & (0, 1, 0) \end{array}$$

To understand this table, first consider a nonvertical affine line L defined by $y = mx + b$. Under the map (1), a point (x, y) on L maps to a point $(x, y, 1)$ of the projective line $\overline{L}$ defined by $y = mx + bz$. Thus, L can be regarded as subset of $\overline{L}$. By Proposition 4, the remaining points of $\overline{L}$ come from where it meets $z = 0$. But the equations $z = 0$ and $y = mx + bz$ clearly imply $y = mx$, so that the solutions are $(x, mx, 0)$. We have $x \neq 0$ since homogeneous coordinates never simultaneously vanish, and dividing by x shows that $(1, m, 0)$ is the unique point of $\overline{L} \cap H_\infty$. The case of vertical lines is left as an exercise.

The table shows that two lines in $\mathbb{R}^2$ meet at the same point at ∞ if and only if they are parallel. For nonvertical lines, the point at ∞ encodes the slope, and for vertical lines, there is a single (but different) point at ∞. Be sure you understand this. In the exercises, you will check that the points listed in the table exhaust all of H_∞. Consequently, H_∞ consists of a unique point at ∞ for every equivalence class of parallel lines. Then $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup H_\infty$ shows that the projective planes of Definitions 1 and 2 are the same object.

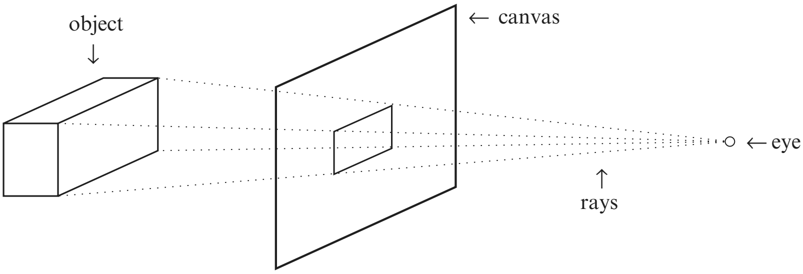
We next introduce a more geometric way of thinking about points in the projective plane. Let (x, y, z) be homogeneous coordinates of a point p in $\mathbb{P}^2(\mathbb{R})$, so that all other homogeneous coordinates for p are given by $\lambda(x, y, z)$ for $\lambda \in \mathbb{R} - \{0\}$. The crucial observation is that these points all lie on the same line through the origin in $\mathbb{R}^3$:



The requirement in Definition 2 that $(x, y, z) \neq (0, 0, 0)$ guarantees that we get a line in $\mathbb{R}^3$. Conversely, given *any* line L through the origin in $\mathbb{R}^3$, a point (x, y, z) on $L - \{0\}$ gives homogeneous coordinates for a uniquely determined point in $\mathbb{P}^2(\mathbb{R})$ [since any other point on $L - \{0\}$ is a nonzero multiple of (x, y, z)]. This shows that we have a one-to-one correspondence.

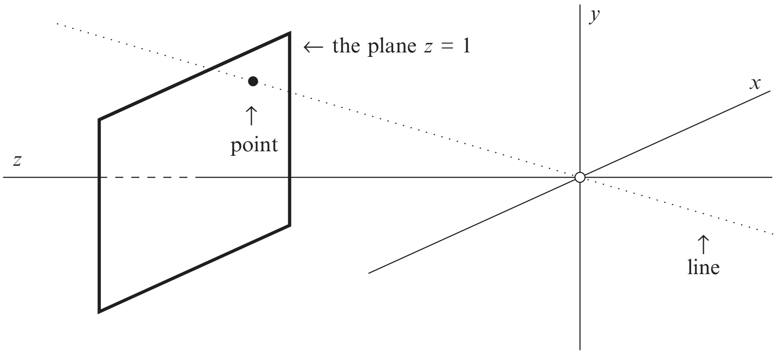
$$\mathbb{P}^2(\mathbb{R}) \cong \{\text{lines through the origin in } \mathbb{R}^3\}. \tag{2}$$

Although it may seem hard to think of a point in $\mathbb{P}^2(\mathbb{R})$ as a line in $\mathbb{R}^3$, there is a strong intuitive basis for this identification. We can see why by studying how to draw a 3-dimensional object on a 2-dimensional canvas. Imagine lines or rays that link our eye to points on the object. Then we draw according to where the rays intersect the canvas:



Renaissance texts on perspective would speak of the "pyramid of rays" connecting the artist's eye with the object being painted. For us, the crucial observation is that each ray hits the canvas exactly once, giving a one-to-one correspondence between rays and points on the canvas.

To make this more mathematical, we will let the "eye" be the origin and the "canvas" be the plane $z = 1$ in the coordinate system pictured at the top of the next page. Rather than work with rays (which are half-lines), we will work with lines through the origin. Then, as the picture indicates, every point in the plane $z = 1$ determines a unique line through the origin. This one-to-one correspondence allows us to think of a *point* in the plane as a *line through the origin* in $\mathbb{R}^3$ [which by (2) is a point in $\mathbb{P}^2(\mathbb{R})$]. There are two interesting things to note about this correspondence:

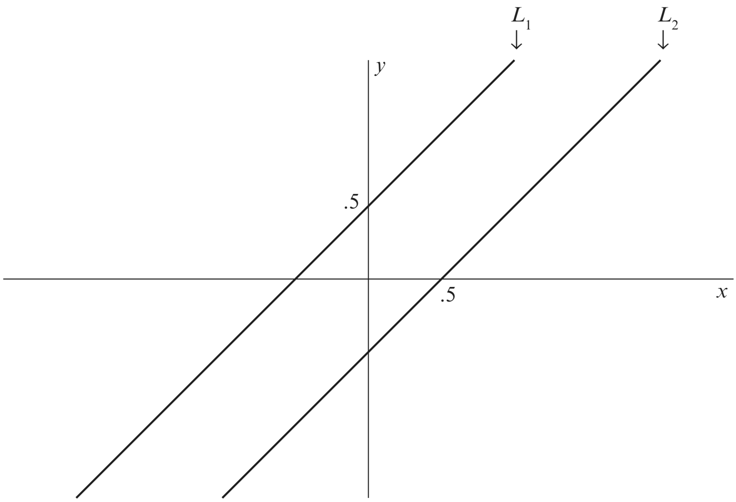


- A point (x, y) in the plane gives the point $(x, y, 1)$ on our "canvas" $z = 1$. The corresponding line through the origin is a point $p \in \mathbb{P}^2(\mathbb{R})$ with homogeneous coordinates $(x, y, 1)$. Hence, the correspondence given above is *exactly* the map $\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R})$ from Proposition 4.
- The correspondence is not onto since this method will never produce a line in the (x, y)-plane. Do you see how these lines can be thought of as the points at ∞?

In many situations, it is useful to be able to think of $\mathbb{P}^2(\mathbb{R})$ both algebraically (in terms of homogeneous coordinates) and geometrically (in terms of lines through the origin).

As the final topic in this section, we will use homogeneous coordinates to examine the line at ∞ more closely. The basic observation is that although we began with coordinates x and y, once we have homogeneous coordinates, there is nothing special about the extra coordinate z—it is no different from x or y. In particular, if we want, we could regard x and z as the original coordinates and y as the extra one.

To see how this can be useful, consider the parallel lines $L_1 : y = x + 1/2$ and $L_2 : y = x - 1/2$ in the (x, y)-plane:

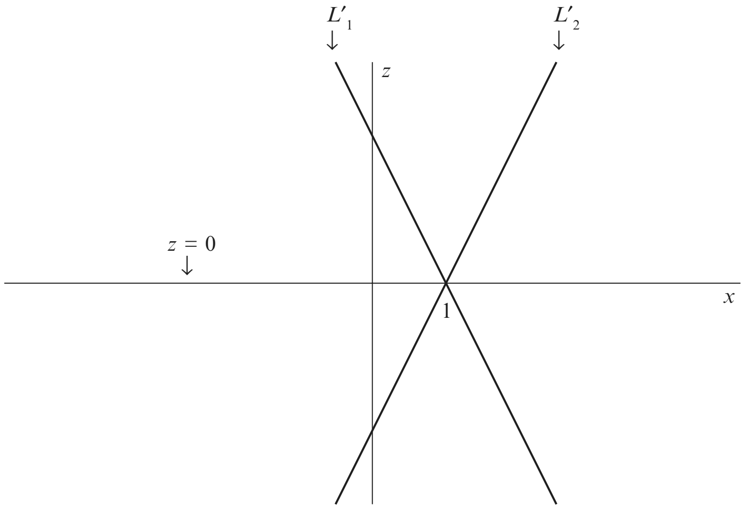


The (x, y)-Plane

We know that these lines intersect at ∞ since they are parallel. But the picture does not show their point of intersection. To view these lines at ∞, consider the projective lines

$$\overline{L}_1 : y = x + (1/2)z,$$
$$\overline{L}_2 : y = x - (1/2)z$$

determined by L_1 and L_2. Now regard x and z as the original variables. Thus, we map the (x, z)-plane $\mathbb{R}^2$ to $\mathbb{P}^2(\mathbb{R})$ via $(x, z) \mapsto (x, 1, z)$. As in Proposition 4, this map is one-to-one, and we can recover the (x, z)-plane inside $\mathbb{P}^2(\mathbb{R})$ by setting $y = 1$. If we do this with the equations of the projective lines $\overline{L}_1$ and $\overline{L}_2$, we get the lines $L'_1 : z = -2x + 2$ and $L'_2 : z = 2x - 2$. This gives the following picture in the (x, z)-plane:



The (x, z)-Plane

In this picture, the x-axis is defined by $z = 0$, which is the line at ∞ as we originally set things up in Proposition 4. Note that L'_1 and L'_2 meet when $z = 0$, which corresponds to the fact that L_1 and L_2 meet at ∞. Thus, the above picture shows how our two lines behave as they approach the line at ∞. In the exercises, we will study what some other common curves look like at ∞.

It is interesting to compare the above picture with the perspective drawing of two roads given earlier in the section. It is no accident that the horizon in the perspective drawing represents the line at ∞. The exercises will explore this idea in more detail.

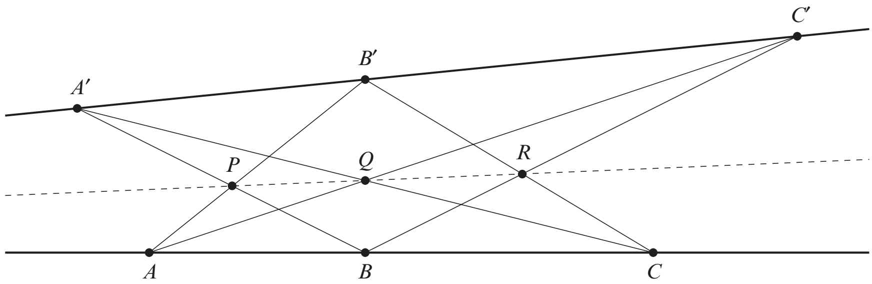
Another interesting observation is that the Euclidean notion of distance does not play a prominent role in the geometry of projective space. For example, the lines L_1 and L_2 in the (x, y)-plane are a constant distance apart, whereas L'_1 and L'_2 get closer and closer in the (x, z)-plane. This explains why the geometry of $\mathbb{P}^2(\mathbb{R})$ is quite different from Euclidean geometry.

EXERCISES FOR §1

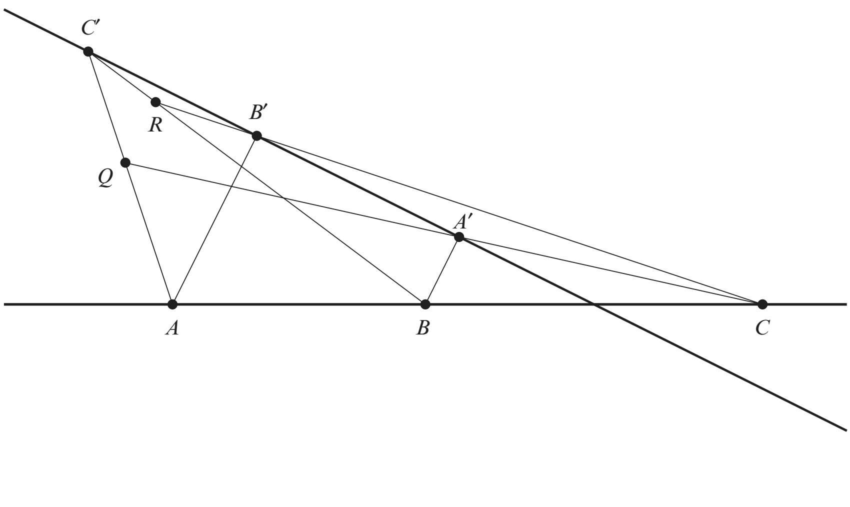
1. Using $\mathbb{P}^2(\mathbb{R})$ as given in Definition 1, we saw that the projective lines in $\mathbb{P}^2(\mathbb{R})$ are $\overline{L} = L \cup [L]_\infty$, and the line at ∞.
   a. Prove that *any* two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line. Hint: There are three cases to consider, depending on how many of the points are points at ∞.
   b. Prove that *any* two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ meet at a unique point. Hint: Do this case-by-case.
2. There are many theorems that initially look like theorems in the plane, but which are really theorems in $\mathbb{P}^2(\mathbb{R})$ in disguise. One classic example is Pappus's Theorem, which goes as follows. Suppose we have two collinear triples of points A, B, C and A', B', C'. Then let

$$
\begin{aligned}
P &= \overline{AB'} \cap \overline{A'B},\\
Q &= \overline{AC'} \cap \overline{A'C},\\
R &= \overline{BC'} \cap \overline{B'C}.
\end{aligned}
$$

   Pappus's Theorem states that P, Q, R are always collinear points. In Exercise 8 of Chapter 6, §4, we drew the following picture to illustrate the theorem:



   a. If we let the points on one of the lines go the other way, then we can get the following configuration of points and lines:



Note that P is now a point at ∞. Is Pappus's Theorem still true [in $\mathbb{P}^2(\mathbb{R})$] for the picture on the preceding page?

b. By moving the point C in the picture for part (a) show that you can also make Q a point at ∞. Is Pappus's Theorem still true? What line do P, Q, R lie on? Draw a picture to illustrate what happens.

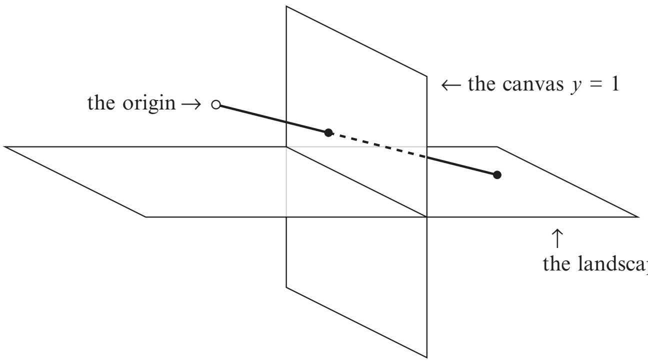
If you made a purely affine version of Pappus's Theorem that took cases (a) and (b) into account, the resulting statement would be rather cumbersome. By working in $\mathbb{P}^2(\mathbb{R})$, we cover these cases simultaneously.

3. We will continue the study of the parametrization $(x, y) = ((1+t^2)/(1-t^2), 2t/(1-t^2))$ of $x^2 - y^2 = 1$ begun in the text.
   a. Given t, show that (x, y) is the point where the hyperbola intersects the line of slope t going through the point $(-1, 0)$. Illustrate your answer with a picture. Hint: Use the parametrization to show that $t = y/(x+1)$.
   b. Use the answer to part (a) to explain why $t = \pm 1$ maps to the points at ∞ corresponding to the asymptotes of the hyperbola. Illustrate your answer with a drawing.
   c. Using homogeneous coordinates, show that we can write the parametrization as
   $$((1+t^2)/(1-t^2), 2t/(1-t^2), 1) = (1+t^2, 2t, 1-t^2),$$
   and use this to explain what happens when $t = \pm 1$. Does this give the same answer as part(b)?
   d. We can also use the technique of part (c) to understand what happens when $t \to \infty$. Namely, in the parametrization $(x, y, z) = (1+t^2, 2t, 1-t^2)$, substitute $t = 1/u$. Then clear denominators (this is legal since we are using homogeneous coordinates) and let $u \to 0$. What point do you get on the hyperbola?
4. This exercise will study what the hyperbola $x^2 - y^2 = 1$ looks like at ∞.
   a. Explain why the equation $x^2 - y^2 = z^2$ gives a well-defined curve C in $\mathbb{P}^2(\mathbb{R})$. Hint: See the discussion following Definition 3.
   b. What are the points at ∞ on C? How does your answer relate to Exercise 3?
   c. In the (x, z) coordinate system obtained by setting $y = 1$, show that C is still a hyperbola.
   d. In the (y, z) coordinate system obtained by setting $x = 1$, show that C is a circle.
   e. Use the parametrization of Exercise 3 to obtain a parametrization of the circle from part (d).
5. Consider the parabola $y = x^2$.
   a. What equation should we use to make the parabola into a curve in $\mathbb{P}^2(\mathbb{R})$?
   b. How many points at ∞ does the parabola have?
   c. By choosing appropriate coordinates (as in Exercise 4), explain why the parabola is tangent to the line at ∞.
   d. Show that the parabola looks like a hyperbola in the (y, z) coordinate system.
6. When we use the (x, y) coordinate system inside $\mathbb{P}^2(\mathbb{R})$, we only view a piece of the projective plane. In particular, we miss the line at ∞. As in the text, we can use (x, z) coordinates to view the line at ∞. Show that there is exactly one point in $\mathbb{P}^2(\mathbb{R})$ that is visible in neither (x, y) nor (x, z) coordinates. How can we view what is happening at this point?
7. In the proof of Proposition 4, show that the image of the map (2) is disjoint from H_∞.
8. As in the text, the line H_∞ is defined by $z = 0$. Thus, points on H_∞ have homogeneous coordinates $(a, b, 0)$, where $(a, b) \neq (0, 0)$.
   a. A vertical affine line $x = c$ gives the projective line $x = cz$. Show that this meets H_∞ at the point $(0, 1, 0)$.

b. Show that a point on H_∞ different from $(0, 1, 0)$ can be written uniquely as $(1, m, 0)$ for some real number m.

9. In the text, we viewed parts of $\mathbb{P}^2(\mathbb{R})$ in the (x, y) and (x, z) coordinate systems. In the (x, z) picture, it is natural to ask what happened to y. To see this, we will study how (x, y) coordinates look when viewed in the (x, z)-plane.
   a. Show that (a, b) in the (x, y)-plane gives the point $(a/b, 1/b)$ in the (x, z)-plane.
   b. Use the formula of part (a) to study what the parabolas $(x, y) = (t, t^2)$ and $(x, y) = (t^2, t)$ look like in the (x, z)-plane. Draw pictures of what happens in both (x, y) and (x, z) coordinates.

10. In this exercise, we will discuss the mathematics behind the perspective drawing given in the text. Suppose we want to draw a picture of a landscape, which we will assume is a horizontal plane. We will make our drawing on a canvas, which will be a vertical plane. Our eye will be a certain distance above the landscape, and to draw, we connect a point on the landscape to our eye with a line, and we put a dot where the line hits the canvas:



To give formulas for what happens, we will pick coordinates (x, y, z) so that our eye is the origin, the canvas is the plane $y = 1$, and the landscape is the plane $z = 1$ (thus, the positive z-axis points down).
   a. Starting with the point $(a, b, 1)$ on the landscape, what point do we get in the canvas $y = 1$?
   b. Explain how the answer to part (a) relates to Exercise 9. Write a brief paragraph discussing the relation between perspective drawings and the projective plane.

11. As in Definition 3, a projective line in $\mathbb{P}^2(\mathbb{R})$ is defined by an equation of the form $Ax + By + Cz = 0$, where $(A, B, C) \neq (0, 0, 0)$.
   a. Why do we need to make the restriction $(A, B, C) \neq (0, 0, 0)$?
   b. Show that (A, B, C) and (A', B', C') define the same projective line if and only if $(A, B, C) = \lambda(A', B', C')$ for some nonzero real number λ. Hint: One direction is easy. For the other direction, take two distinct points (a, b, c) and (a', b', c') on the line $Ax + By + Cz = 0$. Show that (a, b, c) and (a', b', c') are linearly independent and conclude that the equations $Xa + Yb + Zc = Xa' + Yb' + Zc' = 0$ have a 1-dimensional solution space for the variables X, Y, Z.
   c. Conclude that the set of projective lines in $\mathbb{P}^2(\mathbb{R})$ can be identified with the set $\{(A, B, C) \in \mathbb{R}^3 : (A, B, C) \neq (0, 0, 0)\}/\sim$. This set is called the *dual projective plane* and is denoted $\mathbb{P}^2(\mathbb{R})^\vee$.
   d. Describe the subset of $\mathbb{P}^2(\mathbb{R})^\vee$ corresponding to affine lines.

e. Given a point $p \in \mathbb{P}^2(\mathbb{R})$, consider the set $\tilde{p}$ of all projective lines L containing p. We can regard $\tilde{p}$ as a subset of $\mathbb{P}^2(\mathbb{R})^\vee$. Show that $\tilde{p}$ is a projective line in $\mathbb{P}^2(\mathbb{R})^\vee$. We call $\tilde{p}$ the *pencil of lines* through p.

f. The Cartesian product $\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})^\vee$ has the natural subset

$$I = \{(p, L) \in \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})^\vee \ : \ p \in L\}.$$

Show that I is described by the equation $Ax + By + Cz = 0$, where (x, y, z) are homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$ and (A, B, C) are homogeneous coordinates on the dual. We will study varieties of this type in §5.

Parts (d), (e), and (f) of this exercise illustrate how collections of naturally defined geometric objects can be given an algebraic structure.

§2 Projective Space and Projective Varieties

The construction of the real projective plane given in Definition 2 of §1 can be generalized to yield projective spaces of any dimension n over any field k. We define an equivalence relation $\sim$ on the nonzero points of k^{n+1} by setting

$$(x_0', \ldots, x_n') \sim (x_0, \ldots, x_n)$$

if there is a nonzero element $\lambda \in k$ such that $(x_0', \ldots, x_n') = \lambda(x_0, \ldots, x_n)$. If we let 0 denote the origin $(0, \ldots, 0)$ in k^{n+1}, then we define projective space as follows.

Definition 1. ***n*-dimensional projective space over the field** *k, denoted $\mathbb{P}^n(k)$, is the set of equivalence classes of $\sim$ on $k^{n+1} - \{0\}$. Thus,*

$$\mathbb{P}^n(k) = (k^{n+1} - \{0\})/\sim.$$

Each nonzero $(n+1)$-tuple $(x_0, \ldots, x_n) \in k^{n+1}$ defines a point p in $\mathbb{P}^n(k)$, and we say that $(x_0, \ldots, x_n)$ are **homogeneous coordinates** *of p.*

Like $\mathbb{P}^2(\mathbb{R})$, each point $p \in \mathbb{P}^n(k)$ has many sets of homogeneous coordinates. For example, in $\mathbb{P}^3(\mathbb{C})$, the homogeneous coordinates $(0, \sqrt{2}, 0, i)$ and $(0, 2i, 0, -\sqrt{2})$ describe the same point since $(0, 2i, 0, -\sqrt{2}) = \sqrt{2}i(0, \sqrt{2}, 0, i)$. In general, we will write $p = (x_0, \ldots, x_n)$ to denote that $(x_0, \ldots, x_n)$ are homogeneous coordinates of $p \in \mathbb{P}^n(k)$.

As in §1, we can think of $\mathbb{P}^n(k)$ more geometrically as the set of lines through the origin in k^{n+1}. More precisely, you will show in Exercise 1 that there is a one-to-one correspondence

$$\mathbb{P}^n(k) \cong \{\text{lines through the origin in } k^{n+1}\}. \tag{1}$$

Just as the real projective plane contains the affine plane $\mathbb{R}^2$ as a subset, $\mathbb{P}^n(k)$ contains the affine space k^n.

Proposition 2. *Let*

$$U_0 = \{(x_0, \ldots, x_n) \in \mathbb{P}^n(k) : x_0 \neq 0\}.$$

Then the map ϕ taking $(a_1, \ldots, a_n)$ in k^n to the point with homogeneous coordinates $(1, a_1, \ldots, a_n)$ in $\mathbb{P}^n(k)$ is a one-to-one correspondence between k^n and $U_0 \subset \mathbb{P}^n(k)$.

Proof. Since the first component of $\phi(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$ is nonzero, we get a map $\phi : k^n \to U_0$. We can also define an inverse map $\psi : U_0 \to k^n$ as follows. Given $p = (x_0, \ldots, x_n) \in U_0$ since $x_0 \neq 0$ we can multiply the homogeneous coordinates by the nonzero scalar $\lambda = \frac{1}{x_0}$ to obtain $p = (1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. Then set $\psi(p) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in k^n$. We leave it as an exercise for the reader to show that ψ is well-defined and that ϕ and ψ are inverse mappings. This establishes the desired one-to-one correspondence. □

By the definition of U_0, we see that $\mathbb{P}^n(k) = U_0 \cup H$, where

$$H = \{p \in \mathbb{P}^n(k) : p = (0, x_1, \ldots, x_n)\}. \tag{2}$$

If we identify U_0 with the affine space k^n, then we can think of H as the *hyperplane at infinity.* It follows from (2) that the points in H are in one-to-one correspondence with n-tuples $(x_1, \ldots, x_n)$, where two n-tuples represent the same point of H if one is a nonzero scalar multiple of the other (just ignore the first component of points in H). In other words, H is a "copy" of $\mathbb{P}^{n-1}(k)$, the projective space of one smaller dimension. Identifying U_0 with k^n and H with $\mathbb{P}^{n-1}(k)$, we can write

$$\mathbb{P}^n(k) = k^n \cup \mathbb{P}^{n-1}(k). \tag{3}$$

To see what $H = \mathbb{P}^{n-1}(k)$ means geometrically, note that, by (1), a point $p \in \mathbb{P}^{n-1}(k)$ gives a line $L \subset k^n$ going through the origin. Consequently, in the decomposition (3), we should think of p as representing the asymptotic direction of *all* lines in k^n parallel to L. This allows us to regard p as a point at ∞ in the sense of §1, and we recover the intuitive definition of the projective space given there. In the exercises, we will give a more algebraic way of seeing how this works.

A special case worth mentioning is the projective line $\mathbb{P}^1(k)$. Since $\mathbb{P}^0(k)$ consists of a single point (this follows easily from Definition 1), letting $n = 1$ in (3) gives us

$$\mathbb{P}^1(k) = k^1 \cup \mathbb{P}^0(k) = k \cup \{\infty\},$$

where we let ∞ represent the single point of $\mathbb{P}^0(k)$. If we use (1) to think of points in $\mathbb{P}^1(k)$ as lines through the origin in k^2, then the above decomposition reflects the fact these lines are characterized by their slope (where the vertical line has slope ∞). When $k = \mathbb{C}$, it is customary to call

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$$

the *Riemann sphere.* The reason for this name will be explored in the exercises.

For completeness, we mention that there are many other copies of k^n in $\mathbb{P}^n(k)$ besides U_0. Indeed the proof of Proposition 2 may be adapted to yield the following results.

Corollary 3. *For each $i = 0, \ldots n$, let*

$$U_i = \{(x_0, \ldots, x_n) \in \mathbb{P}^n(k) : x_i \neq 0\}.$$

(i) *The points of each U_i are in one-to-one correspondence with the points of k^n.*

(ii) *The complement* $\mathbb{P}^n(k) - U_i$ *may be identified with* $\mathbb{P}^{n-1}(k)$.
(iii) *We have* $\mathbb{P}^n(k) = \cup_{i=0}^n U_i$.

Proof. See Exercise 5. $\square$

Our next goal is to extend the definition of varieties in affine space to projective space. For instance, we can ask whether it makes sense to consider $\mathbf{V}(f)$ for a polynomial $f \in k[x_0, \ldots, x_n]$. A simple example shows that some care must be taken here. For instance, in $\mathbb{P}^2(\mathbb{R})$, we can try to construct $\mathbf{V}(x_1 - x_2^2)$. The point $p = (x_0, x_1, x_2) = (1, 4, 2)$ appears to be in this set since the components of p satisfy the equation $x_1 - x_2^2 = 0$. However, a problem arises when we note that the same point p can be represented by the homogeneous coordinates $p = 2(1, 4, 2) = (2, 8, 4)$. If we substitute these components into our polynomial, we obtain $8 - 4^2 = -8 \neq 0$. We get different results depending on which homogeneous coordinates we choose.

To avoid problems of this type, we use *homogeneous* polynomials when working in $\mathbb{P}^n(k)$. From Definition 6 of Chapter 7, §1, recall that a polynomial is *homogeneous* of total degree d if every term appearing in f has total degree exactly d. The polynomial $f = x_1 - x_2^2$ in the example is not homogeneous, and this is what caused the inconsistency in the values of f on different homogeneous coordinates representing the same point. For a homogeneous polynomial, this does not happen.

Proposition 4. *Let* $f \in k[x_0, \ldots, x_n]$ *be a homogeneous polynomial. If* f *vanishes on any one set of homogeneous coordinates for a point* $p \in \mathbb{P}^n(k)$*, then* f *vanishes for all homogeneous coordinates of* p*. In particular* $\mathbf{V}(f) = \{p \in \mathbb{P}^n(k) : f(p) = 0\}$ *is a well-defined subset of* $\mathbb{P}^n(k)$.

Proof. Let $(a_0, \ldots, a_n)$ and $(\lambda a_0, \ldots, \lambda a_n)$ be homogeneous coordinates for $p \in \mathbb{P}^n(k)$ and assume that $f(a_0, \ldots, a_n) = 0$. If f is homogeneous of total degree k, then every term in f has the form

$$c x_0^{\alpha_0} \cdots x_n^{\alpha_n},$$

where $\alpha_0 + \cdots + \alpha_n = k$. When we substitute $x_i = \lambda a_i$, this term becomes

$$c(\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n} = \lambda^k c a_0^{\alpha_0} \cdots a_n^{\alpha_n}.$$

Summing over the terms in f, we find a common factor of λ^k and, hence,

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^k f(a_0, \ldots, a_n) = 0.$$

This proves the proposition. $\square$

Notice that even if f is homogeneous, the equation $f = a$ does not make sense in $\mathbb{P}^n(k)$ when $0 \neq a \in k$. The equation $f = 0$ is special because it gives a well-defined subset of $\mathbb{P}^n(k)$. We can also consider subsets of $\mathbb{P}^n(k)$ defined by the vanishing of a *system* of homogeneous polynomials (possibly of different total degrees). The correct generalization of the affine varieties introduced in Chapter 1, §2 is as follows.

Definition 5. *Let k be a field and let $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials. We set*

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n(k) : f_i(a_0, \ldots, a_n) = 0 \text{ for all } 1 \le i \le s\}.$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the **projective variety** *defined by $f_1, \ldots, f_s$.*

For example, in $\mathbb{P}^n(k)$, any nonzero homogeneous polynomial of degree 1,

$$\ell(x_0, \ldots, x_n) = c_0x_0 + \cdots + c_nx_n,$$

defines a projective variety $\mathbf{V}(\ell)$ called a *hyperplane.* One example we have seen is the hyperplane at infinity, which was defined as $H = V(x_0)$. When $n = 2$, we call $\mathbf{V}(\ell)$ a projective line, or more simply a *line* in $\mathbb{P}^2(k)$. Similarly, when $n = 3$, we call a hyperplane a *plane* in $\mathbb{P}^3(k)$. Varieties defined by one or more linear polynomials (homogeneous polynomials of degree 1) are called *linear varieties* in $\mathbb{P}^n(k)$. For instance, $\mathbf{V}(x_1, x_2) \subset \mathbb{P}^3(k)$ is a linear variety which is a projective line in $\mathbb{P}^3(k)$.

The projective varieties $\mathbf{V}(f)$ defined by a single nonzero homogeneous equation are known collectively as *hypersurfaces.* However, individual hypersurfaces are usually classified according to the total degree of the defining equation. Thus, if f has total degree 2 in $k[x_0, \ldots, x_n]$, we usually call $\mathbf{V}(f)$ a *quadric hypersurface*, or *quadric* for short. For instance, $\mathbf{V}(-x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^3(\mathbb{R})$ is quadric. Similarly, hypersurfaces defined by equations of total degree 3, 4, and 5 are known as *cubics, quartics*, and *quintics*, respectively.

To get a better understanding of projective varieties, we need to discover what the corresponding algebraic objects are. This leads to the notion of *homogeneous ideal*, which will be discussed in §3. We will see that the entire algebra-geometry correspondence of Chapter 4 can be carried over to projective space.

The final topic we will consider in this section is the relation between affine and projective varieties. As we saw in Corollary 3, the subsets $U_i \subset \mathbb{P}^n(k)$ are copies of k^n. Thus, we can ask how affine varieties in $U_i \cong k^n$ relate to projective varieties in $\mathbb{P}^n(k)$. First, if we take a projective variety V and intersect it with one of the U_i, it makes sense to ask whether we obtain an affine variety. The answer to this question is always *yes*, and the defining equations of the variety $V \cap U_i$ may be obtained by a process called *dehomogenization.* We illustrate this by considering $V \cap U_0$. From the proof of Proposition 2, we know that if $p \in U_0$, then p has homogeneous coordinates of the form $(1, x_1, \ldots, x_n)$. If $f \in k[x_0, \ldots, x_n]$ is one of the defining equations of V, then the polynomial $g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ vanishes at every point of $V \cap U_0$. Setting $x_0 = 1$ in f produces a "dehomogenized" polynomial g which is usually nonhomogeneous. We claim that $V \cap U_0$ is precisely the affine variety obtained by dehomogenizing the equations of V.

Proposition 6. *Let $V = \mathbf{V}(f_1, \ldots, f_s)$ be a projective variety. Then $W = V \cap U_0$ can be identified with the affine variety $\mathbf{V}(g_1, \ldots, g_s) \subset k^n$, where $g_i(y_1, \ldots, y_n) = f_i(1, y_1, \ldots, y_n)$ for each $1 \leq i \leq s$.*

Proof. The comments before the statement of the proposition show that using the mapping $\psi : U_0 \to k^n$ from Proposition 2, $\psi(W) \subset \mathbf{V}(g_1, \ldots, g_s)$. On the other hand, if $(a_1, \ldots, a_n) \in \mathbf{V}(g_1, \ldots, g_s)$, then the point with homogeneous coordinates $(1, a_1, \ldots, a_n)$ is in U_0 and it satisfies the equations

$$f_i(1, a_1, \ldots, a_n) = g_i(a_1, \ldots, a_n) = 0.$$

Thus, $\phi(\mathbf{V}(g_1, \ldots, g_s)) \subset W$. Since the mappings ϕ and ψ are inverses, the points of W are in one-to-one correspondence with the points of $\mathbf{V}(g_1, \ldots, g_s)$. □

For instance, consider the projective variety

$$V = \mathbf{V}(x_1^2 - x_2x_0, x_1^3 - x_3x_0^2) \subset \mathbb{P}^3(\mathbb{R}). \tag{4}$$

To intersect V with U_0, we dehomogenize the defining equations, which gives us the affine variety

$$\mathbf{V}(x_1^2 - x_2, x_1^3 - x_3) \subset \mathbb{R}^3.$$

We recognize this as the familiar twisted cubic in $\mathbb{R}^3$.

We can also dehomogenize with respect to other variables. For example, the above proof shows that, for any projective variety $V \subset \mathbb{P}^3(\mathbb{R})$, $V \cap U_1$ can be identified with the affine variety in $\mathbb{R}^3$ defined by the equations obtained by setting $g_i(x_0, x_2, x_3) = f_i(x_0, 1, x_2, x_3)$. When we do this with the projective variety V defined in (4), we see that $V \cap U_1$ is the affine variety $\mathbf{V}(1 - x_2x_0, 1 - x_3x_0^2)$. See Exercise 9 for a general statement.

Going in the opposite direction, we can ask whether an affine variety in U_i, can be written as $V \cap U_i$ in some projective variety V. The answer is again *yes*, but there is more than one way to do it, and the results can be somewhat unexpected.

One natural idea is to reverse the dehomogenization process described earlier and "homogenize" the defining equations of the affine variety. For example, consider the affine variety $W = \mathbf{V}(x_2 - x_1^3 + x_1^2)$ in $U_0 = \mathbb{R}^2$. The defining equation is not homogeneous, so we do not get a projective variety in $\mathbb{P}^2(\mathbb{R})$ directly from this equation. But we can use the extra variable x_0 to make $f = x_2 - x_1^3 + x_1^2$ homogeneous. Since f has total degree 3, we modify f so that every term has total degree 3. This leads to the homogeneous polynomial

$$f^h = x_2x_0^2 - x_1^3 + x_1^2x_0.$$

Moreover, note that dehomogenizing f^h gives back the original polynomial f in x_1, x_2. The general pattern is the same.

Proposition 7. *Let $g(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ be a polynomial of total degree d.*

(i) *Let $g = \sum_{i=0}^{d} g_i$ be the expansion of g as the sum of its homogeneous components where g_i has total degree i. Then*

$$\begin{aligned} g^h(x_0, \ldots, x_n) &= \sum_{i=0}^{d} g_i(x_1, \ldots, x_n) x_0^{d-i} \\ &= g_d(x_1, \ldots, x_n) + g_{d-1}(x_1, \ldots x_n)x_0 \\ &\quad + \cdots + g_0(x_1, \ldots x_n)x_0^d \end{aligned}$$

is a homogeneous polynomial of total degree d in $k[x_0, \ldots, x_n]$. We will call g^h the **homogenization** *of g.*

(ii) *The homogenization of g can be computed using the formula*

$$g^h = x_0^d \cdot g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

(iii) *Dehomogenizing g^h yields g. That is, $g^h(1, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.*

(iv) *Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial and let x_0^e be the highest power of x_0 dividing F. If $f = F(1, x_1, \ldots, x_n)$ is a dehomogenization of F, then $F = x_0^e \cdot f^h$.*

Proof. We leave the proof to the reader as Exercise 10. □

As a result of Proposition 7, given any affine variety $W = \mathbf{V}(g_1, \ldots, g_s) \subset k^n$, we can homogenize the defining equations of W to obtain a projective variety $V = \mathbf{V}(g_1^h, \ldots, g_s^h) \subset \mathbb{P}^n(k)$. Moreover, by part (iii) and Proposition 6, we see that $V \cap U_0 = W$. Thus, our original affine variety W is the *affine portion* of the projective variety V.

As we mentioned before, though, there are some unexpected possibilities.

Example 8. In this example, we will write the homogeneous coordinates of points in $\mathbb{P}^2(k)$ as (x, y, z). Numbering them as $0, 1, 2$, we see that U_2 is the set of points with homogeneous coordinates $(x, y, 1)$, and x and y are coordinates on $U_2 \cong k^2$. Now consider the affine variety $W = \mathbf{V}(g) = \mathbf{V}(y - x^3 + x) \subset U_2$. We know that W is the affine portion $V \cap U_2$ of the projective variety $V = \mathbf{V}(g^h) = \mathbf{V}(yz^2 - x^3 + xz^2)$.

The variety V consists of W together with the points at infinity $V \cap \mathbf{V}(z)$. The affine portion W is the graph of a cubic polynomial, which is a nonsingular plane curve. The points at infinity, which form the complement of W in V, are given by the solutions of the equations

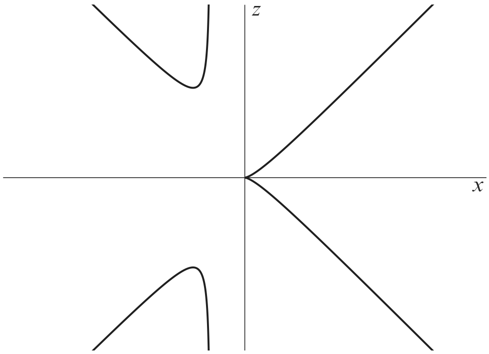
$$\begin{aligned} 0 &= yz^2 - x^3 + xz^2, \\ 0 &= z. \end{aligned}$$

It is easy to see that the solutions are $z = x = 0$ and since we are working in $\mathbb{P}^2(k)$, we get the unique point $p = (0, 1, 0)$ in $V \cap \mathbf{V}(z)$. Thus, $V = W \cup \{p\}$. An unexpected feature of this example is the nature of the extra point p.

To see what V looks like at p, let us dehomogenize the equation of V with respect to y and study the intersection $V \cap U_1$. We find

$$W' = V \cap U_1 = \mathbf{V}(g^h(x, 1, z)) = \mathbf{V}(z^2 - x^3 + xz^2).$$

From the discussion of singularities in §4 of Chapter 3, one can easily check that p, which becomes the point $(x, z) = (0, 0) \in W'$, is a singular point on W':



Thus, even if we start from a nonsingular affine variety (that is, one with no singular points), homogenizing the equations and taking the corresponding projective variety may yield a more complicated geometric object. In effect, we are not "seeing the whole picture" in the original affine portion of the variety. In general, given a projective variety $V \subset \mathbb{P}^n(k)$, since $\mathbb{P}^n(k) = \cup_{i=0}^n U_i$, we may need to consider $V \cap U_i$ for each $i = 0, \ldots, n$ to see the whole picture of V.

Our next example shows that simply homogenizing the defining equations can lead to the "wrong" projective variety.

Example 9. Consider the affine twisted cubic $W = \mathbf{V}(x_2 - x_1^2, x_3 - x_1^3)$ in $\mathbb{R}^3$. By Proposition 7, $W = V \cap U_0$ for the projective variety $V = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3) \subset \mathbb{P}^3(\mathbb{R})$. As in Example 8, we can ask what part of V we are "missing" in the affine portion W. The complement of W in V is $V \cap H$, where $H = \mathbf{V}(x_0)$ is the plane at infinity. Thus, $V \cap H = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_0)$, and one easily sees that these equations reduce to

$$\begin{aligned} x_1^2 &= 0,\\ x_1^3 &= 0,\\ x_0 &= 0. \end{aligned}$$

The coordinates x_2 and x_3 are arbitrary here, so $V \cap H$ is the projective line $\mathbf{V}(x_0, x_1) \subset \mathbb{P}^3(\mathbb{R})$. Thus we have $V = W \cup \mathbf{V}(x_0, x_1)$.

Since the twisted cubic W is a curve in $\mathbb{R}^3$, our intuition suggests that it should only have a finite number of points at infinity (in the exercises, you will see that this is indeed the case). This indicates that V is probably too big; there should be a smaller projective variety V' containing W. One way to create such a V' is to homogenize other polynomials that vanish on W. For example, the parametrization (t, t^2, t^3) of W shows that $x_1x_3 - x_2^2 \in \mathbf{I}(W)$. Since $x_1x_3 - x_2^2$ is already homogeneous, we can add it to the defining equations of V to get

$$V' = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_1x_3 - x_2^2) \subset V.$$

Then V' is a projective variety with the property that $V' \cap U_0 = W$, and in the exercises you will show that $V' \cap H$ consists of the single point $p = (0, 0, 0, 1)$. Thus, $V' = W \cup \{p\}$, so that we have a smaller projective variety that restricts to the twisted cubic. The difference between V and V' is that V has an extra component at infinity. In §4, we will show that V' is the smallest projective variety containing W.

In Example 9, the process by which we obtained V was completely straightforward (we homogenized the defining equations of W), yet it gave us a projective variety that was too big. This indicates that something more subtle is going on. The complete answer will come in §4, where we will learn an algorithm for finding the smallest projective variety containing $W \subset k^n \cong U_i$.

EXERCISES FOR §2

1. In this exercise, we will give a more geometric way to describe the construction of $\mathbb{P}^n(k)$. Let $\mathcal{L}$ denote the set of lines through the origin in k^{n+1}.
   a. Show that every element of $\mathcal{L}$ can be represented as the set of scalar multiples of some nonzero vector in k^{n+1}.
   b. Show that two nonzero vectors v' and v in k^{n+1} define the same element of $\mathcal{L}$ if and only if $v' \sim v$ as in Definition 1.
   c. Show that there is a one-to-one correspondence between $\mathbb{P}^n(k)$ and $\mathcal{L}$.
2. Complete the proof of Proposition 2 by showing that the mappings ϕ and ψ defined in the proof are inverses.
3. In this exercise, we will study how lines in $\mathbb{R}^n$ relate to points at infinity in $\mathbb{P}^n(\mathbb{R})$. We will use the decomposition $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{P}^{n-1}(\mathbb{R})$ given in (3). Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a + bt$, where $a \in L$ and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $(a_1 + b_1t, \ldots, a_n + b_nt)$.
   a. We can regard L as lying in $\mathbb{P}^n(\mathbb{R})$ using the homogeneous coordinates
   $$(1, a_1 + b_1t, \ldots, a_n + b_nt).$$
   To find out what happens as t $\to \pm\infty$, divide by t to obtain
   $$\left(\frac{1}{t}, \frac{a_1}{t} + b_1, \ldots, \frac{a_n}{t} + b_n\right).$$
   As $t \to \pm\infty$, what point of $H = \mathbb{P}^{n-1}(\mathbb{R})$ do you get?
   b. The line L will have many parametrizations. Show that the point of $\mathbb{P}^{n-1}(\mathbb{R})$ given by part (a) is the same for all parametrizations of L. Hint: Two nonzero vectors are parallel if and only if one is a scalar multiple of the other.
   c. Parts (a) and (b) show that a line L in $\mathbb{R}^n$ has a well-defined point at infinity in $H = \mathbb{P}^{n-1}(\mathbb{R})$. Show that two lines in $\mathbb{R}^n$ are parallel if and only if they have the same point at infinity.
4. When $k = \mathbb{R}$ or $\mathbb{C}$, the projective line $\mathbb{P}^1(k)$ is easy to visualize.

a. In the text, we called $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. To see why this name is justified, use the parametrization from Exercise 6 of Chapter 1, §3 to show how the plane corresponds to the sphere minus the north pole. Then explain why we can regard $\mathbb{C} \cup \{\infty\}$ as a sphere.

b. What common geometric object can we use to represent $\mathbb{P}^1(\mathbb{R})$? Illustrate your reasoning with a picture.

5. Prove Corollary 3.
6. This problem studies the subsets $U_i \subset \mathbb{P}^n(k)$.

a. In $\mathbb{P}^4(k)$, identify the points that are in the subsets U_2, $U_2 \cap U_3$, and $U_1 \cap U_3 \cap U_4$.

b. Give an identification of $\mathbb{P}^4(k) - U_2$, $\mathbb{P}^4(k) - (U_2 \cup U_3)$, and $\mathbb{P}^4(k) - (U_1 \cup U_3 \cup U_4)$ as a "copy" of another projective space.

c. In $\mathbb{P}^4(k)$, which points are $\cap_{i=0}^4 U_i$?

d. In general, describe the subset $U_{i_1}, \cap \ldots \cap U_{i_s} \subset \mathbb{P}^n(k)$, where

$$1 \le i_1 < i_2 < \cdots < i \le n.$$

7. In this exercise, we will study when a nonhomogeneous polynomial has a well-defined zero set in $\mathbb{P}^n(k)$. Let k be an infinite field. We will show that if $f \in k[x_0, \ldots, x_n]$ is *not* homogeneous, but f vanishes on all homogeneous coordinates of some $p \in \mathbb{P}^n(k)$, then each of the homogeneous components f_i of f (see Definition 6 of Chapter 7, §1) must vanish at p.

a. Write f as a sum of its homogeneous components $f = \sum_i f_i$. If $p = (a_0, \ldots, a_n)$, then show that

$$\begin{aligned} f(\lambda a_0, \ldots, \lambda a_n) &= \sum_i f_i(\lambda a_0, \ldots, \lambda a_n) \\ &= \sum_i \lambda^i f_i(a_0, \ldots, a_n). \end{aligned}$$

b. Deduce that if f vanishes for all $\lambda \neq 0 \in k$, then $f_i(a_0, \ldots, a_n) = 0$ for all i.

8. By dehomogenizing the defining equations of the projective variety V, find equations for the indicated affine varieties.

a. Let $\mathbb{P}^2(\mathbb{R})$ have homogeneous coordinates (x, y, z) and let $V = \mathbf{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2(\mathbb{R})$. Find equations for $V \cap U_0$, $V \cap U_2$. (Here U_0 is where $x \neq 0$ and U_2 is where $z \neq 0$.) Sketch each of these curves and think about what this says about the projective variety V.

b. $V = \mathbf{V}(x_0x_2 - x_3x_4, x_0^2x_3 - x_1x_2^2) \subset \mathbb{P}^4(k)$ and find equations for the affine variety $V \cap U_0 \subset k^4$. Do the same for $V \cap U_3$.

9. Let $V = \mathbf{V}(f_1, \ldots, f_s)$ be a projective variety defined by homogeneous polynomials $f_i \in k[x_0, \ldots, x_n]$. Show that the subset $W = V \cap U_i$, can be identified with the affine variety $\mathbf{V}(g_1, \ldots, g_s) \subset k^n$ defined by the dehomogenized polynomials

$$g_j(x_1, \ldots x_i, x_{i+1}, \ldots, x_n) = f_j(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n).$$

Hint: Follow the proof of Proposition 6, using Corollary 3.

10. Prove Proposition 7.
11. Using part (iv) of Proposition 7, show that if $f \in k[x_1, \ldots, x_n]$ and $F \in k[x_0, \ldots, x_n]$ is any homogeneous polynomial satisfying $F(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$, then $F = x_0^e f^h$ for some $e \ge 0$.
12. What happens if we apply the homogenization process of Proposition 7 to a polynomial g that is itself homogeneous?

13. In Example 8, we were led to consider the variety $W' = \mathbf{V}(z^2 - x^3 + xz^2) \subset k^2$. Show that $(x, z) = (0, 0)$ is a singular point of W'. Hint: Use Definition 3 from Chapter 3, §4.
14. For each of the following affine varieties W, apply the homogenization process given in Proposition 7 to write $W = V \cap U_0$, where V is a projective variety. In each case identify $V - W = V \cap H$, where H is the hyperplane at infinity.
    a. $W = \mathbf{V}(y^2 - x^3 - ax - b) \subset \mathbb{R}^2$, $a, b \in \mathbb{R}$. Is the point $V \cap H$ singular here? Hint: Let the homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$ be (z, x, y), so that U_0 is where $z \neq 0$.
    b. $W = \mathbf{V}(x_1x_3 - x_2^2, x_1^2 - x_2) \subset \mathbb{R}^3$. Is there an extra component at infinity here?
    c. $W = \mathbf{V}(x_3^2 - x_1^2 - x_2^2) \subset \mathbb{R}^3$.
15. From Example 9, consider the twisted cubic $W = \mathbf{V}(x_2 - x_1^2, x_3 - x_1^3) \subset \mathbb{R}^3$.
    a. If we parametrize W by (t, t^2, t^3) in $\mathbb{R}^3$, show that as $t \to \pm\infty$, the point $(1, t, t^2, t^3)$ in $\mathbb{P}^3(\mathbb{R})$ approaches $(0, 0, 0, 1)$. Thus, we expect W to have one point at infinity.
    b. Now consider the projective variety

    $$V' = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3, x_1x_3 - x_2^2) \subset \mathbb{P}^3(\mathbb{R}).$$

    Show that $V' \cap U_0 = W$ and that $V' \cap H = \{(0, 0, 0, 1)\}$.
    c. Let $V = \mathbf{V}(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3)$ be as in Example 9. Prove that $V = V' \cup \mathbf{V}(x_0, x_1)$. This shows that V is a union of two proper projective varieties.
16. A homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ can also be used to define the *affine* variety $C = \mathbf{V}_a(f)$ in k^{n+1}, where the subscript denotes we are working in affine space. We call C the *affine cone* over the projective variety $V = \mathbf{V}(f) \subset \mathbb{P}^n(k)$. We will see why this is so in this exercise.
    a. Show that if C contains the point $P \neq (0, \ldots, 0)$, then C contains the whole line through the origin in k^{n+1} spanned by P.
    b. Now consider the point p in $\mathbb{P}^n(k)$ with homogeneous coordinates P. Show that p is in the projective variety V if and only if the line through the origin determined by P is contained in C. Hint: See (1) and Exercise 1.
    c. Deduce that C is the union of the collection of lines through the origin in k^{n+1} corresponding to the points in V via (1). This explains the reason for the "cone" terminology since an ordinary cone is also a union of lines through the origin. Such a cone is given by part (c) of Exercise 14.
17. Homogeneous polynomials satisfy an important relation known as *Euler's Formula.* Let $f \in k[x_0, \ldots, x_n]$ be homogeneous of total degree d. Then Euler's Formula states that

    $$\sum_{i=0}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = d \cdot f.$$

    a. Verify Euler's Formula for the homogeneous polynomial $f = x_0^3 - x_1x_2^2 + 2x_1x_3^2$.
    b. Prove Euler's Formula (in the case $k = \mathbb{R}$) by considering $f(\lambda x_0, \ldots, \lambda x_n)$ as a function of λ and differentiating with respect to λ using the chain rule.
18. In this exercise, we will consider the set of hyperplanes in $\mathbb{P}^n(k)$ in greater detail.
    a. Show that two homogeneous linear polynomials,

    $$\begin{aligned} 0 &= a_0x_0 + \cdots + a_nx_n, \\ 0 &= b_0x_0 + \cdots + b_nx_n, \end{aligned}$$

    define the same hyperplane in $\mathbb{P}^n(k)$ if and only if there is some $\lambda \neq 0 \in k$ such that $b_i = \lambda a_i$ for all $i = 0, \ldots, n$. Hint: Generalize the argument given for Exercise 11 of §1.

b. Show that the map sending the hyperplane with equation $a_0x_0 + \cdots + a_nx_n = 0$ to the vector $(a_0, \ldots, a_n)$ gives a one-to-one correspondence

$$\phi : \{\text{hyperplanes in } \mathbb{P}^n(k)\} \to (k^{n+1} - \{0\})/\sim,$$

where $\sim$ is the equivalence relation of Definition 1. The set on the left is called the *dual projective space* and is denoted $\mathbb{P}^n(k)^\vee$. Geometrically, the points of $\mathbb{P}^n(k)^\vee$ are hyperplanes in $\mathbb{P}^n(k)$.

c. Describe the subset of $\mathbb{P}^n(k)^\vee$ corresponding to the hyperplanes containing $p = (1, 0, \ldots, 0)$.

19. Let k be an algebraically closed field ($\mathbb{C}$, for example). Show that every homogeneous polynomial $f(x_0, x_1)$ in two variables with coefficients in k can be completely factored into linear homogeneous polynomials in $k[x_0, x_1]$:

$$f(x_0, x_1) = \prod_{i=1}^{d}(a_ix_0 + b_ix_1),$$

where d is the total degree of f. Hint: First dehomogenize f.

20. In §4 of Chapter 5, we introduced the *pencil* defined by two hypersurfaces $V = \mathbf{V}(f)$, $W = \mathbf{V}(g)$. The elements of the pencil were the hypersurfaces $\mathbf{V}(f+cg)$ for $c \in k$. Setting $c = 0$, we obtain V as an element of the pencil. However, W is not (usually) an element of the pencil when it is defined in this way. To include W in the pencil, we can proceed as follows.

a. Let (a, b) be homogeneous coordinates in $\mathbb{P}^1(k)$. Show that $\mathbf{V}(af + bg)$ is well-defined in the sense that all homogeneous coordinates (a, b) for a given point in $\mathbb{P}^1(k)$ yield the same variety $\mathbf{V}(af + bg)$. Thus, we obtain a family of varieties parametrized by $\mathbb{P}^1(k)$, which is also called the *pencil* of varieties defined by V and W.

b. Show that both V and W are contained in the pencil $\mathbf{V}(af + bg)$.

c. Let $k = \mathbb{C}$. Show that every affine curve $\mathbf{V}(f) \subset \mathbb{C}^2$ defined by a polynomial f of total degree d is contained in a pencil of curves $\mathbf{V}(aF + bG)$ parametrized by $\mathbb{P}^1(\mathbb{C})$, where $\mathbf{V}(F)$ is a union of lines and G is a polynomial of degree strictly less than d. Hint: Consider the homogeneous components of f. Exercise 19 will be useful.

21. When we have a curve parametrized by $t \in k$, there are many situations where we want to let $t \to \infty$. Since $\mathbb{P}^1(k) = k \cup \{\infty\}$, this suggests that we should let our parameter space be $\mathbb{P}^1(k)$. Here are two examples of how this works.

a. Consider the parametrization $(x, y) = ((1+t^2)/(1-t^2), 2t/(1-t^2))$ of the hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$. To make this projective, we first work in $\mathbb{P}^2(\mathbb{R})$ and write the parametrization as

$$((1+t^2)/(1-t^2), 2t/(1-t^2), 1) = (1+t^2, 2t, 1-t^2)$$

(see Exercise 3 of §1). The next step is to make t projective. Given $(a, b) \in \mathbb{P}^1(\mathbb{R})$, we can write it as $(1, t) = (1, b/a)$ provided $a \neq 0$. Now substitute $t = b/a$ into the right-hand side and clear denominators. Explain why this gives a well-defined map $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.

b. The twisted cubic in $\mathbb{R}^3$ is parametrized by (t, t^2, t^3). Apply the method of part (a) to obtain a projective parametrization $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$ and show that the image of this map is precisely the projective variety V' from Example 9.

§3 The Projective Algebra–Geometry Dictionary

In this section, we will study the algebra-geometry dictionary for projective varieties. Our goal is to generalize the theorems from Chapter 4 concerning the $\mathbf{V}$ and $\mathbf{I}$ correspondences to the projective case, and, in particular, we will prove a projective version of the Nullstellensatz.

To begin, we note one difference between the affine and projective cases on the algebraic side of the dictionary. Namely, in Definition 5 of §2, we introduced projective varieties as the common zeros of collections of *homogeneous* polynomials. But being homogeneous is *not* preserved under the sum operation in $k[x_0, \ldots, x_n]$. For example, if we add two homogeneous polynomials of different total degrees, the sum will *never* be homogeneous. Thus, if we form the ideal $I = \langle f_1, \ldots, f_s \rangle \subset k[x_0, \ldots, x_n]$ generated by a collection of homogeneous polynomials, I will contain many *nonhomogeneous* polynomials and these would not be candidates for the defining equations of a projective variety.

Nevertheless, each element of I vanishes on all homogeneous coordinates of every point of $V = \mathbf{V}(f_1, \ldots, f_s)$. This follows because each $g \in I$ has the form

$$(1) \qquad g = \sum_{j=1}^{s} A_j f_j$$

for some $A_j \in k[x_0, \ldots, x_n]$. Substituting any homogeneous coordinates of a point in V into g will yield zero since each f_i is zero there.

A more important observation concerns the homogeneous components of g. Suppose we expand each A_j as the sum of its homogeneous components:

$$A_j = \sum_{i=1}^{d} A_{ji}.$$

If we substitute these expressions into (1) and collect terms of the same total degree, it can be shown that the homogeneous components of g also lie in the ideal $I = \langle f_1, \ldots, f_s \rangle$. You will prove this claim in Exercise 2.

Thus, although I contains nonhomogeneous elements g, we see that I also contains the homogeneous components of g. This observation motivates the following definition of a special class of ideals in $k[x_0, \ldots, x_n]$.

Definition 1. *An ideal I in $k[x_0, \ldots, x_n]$ is said to be* **homogeneous** *if for each $f \in I$, the homogeneous components f_i of f are in I as well.*

Most ideals *do not* have this property. For instance, let $I = \langle y - x^2 \rangle \subset k[x, y]$. The homogeneous components of $f = y - x^2$ are $f_1 = y$ and $f_2 = -x^2$. Neither of these polynomials is in I since neither is a multiple of $y - x^2$. Hence, I is not a homogeneous ideal. However, we have the following useful characterization of when an ideal is homogeneous.

Theorem 2. *Let $I \subset k[x_0, \ldots, x_n]$ be an ideal. Then the following are equivalent:*

(i) *I is a homogeneous ideal of $k[x_0, \ldots, x_n]$.*
(ii) *$I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous polynomials.*
(iii) *A reduced Groebner basis of I (with respect to any monomial ordering) consists of homogeneous polynomials.*

Proof. The proof of (ii) $\Rightarrow$ (i) was sketched above (see also Exercise 2). To prove (i) $\Rightarrow$ (ii), let I be a homogeneous ideal. By the Hilbert Basis Theorem, we have $I = \langle F_1, \ldots, F_t \rangle$ for some polynomials $F_j \in k[x_0, \ldots, x_n]$ (not necessarily homogeneous). If we write F_j as the sum of its homogeneous components, say $F_j = \sum_i F_{ji}$, then each $F_{ji} \in I$ since I is homogeneous. Let I' be the ideal generated by the homogeneous polynomials F_{ji}. Then $I \subset I'$ since each F_j is a sum of generators of I'. On the other hand, $I' \subset I$ since each of the homogeneous components of F_j is in I. This proves $I = I'$ and it follows that I has a basis of homogeneous polynomials. Finally, the equivalence (ii) $\Leftrightarrow$ (iii) will be covered in Exercise 3. □

As a result of Theorem 2, for any homogeneous ideal $I \subset k[x_0, \ldots, x_n]$ we may define

$$\mathbf{V}(I) = \{p \in \mathbb{P}^n(k) : \; f(p) = 0 \text{ for all } f \in I\},$$

as in the affine case. We can prove that $\mathbf{V}(I)$ is a projective variety as follows.

Proposition 3. *Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal and suppose that $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous. Then*

$$\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s),$$

so that $\mathbf{V}(I)$ is a projective variety.

Proof. We leave the easy proof as an exercise. □

One way to create a homogeneous ideal is to consider the ideal generated by the defining equations of a projective variety. But there is another way that a projective variety can give us a homogeneous ideal.

Proposition 4. *Let $V \subset \mathbb{P}^n(k)$ be a projective variety and let*

$$\mathbf{I}(V) = \{f \in k[x_0, \ldots, x_n] \; : f(a_0, \ldots, a_n) \; = 0 \textit{ for all } (a_0, \ldots, a_n) \in \mathbf{V}\}.$$

(This means that f must be zero for all homogeneous coordinates of all points in V.) If k is infinite, then $\mathbf{I}(V)$ is a homogeneous ideal in $k[x_0, \ldots, x_n]$.

Proof. $\mathbf{I}(V)$ is closed under sums and closed under products by elements of $k[x_0, \ldots, x_n]$ by an argument exactly parallel to the one for the affine case. Thus, $\mathbf{I}(V)$ is an ideal in $k[x_0, \ldots, x_n]$. Now take $f \in \mathbf{I}(V)$ and fix a point $p \in V$. By assumption, f vanishes at all homogeneous coordinates $(a_0, \ldots, a_n)$ of p. Since k is infinite, Exercise 7 of §2 then implies that each homogeneous component f_i of f vanishes at $(a_0, \ldots, a_n)$. This shows that $f_i \in \mathbf{I}(V)$ and, hence, $\mathbf{I}(V)$ is homogeneous. □

Thus, we have all the ingredients of a dictionary relating projective varieties in $\mathbb{P}^n(k)$ and homogeneous ideals in $k[x_0, \ldots, x_n]$. The following theorem is a direct generalization of part (i) of Theorem 7 of Chapter 4, §2 (the affine ideal-variety correspondence).

Theorem 5. *Let k be an infinite field. Then the maps*

$$\text{projective varieties} \xrightarrow{\mathbf{I}} \text{homogeneous ideals}$$

and

$$\text{homogeneous ideals} \xrightarrow{\mathbf{V}} \text{projective varieties}$$

are inclusion-reversing. Furthermore, for any projective variety, we have

$$\mathbf{V}(\mathbf{I}(V)) = V.$$

so that $\mathbf{I}$ *is always one-to-one.*

Proof. The proof is the same as in the affine case. □

To illustrate the use of this theorem, let us show that every projective variety can be decomposed to irreducible components. As in the affine case, a variety $V \subset \mathbb{P}^n(k)$ is *irreducible* if it cannot be written as a union of two strictly smaller projective varieties.

Theorem 6. *Let k be an infinite field.*

(i) *Given a descending chain of projective varieties in* $\mathbb{P}^n(k)$,

$$V_1 \supset V_2 \supset V_3 \supset \cdots,$$

there is an integer N such that $V_N = V_{N+1} = \cdots$.

(ii) *Every projective variety* $V \subset \mathbb{P}^n(k)$ *can be written uniquely as a finite union of irreducible projective varieties*

$$V = V_1 \cup \cdots \cup V_m,$$

where $V_i \not\subset V_j$ *for* $i \neq j$.

Proof. Since $\mathbf{I}$ is inclusion-reversing, we get the ascending chain of homogeneous ideals

$$\mathbf{I}(V_1) \subset \mathbf{I}(V_2) \subset \mathbf{I}(V_3) \subset \cdots$$

in $k[x_0, \ldots, x_n]$. Then the Ascending Chain Condition (Theorem 7 of Chapter 2, §5) implies that $\mathbf{I}(V_N) = \mathbf{I}(V_{N+1}) = \cdots$ for some N. By Theorem 5, $\mathbf{I}$ is one-to-one and (i) follows immediately.

As in the affine case, (ii) is an immediate consequence of (i). See Theorems 2 and 4 of Chapter 4, §6. □

The relation between operations such as sums, products, and intersections of homogeneous ideals and the corresponding operations on projective varieties is also the same as in affine space. We will consider these topics in more detail in the exercises below.

We define the radical of a homogeneous ideal as usual:

$$\sqrt{I} = \{f \in k[x_0, \ldots, x_n] \ : \ f^m \in I \text{ for some } m \geq 1\}.$$

As we might hope, the radical of a homogeneous ideal is always itself homogeneous.

Proposition 7. *Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then $\sqrt{I}$ is also a homogeneous ideal.*

Proof. If $f \in \sqrt{I}$, then $f^m \in I$ for some $m \geq 1$. If $f \neq 0$, decompose f into its homogeneous components

$$f = \sum_i f_i = f_{max} + \sum_{i<max} f_i,$$

where f_{max} is the nonzero homogeneous component of maximal total degree in f. Expanding the power f^m, it is easy to show that

$$(f^m)_{max} = (f_{max})^m.$$

Since I is a homogeneous ideal, $(f^m)_{max} \in I$. Hence, $(f_{max})^m \in I$, which shows that $f_{max} \in \sqrt{I}$.

If we let $g = f - f_{max} \in \sqrt{I}$ and repeat the argument, we get $g_{max} \in \sqrt{I}$. But g_{max} is also one of the homogeneous components of f. Applying this reasoning repeatedly shows that all homogeneous components of f are in $\sqrt{I}$. Since this is true for all $f \in \sqrt{I}$, Definition 1 implies that $\sqrt{I}$ is a homogeneous ideal. □

The final part of the algebra–geometry dictionary concerns what happens over an algebraically closed field k. Here, we expect an especially close relation between projective varieties and homogeneous ideals. In the affine case, the link was provided by two theorems proved in Chapter 4, the Weak Nullstellensatz and the Strong Nullstellensatz. Let us recall what these theorems tell us about an ideal $I \subset k[x_1, \ldots, x_n]$:

- (The Weak Nullstellensatz) $\mathbf{V}_a(I) = \emptyset$ in $k^n \iff I = k[x_1, \ldots, x_n]$.
- (The Strong Nullstellensatz) $\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I))$ in $k[x_1, \ldots, x_n]$.

(To prevent confusion, we use $\mathbf{I}_a$ and $\mathbf{V}_a$ to denote the affine versions of $\mathbf{I}$ and $\mathbf{V}$.) It is natural to ask if these results extend to projective varieties and homogeneous ideals.

The answer, surprisingly, is *no*. In particular, the Weak Nullstellensatz fails for certain homogeneous ideals. To see how this can happen, consider the ideal $I = \langle x_0, \ldots, x_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$. Then $\mathbf{V}(I) \subset \mathbb{P}^n(\mathbb{C})$ is defined by the equations $x_0 = \cdots = x_n = 0$. The only solution is $(0, \ldots, 0)$, but this is impossible since we never allow all homogeneous coordinates to vanish simultaneously. It follows that $\mathbf{V}(I) = \emptyset$, yet $I \neq \mathbb{C}[x_0, \ldots, x_n]$.

Fortunately, $I = \langle x_0, \ldots, x_n \rangle$ is one of the few ideals for which $\mathbf{V}(I) = \emptyset$. The following projective version of the Weak Nullstellensatz describes *all* homogeneous ideals with no projective solutions.

Theorem 8 (The Projective Weak Nullstellensatz). *Let k be algebraically closed and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. Then the following are equivalent:*

(i) $\mathbf{V}(I) \subset \mathbb{P}^n(k)$ *is empty.*

(ii) *Let G be a reduced Groebner basis for I (with respect to some monomial ordering). Then for each $0 \leq i \leq n$, there is $g \in G$ such that $\mathrm{LT}(g)$ is a nonnegative power of x_i.*

(iii) *For each $0 \leq i \leq n$, there is an integer $m_i \geq 0$ such that $x_i^{m_i} \in I$.*

(iv) *There is some $r \geq 1$ such that $\langle x_0, \ldots, x_n \rangle^r \subset I$.*

Proof. The ideal I gives us the projective variety $V = \mathbf{V}(I) \subset \mathbb{P}^n(k)$. In this proof, we will also work with the *affine* variety $C_V = \mathbf{V}_a(I) \subset k^{n+1}$. Note that C_V uses the same ideal I, but now we look for solutions in the affine space k^{n+1}. We call C_V the *affine cone* of V. If we interpret points in $\mathbb{P}^n(k)$ as lines through the origin in k^{n+1}, then C_V is the union of the lines determined by the points of V (see Exercise 16 of §2 for the details of how this works). In particular, C_V contains all homogeneous coordinates of the points in V.

To prove (ii) $\Rightarrow$ (i), first suppose that we have a Groebner basis where, for each i, there is $g \in G$ with $\mathrm{LT}(g) = x_i^{m_i}$ for some $m_i \geq 0$. Then Theorem 6 of Chapter 5, §3 implies that C_V is a finite set. But suppose there is a point $p \in V$. Then *all* homogeneous coordinates of p lie in C_V. If we write these in the form $\lambda(a_0, \ldots, a_n)$, we see that there are infinitely many since k is algebraically closed and hence infinite. This contradiction shows that $V = \mathbf{V}(I) = \emptyset$.

Turning to (iii) $\Rightarrow$ (ii), let G be a reduced Groebner basis for I. Then $x_i^{m_i} \in I$ implies that the leading term of some $g \in G$ divides $x_i^{m_i}$, so that $\mathrm{LT}(g)$ must be a power of x_i.

The proof of (iv) $\Rightarrow$ (iii) is obvious since $\langle x_0, \ldots, x_n \rangle^r \subset I$ implies $x_i^r \in I$ for all i. It remains to prove (i) $\Rightarrow$ (iv). We first observe that $V = \emptyset$ implies

$$C_V \subset \{(0, \ldots, 0)\} \text{ in } k^{n+1}.$$

This follows because a nonzero point $(a_0, \ldots, a_n)$ in the affine cone C_V would give homogeneous coordinates of a point in $V \subset \mathbb{P}^n(k)$, which would contradict $V = \emptyset$. Then, applying $\mathbf{I}_a$, we obtain

$$\mathbf{I}_a(\{(0, \ldots, 0)\}) \subset \mathbf{I}_a(C_V).$$

We know $\mathbf{I}_a(\{(0, \ldots, 0)\}) = \langle x_0, \ldots, x_n \rangle$ (see Exercise 7 of Chapter 4, §5) and the affine version of the Strong Nullstellensatz implies $\mathbf{I}_a(C_V) = \mathbf{I}_a(\mathbf{V}_a(I)) = \sqrt{I}$ since k is algebraically closed. Combining these facts, we conclude that

$$\langle x_0, \ldots, x_n \rangle \subset \sqrt{I}.$$

However, in Exercise 12 of Chapter 4, §3 we showed that if some ideal is contained in $\sqrt{I}$, then a power of the ideal lies in I. This completes the proof of the theorem. □

From part (ii) of the theorem, we get an algorithm for determining if a homogeneous ideal has projective solutions over an algebraically closed field. In Exercise 10, we will discuss other conditions which are equivalent to $\mathbf{V}(I) = \emptyset$ in $\mathbb{P}^n(k)$.

Once we exclude the ideals described in Theorem 8, we get the following form of the Nullstellensatz for projective varieties.

Theorem 9 (The Projective Strong Nullstellensatz). *Let k be an algebraically closed field and let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. If $V = \mathbf{V}(I)$ is a nonempty projective variety in $\mathbb{P}^n(k)$, then we have*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Proof. As in the proof of Theorem 8, we will work with the projective variety $V = \mathbf{V}(I) \subset \mathbb{P}^n(k)$ and its affine cone $C_V = \mathbf{V}_a(I) \subset k^{n+1}$. We first claim that

$$\mathbf{I}_a(C_V) = \mathbf{I}(V) \tag{2}$$

when $V \neq \emptyset$. To see this, suppose that $f \in \mathbf{I}_a(C_V)$. Given $p \in V$, any homogeneous coordinates of p lie in C_V, so that f vanishes at all homogeneous coordinates of p. By definition, this implies $f \in \mathbf{I}(V)$. Conversely, take $f \in \mathbf{I}(V)$. Since any nonzero point of C_V gives homogeneous coordinates for a point in V, it follows that f vanishes on $C_V - \{0\}$. It remains to show that f vanishes at the origin. Since $\mathbf{I}(V)$ is a homogeneous ideal, we know that the homogeneous components f_i of f also vanish on V. In particular, the constant term of f, which is the homogeneous component f_0 of total degree 0, must vanish on V. Since $V \neq \emptyset$, this forces $f_0 = 0$, which means that f vanishes at the origin. Hence, $f \in \mathbf{I}_a(C_V)$ and (2) is proved.

By the affine form of the Strong Nullstellensatz, we know that $\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I))$. Then, using (2), we obtain

$$\sqrt{I} = \mathbf{I}_a(\mathbf{V}_a(I)) = \mathbf{I}_a(C_V) = \mathbf{I}(V) = \mathbf{I}(\mathbf{V}(I)).$$

which completes the proof of the theorem. □

Now that we have the Nullstellensatz, we can complete the projective ideal–variety correspondence begun in Theorem 5. A radical homogeneous ideal in $k[x_0, \ldots, x_n]$ is a homogeneous ideal satisfying $\sqrt{I} = I$. As in the affine case, we have a one-to-one correspondence between projective varieties and radical homogeneous ideals, provided we exclude the cases $\sqrt{I} = \langle x_0, \ldots, x_n \rangle$ and $\sqrt{I} = \langle 1 \rangle$.

Theorem 10. *Let k be an algebraically closed field. If we restrict the correspondences of Theorem 5 to nonempty projective varieties and radical homogeneous ideals properly contained in $\langle x_0, \ldots, x_n \rangle$, then*

$$\{\text{nonempty projective varieties}\} \xrightarrow{\mathrm{I}} \left\{\begin{matrix}\text{radical homogeneous ideals} \\ \text{properly contained in} \langle x_0, \ldots, x_n \rangle\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\text{radical homogeneous ideals} \\ \text{properly contained in} \langle x_0, \ldots, x_n \rangle\end{matrix}\right\} \xrightarrow{\mathrm{V}} \{\text{nonempty projective varieties}\}$$

are inclusion-reversing bijections which are inverses of each other.

Proof. First, it is an easy consequence of Theorem 8 that the only radical homogeneous ideals I with $\mathbf{V}(I) = \emptyset$ are $\langle x_0, \dots, x_n \rangle$ and $k[x_0, \dots, x_n]$. See Exercise 10 for the details. A second observation is that if I is a homogeneous ideal different from $k[x_0, \dots, x_n]$, then $I \subset \langle x_0, \dots, x_n \rangle$. This will also be covered in Exercise 9.

These observations show that the radical homogeneous ideals with $\mathbf{V}(I) \neq \emptyset$ are precisely those which satisfy $I \subsetneqq \langle x_0, \dots, x_n \rangle$. Then the rest of the theorem follows as in the affine case, using Theorem 9. □

We also have a correspondence between *irreducible* projective varieties and homogeneous *prime* ideals, which will be studied in the exercises.

EXERCISES FOR §3

1. In this exercise, you will study the question of determining when a principal ideal $I = \langle f \rangle$ is homogeneous by elementary methods.
   a. Show that $I = \langle x^2y - x^3 \rangle$ is a homogeneous ideal in $k[x, y]$ without appealing to Theorem 2. Hint: Each element of I has the form $g = A \cdot (x^2y - x^3)$. Write A as the sum of its homogeneous components and use this to determine the homogeneous components of g.
   b. Show that $\langle f \rangle \subset k[x_0, \dots, x_n]$ is a homogeneous ideal if and only if f is a homogeneous polynomial without using Theorem 2.
2. This exercise gives some useful properties of the homogeneous components of polynomials.
   a. Show that if $f = \sum_i f_i$ and $g = \sum_i g_i$ are the expansions of two polynomials as the sums of their homogeneous components, then $f = g$ if and only if $f_i = g_i$ for all i.
   b. Show that if $f = \sum_i f_i$ and $g = \sum_j g_j$ are the expansions of two polynomials as the sums of their homogeneous components, then the homogeneous components h_k of the product $h = f \cdot g$ are given by $h_k = \sum_{i+j=k} f_i \cdot g_j$.
   c. Use parts (a) and (b) to carry out the proof (sketched in the text) of the implication (ii) $\Rightarrow$ (i) from Theorem 2.
3. This exercise will study how the algorithms of Chapter 2 interact with homogeneous polynomials.
   a. Suppose we use the division algorithm to divide a homogeneous polynomial f by homogeneous polynomials $f_1, \dots, f_s$. This gives an expression of the form $f = a_1 f_1 + \cdots + a_s f_s + r$. Prove that the quotients $a_1, \dots, a_s$ and remainder r are homogeneous polynomials (possibly zero). What is the total degree of r?
   b. If f, g are homogeneous polynomials, prove that the S-polynomial $S(f, g)$ is homogeneous.
   c. By analyzing the Buchberger algorithm, show that a homogeneous ideal has a homogeneous Groebner basis.
   d. Prove the implication (ii) $\Leftrightarrow$ (iii) of Theorem 2.
4. Suppose that an ideal $I \subset k[x_0, \dots, x_n]$ has a basis G consisting of homogeneous polynomials.
   a. Prove that G is a Groebner basis for I with respect to lex order if and only if it is a Groebner basis for I with respect to grlex (assuming that the variables are ordered the same way).
   b. Conclude that, for a homogeneous ideal, the reduced Groebner basis for lex and grlex are the same.
5. Prove Proposition 3.

6. In this exercise we study the algebraic operations on ideals introduced in Chapter 4 for homogeneous ideals. Let $I_1, \ldots, I_l$ be homogeneous ideals in $k[x_0, \ldots, x_n]$.
   a. Show that the ideal sum $I_1 + \cdots + I_l$ is also homogeneous. Hint: Use Theorem 2.
   b. Show that the intersection $I_l \cap \cdots \cap I_l$ is also a homogeneous ideal.
   c. Show that the ideal product $I_1 \cdots I_l$ is a homogeneous ideal.
7. The interaction between the algebraic operations on ideals in Exercise 6 and the corresponding operations on projective varieties is the same as in the affine case. Let $I_1, \ldots, I_l$ be homogeneous ideals in $k[x_0, \ldots, x_n]$ and let $V_i = \mathbf{V}(I_i)$ be the corresponding projective variety in $\mathbb{P}^n(k)$.
   a. Show that $\mathbf{V}(I_1 + \cdots + I_l) = \bigcap_{i=1}^l V_i$.
   b. Show that $\mathbf{V}(I_1 \cap \cdots \cap I_l) = \mathbf{V}(I_1 \cdots I_l) = \bigcup_{i=1}^l V_i$.
8. Let $f_1, \ldots, f_s$ be homogeneous polynomials of total degrees $d_1 < d_2 \leq \cdots \leq d_s$ and let $I = \langle f_1, \ldots, f_s \rangle \subset k[x_0, \ldots, x_n]$.
   a. Show that if g is another homogeneous polynomial of degree d_1 in I, then g must be a constant multiple of f_1. Hint: Use parts (a) and (b) of Exercise 2.
   b. More generally, show that if the total degree of g is d, then g must be an element of the ideal $I_d = \langle f_i : \deg(f_i) \leq d \rangle \subset I$.
9. This exercise will study some properties of the ideal $I_0 = \langle x_0, \ldots, x_n \rangle \subset k[x_0, \ldots, x_n]$.
   a. Show that every proper homogeneous ideal in $k[x_0, \ldots, x_n]$ is contained in I_0.
   b. Show that the r-th power I_0^r is the ideal generated by the collection of monomials in $k[x_0, \ldots, x_n]$ of total degree exactly r and deduce that every homogeneous polynomial of degree $\geq r$ is in I_0^r.
   c. Let $V = \mathbf{V}(I_0) \subset \mathbb{P}^n(k)$ and $C_V = \mathbf{V}_a(I_0) \subset k^{n+1}$. Show that $\mathbf{I}_a(C_V) \neq \mathbf{I}(V)$, and explain why this does not contradict equation (2) in the text.
10. Given a homogeneous ideal $I \subset k[x_0, \ldots, x_n]$, where k is algebraically closed, prove that $\mathbf{V}(I) = \emptyset$ in $\mathbb{P}^n(k)$ is equivalent to either of the following two conditions:
    (i) There is an $r \geq 1$ such that every homogeneous polynomial of total degree $\geq r$ is contained in I.
    (ii) The radical of I is either $\langle x_0, \ldots, x_n \rangle$ or $k[x_0, \ldots, x_n]$.
    Hint: For (i), use Exercise 9, and for (ii), use the proof of Theorem 8 to show that $\langle x_0, \ldots, x_0 \rangle \subset \sqrt{I}$.
11. A homogeneous ideal is said to be *prime* if it is prime as an ideal in $k[x_0, \ldots, x_n]$.
    a. Show that a homogeneous ideal $I \subset k[x_0, \ldots, x_n]$ is prime if and only if whenever the product of two *homogeneous* polynomials F, G satisfies $F \cdot G \in I$, then $F \in I$ or $G \in I$.
    b. Let k be algebraically closed. Let I be a homogeneous ideal. Show that the projective variety $\mathbf{V}(I)$ is irreducible if I is prime. Also, when I is radical, prove that the converse holds, i.e., that I is prime if $\mathbf{V}(I)$ is irreducible. Hint: Consider the proof of the corresponding statement in the affine case (Proposition 3 of Chapter 4, §5).
    c. Let k be algebraically closed. Show that the mappings $\mathbf{V}$ and $\mathbf{I}$ induce one-to-one correspondence between homogeneous prime ideals in $k[x_0, \ldots, x_n]$ properly contained in $\langle x_0, \ldots, x_n \rangle$ and nonempty irreducible projective varieties in $\mathbb{P}^n(k)$.
12. Prove that a homogeneous prime ideal is a radical ideal in $k[x_0, \ldots, x_n]$.

§4 The Projective Closure of an Affine Variety

In §2, we showed that any affine variety could be regarded as the affine portion of a projective variety. Since this can be done in more than one way (see Example 9 of §2), we would like to find the *smallest* projective variety containing a given affine variety. As we will see, there is an algorithmic way to do this.

Given homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n(k)$, we have the subset $U_0 \subset \mathbb{P}^n(k)$ defined by $x_0 \neq 0$. If we identify U_0 with k^n using Proposition 2 of §2, then we get coordinates $x_1, \ldots, x_n$ on k^n. As in §3, we will use $\mathbf{I}_a$ and $\mathbf{V}_a$ for the affine versions of $\mathbf{I}$ and $\mathbf{V}$.

We first discuss how to homogenize an ideal of $k[x_1, \ldots, x_n]$. Given $I \subset k[x_1, \ldots, x_n]$, the standard way to produce a homogeneous ideal $I^h \subset k[x_0, \ldots, x_n]$ is as follows.

Definition 1. *Let I be an ideal in $k[x_1, \ldots, x_n]$. We define the* **homogenization of** *I to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset k[x_0, \ldots, x_n],$$

where f^h is the homogenization of f as in Proposition 7 of §2.

Naturally enough, we have the following result.

Proposition 2. *For any ideal $I \subset k[x_1, \ldots, x_n]$, the homogenization I^h is a homogeneous ideal in $k[x_0, \ldots, x_n]$.*

Proof. See Exercise 1. □

Definition 1 is not entirely satisfying as it stands because it does not give us a *finite* generating set for the ideal I^h. There is a subtle point here. Given a particular finite generating set $f_1, \ldots, f_s$ for $I \subset k[x_1, \ldots, x_n]$, it is always true that $\langle f_1^h, \ldots, f_s^h \rangle$ is a homogeneous ideal contained in I^h. However, as the following example shows, I^h can be *strictly larger* than $\langle f_1^h, \ldots, f_s^h \rangle$.

Example 3. Consider $I = \langle f_1, f_2 \rangle = \langle x_2 - x_1^2, x_3 - x_1^3 \rangle$, the ideal of the affine twisted cubic in $\mathbb{R}^3$. If we homogenize f_1, f_2, then we get the ideal $J = \langle x_2 x_0 - x_1^2, x_3 x_0^2 - x_1^3 \rangle$ in $\mathbb{R}[x_0, x_1, x_2, x_3]$. We claim that $J \neq I^h$. To prove this, consider the polynomial

$$f_3 = f_2 - x_1 f_1 = x_3 - x_1^3 - x_1(x_2 - x_1^2) = x_3 - x_1 x_2 \in I.$$

Then $f_3^h = x_0 x_3 - x_1 x_2$ is a homogeneous polynomial of degree 2 in I^h. Since the generators of J are also homogeneous, of degrees 2 and 3, respectively, if we had an equation $f_3^h = A_1 f_1^h + A_2 f_2^h$, then using the expansions of A_1 and A_2 into homogeneous components, we would see that f_3^h was a constant multiple of f_1^h. (See Exercise 8 of §3 for a general statement along these lines.) Since this is clearly false, we have $f_3^h \notin J$, and thus, $J \neq I^h$.

Hence, we may ask whether there is some reasonable method for computing a finite generating set for the ideal I^h. The answer is given in the following theorem. A *graded* monomial order in $k[x_1, \ldots, x_n]$ is one that orders first by total degree:

$$x^\alpha > x^\beta$$

whenever $|\alpha| > |\beta|$. Note that grlex and grevlex are graded orders, whereas lex is not.

Theorem 4. *Let I be an ideal in $k[x_1, \ldots, x_n]$ and let $G = \{g_1, \ldots, g_t\}$ be a Groebner basis for I with respect to a graded monomial order in $k[x_1, \ldots, x_n]$. Then $G^h = \{g_1^h, \ldots, g_t^h\}$ is a basis for $I^h \subset k[x_0, \ldots, x_n]$.*

Proof. We will prove the theorem by showing the stronger statement that G^h is actually a Groebner basis for I^h with respect to an appropriate monomial order in $k[x_0, \ldots, x_n]$.

Every monomial in $k[x_0, \ldots, x_n]$ can be written

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_0^d = x^\alpha x_0^d,$$

where x^α contains no x_0 factors. Then we can extend the graded order $>$ on monomials in $k[x_1, \ldots, x_n]$ to a monomial order $>_h$ in $k[x_0, \ldots, x_n]$ as follows:

$$x^\alpha x_0^d >_h x^\beta x_0^e \iff x^\alpha > x^\beta \quad \text{or} \quad x^\alpha = x^\beta \text{ and } d > e.$$

In Exercise 2, you will show that this defines a monomial order in $k[x_0, \ldots, x_n]$. Note that under this ordering, we have $x_i >_h x_0$ for all $i \geq 1$.

For us, the most important property of the order $>_h$ is given in the following lemma.

Lemma 5. *If $f \in k[x_1, \ldots, x_n]$ and $>$ is a graded order on $k[x_1, \ldots, x_n]$, then*

$$\mathrm{LM}_{>_h}(f^h) = \mathrm{LM}_>(f).$$

Proof of Lemma. Since $>$ is a graded order, for any $f \in k[x_1, \ldots, x_n]$, $\mathrm{LM}_>(f)$ is one of the monomials x^α appearing in the homogeneous component of f of *maximal* total degree. When we homogenize, this term is unchanged. If $x^\beta x_0^e$ is any one of the other monomials appearing in f^h, then $\alpha > \beta$. By the definition of $>_h$, it follows that $x^\alpha >_h x^\beta x_0^e$. Hence, $x^\alpha = \mathrm{LM}_{>_h}(f^h)$, and the lemma is proved. □

We will now show that G^h forms a Groebner basis for the ideal I^h with respect to the monomial order $>_h$. Each $g_i^h \in I^h$ by definition. Thus, it suffices to show that the ideal of leading terms $\langle \mathrm{LT}_{>_h}(I^h) \rangle$ is generated by $\mathrm{LT}_{>_h}(G^h)$. To prove this, consider $F \in I^h$. Since I^h is a homogeneous ideal, each homogeneous component of F is in I^h and, hence, we may assume that F is homogeneous. Because $F \in I^h$, by definition we have

$$F = \sum_j A_j f_j^h, \tag{1}$$

where $A_j \in k[x_0, \ldots, x_n]$ and $f_j \in I$, We will let $f = F(1, x_1, \ldots, x_n)$ denote the dehomogenization of F. Then setting $x_0 = 1$ in (1) yields

$$\begin{aligned} f = F(1, x_1, \ldots, x_n) &= \sum_j A_j(1, x_1, \ldots, x_n) f_j^h(1, x_1, \ldots, x_n) \\ &= \sum_j A_j(1, x_1, \ldots, x_n) f_j \end{aligned}$$

since $f_j^h(1, x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n)$ by part (iii) of Proposition 7 from §2. This shows that $f \in I \subset k[x_1, \ldots, x_n]$. If we homogenize f, then part (iv) of Proposition 7 in §2 implies that

$$F = x_0^e \cdot f^h$$

for some $e \geq 0$. Thus,

$$\text{(2)} \qquad \text{LM}_{>_h}(F) = x_0^e \cdot \text{LM}_{>_h}(f^h) = x_0^e \cdot \text{LM}_{>}(f),$$

where the last equality is by Lemma 5. Since G is a Groebner basis for I, we know that $\text{LM}_{>}(f)$ is divisible by some $\text{LM}_{>}(g_i) = \text{LM}_{>_h}(g_i^h)$ (using Lemma 5 again). Then (2) shows that $\text{LM}_{>_h}(F)$ is divisible by $\text{LM}_{>_h}(g_i^h)$, as desired. This completes the proof of the theorem. □

In Exercise 5, you will see that there is a more elegant formulation of Theorem 4 for the special case of grevlex order.

To illustrate the theorem, consider the ideal $I = \langle x_2 - x_1^2, x_3 - x_1^3 \rangle$ of the affine twisted cubic $W \subset \mathbb{R}^3$ once again. Computing a Groebner basis for I with respect to grevlex order, we find

$$G = \left\{ x_1^2 - x_2, x_1x_2 - x_3, x_1x_3 - x_2^2 \right\}.$$

By Theorem 4, the homogenizations of these polynomials generate I^h. Thus,

$$\text{(3)} \qquad I^h = \left\langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_2^2 \right\rangle.$$

Note that this ideal gives us the projective variety $V' = \mathbf{V}(I^h) \subset \mathbb{P}^3(\mathbb{R})$ which we discovered in Example 9 of §2.

For the remainder of this section, we will discuss the geometric meaning of the homogenization of an ideal. We will begin by studying what happens when we homogenize the ideal $\mathbf{I}_a(W)$ of all polynomials vanishing on an affine variety W. This leads to the following definition.

Definition 6. *Given an affine variety $W \subset k^n$, the* **projective closure** *of W is the projective variety $\overline{W} = \mathbf{V}(\mathbf{I}_a(W)^h) \subset \mathbb{P}^n(k)$, where $\mathbf{I}_a(W)^h \subset k[x_0, \ldots, x_n]$ is the homogenization of the ideal $\mathbf{I}_a(W) \subset k[x_1, \ldots, x_n]$ as in Definition 1.*

The projective closure has the following important properties.

Proposition 7. *Let $W \subset k^n$ be an affine variety and let $\overline{W} \subset \mathbb{P}^n(k)$ be its projective closure. Then:*

(i) $\overline{W} \cap U_0 = \overline{W} \cap k^n = W$.
(ii) *$\overline{W}$ is the smallest projective variety in $\mathbb{P}^n(k)$ containing W.*
(iii) *If W is irreducible, then so is $\overline{W}$.*
(iv) *No irreducible component of $\overline{W}$ lies in the hyperplane at infinity $\mathbf{V}(x_0) \subset \mathbb{P}^n(k)$.*

Proof. (i) Let G be a Groebner basis of $\mathbf{I}_a(W)$ with respect to a graded order on $k[x_1, \ldots, x_n]$. Then Theorem 4 implies that $\mathbf{I}_a(W)^h = \langle g^h : g \in G \rangle$. We know that

$k^n \cong U_0$ is the subset of $\mathbb{P}^n(k)$, where $x_0 = 1$. Thus, we have

$$\overline{W} \cap U_0 = \mathbf{V}(g^h : g \in G) \cap U_0 = \mathbf{V}_a(g^h(1, x_1, \ldots, x_n) : g \in G).$$

Since $g^h(1, x_1, \ldots, x_n) = g$ by part (iii) of Proposition 7 from §2, we get $\overline{W} \cap U_0 = W$.

(ii) We need to prove that if V is a projective variety containing W, then $\overline{W} \subset V$. Suppose that $V = \mathbf{V}(F_1, \ldots, F_s)$. Then F_i vanishes on V, so that its dehomogenization $f_i = F_i(1, x_1, \ldots, x_n)$ vanishes on W. Thus, $f_i \in \mathbf{I}_a(W)$ and, hence, $f_i^h \in \mathbf{I}_a(W)^h$. This shows that f_i^h vanishes on $\overline{W} = \mathbf{V}(\mathbf{I}_a(W)^h)$. But part (iv) of Proposition 7 from §2 implies that $F_i = x_0^{e_i} f_i^h$ for some integer e_i. Thus, F_i vanishes on $\overline{W}$, and since this is true for all i, it follows that $\overline{W} \subset V$.

The proof of (iii) will be left as an exercise. To prove (iv), let $\overline{W} = V_1 \cup \cdots \cup V_m$ be the decomposition of $\overline{W}$ into irreducible components. Suppose that one of them, V_1, was contained in the hyperplane at infinity $\mathbf{V}(x_0)$. Then

$$\begin{aligned} W = \overline{W} \cap U_0 &= (V_1 \cup \cdots \cup V_m) \cap U_0 \\ &= (V_1 \cap U_0) \cup ((V_2 \cup \cdots \cup V_m) \cap U_0) \\ &= (V_2 \cup \cdots \cup V_m) \cap U_0. \end{aligned}$$

This shows that $V_2 \cup \cdots \cup V_m$ is a projective variety containing W. By the minimality of $\overline{W}$, it follows that $\overline{W} = V_2 \cup \cdots \cup V_m$ and, hence, $V_1 \subset V_2 \cup \cdots \cup V_m$. We will leave it as an exercise to show that this is impossible since V_1 is an irreducible component of $\overline{W}$. This contradiction completes the proof. $\square$

For an example of how the projective closure works, consider the affine twisted cubic $W \subset \mathbb{R}^3$. In §4 of Chapter 1, we proved that

$$\mathbf{I}_a(W) = \langle x_2 - x_1^2, x_3 - x_1^3 \rangle.$$

Using Theorem 4, we proved in (3) that

$$\mathbf{I}_a(W)^h = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_2^2 \rangle.$$

It follows that the variety $V' = \mathbf{V}(x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_2^2)$ discussed in Example 9 of §2 is the projective closure of the affine twisted cubic.

The main drawback of the definition of projective closure is that it requires that we know $\mathbf{I}_a(W)$. It would be much more convenient if we could compute the projective closure directly from *any* defining ideal of W. When the field k is algebraically closed, this can always be done.

Theorem 8. *Let k be an algebraically closed field, and let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Then $\mathbf{V}(I^h) \subset \mathbb{P}^n(k)$ is the projective closure of $\mathbf{V}_a(I) \subset k^n$.*

Proof. Let $W = \mathbf{V}_a(I) \subset k^n$ and $Z = \mathbf{V}(I^h) \subset \mathbb{P}^n(k)$. The proof of part (i) of Proposition 7 shows that Z is a projective variety containing W.

To prove that Z is the smallest such variety, we proceed as in part (ii) of Proposition 7. Thus, let $V = \mathbf{V}(F_1, \ldots, F_s)$ be any projective variety containing W. As in the earlier argument, the dehomogenization $f_i = F_i(1, x_1, \ldots, x_n)$ is in $\mathbf{I}_a(W)$. Since k is algebraically closed, the Nullstellensatz implies that $\mathbf{I}_a(W) = \sqrt{I}$, so that $f_i^m \in I$ for some integer m. This tells us that

$$(f_i^m)^h \in I^h$$

and, consequently, $(f_i^m)^h$ vanishes on Z. In the exercises, you will show that

$$(f_i^m)^h = (f_i^h)^m,$$

and it follows that f_i^h vanishes on Z. Then $F_i = x_0^{e_i} f_i^h$ shows that F_i is also zero on Z. As in Proposition 7, we conclude that $Z \subset V$.

This shows that Z is the smallest projective variety containing W. Since the projective closure $\overline{W}$ has the same property by Proposition 7, we see that $Z = \overline{W}$. □

If we combine Theorems 4 and 8, we get an **algorithm for computing the projective closure of an affine variety** over an algebraically closed field k: given $W \subset k^n$ defined by $f_1 = \cdots = f_s = 0$, compute a Groebner basis G of $\langle f_1, \ldots, f_s\rangle$ with respect to a graded order, and then the projective closure in $\mathbb{P}^n(k)$ is defined by $g^h = 0$ for $g \in G$.

Unfortunately, Theorem 8 can fail over fields that are not algebraically closed. Here is an example that shows what can go wrong.

Example 9. Consider $I = \langle x_1^2 + x_2^4\rangle \subset \mathbb{R}[x_1, x_2]$. Then $W = \mathbf{V}_a(I)$ consists of the single point $(0, 0)$ in $\mathbb{R}^2$, and hence, the projective closure is the single point $\overline{W} = \{(1, 0, 0)\} \subset \mathbb{P}^2(\mathbb{R})$ (since this is obviously the smallest projective variety containing W). On the other hand, $I^h = \langle x_1^2 x_0^2 + x_2^4\rangle$, and it is easy to check that

$$\mathbf{V}(I^h) = \{(1, 0, 0), (0, 1, 0)\} \subset \mathbb{P}^2(\mathbb{R}).$$

This shows that $\mathbf{V}(I^h)$ is strictly larger than the projective closure of $W = \mathbf{V}_a(I)$.

EXERCISES FOR §4

1. Prove Proposition 2.
2. Show that the order $>_h$ defined in the proof of Theorem 4 is a monomial order on $k[x_0, \ldots, x_0]$. Hint: This can be done directly or by using the mixed orders defined in Exercise 10 of Chapter 2, §4.
3. Show by example that the conclusion of Theorem 4 is *not true* if we use an arbitrary monomial order in $k[x_1, \ldots, x_n]$ and homogenize a Groebner basis with respect to that order. Hint: One example can be obtained using the ideal of the affine twisted cubic and computing a Groebner basis with respect to a nongraded order.
4. Let $>$ be a graded monomial order on $k[x_1, \ldots, x_n]$ and let $>_h$ be the order defined in the proof of Theorem 4. In the proof of the theorem, we showed that if G is a Groebner basis for $I \subset k[x_1, \ldots, x_n]$ with respect to $>$, then G^h was a Groebner basis for I^h with respect to $>_h$. In this exercise, we will explore other monomial orders on $k[x_0, \ldots, x_n]$ that have this property.
   a. Define a graded version of $>_h$ by setting
   $$x^\alpha x_0^d >_{gh} x^\beta x_0^e \iff |\alpha| + d > |\beta| + e \quad \text{or} \quad |\alpha| + d = |\beta| + e \text{ and } x^\alpha x_0^d >_h x^\beta x_0^e.$$
   Show that G^h is a Groebner basis with respect to $>_{gh}$.

b. More generally, let $>'$ be any monomial order on $k[x_0, \ldots, x_n]$ which extends $>$ and which has the property that among monomials of the same total degree, any monomial containing x_0 is smaller than all monomials containing only $x_1, \ldots, x_n$. Show that G^h is a Groebner basis for $>'$.

5. Let $>$ denote grevlex order in the ring $S = k[x_1, \ldots, x_n, x_{n+1}]$. Consider $R = k[x_1, \ldots, x_n] \subset S$. For $f \in R$, let f^h denote the homogenization of f with respect to the variable x_{n+1}.
   a. Show that if $f \in R \subset S$ (that is, f depends only on $x_1, \ldots, x_n$), then $\mathrm{LT}_>(f) = \mathrm{LT}_>(f^h)$.
   b. Use part (a) to show that if G is a Groebner basis for an ideal $I \subset R$ with respect to grevlex, then G^h is a Groebner basis for the ideal I^h in S with respect to grevlex.
6. Prove that the homogenization has the following properties for polynomials $f, g \in k[x_1, \ldots, x_n]$:
$$(fg)^h = f^h g^h,$$
$$(f^m)^h = (f^h)^m \quad \text{for any integer } m \geq 0.$$
Hint: Use the formula for homogenization given by part (ii) of Proposition 7 from §2.
7. Show that $I \subset k[x_1, \ldots, x_n]$ is a prime ideal if and only if I^h is a prime ideal in $k[x_0, \ldots, x_0]$. Hint: For the $\Rightarrow$ implication, use part (*a*) of Exercise 11 of §3; for the converse implication, use Exercise 6.
8. Adapt the proof of part (ii) of Proposition 7 to show that $\mathbf{I}(\overline{W}) = \mathbf{I}_a(W)^h$ for any affine variety $W \subset k^n$.
9. Prove that an affine variety W is irreducible if and only if its projective closure $\overline{W}$ is irreducible.
10. Let $W = V_1 \cup \cdots \cup V_m$ be the decomposition of a projective variety into its irreducible components such that $V_i \not\subset V_j$ for $i \neq j$. Prove that $V_1 \not\subset V_2 \cup \cdots \cup V_m$.

In Exercises 11–14, we will explore some interesting varieties in projective space. For ease of notation, we will write $\mathbb{P}^n$ rather than $\mathbb{P}^n(k)$. We will also assume that k is algebraically closed so that we can apply Theorem 8.

11. The twisted cubic that we have used repeatedly for examples is one member of an infinite family of curves known as the *rational normal curves*. The rational normal curve in k^n is the image of the polynomial parametrization $\phi : k \to k^n$ given by
$$\phi(t) = (t, t^2, t^3, \ldots, t^n).$$
By our general results on implicitization from Chapter 3, we know the rational normal curves are affine varieties. Their projective closures in $\mathbb{P}^n$ are also known as rational normal curves.
   a. Find affine equations for the rational normal curves in k^4 and k^5.
   b. Homogenize your equations from part a and consider the projective varieties defined by these homogeneous polynomials. Do your equations define the projective closure of the affine curve? Are there any “extra” components at infinity?
   c. Using Theorems 4 and 8, find a set of homogeneous equations defining the projective closures of these rational normal curves in $\mathbb{P}^4$ and $\mathbb{P}^5$, respectively. Do you see a pattern?
   d. Show that the rational normal curve in $\mathbb{P}^n$ is the variety defined by the set of homogeneous quadrics obtained by taking all possible 2×2 subdeterminants of the $2 \times n$ matrix:
$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

12. The affine Veronese surface $S \subset k^5$ was introduced in Exercise 6 of Chapter 5, §1. It is the image of the polynomial parametrization $\phi : k^2 \to k^5$ given by
$$\phi(x_1, x_2) = (x_1, x_2, x_1^2, x_1x_2, x_2^2).$$
The projective closure of S is a projective variety known as the *projective Veronese surface*.
    a. Find a set of homogeneous equations for the projective Veronese surface in $\mathbb{P}^5$.
    b. Show that the parametrization of the affine Veronese surface above can be extended to a mapping from $\tilde{\phi} : \mathbb{P}^2 \to \mathbb{P}^5$ whose image coincides with the entire projective Veronese surface. Hint: You must show that $\tilde{\phi}$ is well-defined (i.e., that it yields the same point in $\mathbb{P}^5$ for any choice of homogeneous coordinates for a point in $\mathbb{P}^2$).
13. The Cartesian product of two affine spaces is simply another affine space: $k^n \times k^m = k^{m+n}$. If we use the standard inclusions $k^n \subset \mathbb{P}^n, k^m \subset \mathbb{P}^m$, and $k^{n+m} \subset \mathbb{P}^{n+m}$ given by Proposition 2 of §2, how is $\mathbb{P}^{n+m}$ different from $\mathbb{P}^n \times \mathbb{P}^m$ (as a set)?
14. In this exercise, we will see that $\mathbb{P}^n \times \mathbb{P}^m$ can be identified with a certain projective variety in $\mathbb{P}^{n+m+nm}$ known as a *Segre variety*. The idea is as follows. Let $p = (x_0, \ldots, x_n)$ be homogeneous coordinates of $p \in \mathbb{P}^n$ and let $q = (y_0, \ldots, y_m)$ be homogeneous coordinates of $q \in \mathbb{P}^m$. The Segre mapping $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+nm}$ is defined by taking the pair $(p, q) \in \mathbb{P}^n \times \mathbb{P}^m$ to the point in $\mathbb{P}^{n+m+nm}$ with homogeneous coordinates
$$(x_0y_0, x_0y_1, \ldots, x_0y_m, x_1y_0, \ldots, x_1y_m, \ldots, x_ny_0, \ldots, x_ny_m).$$
The components are all the possible products x_iy_j where $0 \leq i \leq n$ and $0 \leq j \leq m$. The image is a projective variety called a Segre variety.
    a. Show that σ is a well-defined mapping. (That is, show that we obtain the same point in $\mathbb{P}^{n+m+nm}$ no matter what homogeneous coordinates for p, q we use.)
    b. Show that σ is a one-to-one mapping and that the "affine part" $k^n \times k^m$ maps to an affine variety in $k^{n+m+nm} = U_0 \subset \mathbb{P}^{n+m+nm}$ that is *isomorphic* to k^{n+m}. (See Chapter 5, §4.)
    c. Taking $n = m = 1$ above, write out $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ explicitly and find homogeneous equation(s) for the image. Hint: You should obtain a single quadratic equation. This Segre variety is a quadric surface in $\mathbb{P}^3$.
    d. Now consider the case $n = 2, m = 1$ and find homogeneous equations for the Segre variety in $\mathbb{P}^5$.
    e. What is the intersection of the Segre variety in $\mathbb{P}^5$ and the Veronese surface in $\mathbb{P}^5$? (See Exercise 12.)

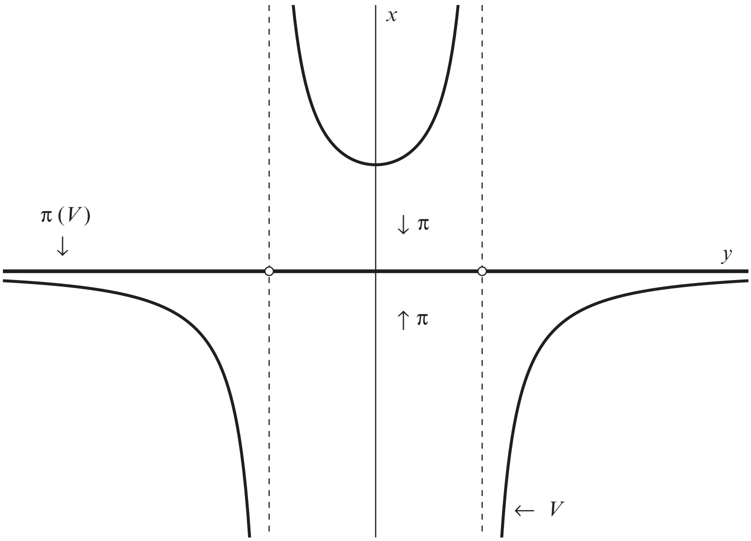
§5 Projective Elimination Theory

In Chapter 3, we encountered numerous instances of "missing points" when studying the geometric interpretation of elimination theory. Since our original motivation for projective space was to account for "missing points," it makes sense to look back at elimination theory using what we know about $\mathbb{P}^n(k)$. You may want to review the first two sections of Chapter 3 before reading further.

We begin with the following example.

Example 1. Consider the variety $V \subset \mathbb{C}^2$ defined by the equation
$$xy^2 = x - 1.$$

To eliminate x, we use the elimination ideal $I_1 = \langle xy^2 - x + 1\rangle \cap \mathbb{C}[y]$, and it is easy to show that $I_1 = \{0\} \subset \mathbb{C}[y]$. In Chapter 3, we observed that eliminating x corresponds geometrically to the projection $\pi(V) \subset \mathbb{C}$, where $\pi : \mathbb{C}^2 \to \mathbb{C}$ is defined by $\pi(x, y) = y$. We know that $\pi(V) \subset \mathbf{V}(I_1) = \mathbb{C}$, but as the following picture shows, $\pi(V)$ does not fill up all of $\mathbf{V}(I_1)$:



We can control the missing points using the Geometric Extension Theorem (Theorem 2 of Chapter 3, §2). Recall how this works: if we write the defining equation of V as $(y^2 - 1)x + 1 = 0$, then the Extension Theorem guarantees that we can solve for x whenever the leading coefficient of x does not vanish. Thus, $y = \pm 1$ are the only missing points.

To reinterpret the Geometric Extension Theorem in terms of projective space, first observe that the standard projective plane $\mathbb{P}^2(\mathbb{C})$ is not quite what we want. We are really only interested in directions along the projection (i.e., parallel to the x-axis) since all of our missing points lie in this direction. So we do not need all of $\mathbb{P}^2(\mathbb{C})$. A more serious problem is that in $\mathbb{P}^2(\mathbb{C})$, all lines parallel to the x-axis correspond to a *single* point at infinity, yet we are missing *two* points.

To avoid this difficulty, we will use something besides $\mathbb{P}^2(\mathbb{C})$. If we write π as $\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, the idea is to make the first factor projective rather than the whole thing. This gives us $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$, and we will again use π to denote the projection $\pi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}$ onto the second factor.

We will use coordinates (t, x, y) on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$, where (t, x) are homogeneous coordinates on $\mathbb{P}^1(\mathbb{C})$ and y is the usual coordinate on $\mathbb{C}$. Thus, (in analogy with Proposition 2 of §2) a point $(1, x, y) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$ corresponds to $(x, y) \in \mathbb{C} \times \mathbb{C}$. We will regard $\mathbb{C} \times \mathbb{C}$ as a subset of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$ and you should check that the complement consists of the "points at infinity" $(0, 1, y)$.

We can extend $V \subset \mathbb{C} \times \mathbb{C}$ to $\overline{V} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$ by making the equation of V homogeneous with respect to t and x. Thus, $\overline{V}$ is defined by

$$xy^2 = x - t.$$

In Exercise 1, you will check that this equation is well-defined on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$. To find the solutions of this equation, we first set $t = 1$ to get the affine portion and then we set $t = 0$ to find the points at infinity. This leads to

$$\overline{V} = V \cup \{(0, 1, \pm 1)\}$$

(remember that t and x cannot simultaneously vanish since they are homogeneous coordinates). Under the projection $\pi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}$, it follows that $\pi(\overline{V}) = \mathbb{C} = \mathbf{V}(I_1)$ because the two points at infinity map to the "missing points" $y = \pm 1$. As we will soon see, the equality $\pi(\overline{V}) = \mathbf{V}(I_1)$ is a special case of the projective version of the Geometric Extension Theorem.

We will use the following general framework for generalizing the issues raised by Example 1. Suppose we have equations

$$\begin{aligned} f_1(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0, \\ &\vdots \\ f_s(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0, \end{aligned}$$

where $f_1, \ldots, f_s \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Algebraically, we can eliminate $x_1, \ldots, x_n$ by computing the ideal $I_n = \langle f_1, \ldots, f_s \rangle \cap k[y_1, \ldots, y_m]$ (the Elimination Theorem from Chapter 3, §1 tells us how to do this). If we think geometrically, the above equations define a variety $V \subset k^n \times k^m$, and eliminating $x_1, \ldots, x_n$ corresponds to considering $\pi(V)$, where $\pi : k^n \times k^m \to k^m$ is projection onto the last m coordinates. Our goal is to describe the relation between $\pi(V)$ and $\mathbf{V}(I_n)$.

The basic idea is to make the first factor projective. To simplify notation, we will write $\mathbb{P}^n(k)$ as $\mathbb{P}^n$ when there is no confusion about what field we are dealing with. A point in $\mathbb{P}^n \times k^m$ will have coordinates $(x_0, \ldots, x_n, y_1, \ldots, y_m)$, where $(x_0, \ldots, x_n)$ are homogeneous coordinates in $\mathbb{P}^n$ and $(y_1, \ldots, y_m)$ are usual coordinates in k^m. Thus, $(1, 1, 1, 1)$ and $(2, 2, 1, 1)$ are coordinates for the same point in $\mathbb{P}^1 \times k^2$, whereas $(2, 2, 2, 2)$ gives a different point. As in Proposition 2 of §2, we will use the map

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (1, x_1, \ldots, x_n, y_1, \ldots, y_m)$$

to identify $k^n \times k^m$ with the subset of $\mathbb{P}^n \times k^m$ where $x_0 \neq 0$.

We can define varieties in $\mathbb{P}^n \times k^m$ using "partially" homogeneous polynomials as follows.

Definition 2. *Let k be a field.*

(i) *A polynomial* $F \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ *is* $(x_0, \ldots, x_n)$-**homogeneous** *provided there is an integer* $l \geq 0$ *such that*

$$F = \sum_{|\alpha| = l} h_\alpha(y_1, \ldots, y_m) x^\alpha,$$

where x^α *is a monomial in* $x_0, \ldots, x_n$ *of multidegree* α *and* $h_\alpha \in k[y_1, \ldots, y_m]$.

(ii) *The* **variety** $\mathbf{V}(F_1, \ldots, F_s) \subset \mathbb{P}^n \times k^m$ *defined by* $(x_0, \ldots, x_n)$*-homogeneous polynomials* $F_1, \ldots, F_s \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ *is the set*

$$\{(a_0, \ldots, a_n, b_1, \ldots, b_m) \in \mathbb{P}^n \times k^m : F_i(a_0, \ldots, a_n, b_1, \ldots, b_m) = 0 \\ \text{for } 1 \leq i \leq s\}.$$

In the exercises, you will show that if a $(x_0, \ldots, x_n)$-homogeneous polynomial vanishes at one set of coordinates for a point in $\mathbb{P}^n \times k^m$, then it vanishes for *all* coordinates of the point. This shows that the variety $\mathbf{V}(F_1, \ldots, F_s)$ is a well-defined subset of $\mathbb{P}^n \times k^m$ when $F_1, \ldots, F_s$ are $(x_0, \ldots, x_n)$-homogeneous.

We can now discuss what elimination theory means in this context. Suppose we have $(x_0, \ldots, x_n)$-homogeneous equations

$$\begin{aligned} F_1(x_0, \ldots, x_n, y_1, \ldots, y_m) &= 0, \\ &\vdots \\ F_s(x_0, \ldots, x_n, y_1, \ldots, y_m) &= 0. \end{aligned} \tag{1}$$

These define the variety $V = \mathbf{V}(F_1, \ldots, F_s) \subset \mathbb{P}^n \times k^m$. We also have the projection map

$$\pi : \mathbb{P}^n \times k^m \to k^m$$

onto the last m coordinates. Then we can interpret $\pi(V) \subset k^m$ as the set of all m-tuples $(y_1, \ldots, y_m)$ for which the equations (1) have a *nontrivial* solution in $x_0, \ldots, x_n$ (which means that at least one x_i is nonzero).

To understand what this means algebraically, let us work out an example.

Example 3. In this example, we will use (u, v, y) as coordinates on $\mathbb{P}^1 \times k$. Then consider the equations

$$\begin{aligned} F_1 &= u + vy = 0, \\ F_2 &= u + uy = 0. \end{aligned} \tag{2}$$

Since (u, v) are homogeneous coordinates on $\mathbb{P}^1$, it is straightforward to show that

$$V = \mathbf{V}(F_1, F_2) = \{(0, 1, 0), (1, 1, -1)\}.$$

Under the projection $\pi : \mathbb{P}^1 \times k \to k$, we have $\pi(V) = \{0, -1\}$, so that for a given y, the equations (2) have a nontrivial solution if and only if $y = 0$ or -1. Thus, (2) implies that $y(1 + y) = 0$.

Ideally, there should be a purely algebraic method of "eliminating" u and v from (2) to obtain $y(1 + y) = 0$. Unfortunately, the kind of elimination we did in Chapter 3 does not work. To see why, let $I = \langle F_1, F_2 \rangle \subset k[u, v, y]$ be the ideal generated by F_1 and F_2. Since every term of F_1 and F_2 contains u or v, it follows that

$$I \cap k[y] = \{0\}.$$

From the affine point of view, this is the correct answer since the *affine* variety

$$\mathbf{V}_a(F_1, F_2) \subset k^2 \times k$$

contains the trivial solutions $(0, 0, y)$ for *all* $y \in k$. Thus, the affine methods of Chapter 3 will be useful only if we can find an algebraic way of excluding the solutions where $u = v = 0$.

We can shed some light on the matter by computing Groebner bases for $I = \langle F_1, F_2 \rangle$ using various lex orders:

$$\text{using } u > v > y : I = \langle u + vy, vy^2 + vy \rangle,$$
$$\text{using } v > u > y : I = \langle vu - u^2, vy + u, u + uy \rangle.$$

The last entries in each basis show that our desired answer $y(1 + y)$ is *almost* in I, in the sense that

$$u \cdot y(1 + y), v \cdot y(1 + y) \in I. \tag{3}$$

In the language of ideal quotients from §4 of Chapter 4, this implies that

$$y(1 + y) \in I : \langle u, v \rangle.$$

Recall from Chapter 4 that for affine varieties, the ideal quotient corresponds (roughly) to the difference of varieties (see Theorem 7 of Chapter 4, §4 for a precise statement). Thus, the ideal $I : \langle u, v \rangle$ is closely related to the difference

$$\mathbf{V}_a(F_1, F_2) - \mathbf{V}_a(u, v) \subset k^2 \times k.$$

This set consists *exactly* of the nontrivial solutions of (2). Hence, the ideal quotient enters in a natural way.

Thus, our goal of eliminating u and v projectively from (2) leads to the polynomial

$$y(1 + y) \in \tilde{I} = (I : \langle u, v \rangle) \cap k[y].$$

Using the techniques of Chapter 4, it can be shown that $\tilde{I} = \langle y(1 + y) \rangle$ in this case.

With this example, we are very close to the definition of projective elimination. The only difference is that in general, higher powers of the variables may be needed in (3) (see Exercise 7 for an example). Hence, we get the following definition of the projective elimination ideal.

Definition 4. *Given an ideal $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ generated by $(x_0, \ldots, x_n)$-homogeneous polynomials, the* **projective elimination ideal of** *I is the set*

$$\hat{I} = \{f \in k[y_1, \ldots, y_m] : \textit{for each } 0 \leq i \leq n, \textit{there is } e_i \geq 0 \text{ with } x_i^{e_i} f \in I\}.$$

It is an easy exercise to show that $\hat{I}$ is, in fact, an ideal of $k[y_1, \ldots, y_m]$. To begin to see the role played by $\hat{I}$, we have the following result.

Proposition 5. *Let $V = \mathbf{V}(F_1, \ldots, F_s) \subset \mathbb{P}^n \times k^m$ be a variety defined by $(x_0, \ldots, x_n)$-homogeneous polynomials and let $\pi : \mathbb{P}^n \times k^m \to k^m$ be the projection* *map. Then in k^m, we have*

$$\pi(V) \subset \mathbf{V}(\hat{I}),$$

where $\hat{I}$ is the projective elimination ideal of $I = \langle F_1, \ldots, F_s \rangle$.

Proof. Suppose that we have $(a_0, \ldots, a_n, b_1, \ldots, b_m) \in V$ and $f \in \hat{I}$. Then $x_i^{e_i} f(y_1, \ldots, y_m) \in I$ implies that this polynomial vanishes on V, and hence,

$$a_i^{e_i} f(b_1, \ldots, b_m) = 0$$

for all i. Since $(a_0, \ldots, a_n)$ are homogeneous coordinates, at least one $a_i \neq 0$ and, thus, $f(b_1, \ldots, b_m) = 0$. This proves that f vanishes on $\pi(V)$ and the proposition follows. $\square$

When the field is algebraically closed, we also have the following projective version of the Extension Theorem.

Theorem 6 (The Projective Extension Theorem). *Assume that k is algebraically closed and that $V = \mathbf{V}(F_1, \ldots, F_s) \in \mathbb{P}^n \times k^m$ is defined by $(x_0, \ldots, x_n)$-homogeneous polynomials in $k[x_0, \ldots, x_n, y_1, \ldots, y_m]$. Let $I = \langle F_1, \ldots, F_s \rangle$ and let $\hat{I} \subset k[y_1, \ldots, y_m]$ be the projective elimination ideal of I. If*

$$\pi : \mathbb{P}^n \times k^m \longrightarrow k^m$$

is projection onto the last m coordinates, then

$$\pi(V) = \mathbf{V}(\hat{I}).$$

Proof. The inclusion $\pi(V) \subset \mathbf{V}(\hat{I})$ follows from Proposition 5. For the opposite inclusion, let $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbf{V}(\hat{I})$ and set $F_i(x_0, \ldots, x_n, \mathbf{c}) = F_i(x_0, \ldots, x_n, c_1, \ldots, c_m)$. This is a homogeneous polynomial in $x_0, \ldots, x_n$, say of total degree $= d_i$ [equal to the total degree of $F_i(x_0, \ldots, x_n, y_1, \ldots, y_m)$ in $x_0, \ldots, x_n$].

If $\mathbf{c} \notin \pi(V)$, then it follows that the equations

$$F_i(x_0, \ldots, x_n, \mathbf{c}) = \cdots = F_s(x_0, \ldots, x_n, \mathbf{c}) = 0$$

define the empty variety in $\mathbb{P}^n$. Since the field k is algebraically closed, the Projective Weak Nullstellensatz (Theorem 8 of §3) implies that for some $r \geq 1$, we have

$$\langle x_0, \ldots, x_n \rangle^r \subset \langle F_1(x_0, \ldots, x_n, \mathbf{c}), \ldots, F_s(x_0, \ldots, x_n, \mathbf{c}) \rangle.$$

This means that the monomials x^α, $|\alpha| = r$, can be written as a polynomial linear combination of the $F_i(x_0, \ldots, x_n, \mathbf{c})$, say

$$x^\alpha = \sum_{i=1}^{s} H_i(x_0, \ldots, x_n) F_i(x_0, \ldots, x_n, \mathbf{c}).$$

By taking homogeneous components, we can assume that each H_i is homogeneous of total degree $r - d_i$ [since d_i is the total degree of $F_i(x_0, \ldots, x_n, \mathbf{c})$]. Then, writing each H_i as a linear combination of monomials x^{β_i} with $|\beta_i| = r - d_i$, we see that the polynomials

$$x^{\beta_i} F_i(x_0, \ldots, x_n, \mathbf{c}), i = 1, \ldots, s, |\beta_i| = r - d_i$$

span the vector space of *all* homogeneous polynomials of total degree r in $x_0, \ldots, x_n$. If the dimension of this space is denoted N_r, then by standard results in linear algebra, we can find N_r of these polynomials which form a basis for this space. We will denote this basis as

$$G_j(x_0, \ldots, x_n, \mathbf{c}), \ j = 1, \ldots, N_r.$$

To see why this leads to a contradiction, we will use linear algebra and the properties of determinants to create an interesting element of the elimination ideal $\hat{I}$. The polynomial $G_j(x_0, \ldots, x_n, \mathbf{c})$ comes from a polynomial $G_j = G_j(x_0, \ldots, x_n, y_1, \ldots, y_m) \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$. Each G_j is of the form $x^{\beta_i} F_i$, for some i and β_i, and is homogeneous in $x_0, \ldots, x_n$ of total degree r. Thus, we can write

$$G_j = \sum_{|\alpha|=r} a_{j\alpha}(y_1, \ldots, y_m) x^\alpha. \tag{4}$$

Since the x^α with $|\alpha| = r$ form a basis of all homogeneous polynomials of total degree r, there are N_r such monomials. Hence we get a square matrix of polynomials $a_{j\alpha}(y_1, \ldots, y_m)$. Then let

$$D(y_1, \ldots, y_m) = \det(a_{j\alpha}(y_1, \ldots, y_m) : 1 \le j \le N_r, |\alpha| = r)$$

be the corresponding determinant. If we substitute $\mathbf{c}$ into (4), we obtain

$$G_j(x_0, \ldots, x_n, \mathbf{c}) = \sum_{|\alpha|=r} a_{j\alpha}(\mathbf{c}) x^\alpha,$$

and since the $G_j(x_0, \ldots, x_n, \mathbf{c})$'s and x^α's are bases of the same vector space, we see that

$$D(\mathbf{c}) \neq 0.$$

In particular, this shows that $D(y_1, \ldots, y_m) \neq 0$ in $k[y_1, \ldots, y_m]$.

Working over the function field $k(y_1, \ldots, y_m)$ (see Chapter 5, §5), we can regard (4) as a system of linear equations over $k(y_1, \ldots, y_m)$ with the x^α as variables. Applying Cramer's Rule (Proposition 3 of Appendix A, §3), we conclude that

$$x^\alpha = \frac{\det(M_\alpha)}{D(y_1, \ldots, y_m)},$$

where M_α is the matrix obtained from $(a_{j\alpha})$ by replacing the α column by $G_1, \ldots, G_{N_r}$. If we multiply each side by $D(y_1, \ldots, y_m)$ and expand $\det(M_\alpha)$ along this column, we get an equation of the form

$$x^\alpha D(y_1, \ldots, y_m) = \sum_{j=1}^{N_r} H_{j\alpha}(y_1, \ldots, y_m) G_j(x_0, \ldots, x_n, y_1, \ldots, y_m).$$

However, every G_j is of the form $x^{\beta_i} F_i$, and if we make this substitution and write the sum in terms of the F_i, we obtain

$$x^\alpha D(y_1, \ldots, y_m) \in \langle F_1, \ldots, F_s \rangle = I.$$

This shows that $D(y_1, \ldots, y_m)$ is in the projective elimination ideal $\hat{I}$, and since $\mathbf{c} \in \mathbf{V}(\hat{I})$, we conclude that $D(\mathbf{c}) = 0$. This contradicts what we found above, which proves that $\mathbf{c} \in \pi(V)$, as desired. □

Theorem 6 tells us that when we project a variety $V \subset \mathbb{P}^n \times k^m$ into k^m, the result is again a variety. This has the following nice interpretation: if we think of the variables $y_1, \ldots, y_m$ as parameters in the system of equations

$$F_1(x_0, \ldots, x_n, y_1, \ldots, y_m) = \cdots = F_s(x_0, \ldots, x_n, y_1, \ldots, y_m) = 0,$$

then the equations defining $\pi(V) = \mathbf{V}(\hat{I})$ in k^m tell us what conditions the parameters must satisfy in order for the above equations to have a nontrivial solution (i.e., a solution different from $x_0 = \cdots = x_n = 0$).

For the elimination theory given in Theorem 6 to be useful, we need to be able to compute the elimination ideal $\hat{I}$. We will explore this question in the following two propositions. We first show how to represent $\hat{I}$ as an ideal quotient.

Proposition 7. *If $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ is an ideal, then, for all sufficiently large integers e, we have*

$$\hat{I} = (I : \langle x_0^e, \ldots, x_n^e \rangle) \cap k[y_1, \ldots, y_m].$$

Proof. The definition of ideal quotient shows that

$$f \in I : \langle x_0^e, \ldots, x_n^e \rangle \Longrightarrow x_i^e f \in I \quad \text{for all } 0 \leq i \leq n.$$

It follows immediately that $(I : \langle x_0^e, \ldots, x_n^e \rangle) \cap k[y_1, \ldots, y_m] \subset \hat{I}$ for all $e \geq 0$.

We need to show that the opposite inclusion occurs for large e. First, observe that we have an ascending chain of ideals

$$I : \langle x_0, \ldots, x_n \rangle \subset I : \langle x_0^2, \ldots, x_n^2 \rangle \subset \cdots .$$

Then the ascending chain condition (Theorem 7 of Chapter 2, §5) implies that

$$I : \langle x_0^e, \ldots, x_n^e \rangle = I : \langle x_0^{e+1}, \ldots, x_n^{e+1} \rangle = \cdots$$

for some integer e. If we fix such an e, it follows that

$$I : \langle x_0^d, \ldots, x_n^d \rangle \subset I : \langle x_0^e, \ldots, x_n^e \rangle \tag{5}$$

for *all* integers $d \geq 0$.

Now suppose $f \in \hat{I}$. For each $0 \leq i \leq n$, this means $x_i^{e_i} f \in I$ for some $e_i \geq 0$. Let $d = \max(e_0, \ldots, e_n)$. Then $x_i^d f \in I$ for all i, which implies $f \in I : \langle x_0^d, \ldots, x_n^d \rangle$. By (5), it follows that $f \in (I : \langle x_0^e, \ldots, x_n^e \rangle) \cap k[y_1, \ldots, y_m]$, and the proposition is proved. □

We next relate $\hat{I}$ to the kind of elimination we did in Chapter 3. The basic idea is to reduce to the affine case by dehomogenization. If we fix $0 \leq i \leq n$, then setting $x_i = 1$ in $F \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ gives the polynomial

$$F^{(i)} = F(x_0, \ldots, 1, \ldots, x_n, y_1, \ldots, y_m) \in k[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m],$$

where $\hat{x}_i$ means that x_i is omitted from the list of variables. Then, given an ideal $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$, we get the *dehomogenization*

$$I^{(i)} = \{F^{(i)} : F \in I\} \subset k[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m].$$

It is easy to show that $I^{(i)}$ is an ideal in $k[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m]$. We also leave it as an exercise to show that if $I = \langle F_1, \ldots, F_s \rangle$, then

$$I^{(i)} = \langle F_1^{(i)}, \ldots, F_s^{(i)} \rangle. \tag{6}$$

Let $V \subset \mathbb{P}^n \times k^m$ be the variety defined by I. One can think of $I^{(i)}$ as defining the affine portion $V \cap (U_i \times k^m)$, where $U_i \cong k^n$ is the subset of $\mathbb{P}^n$ where $x_i = 1$. Since we are now in a purely affine situation, we can eliminate using the methods of Chapter 3. In particular, we get the n-th elimination ideal

$$I_n^{(i)} = I^{(i)} \cap k[y_1, \ldots, y_m],$$

where the subscript n indicates that the n variables $x_0, \ldots, \hat{x}_i, \ldots, x_n$ have been eliminated. We now compute $\hat{I}$ in terms of its dehomogenizations $I^{(i)}$ as follows.

Proposition 8. *Let $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal that is generated by $(x_0, \ldots, x_n)$-homogeneous polynomials. Then*

$$\hat{I} = I_n^{(0)} \cap I_n^{(1)} \cap \cdots \cap I_n^{(n)}.$$

Proof. It suffices to show that

$$\hat{I} = I^{(0)} \cap \cdots \cap I^{(n)} \cap k[y_1, \ldots, y_m].$$

First, suppose that $f \in \hat{I}$. Then $x_i^{e_i} f(y_1, \ldots, y_m) \in I$, so that when we set $x_i = 1$, we get $f(y_1, \ldots, y_m) \in I^{(i)}$. This proves $f \in I^{(0)} \cap \cdots \cap I^{(n)} \cap k[y_1, \ldots, y_m]$.

For the other inclusion, we first study the relation between I and $I^{(i)}$. An element $f \in I^{(i)}$ is obtained from some $F \in I$ by setting $x_i = 1$. We claim that F can be assumed to be $(x_0, \ldots, x_n)$-homogeneous. To prove this, note that F can be written as a sum $F = \sum_{j=0}^{d} F_j$, where F_j is $(x_0, \ldots, x_n)$-homogeneous of total degree j in $x_0, \ldots, x_n$. Since I is generated by $(x_0, \ldots, x_n)$-homogeneous polynomials, the proof of Theorem 2 of §3 can be adapted to show that $F_j \in I$ for all j (see Exercise 4). This implies that

$$\sum_{j=0}^{d} x_i^{d-j} F_j$$

is a $(x_0, \ldots, x_n)$-homogeneous polynomial in I which dehomogenizes to f when $x_i = 1$. Thus, we can assume that $F \in I$ is $(x_0, \ldots, x_n)$-homogeneous.

As in §2, we can define a homogenization operator which takes a polynomial $f \in k[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m]$ and uses the extra variable x_i to produce a

$(x_0, \dots, x_n)$-homogeneous polynomial $f^h \in k[x_0, \dots, x_n, y_1, \dots, y_m]$. We leave it as an exercise to show that if a $(x_0, \dots, x_n)$-homogeneous polynomial F dehomogenizes to f using $x_i = 1$, then

$$F = x_i^e f^h \tag{7}$$

for some integer $e \geq 0$.

Now suppose $f \in I^{(i)} \cap k[y_1, \dots, y_m]$. As we proved earlier, f comes from $F \in I$ which is $(x_0, \dots, x_n)$-homogeneous. Since f does not involve $x_0, \dots, x_n$, we have $f = f^h$, and then (7) implies $x_i^e f \in I$. It follows immediately that $I^{(0)} \cap \cdots \cap I^{(n)} \cap k[y_1, \dots, y_m] \subset \hat{I}$, and the proposition is proved. □

Proposition 8 has a nice interpretation. Namely, $I_n^{(i)}$ can be thought of as eliminating $x_0, \dots, \hat{x}_i, \dots, x_n$ on the affine piece of $\mathbb{P}^n \times k^m$ where $x_i = 1$. Then intersecting these affine elimination ideals (which roughly corresponds to the eliminating on the union of the affine pieces) gives the projective elimination ideal.

We can also use Proposition 8 to give an algorithm for finding $\hat{I}$. If $I = \langle F_1, \dots, F_s \rangle$, we know a basis of $I^{(i)}$ by (6), so that we can compute $I_n^{(i)}$ using the Elimination Theorem of Chapter 3, §1. Then the algorithm for ideal intersections from Chapter 4, §3 tells us how to compute $\hat{I} = I_n^{(0)} \cap \cdots \cap I_n^{(n)}$. A second algorithm for computing $\hat{I}$, based on Proposition 7, will be discussed in the exercises.

To see how this works in practice, consider the equations

$$\begin{aligned} F_1 &= u + vy = 0, \\ F_2 &= u + uy = 0 \end{aligned}$$

from Example 3. If we set $I = \langle u + vy, u + uy \rangle \subset k[u, v, y]$, then we have

$$\begin{aligned} &\text{when } u = 1 : I_1^{(u)} = \langle 1 + vy, 1 + y \rangle \cap k[y] = \langle 1 + y \rangle, \\ &\text{when } v = 1 : I_1^{(v)} = \langle u + y, u + uy \rangle \cap k[y] = \langle y(1 + y) \rangle, \end{aligned}$$

and it follows that $\hat{I} = I_1^{(u)} \cap I_1^{(v)} = \langle y(1 + y) \rangle$. Can you explain why $I_1^{(u)}$ and $I_1^{(v)}$ are different?

We next return to a question posed earlier concerning the missing points that can occur in the affine case. An ideal $I \subset k[x_1, \dots, x_n, y_1, \dots, y_m]$ gives a variety $V = \mathbf{V}_a(I) \subset k^n \times k^m$, and under the projection $\pi : k^n \times k^m \to k^m$, we know that $\pi(V) \subset \mathbf{V}(I_n)$, where I_n is the n-th elimination ideal of I. We want to show that points in $\mathbf{V}(I_n) - \pi(V)$ come from points at infinity in $\mathbb{P}^n \times k^m$.

To decide what variety in $\mathbb{P}^n \times k^m$ to use, we will homogenize with respect to x_0. Recall from the proof of Proposition 8 that $f \in k[x_1, \dots, x_n, y_1, \dots, y_m]$ gives us a $(x_0, \dots, x_n)$-homogeneous polynomial $f^h \in k[x_0, \dots, x_n, y_1, \dots, y_m]$. Exercise 12 will study homogenization in more detail. Then the *$(x_0, \dots, x_n)$-homogenization of I* is defined to be the ideal

$$I^h = \langle f^h : f \in I \rangle \subset k[x_0, \dots, x_n, y_1, \dots, y_m].$$

Using the Hilbert Basis Theorem, it follows easily that I^h is generated by finitely many $(x_0, \ldots, x_n)$-homogeneous polynomials.

The following proposition gives the main properties of I^h.

Proposition 9. *Given an ideal $I \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, let I^h be its $(x_0, \ldots, x_n)$-homogenization. Then:*

(i) *The projective elimination ideal of I^h equals the n-th elimination ideal of I. Thus, $\widehat{I^h} = I_n \subset k[y_1, \ldots, y_m]$.*

(ii) *If k is algebraically closed, then the variety $\overline{V} = \mathbf{V}(I^h)$ is the smallest variety in $\mathbb{P}^n \times k^m$ containing the affine variety $V = \mathbf{V}_a(I) \subset k^n \times k^m$. We call $\overline{V}$ the* **projective closure of V in $\mathbb{P}^n \times k^m$.**

Proof. (i) It is straightforward to show that dehomogenizing I^h with respect to x_0 gives $(I^h)^{(0)} = I$. Then the proof of Proposition 8 implies that $\widehat{I^h} \subset I_n$. Going the other way, take $f \in I_n$. Since $f \in k[y_1, \ldots, y_m]$, it is already $(x_0, \ldots, x_n)$-homogeneous. Hence, $f = f^h \in I^h$ and it follows that $x_i^0 f \in I^h$ for all i. This shows that $f \in \widehat{I^h}$, and (i) is proved.

Part (ii) is similar to Theorem 8 of §4 and is left as an exercise. □

Using Theorem 6 and Proposition 9 together, we get the following nice result.

Corollary 10. *Assume that k is algebraically closed and let $V = \mathbf{V}_a(I) \subset k^n \times k^m$, where $I \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is an ideal. Then*

$$\mathbf{V}(I_n) = \pi(\overline{V}),$$

where $\overline{V} \subset \mathbb{P}^n \times k^m$ is the projective closure of V and $\pi\colon \mathbb{P}^n \times k^m \to k^m$ is the projection

Proof. Since Proposition 9 tells us that $\overline{V} = \mathbf{V}(I^h)$ and $\widehat{I^h} = I_n$, the corollary follows immediately from Theorem 6. □

In Chapter 3, points of $\mathbf{V}(I_n)$ were called "partial solutions." The partial solutions which do not extend to solutions in V give points of $\mathbf{V}(I_n) - \pi(V)$, and the corollary shows that these points come from points at infinity in the projective closure $\overline{V}$ of V.

To use Corollary 10, we need to be able to compute I^h. As in §4, the difficulty is that $I = \langle f_1, \ldots, f_s\rangle$ need not imply $I^h = \langle f_s^h, \ldots, f_s^h\rangle$. But if we use an appropriate Groebner basis, we get the desired equality.

Proposition 11. *Let $>$ be a monomial order on $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ such that for all monomials $x^\alpha y^\gamma$, $x^\beta y^\delta$ in $x_1, \ldots, x_n, y_1, \ldots, y_m$, we have*

$$|\alpha| > |\beta| \Longrightarrow x^\alpha y^\gamma > x^\beta y^\delta.$$

If $G = \{g_1, \ldots, g_s\}$ is a Groebner basis for $I \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with respect to $>$, then $G^h = \{g_1^h, \ldots, g_s^h\}$ is a basis for $I^h \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$.

Proof. This is similar to Theorem 4 of §4 and is left as an exercise. □

In Example 1, we considered $I = \langle xy^2 - x + 1\rangle \subset \mathbb{C}[x, y]$. This is a principal ideal and, hence, $xy^2 - x + 1$ is a Groebner basis for any monomial ordering (see Exercise 10 of Chapter 2, §5). If we homogenize with respect to the new variable t, Proposition 11 tells us that I^h is generated by the (t, x)-homogeneous polynomial $xy^2 - x + t$. Now let $\overline{V} = \mathbf{V}(I^h) \subset \mathbb{P}^1 \times \mathbb{C}$. Then Corollary 10 shows $\pi(\overline{V}) = \mathbf{V}(I_1) = \mathbb{C}$, which agrees with what we found in Example 1.

Using Corollary 10 and Proposition 11, we can point out a weakness in the Geometric Extension Theorem given in Chapter 3. This theorem stated that if $I = \langle f_1, \ldots, f_s\rangle$, then

$$\mathbf{V}(I_1) = \pi(V) \cup (\mathbf{V}(g_1, \ldots, g_s) \cap \mathbf{V}(I_1)), \tag{8}$$

where $V = \mathbf{V}_a(I)$ and $g_i \in k[x_2, \ldots, x_n]$ is the leading coefficient of f_i with respect to x_1. From the projective point of view, $\{(0, 1)\} \times V(g_1, \ldots, g_s)$ are the points at infinity in $Z = \mathbf{V}(f_1^h, \ldots, f_s^h)$ (this follows from the proof of Theorem 6). Since $f_1, \ldots, f_s$ was an *arbitrary* basis of I, Z may *not* be the projective closure of V and, hence, $\mathbf{V}(g_1, \ldots, g_s)$ may be too large. To get the smallest possible $\mathbf{V}(g_1, \ldots, g_s) \cap \mathbf{V}(I_1)$ in (8), we should use a Groebner basis for I with respect to a monomial ordering of the type described in Proposition 11.

We will end the section with a study of maps between projective spaces. Suppose that $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ are homogeneous polynomials of total degree d such that $\mathbf{V}(f_0, \ldots, f_m) = \emptyset$ in $\mathbb{P}^n$. Then we can define a map $F : \mathbb{P}^n \to \mathbb{P}^m$ by the formula

$$F(x_0, \ldots, x_n) = (f_0(x_0, \ldots, x_n), \ldots, f_m(x_0, \ldots, x_n)).$$

Since $f_0, \ldots, f_m$ never vanish simultaneously on $\mathbb{P}^n$, $F(x_0, \ldots, x_n)$ always gives a point in $\mathbb{P}^n$. Furthermore, since the f_i are all homogeneous of total degree d, it follows that

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$$

for all $\lambda \in k - \{0\}$. Thus, F is a well-defined function from $\mathbb{P}^n$ to $\mathbb{P}^m$.

We have already seen examples of such maps between projective spaces. For instance, Exercise 21 of §2 studied the map $F : \mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$F(a, b) = (a^2 + b^2, 2ab, a^2 - b^2).$$

This is a projective parametrization of $\mathbf{V}(x^2 - y^2 - z^2)$. Also, Exercise 12 of §4 discussed the Veronese map $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ defined by

$$\phi(x_0, x_1, x_2) = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$

The image of this map is called the Veronese surface in $\mathbb{P}^5$.

Over an algebraically closed field, we can describe the image of $F : \mathbb{P}^n \to \mathbb{P}^m$ using elimination theory as follows.

Theorem 12. *Let k be algebraically closed and let $F : \mathbb{P}^n \to \mathbb{P}^m$ be defined by homogeneous polynomials $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ which have the same total* *degree* >0 *and no common zeros in* $\mathbb{P}^n$. *In* $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$, *let* I *be the ideal* $\langle y_0 - f_0, \ldots, y_m - f_m \rangle$ *and let* $I_{n+1} = I \cap k[y_0, \ldots, y_m]$. *Then* I_{n+1} *is a homogeneous ideal in* $k[y_0, \ldots, y_m]$ *and*

$$F(\mathbb{P}^n) = \mathbf{V}(I_{n+1}).$$

Proof. We will first show that I_{n+1} is a homogeneous ideal. Suppose that the f_i have total degree d. Since the generators $y_i - f_i$ of I are not homogeneous (unless $d = 1$), we will introduce *weights* on the variables $x_0, \ldots, x_n, y_0, \ldots, y_m$. We say that each x_i has weight 1 and each y_j has weight d. Then a monomial $x^\alpha y^\beta$ has weight $|\alpha| + d|\beta|$, and a polynomial $f \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is *weighted homogeneous* provided every monomial in f has the same weight.

The generators $y_i - f_i$ of I all have weight d, so that I is a weighted homogeneous ideal. If we compute a reduced Groebner basis G for I with respect to any monomial order, an argument similar to the proof of Theorem 2 of §3 shows that G consists of weighted homogeneous polynomials. For an appropriate lex order, the Elimination Theorem from Chapter 3 shows that $G \cap k[y_0, \ldots, y_m]$ is a basis of $I_{n+1} = I \cap k[y_0, \ldots, y_m]$. Thus, I_{n+1} has a weighted homogeneous basis. Since the y_i's all have the same weight, a polynomial in $k[y_0, \ldots, y_m]$ is weighted homogeneous if and only if it is homogeneous in the usual sense. This proves that I_{n+1} is a homogeneous ideal.

To study the image of F, we need to consider varieties in the product $\mathbb{P}^n \times \mathbb{P}^m$. A polynomial $h \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is *bihomogeneous* if it can be written as

$$h = \sum_{|\alpha|=k, |\beta|=l} a_{\alpha\beta} x^\alpha y^\beta.$$

If $h_1, \ldots, h_s$ are bihomogeneous, we get a well-defined set

$$\mathbf{V}(h_1, \ldots, h_s) \subset \mathbb{P}^n \times \mathbb{P}^m$$

which is the *variety* defined by $h_1, \ldots, h_s$. Similarly, if $J \subset k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is generated by bihomogeneous polynomials, then we get a variety $\mathbf{V}(J) \subset \mathbb{P}^n \times \mathbb{P}^m$. (See Exercise 16 for the details.)

Elimination theory applies nicely to this situation. The projective elimination ideal $\hat{J} \subset k[y_0, \ldots, y_m]$ is a homogeneous ideal (see Exercise 16). Then, using the projection $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, it is an easy corollary of Theorem 6 that

$$\pi(\mathbf{V}(J)) = \mathbf{V}(\hat{J}) \tag{9}$$

in $\mathbb{P}^m$ (see Exercise 16). As in Theorem 6, this requires that k be algebraically closed.

We cannot apply this theory to I because it is not generated by bihomogeneous polynomials. So we will work with the bihomogeneous ideal $J = \langle y_i f_j - y_j f_i \rangle$. Let us first show that $\mathbf{V}(J) \subset \mathbb{P}^n \times \mathbb{P}^m$ is the graph of $F : \mathbb{P}^n \to \mathbb{P}^m$. Given $p \in \mathbb{P}^n$, we have $(p, F(p)) \in \mathbf{V}(J)$ since $y_i = f_i(p)$ for all i. Conversely, suppose that $(p, q) \in \mathbf{V}(J)$. Then $q_i f_j(p) = q_j f_i(p)$ for all i, j, where q_i is the i-th homogeneous coordinate of q. We can find j with $q_j \neq 0$, and by our assumption on $f_0, \ldots, f_m$, there is i with $f_i(p) \neq 0$. Then $q_i f_j(p) = q_j f_i(p) \neq 0$ shows that $q_i \neq 0$. Now let $\lambda = q_i / f_i(p)$, which is a nonzero element of k. From the defining equations of $\mathbf{V}(J)$, it follows easily that $q = \lambda F(p)$, which shows that (p, q) is in the graph of F in $\mathbb{P}^n \times \mathbb{P}^m$.

As we saw in §3 of Chapter 3, the projection of the graph is the image of the function. Thus, under $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, we have $\pi(\mathbf{V}(J)) = F(\mathbb{P}^n)$. If we combine this with (9), we get $F(\mathbb{P}^n) = \mathbf{V}(\hat{J})$ since k is algebraically closed. This proves that the image of F is a variety in $\mathbb{P}^m$.

Since we know an algorithm for computing $\hat{J}$, we could stop here. The problem is that $\hat{J}$ is somewhat complicated to compute. It is much simpler to work with $I_{n+1} = I \cap k[y_0, \ldots, y_m]$, which requires nothing more than the methods of Chapter 3. So the final step in the proof is to show that $\mathbf{V}(\hat{J}) = \mathbf{V}(I_{n+1})$ in $\mathbb{P}^m$.

It suffices to work in affine space k^{m+1} and prove that $\mathbf{V}_a(\hat{J}) = \mathbf{V}_a(I_{n+1})$. Observe that the variety $\mathbf{V}_a(I) \subset k^{n+1} \times k^{m+1}$ is the graph of the map $k^{n+1} \to k^{m+1}$ defined by $(f_0, \ldots, f_m)$. Under the projection $\pi : k^{n+1} \times k^{m+1} \to k^{m+1}$, we claim that $\pi(\mathbf{V}_a(I)) = \mathbf{V}_a(\hat{J})$. We know that $\mathbf{V}(\hat{J})$ is the image of F in $\mathbb{P}^m$. Once we exclude the origin, this means that $q \in \mathbf{V}_a(\hat{J})$ if and only if there is a some $p \in k^{n+1}$ such that q equals $F(p)$ in $\mathbb{P}^m$. Hence, $q = \lambda F(p)$ in k^{m+1} for some $\lambda \neq 0$. If we set $\lambda' = \sqrt[d]{\lambda}$, then $q = F(\lambda' p)$, which is equivalent to $q \in \pi(\mathbf{V}_a(I))$. The claim now follows easily.

By the Closure Theorem (Theorem 3 of Chapter 3, §2), $\mathbf{V}_a(I_{n+1})$ is the smallest variety containing $\pi(\mathbf{V}_a(I))$. Since this projection equals the variety $\mathbf{V}_a(\hat{J})$, it follows immediately that $\mathbf{V}_a(I_{n+1}) = \mathbf{V}_a(\hat{J})$. This completes the proof of the theorem. □

EXERCISES FOR §5

1. In Example 1, explain why $xy^2 - x + t = 0$ determines a well-defined subset of $\mathbb{P}^1 \times \mathbb{C}$, where (t, x) are homogeneous coordinates on $\mathbb{P}^1$ and y is a coordinate on $\mathbb{C}$. Hint: See Exercise 2.
2. Suppose $F \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ is $(x_0, \ldots, x_n)$-homogeneous. Show that if F vanishes at one set of coordinates for a point in $\mathbb{P}^n \times k^m$, then F vanishes at *all* coordinates for the point.
3. In Example 3, show that $\mathbf{V}(F_1, F_2) = \{(0, 1, 0), (1, 1, -1)\}$.
4. This exercise will study ideals generated by $(x_0, \ldots, x_n)$-homogeneous polynomials.
   a. Prove that every $F \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ can be written uniquely as a sum $\sum_i F_i$ where F_i is a $(x_0, \ldots, x_n)$-homogeneous polynomial of degree i in $x_0, \ldots, x_n$. We call these the $(x_0, \ldots, x_n)$-*homogeneous components* of F.
   b. Prove that an ideal $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ is generated by $(x_0, \ldots, x_n)$-homogeneous polynomials if and only if I contains the $(x_0, \ldots, x_n)$-homogeneous components of each of its elements.
5. Let $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal generated by $(x_0, \ldots, x_n)$-homogeneous polynomials. We will discuss a method for computing the ideal $(I : x_i) \cap k[y_1, \ldots, y_m]$. For convenience, we will concentrate on the case $i = 0$. Let $>$ be lex order with $x_1 > \cdots > x_n > x_0 > y_1 > \cdots > y_m$ and let G be a reduced Groebner basis for I.
   a. Suppose that $g \in G$ has $\text{LT}(g) = x_0 y^\alpha$. Prove that $g = x_0 h_1(y_1, \ldots, y_m) + h_0(y_1, \ldots, y_m)$.
   b. If $g \in G$ has $\text{LT}(g) = x_0 y^\alpha$. Prove that $g = x_0 h_1(y_1, \ldots, y_m)$. Hint: Use part (b) of Exercise 4 and the fact that $G \cap k[y_1, \ldots, y_m]$ is a Groebner basis of $I \cap k[y_1, \ldots, y_m]$. Remember that G is reduced.

c. Let $G' = \{g \in k[y_1, \ldots, y_m] : \text{either } g \text{ or } x_0 g \in G\}$. Show that $G' \subset (I : x_0) \cap k[y_1, \ldots, y_m]$ and that the leading term of every element of $(I : x_0) \cap k[y_1, \ldots, y_m]$ is divisible by the leading term of some element of G'. This shows that G' is a Groebner basis.

d. Explain how to compute $(I : x_0^e) \cap k[y_1, \ldots, y_m]$.

6. In Example 3, we claimed that $(I : \langle u, v\rangle) \cap k[y] = \langle y(1+y)\rangle$ when $I = \langle u+vy, u+uy\rangle \subset k[u, v, y]$. Prove this using the method of Exercise 5. Hint: $I : \langle u, v\rangle = \langle I : u\rangle \cap (I : v)$. Also, the needed Groebner bases have already been computed in Example 3.

7. As in Example 3, we will use (u, v, y) as coordinates on $\mathbb{P}^1 \times k$. Let $F_1 = u - vy$ and $F_2 = u^2 - v^2 y$ in $k[u, v, y]$.
   a. Compute $\mathbf{V}(F_1, F_2)$ and explain geometrically why eliminating u and v should lead to the equation $y(1 - y) = 0$.
   b. By computing appropriate Groebner bases, show that $u^2 y(1 - y)$ and $v^2 y^2(1 - y)$ lie in $I = \langle F_1, F_2\rangle$, whereas $uy(1 - y)$ and $vy(1 - y)$ do not.
   c. Show that $(I : \langle u, v\rangle) \cap k[y] = \{0\}$ and that $(I : \langle u^2, v^2\rangle) \cap k[y] = \langle y(1 - y)\rangle$. Hint: Use Exercise 5.

8. Prove that the set $\hat{I}$ defined in Definition 4 is an ideal of $k[y_1, \ldots, y_m]$. Note: Although this follows from Proposition 7, you should give a direct argument using the definition.

9. Let $I \subset k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal. Adapt the argument of Proposition 7 to show that

$$\hat{I} = (I : \langle x_0, \ldots, x_n\rangle^e) \cap k[y_1, \ldots, y_m]$$

   for all sufficiently large integers e. Hint: By Exercise 8 of §3, $\langle x_0, \ldots, x_n\rangle^e$ is generated by all monomials x^α of total degree e.

10. In this exercise, we will use Proposition 7 to describe an algorithm for computing the projective elimination ideal $\hat{I}$.
   a. Show that if $I : \langle x_0^e, \ldots, x_n^e\rangle = I : \langle x_0^{e+1}, \ldots, x_n^{e+1}\rangle$ for $e \geq 0$, then $I : \langle x_0^e, \ldots, x_n^e\rangle = (I : \langle x_0^d, \ldots, x_n^d\rangle)$ for all $d \geq e$.
   b. Use part (a) to describe an algorithm for finding an integer e such that $\hat{I}$ is given by $(I : \langle x_0^e, \ldots, x_n^e\rangle) \cap k[y_1, \ldots, y_m]$.
   c. Once we know e, use algorithms from Chapters 3 and 4 to explain how we can compute $\hat{I}$ using Proposition 7.

11. In this exercise, we will use dehomogenization operator $F \mapsto F^{(i)}$ defined in the discussion preceding Proposition 8.
   a. Prove that $I^{(i)} = \{F^i : F \in I\}$ is an ideal in $k[x_0, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m]$.
   b. If $I = \langle F_1, \ldots, F_s\rangle$, then show that $I^{(i)} = \langle F_1^{(i)}, \ldots, F_s^{(i)}\rangle$.

12. In the proof of Proposition 8, we needed the homogenization operator, which makes a polynomial $f \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ into a $(x_0, \ldots, x_n)$-homogeneous polynomial f^h using the extra variable x_0.
   a. Give a careful definition of f^h.
   b. If we dehomogenize f^h by setting $x_0 = 1$, show that we get $(f^h)^{(0)} = f$.
   c. Let $f = F^{(0)}$ be the dehomogenization of a $(x_0, \ldots, x_n)$-homogeneous polynomial F. Then prove that $F = x_0^e f^h$ for some integer $e \geq 0$.

13. Prove part (ii) of Proposition 9.

14. Prove Proposition 11. Also give an example of a monomial order which satisfies the hypothesis of the proposition. Hint: You can use an appropriate weight order from Exercise 12 of Chapter 2, §4.

15. The proof of Theorem 12 used weighted homogeneous polynomials. The general setup is as follows. Given variables $x_0, \ldots, x_n$, we assume that each variable has a weight q_i, which

we assume to be a positive integer. Then the *weight* of a monomial x^α is $\sum_{i=0}^n q_i\alpha_i$, where $\alpha = (\alpha_0, \ldots, \alpha_n)$. A polynomial is weighted homogeneous if all of its monomials have the same weight.

a. Show that every $f \in k[x_0, \ldots, x_n]$ can be written uniquely as a sum of weighted homogeneous polynomials $\sum_i f_i$, where f_i is weighted homogeneous of weight i. These are called the *weighted homogeneous components* of f.

b. Define what it means for an ideal $I \subset k[x_0, \ldots, x_n]$ to be a *weighted homogeneous ideal.* Then formulate and prove a version of Theorem 2 of §3 for weighted homogeneous ideals.

16. This exercise will study the elimination theory of $\mathbb{P}^n \times \mathbb{P}^m$. We will use the polynomial ring $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$, where $(x_0, \ldots, x_m)$ are homogeneous coordinates on $\mathbb{P}^n$ and $(y_0, \ldots, y_m)$ are homogeneous coordinates on $\mathbb{P}^m$.

a. As in the text, $h \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is *bihomogeneous* if it can be written in the form
$$h = \sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta} x^\alpha y^\beta.$$
We say that h has *bidegree* (k, l). If $h_1, \ldots, h_s$ are bihomogeneous, show that we get a well-defined variety
$$\mathbf{V}(h_1, \ldots, h_s) \subset \mathbb{P}^n \times \mathbb{P}^m.$$
Also, if $J \subset k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is an ideal generated by bihomogeneous polynomials, explain how to define $\mathbf{V}(J) \subset \mathbb{P}^n \times \mathbb{P}^m$ and prove that $\mathbf{V}(J)$ is a variety.

b. If J is generated by bihomogeneous polynomials, we have $V = \mathbf{V}(J) \subset \mathbb{P}^n \times \mathbb{P}^m$. Since J is also $(x_0, \ldots, x_n)$-homogeneous, we can form its projective elimination ideal $\hat{J} \subset k[y_0, \ldots, y_m]$. Prove that $\hat{J}$ is a homogeneous ideal.

c. Now assume that k is algebraically closed. Under the projection $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, prove that
$$\pi(V) = \mathbf{V}(\hat{J})$$
in $\mathbb{P}^m$. This is the main result in the elimination theory of varieties in $\mathbb{P}^n \times \mathbb{P}^m$. Hint: J also defines a variety in $\mathbb{P}^n \times k^{m+1}$, so that you can apply Theorem 6 to the projection $\mathbb{P}^n \times k^{m+1} \to k^{m+1}$.

17. For the two examples of maps between projective spaces given in the discussion preceding Theorem 12, compute defining equations for the images of the maps.

18. In Exercise 11 of §1, we considered the projective plane $\mathbb{P}^2$, with coordinates (x, y, z), and the dual projective plane $\mathbb{P}^{2\vee}$, where $(A, B, C) \in \mathbb{P}^{2\vee}$ corresponds to the projective line L defined by $Ax + By + Cz = 0$ in $\mathbb{P}^2$. Show that the subset
$$\{(p, l) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} : p \in L\} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$$
is the variety defined by a bihomogeneous polynomial in $k[x, y, z, A, B, C]$ of bidegree $(1, 1)$. Hint: See part (f) of Exercise 11 of §1.

§6 The Geometry of Quadric Hypersurfaces

In this section, we will study quadric hypersurfaces in $\mathbb{P}^n(k)$. These varieties generalize conic sections in the plane and their geometry is quite interesting. To simplify notation, we will write $\mathbb{P}^n$ rather than $\mathbb{P}^n(k)$, and we will use $x_0, \ldots, x_n$ as homogeneous coordinates. Throughout this section, we will assume that k is a field not of characteristic 2. This means that $2 = 1 + 1 \neq 0$ in k, so that in particular we can divide by 2.

Before introducing quadric hypersurfaces, we need to understand the notion of *projective equivalence.* Let $\mathrm{GL}(n+1, k)$ be the set of invertible $(n+1)\times(n+1)$ matrices with entries in k. We can use elements $A \in \mathrm{GL}(n+1, k)$ to create transformations of $\mathbb{P}^n$ as follows. Under matrix multiplication, A induces a linear map $A : k^{n+1} \to k^{n+1}$ which is an isomorphism since A is invertible. This map takes subspaces of k^{n+1} to subspaces of the same dimension, and restricting to 1-dimensional subspaces, it follows that A takes a line through the origin to a line through the origin. Thus A induces a map $A : \mathbb{P}^n \to \mathbb{P}^n$ [see (1) from §2]. We call such a map a *projective linear transformation.*

In terms of homogeneous coordinates, we can describe $A : \mathbb{P}^n \to \mathbb{P}^n$ as follows. Suppose that $A = (a_{ij})$, where $0 \leq i, j \leq n$. If $(b_0, \ldots, b_n)$ are homogeneous coordinates of a point $p \in \mathbb{P}^n$, it follows by matrix multiplication that

$$(1) \qquad A(p) = (a_{00}b_0 + \cdots + a_{0n}b_n, \ldots, a_{n0}b_0 + \cdots + a_{nn}b_n)$$

are homogeneous coordinates for $A(p)$. This formula makes it easy to work with projective linear transformations. Note that $A : \mathbb{P}^n \to \mathbb{P}^n$ is a bijection, and its inverse is given by the matrix $A^{-1} \in \mathrm{GL}(n+1, k)$. In Exercise 1, you will study the set of *all* projective linear transformations in more detail.

Given a variety $V \subset \mathbb{P}^n$ and an element $A \in \mathrm{GL}(n+1, k)$, we can apply A to all points of V to get the subset $A(V) = \{A(p) : p \in V\} \subset \mathbb{P}^n$.

Proposition 1. *If $A \in \mathrm{GL}(n+1, k)$ and $V \subset \mathbb{P}^n$ is a variety, then $A(V) \subset \mathbb{P}^n$ is also a variety. We say that V and $A(V)$ are* **projectively equivalent.**

Proof. Suppose that $V = \mathbf{V}(f_1, \ldots, f_s)$, where each f_i is a homogeneous polynomial. Since A is invertible, it has an inverse matrix $B = A^{-1}$. Then for each i, let $g_i = f_i \circ B$. If $B = (b_{ij})$, this means

$$g_i(x_0, \ldots, x_n) = f_i(b_{00}x_0 + \cdots + b_{0n}x_n, \ldots, b_{n0}x_0 + \cdots + b_{nn}x_n).$$

It is easy to see that g_i is homogeneous of the same total degree as f_i, and we leave it as an exercise to show that

$$(2) \qquad A(\mathbf{V}(f_1, \ldots, f_s)) = \mathbf{V}(g_1, \ldots, g_s).$$

This equality proves the proposition. $\square$

We can regard $A = (a_{ij})$ as transforming $x_0, \ldots, x_n$ into new coordinates $X_0, \ldots, X_n$ defined by

$$(3) \qquad X_i = \sum_{j=0}^{n} a_{ij}x_j.$$

These give homogeneous coordinates on $\mathbb{P}^n$ because $A \in \mathrm{GL}(n+1, k)$. It follows from (1) that we can think of $A(V)$ as the original V viewed using the new homogeneous coordinates $X_0, \ldots, X_n$. An example of how this works will be given in Proposition 2.

In studying $\mathbb{P}^n$, an important goal is to classify varieties up to projective equivalence. In the exercises, you will show that projective equivalence is an equivalence relation. As an example of how this works, let us classify hyperplanes $H \subset \mathbb{P}^n$ up to projective equivalence. Recall from §2 that a *hyperplane* is defined by a linear equation of the form

$$a_0x_0 + \cdots + a_nx_n = 0$$

where $a_0, \ldots, a_n$ are not all zero.

Proposition 2. *All hyperplanes $H \subset \mathbb{P}^n$ are projectively equivalent.*

Proof. We will show that H is projectively equivalent to $\mathbf{V}(x_0)$. Since projective equivalence is an equivalence relation, this will prove the proposition.

Suppose that H is defined by $f = a_0x_0 + \cdots + a_nx_n$, and assume in addition that $a_0 \neq 0$. Now consider the new homogeneous coordinates

$$\begin{aligned} X_0 &= a_0x_0 + a_1x_1 + \cdots + a_nx_n, \\ X_1 &= x_1 \\ &\ \ \vdots \\ X_n &= x_n. \end{aligned} \tag{4}$$

Then it is easy to see that $\mathbf{V}(f) = \mathbf{V}(X_0)$.

Thus, in the $X_0, \ldots, X_n$ coordinate system, $\mathbf{V}(f)$ is defined by the vanishing of the first coordinate. As explained in (3), this is the same as saying that $\mathbf{V}(f)$ and $\mathbf{V}(x_0)$ are projectively equivalent via the coefficient matrix

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

from (4). This is invertible since $a_0 \neq 0$. You should check that $A(\mathbf{V}(f)) = \mathbf{V}(x_0)$, so that we have the desired projective equivalence.

More generally, if $a_i \neq 0$ in f, a similar argument shows that $\mathbf{V}(f)$ is projectively equivalent to $\mathbf{V}(x_i)$. We leave it as an exercise to show that $\mathbf{V}(x_i)$ is projectively equivalent to $\mathbf{V}(x_0)$ for all i, and the proposition is proved. □

In §2, we observed that $\mathbf{V}(x_0)$ can be regarded as a copy of the projective space $\mathbb{P}^{n-1}$. It follows from Proposition 2 that *all* hyperplanes in $\mathbb{P}^n$ look like $\mathbb{P}^{n-1}$.

Now that we understand hyperplanes, we will study the next simplest case, hypersurfaces defined by a homogeneous polynomial of total degree 2.

Definition 3. *A variety $V = \mathbf{V}(f) \subset \mathbb{P}^n$, where f is a nonzero homogeneous polynomial of total degree 2, is called a* **quadric hypersurface***, or more simply, a* **quadric***.*

The simplest examples of quadrics come from analytic geometry. Recall that a conic section in $\mathbb{R}^2$ is defined by an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To get the projective closure in $\mathbb{P}^2(\mathbb{R})$, we homogenize with respect to z to get

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0,$$

which is homogeneous of total degree 2. For this reason, quadrics in $\mathbb{P}^2$ are called *conics.*

We can classify quadrics up to projective equivalence as follows.

Theorem 4 (Normal Form for Quadrics). *Let $f = \sum_{i,j=0}^n a_{ij}x_ix_j \in k[x_0, \dots, x_n]$ be a nonzero homogeneous polynomial of total degree 2, and assume that k is a field not of characteristic 2. Then $\mathbf{V}(f)$ is projectively equivalent to a quadric defined by an equation of the form*

$$c_0x_0^2 + c_1x_1^2 + \cdots + c_nx_n^2 = 0,$$

where $c_0, \dots, c_n$ are elements of k, not all zero.

Proof. Our strategy will be to find a change of coordinates $X_i = \sum_{j=0}^n b_{ij}x_j$ such that f has the form

$$c_0X_0^2 + c_1X_1^2 + \cdots + c_nX_n^2.$$

As in Proposition 2, this will give the desired projective equivalence. Our proof will be an elementary application of completing the square.

We will use induction on the number of variables. For one variable, the theorem is trivial since $a_{00}x_0^2$ is the only homogeneous polynomial of total degree 2. Now assume that the theorem is true when there are n variables.

Given $f = \sum_{i,j=0}^n a_{ij}x_ix_j$, we first claim that by a change of coordinates, we can assume $a_{00} \neq 0$. To see this, first suppose that $a_{00} = 0$ and $a_{jj} \neq 0$ for some $1 \leq j \leq n$. In this case, we set

$$X_0 = x_j, X_j = x_0, \quad \text{and} \quad X_i = x_i \quad \text{for} \quad i \neq 0, j. \tag{5}$$

Then the coefficient of X_0^2 in the expansion of f in terms of $X_0, \dots, X_n$ is nonzero. On the other hand, if all $a_{ii} = 0$, then since $f \neq 0$, we must have $a_{ij} \neq -a_{ji}$ for some $i \neq j$. Making a change of variables as in (5), we may assume that $a_{01} \neq -a_{10}$. Now set

$$X_0 = x_0, X_1 = x_1 - x_0, \quad \text{and} \quad X_i = x_i \quad \text{for} \quad i \geq 2. \tag{6}$$

We leave it as an easy exercise to show that in terms of $X_0, \dots, X_n$, the polynomial f has the form $\sum_{i,j=0}^n c_{ij}X_iX_j$ where $c_{00} = a_{01} + a_{10} \neq 0$. This establishes the claim.

Now suppose that $f = \sum_{i,j=0}^n a_{ij}x_ix_j$ where $a_{00} \neq 0$. Let $b_i = a_{i0} + a_{0i}$ and note that

$$\frac{1}{a_{00}}\left(a_{00}x_0 + \sum_{i=1}^n \frac{b_i}{2}x_i\right)^2 = a_{00}x_0^2 + \sum_{i=1}^n b_ix_0x_i + \sum_{i,j=1}^n \frac{b_ib_j}{4a_{00}}x_ix_j.$$

Since the characteristic of k is not 2, we know that $2 = 1 + 1 \neq 0$ and, thus, division by 2 is possible in k. Now we introduce new coordinates $X_0, \ldots, X_n$, where

$$(7) \qquad X_0 = x_0 + \frac{1}{a_{00}} \sum_{i=1}^{n} \frac{b_i}{2} x_i \quad \text{and} \quad X_i = x_i \quad \text{for} \quad i \geq 1.$$

Writing f in terms of $X_0, \ldots, X_n$, all of the terms $X_0 X_i$ cancel for $1 \leq i \leq n$ and, hence, we get a sum of the form

$$a_{00} X_0^2 + \sum_{i,j=1}^{n} d_{ij} X_i X_j.$$

The sum $\sum_{i,j=1}^{n} d_{ij} X_i X_j$ involves the n variables $X_1, \ldots, X_n$, so that by our inductive assumption, we can find a change of coordinates (only involving $X_1, \ldots, X_n$) which transforms $\sum_{i,j=1}^{n} d_{ij} X_i X_j$ into $e_1 X_1^2 + \cdots + e_n X_n^2$. We can regard this as a coordinate change for $X_0, X_1, \ldots X_n$ which leaves X_0 fixed. Then we have a coordinate change that transforms $a_{00} X_0^2 + \sum_{i,j=1}^{n} d_{ij} X_i X_j$ into the desired form. This completes the proof of the theorem. □

In the normal form $c_0 x_0^2 + \cdots + c_n x_n^2$ given by Theorem 4, some of the coefficients c_i may be zero. By relabeling coordinates, we may assume that $c_i \neq 0$ if $0 \leq i \leq p$ and $c_i = 0$ for $i > p$. Then the quadric is projectively equivalent to one given by the equation

$$(8) \qquad c_0 x_0^2 + \cdots + c_p x_p^2 = 0, \quad c_0, \ldots, c_p \text{ nonzero}.$$

There is a special name for the number of nonzero coefficients.

Definition 5. *Let $V \subset \mathbb{P}^n$ be a quadric hypersurface.*
(i) *If V is defined by an equation of the form* (8)*, then V has* **rank** $p + 1$.
(ii) *More generally, if V is an arbitrary quadric, then V has* **rank** $p + 1$ *if V is projectively equivalent to a quadric defined by an equation of the form* (8).

For example, suppose we use homogeneous coordinates (x, y, z) in $\mathbb{P}^2(\mathbb{R})$. Then the three conics defined by

$$x^2 + y^2 - z^2 = 0, \; x^2 - z^2 = 0, \; x^2 = 0$$

have ranks 3, 2 and 1, respectively. The first conic is the projective version of the circle, whereas the second is the union of two projective lines $\mathbf{V}(x - z) \cap \mathbf{V}(x + z)$, and the third is the projective line $\mathbf{V}(x)$, which we regard as a degenerate conic of multiplicity two. (In general, we can regard any rank 1 quadric as a hyperplane of multiplicity two.)

In the second part of Definition 5, we need to show that the rank is well-defined. Given a quadric V, this means showing that for all projectively equivalent quadrics defined by an equation of the form (8), the number of nonzero coefficients is always the same. We will prove this by computing the rank directly from the defining polynomial $f = \sum_{i,j=0}^{n} a_{ij} x_i x_j$ of V.

A first observation is that we can assume $a_{ij} = a_{ji}$ for all i, j. This follows by setting $b_{ij} = (a_{ij} + a_{ji})/2$ (remember that k has characteristic different from 2). An easy computation shows that $f = \sum_{i,j=0}^{n} b_{ij} x_i x_j$, and our claim follows since $b_{ij} = b_{ji}$.

A second observation is that we can use matrix multiplication to represent f. The coefficients of f form an $(n+1)\times(n+1)$ matrix $Q = (a_{ij})$, which we will assume to be symmetric by our first observation. Let $\mathbf{x}$ be the column vector with entries $x_0, \ldots, x_n$. We leave it as an exercise to show

$$f(\mathbf{x}) = \mathbf{x}^t Q\mathbf{x},$$

where $\mathbf{x}^t$ is the transpose of $\mathbf{x}$.

We can compute the rank of $\mathbf{V}(f)$ in terms of Q as follows.

Proposition 6. *Let $f = \mathbf{x}^t Q\mathbf{x}$, where Q is an $(n+1)\times(n+1)$ symmetric matrix.*

(i) *Given an element $A \in \mathrm{GL}(n+1, k)$, let $B = A^{-1}$. Then*

$$A(\mathbf{V}(f)) = \mathbf{V}(g).$$

where $g(\mathbf{x}) = \mathbf{x}^t B^t Q B\mathbf{x}$.

(ii) *The rank of the quadric hypersurface $\mathbf{V}(f)$ equals the rank of the matrix Q.*

Proof. To prove (i), we note from (2) that $A(\mathbf{V}(f)) = \mathbf{V}(g)$, where $g = f \circ B$. We compute g as follows:

$$g(\mathbf{x}) = f(B\mathbf{x}) = (B\mathbf{x})^t Q(B\mathbf{x}) = \mathbf{x}^t B^t Q B\mathbf{x},$$

where we have used the fact that $(UV)^t = V^t U^t$ for all matrices U, V such that UV is defined. This completes the proof of (i).

To prove (ii), first note that Q and $B^t Q B$ have the same rank. This follows since multiplying a matrix on the right or left by an invertible matrix does not change the rank [see Theorem 4.12 from FINKBEINER (1978)].

Now suppose we have used Theorem 4 to find a matrix $A \in \mathrm{GL}(n+1, k)$ such that $g = c_0 x_0^2 + \cdots + c_p x_p^2$ with $c_0, \ldots, c_p$ nonzero. The matrix of g is a diagonal matrix with $c_0, \ldots, c_p$ on the main diagonal. If we combine this with part (i), we see that

$$B^t Q B = \begin{pmatrix} c_0 & & & & & \\ & \ddots & & & & \\ & & c_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where $B = A^{-1}$. The rank of a matrix is the maximum number of linearly independent columns and it follows that $B^t Q B$ has rank $p+1$. The above observation then implies that Q also has rank $p+1$, as desired. $\square$

When k is an algebraically closed field (such as $k = \mathbb{C}$), Theorem 4 and Proposition 6 show that quadrics are completely classified by their rank.

Proposition 7. *If k is algebraically closed (and not of characteristic 2), then a quadric hypersurface of rank $p+1$ is projectively equivalent to the quadric defined by the equation*

$$\sum_{i=0}^{p} x_i^2 = 0.$$

In particular, two quadrics are projectively equivalent if and only if they have the same rank.

Proof. By Theorem 4, we can assume that we have a quadric defined by a polynomial of the form $c_0 x_0^2 + \cdots + c_p x_p^2 = 0$, where $p+1$ is the rank. Since k is algebraically closed, the equation $x^2 - c_i = 0$ has a root in k. Pick a root and call it $\sqrt{c_i}$. Note that $\sqrt{c_i} \neq 0$ since c_i is nonzero. Then set

$$\begin{aligned} X_i &= \sqrt{c_i}\, x_i, \quad 0 \le i \le p,\\ X_i &= x_i, \quad p < i \le n. \end{aligned}$$

This gives the desired form and it follows that quadrics of the same rank are projectively equivalent. To prove the converse, suppose that $\mathbf{V}(f)$ and $\mathbf{V}(g)$ are projectively equivalent. By Proposition 6, we can assume that f and g have matrices Q and $B^t QB$, respectively, where B is invertible. As noted in the proof of Proposition 6, Q and $B^t QB$ have the same rank, which implies the same for the quadrics $\mathbf{V}(f)$ and $\mathbf{V}(g)$. □

Over the real numbers, the rank is not the only invariant of a quadric hypersurface. For example, in $\mathbb{P}^2(\mathbb{R})$, the conics $V_1 = \mathbf{V}(x^2 + y^2 + z^2)$ and $V_2 = \mathbf{V}(x^2 + y^2 - z^2)$ have rank 3 but cannot be projectively equivalent since V_1 is empty, yet V_2 is not. In the exercises, you will show given any quadric $\mathbf{V}(f)$ with coefficients in $\mathbb{R}$, there are integers $r \ge -1$ and $s \ge 0$ with $0 \le r + s \le n$ such that $\mathbf{V}(f)$ is projectively equivalent over $\mathbb{R}$ to a quadric of the form

$$x_0^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2 = 0.$$

(The case $r = -1$ corresponds to when all of the signs are negative.)

We are most interested in quadrics of maximal rank in $\mathbb{P}^n$.

Definition 8. *A quadric hypersurface in $\mathbb{P}^n$ is* **nonsingular** *if it has rank $n+1$.*

A nonsingular quadric is defined by an equation $f = \mathbf{x}^t Q \mathbf{x} = 0$ where Q has rank $n+1$. Since Q is an $(n+1) \times (n+1)$ matrix, this is equivalent to Q being invertible. An immediate consequence of Proposition 7 is the following.

Corollary 9. *Let k be an algebraically closed field. Then all nonsingular quadrics in $\mathbb{P}^n$ are projectively equivalent.*

In the exercises, you will show that a quadric in $\mathbb{P}^n$ of rank $p + 1$ can be represented as the join of a nonsingular quadric in $\mathbb{P}^p$ with a copy of $\mathbb{P}^{n-p-1}$. Thus, we can understand all quadrics once we know the nonsingular ones.

For the remainder of the section, we will discuss some interesting properties of nonsingular quadrics in $\mathbb{P}^2, \mathbb{P}^3$, and $\mathbb{P}^5$. For the case of $\mathbb{P}^2$, consider the mapping $F : \mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$F(u, v) = (u^2, uv, v^2),$$

where (u, v) are homogeneous coordinates on $\mathbb{P}^1$. Using elimination theory, it is easy to see that the image of F is contained in the nonsingular conic $\mathbf{V}(x_0x_2 - x_1^2)$. In fact, the map $F : \mathbb{P}^1 \to \mathbf{V}(x_0x_2 - x_1^2)$ is a bijection (see Exercise 11), so that this conic looks like a copy of $\mathbb{P}^1$. When k is algebraically closed, it follows that *all* nonsingular conics in $\mathbb{P}^2$ look like $\mathbb{P}^1$.

When we move to quadrics in $\mathbb{P}^3$, the situation is more interesting. Consider the mapping

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

which takes a point $(x_0, x_1, y_0, y_1) \in \mathbb{P}^1 \times \mathbb{P}^1$ to the point $(x_0y_0, x_0y_1, x_1y_0, x_1y_1) \in \mathbb{P}^3$. This map is called a *Segre map* and its properties were studied in Exercise 14 of §4. For us, the important fact is that the image of F is a nonsingular quadric.

Proposition 10. *The Segre map* $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ *is one-to-one and its image is the nonsingular quadric* $\mathbf{V}(z_0z_3 - z_1z_2)$.

Proof. We will use (z_0, z_1, z_2, z_3) as homogeneous coordinates on $\mathbb{P}^3$. If we eliminate x_0, x_1, y_0, y_1 from the equations

$$\begin{aligned} x_0y_0 &= z_0,\\ x_0y_1 &= z_1,\\ x_1y_0 &= z_2,\\ x_1y_1 &= z_3, \end{aligned}$$

then it follows easily that

$$\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbf{V}(z_0z_3 - z_1z_2). \tag{9}$$

To prove equality, suppose that $(w_0, w_1, w_2, w_3) \in \mathbf{V}(z_0z_3 - z_1z_2)$. If $w_0 \neq 0$, then $(w_0, w_2, w_0, w_1) \in \mathbb{P}^1 \times \mathbb{P}^1$ and

$$\sigma(w_0, w_2, w_0, w_1) = (w_0^2, w_0w_1, w_0w_2, w_1w_2).$$

However, since $w_0w_3 - w_1w_2 = 0$, we can write this as

$$\sigma(w_0, w_2, w_0, w_1) = (w_0^2, w_0w_1, w_0w_2, w_0w_3) = (w_0, w_1, w_2, w_3).$$

When a different coordinate is nonzero, the proof is similar and it follows that (9) is an equality. The above argument can be adapted to show that σ is one-to-one (we leave the details as an exercise) and it is also easy to see that $\mathbf{V}(z_0z_3 - z_1z_2)$ is nonsingular. This proves the proposition. □

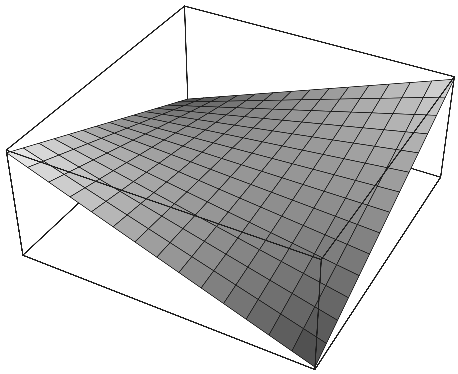
Proposition 10 has some nice consequences concerning lines on the quadric surface $\mathbf{V}(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$. But before we can discuss this, we need to learn how to describe projective lines in $\mathbb{P}^3$.

Two points $p \neq q$ in $\mathbb{P}^3$ give linearly independent vectors $p = (a_0, a_1, a_2, a_3)$ and $q = (b_0, b_1, b_2, b_3)$ in k^4. Now consider the map $F : \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$F(u, v) = (a_0u - b_0v, a_1u - b_1v, a_2u - b_2v, a_3u - b_3v). \tag{10}$$

Since p and q are linearly independent, $a_0u - b_0v, \ldots, a_3u - b_3v$ cannot vanish simultaneously, so that F is defined on all of $\mathbb{P}^1$. In Exercise 13, you will show that the image of F is a variety $L \subset \mathbb{P}^3$ defined by linear equations. We call L the *projective line* (or more simply, the *line*) determined by p and q. Note that L contains both p and q. In the exercises, you will show that all lines in $\mathbb{P}^3$ are projectively equivalent and that they can be regarded as copies of $\mathbb{P}^1$ sitting inside $\mathbb{P}^3$.

Using the Segre map σ, we can identify the quadric $V = \mathbf{V}(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$ with $\mathbb{P}^1 \times \mathbb{P}^1$. If we fix $b = (b_0, b_1) \in \mathbb{P}^1$, the image in V of $\mathbb{P}^1 \times \{b\}$ under σ consists of the points (ub_0, ub_1, vb_0, vb_1) as (u, v) ranges over $\mathbb{P}^1$. By (10), this is the projective line through the points $(b_0, b_1, 0, 0)$ and $(0, 0, b_0, b_1)$. Hence, $b \in \mathbb{P}^1$ determines a line L_b lying on the quadric V. If $b \neq b'$, one can easily show that L_b does not intersect $L_{b'}$ and that every point on V lies on a unique such line. Thus, V is swept out by the family $\{L_b : b \in \mathbb{P}^1\}$ of nonintersecting lines. Such a surface is called a *ruled surface.* In the exercises, you will show that $\{\sigma(\{a\} \times \mathbb{P}^1) : a \in \mathbb{P}^1\}$ is a second family of lines that sweeps out V. If we look at V in the affine space where $z_0 = 1$, then V is defined by $z_3 = z_1z_2$, and we get the following graph:



The two families of lines on V are clearly visible in the above picture. Over an algebraically closed field, Corollary 9 implies that all nonsingular quadrics in $\mathbb{P}^3$ look like this (up to projective equivalence). Over the real numbers, however, there are more possibilities.

For our final example, we will show that the problem of describing lines in $\mathbb{P}^3$ leads to an interesting quadric in $\mathbb{P}^5$. To motivate what follows, let us first recall the situation of lines in $\mathbb{P}^2$. Here, a line $L \subset \mathbb{P}^2$ is defined by a single equation $A_0x_0 + A_1x_1 + A_2x_2 = 0$. In Exercise 11 of §1, we showed that (A_0, A_1, A_2) can be regarded as the "homogeneous coordinates" of L and that the set of all lines forms the dual projective space $\mathbb{P}^{2\vee}$.

It makes sense to ask the same questions for $\mathbb{P}^3$. In particular, can we find "homogeneous coordinates" for lines in $\mathbb{P}^3$? We saw earlier that a line $L \subset \mathbb{P}^3$ can be projectively parametrized using two points $p, q \in L$. This is a good start, but there are infinitely many such pairs on L. How do we get something unique out of this? The idea is the following. Suppose that $p = (a_0, a_1, a_2, a_3)$ and $q = (b_0, b_1, b_2, b_3)$ in k^4. Then consider the 2×4 matrix whose rows are p and q:

$$\Omega = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_1 & b_3 \end{pmatrix}.$$

We will create coordinates for L using the determinants of 2×2 submatrices of Ω. If we number the columns of Ω using $0, 1, 2, 3$, then the determinant formed using columns i and j will be denoted w_{ij}. We can assume $0 \le i < j \le 3$, and we get the six determinants

$$\begin{aligned} w_{01} &= a_0b_1 - a_1b_0, \\ w_{02} &= a_0b_2 - a_2b_0, \\ w_{03} &= a_0b_3 - a_3b_0, \\ w_{12} &= a_1b_2 - a_2b_1, \\ w_{13} &= a_1b_3 - a_3b_1, \\ w_{23} &= a_2b_3 - a_3b_2. \end{aligned} \tag{11}$$

We will encode them in the 6-tuple.

$$\omega(p, q) = (w_{01}, w_{02}, w_{03}, w_{12}, w_{13}, w_{23}) \in k^6.$$

The w_{ij} are called the *Plücker coordinates* of the line L. A first observation is that any line has at least one nonzero Plücker coordinate. To see why, note that Ω has row rank 2 since p and q are linearly independent. Hence the column rank is also 2, so that there must be two linearly independent columns. These columns give a nonzero Plücker coordinate.

To see how the Plücker coordinates depend on the chosen points $p, q \in L$, suppose that we pick a different pair $p', q' \in L$. By (10), we see that in terms of homogeneous coordinates, L can be described as the set

$$L = \{up - vq : (u, v) \in \mathbb{P}^1\}.$$

In particular, we can write

$$\begin{aligned} p' &= up - vq, \\ q' &= sp - tq \end{aligned}$$

for distinct points $(u, v), (s, t) \in \mathbb{P}^1$. We leave it as an exercise to show that

$$\omega(p', q') = \omega(up - vq, sp - tq) = (vs - ut)\omega(p, q)$$

in k^6. Further, it is easy to see that $vs - ut \neq 0$ since $(u, v) \neq (s, t)$ in $\mathbb{P}^1$. This shows that $\omega(p, q)$ gives us a point in $\mathbb{P}^5$ which depends *only* on L. Hence, a line L determines a well-defined point $\omega(L) \in \mathbb{P}^5$.

As we vary L over all lines in $\mathbb{P}^3$, the Plücker coordinates $\omega(L)$ will describe a certain subset of $\mathbb{P}^5$. By eliminating the a_i's and b_i's, from (11), it is easy to see that $w_{01}w_{23} - w_{02}w_{13} + w_{03}w_{12} = 0$ for all sets of Plücker coordinates. If we let $z_{ij}, 0 \leq i < j \leq 3$, be homogeneous coordinates on $\mathbb{P}^5$, it follows that the points $\omega(L)$ all lie in the nonsingular quadric $\mathbf{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12}) \subset \mathbb{P}^5$. Let us prove that this quadric is *exactly* the set of lines in $\mathbb{P}^3$.

Theorem 11. *The map*

$$\{\textit{lines in } \mathbb{P}^3\} \to \mathbf{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$$

which sends a line $L \subset \mathbb{P}^3$ to its Plücker coordinates $\omega(L) \in \mathbf{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$ is a bijection.

Proof. The strategy of the proof is to show that a line $L \subset \mathbb{P}^3$ can be reconstructed from its Plücker coordinates. Given two points $p = (a_0, a_1, a_2, a_3)$ and $q = (b_0, b_1, b_2, b_3)$ on L, it is easy to check that we get the following four vectors in k^4:

$$\begin{aligned} b_0 p - a_0 q &= (0, -w_{01}, -w_{02}, -w_{03}), \\ b_1 p - a_1 q &= (w_{01}, 0, -w_{12}, -w_{13}), \\ b_2 p - a_2 q &= (w_{02}, w_{12}, 0, -w_{23}), \\ b_3 p - a_3 q &= (w_{03}, w_{13}, w_{23}, 0). \end{aligned} \tag{12}$$

It may happen that some of these vectors are 0, but whenever they are nonzero, it follows from (10) that they give points of L.

To prove that ω is one-to-one, suppose that we have lines L and L' such that $\omega(L) = \lambda\omega(L')$ for some nonzero λ. In terms of Plücker coordinates, this means that $w_{ij} = \lambda w'_{ij}$ for all $0 \leq i < j \leq 3$. We know that some Plücker coordinate of L is nonzero, and by permuting the coordinates in $\mathbb{P}^3$, we can assume $w_{01} \neq 0$. Then (12) implies that in $\mathbb{P}^3$, the points

$$\begin{aligned} P &= (0, -w'_{01}, -w'_{02}, -w'_{03}) = (0, -\lambda w_{01}, -\lambda w_{02}, -\lambda w_{03}) \\ &= (0, -w_{01}, -w_{02}, -w_{03}), \\ Q &= (w'_{01}, 0, -w'_{12}, -w'_{13}) = (\lambda w_{01}, 0, -\lambda w_{12}, -\lambda w_{13}) \\ &= (w_{01}, 0, -w_{12}, -w_{13}) \end{aligned}$$

lie on both L and L'. Since there is a unique line through two points in $\mathbb{P}^3$ (see Exercise 14), it follows that $L = L'$. This proves that our map is one-to-one.

To see that ω is onto, pick a point

$$(w_{01}, w_{02}, w_{03}, w_{12}, w_{13}, w_{23}) \in \mathbf{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12}).$$

By changing coordinates in $\mathbb{P}^3$, we can assume $w_{01} \neq 0$. Then the first two vectors in (12) are nonzero and, hence, determine a line $L \subset \mathbb{P}^3$. Using the definition of $\omega(L)$ and the relation $w_{01}w_{23} - w_{02}w_{13} + w_{03}w_{12} = 0$, it is straightforward to show that the

w_{ij} are the Plücker coordinates of L (see Exercise 16 for the details). This shows that ω is onto and completes the proof of the theorem. □

A nice consequence of Theorem 11 is that the set of lines in $\mathbb{P}^3$ can be given the structure of a projective variety. As we observed at the end of Chapter 7, an important idea in algebraic geometry is that any set of geometrically interesting objects should form a variety in some natural way.

Theorem 11 can be generalized in many ways. One can study lines in $\mathbb{P}^n$, and it is even possible to define Plücker coordinates for linear varieties in $\mathbb{P}^n$ of arbitrary dimension. This leads to the study of what are called *Grassmannians*. Using Plücker coordinates, a Grassmannian can be given the structure of a projective variety, although there is usually more than one defining equation. See Exercise 17 for the case of lines in $\mathbb{P}^4$.

We can also think of Theorem 11 from an affine point of view. We already know that there is a natural bijection

$$\{\text{lines through the origin in } k^4\} \cong \{\text{points in } \mathbb{P}^3\},$$

and in the exercises, you will describe a bijection

$$\{\text{planes through the origin in } k^4\} \cong \{\text{lines in } \mathbb{P}^3\}.$$

Thus, Theorem 11 shows that planes through the origin in k^4 have the structure of a quadric hypersurface in $\mathbb{P}^5$. In the exercises, you will see that this has a surprising connection with reduced row echelon matrices. More generally, the Grassmannians mentioned in the previous paragraph can be described in terms of subspaces of a certain dimension in affine space k^{n+1}.

This completes our discussion of quadric hypersurfaces, but by no means exhausts the subject. The classic books by ROTH and SEMPLE (1949) and HODGE and PEDOE (1968) contain a wealth of material on quadric hypersurfaces (and many other interesting projective varieties as well).

EXERCISES FOR §6

1. The set $\mathrm{GL}(n+1, k)$ is closed under inverses and matrix multiplication and is a group in the terminology of Appendix A. In the text, we observed that $A \in \mathrm{GL}(n+1, k)$ induces a projective linear transformation $A : \mathbb{P}^n \to \mathbb{P}^n$. To describe the set of all such transformations, we define a relation on $\mathrm{GL}(n+1, k)$ by

$$A' \sim A \longleftrightarrow A' = \lambda A \quad \text{for some} \quad \lambda \neq 0.$$

   a. Prove that $\sim$ is an equivalence relation. The set of equivalence classes for $\sim$ is denoted $\mathrm{PGL}(n+1, k)$.
   b. Show that if $A \sim A'$ and $B \sim B'$, then $AB \sim A'B'$. Hence, the matrix product operation is well-defined on the equivalence classes for $\sim$ and, thus, $\mathrm{PGL}(n+1, k)$ has the structure of a group. We call $\mathrm{PGL}(n+1, k)$ the *projective linear group*.
   c. Show that two matrices $A, A' \in \mathrm{GL}(n+1, k)$ define the same mapping $\mathbb{P}^n \to \mathbb{P}^n$ if and only if $A' \sim A$. It follows that we can regard $\mathrm{PGL}(n+1, k)$ as a set of invertible transformations on $\mathbb{P}^n$.

2. Prove equation (2) in the proof of Proposition 1.
3. Prove that projective equivalence is an equivalence relation on the set of projective varieties in $\mathbb{P}^n$.
4. Prove that the hyperplanes $\mathbf{V}(x_i)$ and $\mathbf{V}(x_0)$ are projectively equivalent. Hint: See (5).
5. This exercise is concerned with the proof of Theorem 4.
   a. If $f = \sum_{i,j=0}^n a_{ij}x_ix_i$ has $a_{01} \neq -a_{10}$ and $a_{ii} = 0$ for all i, prove that the change of coordinates (6) transforms f into $\sum_{i,j=0}^n c_{ij}X_iX_j$ where $c_{00} = a_{01} + a_{10}$.
   b. If $f = \sum_{i,j=0}^n a_{ij}x_ix_j$ has $a_{00} \neq 0$, verify that the change of coordinates (7) transforms f into $a_{00}X_0^2 + \sum_{i,j=1}^n d_{ij}X_iX_j$.
6. If $f = \sum_{i,j=0}^n a_{ij}x_ix_j$, let Q be the $(n+1) \times (n+1)$ matrix (a_{ij}).
   a. Show that $f(\mathbf{x}) = \mathbf{x}^t Q\mathbf{x}$.
   b. Suppose that k has characteristic 2 ($e.g.$, $k = \mathbb{F}_2$), and let $f = x_0x_1$. Show that there is *no* symmetric 2×2 matrix Q with entries in k such that $f(\mathbf{x}) = \mathbf{x}^t Q\mathbf{x}$.
7. Use the proofs of Theorem 4 and Proposition 7 to write each of the following as a sum of squares. Assume that $k = \mathbb{C}$.
   a. $x_0x_1 + x_0x_2 + x_2^2$.
   b. $x_0^2 + 4x_1x_3 + 2x_2x_3 + x_4^2$.
   c. $x_0x_1 + x_2x_3 - x_4x_5$.
8. Given a nonzero polynomial $f = \sum_{i,j=0}^n a_{ij}x_ix_j$ with coefficients in $\mathbb{R}$, show that there are integers $r \geq -1$ and $s \geq 0$ with $0 \leq r + s \leq n$ such that f can be brought to the form
$$x_0^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$$
   by a suitable coordinate change with real coefficients. One can prove that the integers r and s are uniquely determined by f.
9. Let $f = \sum_{i,j=0}^n a_{ij}x_ix_j \in k[x_0, \ldots, x_n]$ be nonzero. In the text, we observed that $\mathbf{V}(f)$ is a nonsingular quadric if and only if $\det(a_{ij}) \neq 0$. We say that $\mathbf{V}(f)$ is *singular* if it is not nonsingular. In this exercise, we will explore a nice way to characterize singular quadrics.
   a. Show that f is singular if and only if there exists a point $a \in \mathbb{P}^n$ with homogeneous coordinates $(a_0, \ldots, a_n)$ such that
$$\frac{\partial f}{\partial x_0}(a) = \cdots = \frac{\partial f}{\partial x_n}(a) = 0.$$
   b. If $a \in \mathbb{P}^n$ has the property described in part (a), prove that $a \in \mathbf{V}(f)$. In general, a point a of a hypersurface $\mathbf{V}(f)$ (quadric or of higher degree) is called a *singular point* of $\mathbf{V}(f)$ provided that all of the partial derivatives of f vanish at a. Hint: Use Exercise 17 of §2.
10. Let $\mathbf{V}(f) \subset \mathbb{P}^n$ be a quadric of rank $p + 1$, where $0 < p < n$. Prove that there are $X, Y \subset \mathbb{P}^n$ such that (1) $X \simeq \mathbf{V}(g) \subset \mathbb{P}^p$ for some nonsingular quadric g, (2) $Y \simeq \mathbb{P}^{n-p-1}$, (3) $X \cap Y = \emptyset$, and (4) $\mathbf{V}(f)$ is the *join* $X * Y$, which is defined to be the set of all lines in $\mathbb{P}^n$ connecting a point of X to a point of Y (and if $X = \emptyset$, we set $X * Y = Y$). Hint: Use Theorem 4.
11. We will study the map $F : \mathbb{P}^1 \to \mathbb{P}^2$ defined by $F(u, v) = (u^2, uv, v^2)$.
   a. Use elimination theory to prove that the image of F lies in $\mathbf{V}(x_0x_2 - x_1^2)$.
   b. Prove that $F : \mathbb{P}^1 \to \mathbf{V}(x_0x_2 - x_1^2)$ is a bijection. Hint: Adapt the methods used in the proof of Proposition 10.
12. This exercise will study the Segre map $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined in the text.
   a. Use elimination theory to prove that the image of σ lies in the quadric $\mathbf{V}(z_0z_3 - z_1z_2)$.
   b. Use the hint given in the text to prove that σ is one-to-one.
13. In this exercise and the next, we will work out some basic facts about lines in $\mathbb{P}^n$. We start with two distinct points $p, q \in \mathbb{P}^n$, which we will think of as linearly independent vectors in k^{n+1}.

a. We can define a map $F : \mathbb{P}^1 \to \mathbb{P}^n$ by $F(u, v) = up - vq$. Show that this map is defined on all of $\mathbb{P}^1$ and is one-to-one.
b. Let $\ell = a_0x_0 + \cdots + a_nx_n$ be a linear homogeneous polynomial. Show that ℓ vanishes on the image of F if and only if $p, q \in \mathbf{V}(\ell)$.
c. Our goal is to show that the image of F is a variety defined by linear equations. Let Ω be the $2 \times (n + 1)$ matrix whose rows are p and q. Note that Ω has rank 2. If we multiply column vectors in k^{n+1} by Ω, we get a linear map $\Omega : k^{n+1} \to k^2$. Use results from linear algebra to show that the kernel (or nullspace) of this linear map has dimension $n - 1$. Pick a basis $v_1, \ldots, v_{n-1}$ of the kernel, and let ℓ_i be the linear polynomial whose coefficients are the entries of v_1. Then prove that the image of F is $\mathbf{V}(\ell_1 \ldots, \ell_{n-1})$. Hint: Study the subspace of k^{n+1} defined by the equations $\ell_1 = \cdots = \ell_{n-1} = 0$.

14. The exercise will discuss some elementary properties of lines in $\mathbb{P}^n$.
a. Given points $p \neq q$ in $\mathbb{P}^n$, prove that there is a unique line through p and q.
b. If L is a line in $\mathbb{P}^n$ and $U_i \cong k^n$ is the affine space where $x_i = 1$, then show that $L \cap U_i$, is either empty or a line in k^n in the usual sense.
c. Show that all lines in $\mathbb{P}^n$ are projectively equivalent. Hint: In part (c) of Exercise 13, you showed that a line L can be written $L = \mathbf{V}(\ell_1, \ldots, \ell_{n-1})$. Show that you can find ℓ_n and ℓ_{n+1} so that $X_0 = \ell_1, \ldots, X_n = \ell_{n+1}$ is a change of coordinates. What does L look like in the new coordinate system?

15. Let $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre map.
a. Show that $L'_a = \sigma(\{a\} \times \mathbb{P}^1)$ is a line in $\mathbb{P}^3$.
b. Show that every point of $\mathbf{V}(z_0z_3 - z_1z_2)$ lies on a unique line L'_a. This proves that the family of lines $\{L'_a : a \in \mathbb{P}^1\}$ sweeps out the quadric.

16. This exercise will deal with the proof of Theorem 11.
a. Prove that $\omega(up - vq, sp - tq) = (vs - ut)\omega(p, q)$. Hint: if $\binom{p}{q}$ is the 2×4 matrix with rows p and q, show that

$$\begin{pmatrix} up - vq \\ sp - tq \end{pmatrix} = \begin{pmatrix} u & -v \\ s & -t \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

b. Apply elimination theory to (11) to show that Plücker coordinates satisfy the relation $w_{01}w_{23} - w_{02}w_{13} + w_{03}w_{12} = 0$.
c. Complete the proof of Theorem 11 by showing that the map ω is onto.

17. In this exercise, we will study Plücker coordinates for lines in $\mathbb{P}^4$
a. Let $L \subset \mathbb{P}^4$ be a line. Using the homogeneous coordinates of two points $p, q \in L$, define Plücker coordinates and show that we get a point $\omega(L) \in \mathbb{P}^9$ that depends only on L.
b. Find the relations between the Plücker coordinates and use these to find a variety $V \subset \mathbb{P}^4$ such that $\omega(L) \in V$ for all lines L.
c. Show that the map sending a line $L \subset \mathbb{P}^4$ to $\omega(L) \in V$ is a bijection.

18. Show that there is a one-to-one correspondence between lines in $\mathbb{P}^3$ and planes through the origin in k^4. This explains why a line in $\mathbb{P}^3$ is different from a line in k^3 or k^4.

19. There is a nice connection between lines in $\mathbb{P}^3$ and 2×4 reduced row echelon matrices of rank 2. Let $V = \mathbf{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$ be the quadric of Theorem 11.
a. Show that there is a one-to-one correspondence between reduced row echelon matrices of the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

and points in the affine portion $V \cap U_{01}$, where U_{01} is the affine space in $\mathbb{P}^5$ defined by $z_{01} = 1$. Hint: The rows of the above matrix determine a line in $\mathbb{P}^3$. What are its Plücker coordinates?

b. The matrices given in part (a) do not exhaust all possible 2×4 reduced row echelon matrices of rank 2. For example, we also have the matrices

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix}.$$

Show that there is a one-to-one correspondence between these matrices and points of $V \cap \mathbf{V}(z_{01}) \cap U_{02}$.

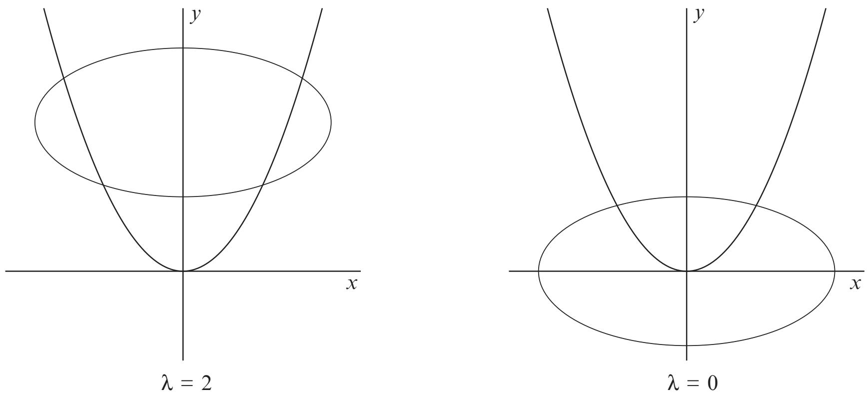
c. Show that there are four remaining types of 2×4 reduced row echelon matrices of rank 2 and prove that each of these is in a one-to-one correspondence with a certain portion of V. Hint: The columns containing the leading 1's will correspond to a certain Plücker coordinate being 1.

d. Explain directly (without using V or Plücker coordinates) why 2×4 reduced row echelon matrices of rank 2 should correspond uniquely to lines in $\mathbb{P}^3$. Hint: See Exercise 18.

§7 Bezout's Theorem

This section will explore what happens when two curves intersect in the plane. We are particularly interested in the *number* of points of intersection. The following examples illustrate why the answer is especially nice when we work with curves in $\mathbb{P}^2(\mathbb{C})$, the projective plane over the complex numbers. We will also see that we need to define the *multiplicity* of a point of intersection. Fortunately, the resultants we learned about in Chapter 3 will make this relatively easy to do.

Example 1. First consider the intersection of a parabola and an ellipse. To allow for explicit calculations, suppose the parabola is $y = x^2$ and the ellipse is $x^2 + 4(y - \lambda)^2 = 4$, where λ is a parameter we can vary. For example, when $\lambda = 2$ or 0, we get the pictures:



Over $\mathbb{R}$, we get different numbers of intersections, and it is clear that there are values of λ for which there are *no* points of intersection (see Exercise 1). What is more interesting is that over $\mathbb{C}$, we have four points of intersection in *both* of the above cases. For example, when $\lambda = 0$, we can eliminate x from $y = x^2$ and $x^2 + 4y^2 = 4$ to obtain $y + 4y^2 = 4$, which has roots

$$y = \frac{-1 \pm \sqrt{65}}{8},$$

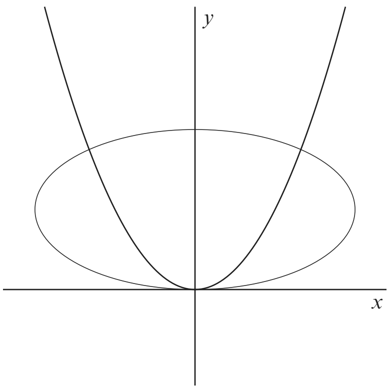
and the corresponding values of x are

$$x = \pm\sqrt{\frac{-1 \pm \sqrt{65}}{8}}.$$

This gives four points of intersection, two real and two complex (since $-1-\sqrt{65} < 0$). You can also check that when $\lambda = 2$, working over $\mathbb{C}$ gives no new solutions beyond the four we see in the above picture (see Exercise 1).

Hence, the number of intersections seems to be more predictable when we work over the complex numbers. As confirmation, you can check that in the cases where there are no points of intersection over $\mathbb{R}$, we still get four points over $\mathbb{C}$ (see Exercise 1).

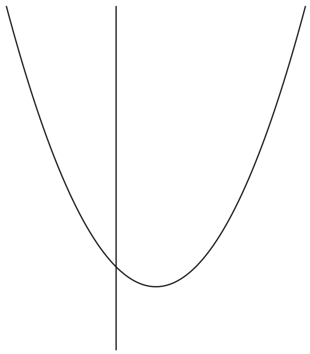
However, even over $\mathbb{C}$, some unexpected things can happen. For example, suppose we intersect the parabola with the ellipse where $\lambda = 1$:



$\lambda = 1$

Here, we see only three points of intersection, and this remains true over $\mathbb{C}$. But the origin is clearly a "special" type of intersection since the two curves are tangent at this point. As we will see later, this intersection has *multiplicity two*, while the other two intersections have *multiplicity one.* If we add up the multiplicities of the points of intersection, we still get four.

Example 2. Now consider the intersection of our parabola $y = x^2$ with a line L. It is easy to see that in most cases, this leads to two points of intersection over $\mathbb{C}$, provided multiplicities are counted properly (see Exercise 2). However, if we intersect with a *vertical* line, then we get the following picture:



There is just one point of intersection, and since multiplicities seem to involve tangency, it should be an intersection of multiplicity one. Yet we want the answer to be two, since this is what we get in the other cases. Where is the other point of intersection?

If we change our point of view and work in the projective plane $\mathbb{P}^2(\mathbb{C})$, the above question is easy to answer: the missing point is "at infinity." To see why, let z be the third variable. Then we homogenize $y = x^2$ to get the projective equation $yz = x^2$, and a vertical line $x = c$ gives the projective line $x = cz$. Eliminating x, we get $yz = c^2z^2$, which is easily solved to obtain $(x, y, z) = (c, c^2, 1)$ or $(0, 1, 0)$ (remember that these are homogeneous coordinates). The first lies in the affine part (where $z = 1$) and is the point we see in the above picture, while the second is on the line at infinity (where $z = 0$).

Example 3. In $\mathbb{P}^2(\mathbb{C})$, consider the two curves given by $C = \mathbf{V}(x^2 - z^2)$ and $D = \mathbf{V}(x^2y - xz^2 - xyz + z^3)$. It is easy to check that $(1, b, 1) \in C \cap D$ for any $b \in \mathbb{C}$, so that the intersection $C \cap D$ is infinite! To see how this could have happened, consider the factorizations

$$x^2 - z^2 = (x - z)(x + z), x^2y - xz^2 - xyz + z^3 = (x - z)(xy - z^2).$$

Thus, C is a union of two projective lines and D is the union of a line and a conic. In fact, these are the irreducible components of C and D in the sense of §3 (see Proposition 4 below). We now see where the problem occurred: C and D have a common irreducible component $\mathbf{V}(x - z)$, so of course their intersection is infinite.

These examples explain why we want to work in $\mathbb{P}^2(\mathbb{C})$. Hence, for the rest of the section, we will use $\mathbb{C}$ and write $\mathbb{P}^2$ instead of $\mathbb{P}^2(\mathbb{C})$. In this context, a *curve* is a projective variety $\mathbf{V}(f)$ defined by a nonzero homogeneous polynomial $f \in \mathbb{C}[x, y, z]$. Our examples also indicate that we should pay attention to multiplicities of intersections and irreducible components of curves. We begin by studying irreducible components.

Proposition 4. *Let $f \in \mathbb{C}[x, y, z]$ be a nonzero homogeneous polynomial. Then the irreducible factors of f are also homogeneous, and if we factor f into irreducibles:*

$$f = f_1^{a_1} \cdots f_s^{a_s},$$

where f_i is not a constant multiple of f_j for $i \neq j$, then

$$\mathbf{V}(f) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_s)$$

is the minimal decomposition of $\mathbf{V}(f)$ into irreducible components in $\mathbb{P}^2$. Furthermore,

$$\mathbf{I}(\mathbf{V}(f)) = \sqrt{\langle f \rangle} = \langle f_1 \cdots f_s \rangle.$$

Proof. First, suppose that f factors as $f = gh$ for some polynomials g, $h \in \mathbb{C}[x, y, z]$. We claim that g and h must be homogeneous since f is. To prove this, write $g = g_m + \cdots + g_0$, where g_i is homogeneous of total degree i and $g_m \neq 0$. Similarly let $h = h_n + \cdots + h_0$. Then

$$\begin{aligned} f = gh &= (g_m + \cdots + g_0)(h_n + \cdots + h_0) \\ &= g_m h_n + \text{terms of lower total degree.} \end{aligned}$$

Since f is homogeneous, we must have $f = g_m h_n$, and with a little more argument, one can conclude that $g = g_m$ and $h = h_n$ (see Exercise 3). Thus g and h are homogeneous. From here, it follows easily that the irreducible factors f are also homogeneous.

Now suppose f factors as above. Then $\mathbf{V}(f) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_s)$ follows immediately, and this is the minimal decomposition into irreducible components by the projective version of Exercise 9 from Chapter 4, §6. Since $\mathbf{V}(f)$ is nonempty (see Exercise 6), the assertion about $\mathbf{I}(\mathbf{V}(f))$ follows from the Projective Nullstellensatz and Proposition 9 of Chapter 4, §2. □

A consequence of Proposition 4 is that every curve $C \subset \mathbb{P}^2$ has a "best" defining equation. If $C = \mathbf{V}(f)$ for some homogeneous polynomial f, then the proposition implies that $\mathbf{I}(C) = \langle f_1 \cdots f_s \rangle$, where $f_1, \ldots, f_s$ are distinct irreducible factors of f. Thus, any other polynomial defining C is a multiple of $f_1 \cdots f_s$, so that $f_1 \cdots f_s = 0$ is the defining equation of smallest total degree. In the language of Chapter 4, §2, $f_1 \cdots f_s$ is a reduced (or square-free) polynomial. Hence, we call $f_1 \cdots f_s = 0$ the *reduced* equation of C. This equation is unique up to multiplication by a nonzero constant.

When we consider the intersection of two curves C and D in $\mathbb{P}^2$, we will assume that C and D have no common irreducible components. This means that their defining polynomials have no common factors. Our goal is to relate the number of points in $C \cap D$ to the degrees of their reduced equations. The following property of resultants will play an important role in our study of this problem.

Lemma 5. *Let $f, g \in \mathbb{C}[x, y, z]$ be homogeneous of total degree m, n respectively. If $f(0, 0, 1)$ and $g(0, 0, 1)$ are nonzero, then the resultant* $\mathrm{Res}(f, g, z)$ *is homogeneous in x and y of total degree mn.*

Proof. First, write f and g as polynomials in z:

$$\begin{aligned} f &= a_0 z^m + \cdots + a_m, \\ g &= b_0 z^n + \cdots + b_n, \end{aligned}$$

and observe that since f is homogeneous of total degree m, each $a_i \in \mathbb{C}[x, y]$ must be homogeneous of degree i. Furthermore, $f(0, 0, 1) \neq 0$ implies that a_0 is a nonzero constant. Similarly, b_i, is homogeneous of degree i and $b_0 \neq 0$.

By Chapter 3, §5, the resultant is given by the $(m+n) \times (m+n)$-determinant

$$\mathrm{Res}(f, g, z) = \det \begin{pmatrix} a_0 & & & b_0 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ a_m & & a_0 & b_n & & b_0 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_m & & & b_n \end{pmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{n \text{ columns}} \qquad \underbrace{\qquad\qquad\qquad}_{m \text{ columns}}$$

where the empty spaces are filled by zeros. To show that $\mathrm{Res}(f, g, z)$ is homogeneous of degree mn, let c_{ij} denote the ij-th entry of the matrix. From the pattern of the above matrix, you can check that the nonzero entries are

$$c_{ij} = \begin{cases} a_{i-j} & \text{if } j \leq n \\ b_{n+i-j} & \text{if } j > n. \end{cases}$$

Thus, a nonzero c_{ij} is homogeneous of total degree $i - j$ (if $j \leq n$) or $n + i - j$ (if $j > n$).

By Proposition 2 of Appendix A, §3, the determinant giving $\mathrm{Res}(f, g, z)$ is a sum of products

$$\pm \prod_{i=1}^{m+n} c_{i\sigma(i)},$$

where σ is a permutation of $\{1, \ldots, m+n\}$. We can assume that each $c_{i\sigma(i)}$ in the product is nonzero. If we write the product as

$$\pm \prod_{\sigma(i) \leq n} c_{i\sigma(i)} \prod_{\sigma(i) > n} c_{i\sigma(i)},$$

then, by the above paragraph, this product is a homogeneous polynomial of degree

$$\sum_{\sigma(i) \leq n} (i - \sigma(i)) + \sum_{\sigma(i) > n} (n + i - \sigma(i)).$$

Since σ is a permutation of $\{1, \ldots, m+n\}$, the first sum has n terms and the second has m, and all i's between 1 and $m + n$ appear exactly once. Thus, we can rearrange the sum to obtain

$$mn + \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) = mn,$$

which proves that $\mathrm{Res}(f, g, z)$ is a sum of homogeneous polynomials of degree mn. □

This lemma shows that the resultant $\mathrm{Res}(f, g, z)$ is homogeneous in x and y. In general, homogeneous polynomials in two variables have an especially simple structure.

Lemma 6. *Let $h \in \mathbb{C}[x, y]$ be a nonzero homogeneous polynomial. Then h can be written in the form*

$$h = c(s_1x - r_1y)^{m_1} \cdots (s_tx - r_ty)^{m_t},$$

where $c \neq 0$ in $\mathbb{C}$ and $(r_1, s_1), \ldots, (r_t, s_t)$ are distinct points of $\mathbb{P}^1$. Furthermore,

$$\mathbf{V}(h) = \{(r_1, s_1), \ldots, (r_t, s_t)\} \subset \mathbb{P}^1.$$

Proof. This follows by observing that the polynomial $h(x, 1) \in \mathbb{C}[x]$ is a product of linear factors since $\mathbb{C}$ is algebraically closed. We leave the details as an exercise. □

As a first application of these lemmas, we show how to bound the number of points in the intersection of two curves using the degrees of their reduced equations.

Theorem 7. *Let C and D be projective curves in $\mathbb{P}^2$ with no common irreducible components. If the degrees of the reduced equations for C and D are m and n respectively, then $C \cap D$ is finite and has at most mn points.*

Proof. Suppose that $C \cap D$ has more than mn points. Choose $mn + 1$ of them, which we label $p_1, \ldots, p_{mn+1}$, and for $1 \leq i < j \leq mn + 1$, let L_{ij} be the line through p_i and p_j. Then pick a point $q \in \mathbb{P}^2$ such that

$$q \notin C \cup D \cup \bigcup_{i<j} L_{ij} \tag{1}$$

(in Exercise 6 you will prove carefully that such points exist).

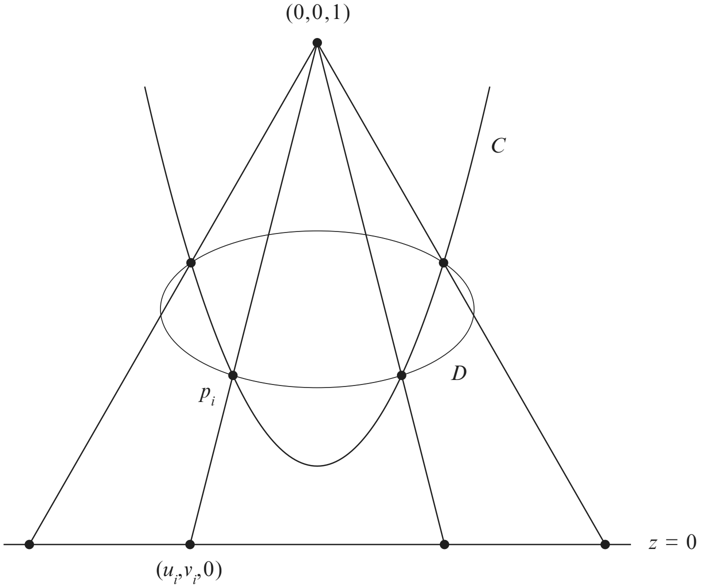
As in §6, a matrix $A \in \mathrm{GL}(3, \mathbb{C})$ gives a map $A : \mathbb{P}^2 \to \mathbb{P}^2$. It is easy to find an A such that $A(q) = (0, 0, 1)$ (see Exercise 6). If we regard A as giving new coordinates for $\mathbb{P}^2$ (see (3) in §6), then the point q has coordinates $(0, 0, 1)$ in the new system. We can thus assume that $q = (0, 0, 1)$ in (1).

Now suppose that $C = \mathbf{V}(f)$ and $D = \mathbf{V}(g)$, where f and g are reduced of degrees m and n respectively. Then (1) implies $f(0, 0, 1) \neq 0$ since $(0, 0, 1) \notin C$, and $g(0, 0, 1) \neq 0$ follows similarly. Thus, by Lemma 5, the resultant $\mathrm{Res}(f, g, z)$ is a homogeneous polynomial of degree mn in x, y. Since f and g have positive degree in z and have no common factors in $\mathbb{C}[x, y, z]$, Proposition 1 of Chapter 3, §6, shows that $\mathrm{Res}(f, g, z)$ is nonzero.

If we let $p_i = (u_i, v_i, w_i)$, then since the resultant is in the ideal generated by f and g (Proposition 1 of Chapter 3, §6), we have

$$\mathrm{Res}(f, g, z)(u_i, v_i) = 0. \tag{2}$$

Note that the line connecting $q = (0, 0, 1)$ to $p_i = (u_i, v_i, w_i)$ intersects $z = 0$ in the point $(u_i, v_i, 0)$ (see Exercise 6). The picture is as follows:



The map taking a point $(u, v, w) \in \mathbb{P}^2 - \{(0, 0, 1)\}$ to $(u, v, 0)$ is an example of a *projection from a point to a line.* Hence, (2) tells us that $\mathrm{Res}(f, g, z)$ vanishes at the points obtained by projecting the $p_i \in C \cap D$ from $(0, 0, 1)$ to the line $z = 0$.

By (1), $(0, 0, 1)$ lies on none of the lines connecting p_i and p_j, which implies that the points $(u_i, v_i, 0)$ are distinct for $i = 1, \ldots, mn + 1$. If we regard $z = 0$ as a copy of $\mathbb{P}^1$ with homogeneous coordinates x, y, then we get distinct points $(u_i, v_i) \in \mathbb{P}^1$, and the homogeneous polynomial $\mathrm{Res}(f, g, z)$ vanishes at all $mn + 1$ of them. By Lemmas 5 and 6, this is impossible since $\mathrm{Res}(f, g, z)$ is nonzero of degree mn, and the theorem follows. $\square$

Now that we have a criterion for $C \cap D$ to be finite, the next step is to define an *intersection multiplicity* for each point $p \in C \cap D$. There are a variety of ways this can be done, but the simplest involves the resultant.

Thus, we define the intersection multiplicity as follows. Let C and D be curves in $\mathbb{P}^2$ with no common components and reduced equations $f = 0$ and $g = 0$. For each pair of points $p \neq q$ in $C \cap D$, let L_{pq} be the projective line connecting p and q. Pick a matrix $A \in \mathrm{GL}(3, \mathbb{C})$ such that in the new coordinate system given by A, we have

$$(3) \qquad (0, 0, 1) \notin C \cup D \cup \bigcup_{p \neq q \text{ in } C \cap D} L_{pq}.$$

(Example 9 below shows how such coordinate changes are done.) As in the proof of Theorem 7, if $p = (u, v, w) \in C \cap D$, then the resultant $\mathrm{Res}(f, g, z)$ vanishes at (u, v), so that by Lemma 6, $vx - uy$ is a factor of $\mathrm{Res}(f, g, z)$.

Definition 8. *Let C and D be curves in $\mathbb{P}^2$ with no common components and reduced defining equations $f = 0$ and $g = 0$. Choose coordinates for $\mathbb{P}^2$ so that (3) is satisfied. Then, given $p = (u, v, w) \in C \cap D$, the* **intersection multiplicity** *$I_p(C, D)$ is defined to be the exponent of $vx - uy$ in the factorization of* $\mathrm{Res}(f, g, z)$.

In order for $I_p(C, D)$ to be well-defined, we need to make sure that we get the same answer no matter what coordinate system satisfying (3) we use in Definition 8. For the moment, we will assume this is true and compute some examples of intersection multiplicities.

Example 9. Consider the following polynomials in $\mathbb{C}[x, y, z]$:

$$\begin{aligned} f &= x^3 + y^3 - 2xyz, \\ g &= 2x^3 - 4x^2y + 3xy^2 + y^3 - 2y^2z. \end{aligned}$$

These polynomials [adapted from WALKER (1950)] define cubic curves $C = \mathbf{V}(f)$ and $D = \mathbf{V}(g)$ in $\mathbb{P}^2$. To study their intersection, we first compute the resultant with respect to z:

$$\mathrm{Res}(f, g, z) = -2y(x - y)^3(2x + y).$$

Since the resultant is in the elimination ideal, points in $C \cap D$ satisfy either $y = 0$, $x - y = 0$ or $2x + y = 0$, and from here, it is easy to show that $C \cap D$ consists of the three points

$$p = (0, 0, 1), \quad q = (1, 1, 1), \quad r = (4/7, -8/7, 1)$$

(see Exercise 7). In particular, this shows that C and D have no common components.

However, the above resultant does *not* give the correct intersection multiplicities since $(0, 0, 1) \in C$ (in fact, it is a point of intersection). Hence, we must change coordinates. Start with a point such as

$$(0, 1, 0) \notin C \cup D \cup L_{pq} \cup L_{pr} \cup L_{qr},$$

and find a coordinate change with $A(0, 1, 0) = (0, 0, 1)$, say $A(x, y, z) = (z, x, y)$. Then

$$(0, 0, 1) \notin A(C) \cup A(D) \cup L_{A(p)A(q)} \cup L_{A(p)A(r)} \cup L_{A(q)A(r)}.$$

To find the defining equation of $A(C)$, note that

$$(u, v, w) \in A(C) \iff A^{-1}(u, v, w) \in C \iff f(A^{-1}(u, v, w)) = 0.$$

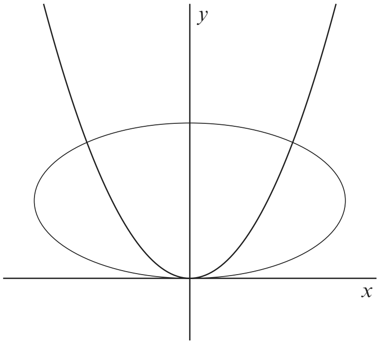
Thus, $A(C)$ is defined by the equation $f \circ A^{-1}(x, y, z) = f(y, z, x) = 0$, and similarly, $A(D)$ is given by $g(y, z, x) = 0$. Then, by Definition 8, the resultant $\mathrm{Res}(f(y, z, x), g(y, z, x), z)$ gives the multiplicities for $A(p) = (1, 0, 0)$, $A(q) = (1, 1, 1)$ and $A(r) = (1, 4/7, -8/7)$. The resultant is

$$\mathrm{Res}(f(y, z, x), g(y, z, x), z) = 8y^5(x - y)^3(4x - 7y).$$

so that in terms of p, q and r, the intersection multiplicities are

$$I_p(C, D) = 5, \quad I_q(C, D) = 3, \quad I_r(C, D) = 1.$$

Example 1. [continued] If we let $\lambda = 1$ in Example 1, we get the curves



In this picture, the point (0, 0, 1) is the origin, so we again must change coordinates before (3) can hold. In the exercises, you will use an appropriate coordinate change to show that the intersection multiplicity at the origin is in fact equal to 2.

Still assuming that the intersection multiplicities in Definition 8 are well-defined, we can now prove Bezout's Theorem.

Theorem 10 (Bezout's Theorem). *Let C and D be curves in $\mathbb{P}^2$ with no common components, and let m and n be the degrees of their reduced defining equations. Then*

$$\sum_{p \in C \cap D} I_p(C, D) = mn,$$

where $I_p(C, D)$ is the intersection multiplicity at p, as defined in Definition 8.

Proof. Let $f = 0$ and $g = 0$ be the reduced equations of C and D, and assume that coordinates have been chosen so that (3) holds. Write $p \in C \cap D$ as $p = (u_p, v_p, w_p)$. Then we claim that

$$\operatorname{Res}(f, g, z) = c \prod_{p \in C \cap D} (v_p x - u_p y)^{I_p(C, D)},$$

where c is a nonzero constant. For each p, it is clear that $(v_p x - u_p y)^{I_p(C,D)}$ is the exact power of $v_p x - u_p y$ dividing the resultant—this follows by the definition of $I_p(C, D)$. We still need to check that this accounts for *all* roots of the resultant. But if $(u, v) \in \mathbb{P}^1$ satisfies $\operatorname{Res}(f, g, z)(u, v) = 0$, then Proposition 3 of Chapter 3, §6, implies that there is some $w \in \mathbb{C}$ such that f and g vanish at (u, v, w). This is because if we write f and g as in the proof of Lemma 5, a_0 and b_0 are nonzero constants by (3). Thus $(u, v, w) \in C \cap D$, and our claim is proved.

By Lemma 5, $\text{Res}(f, g, z)$ is a nonzero homogeneous polynomial of degree mn. Then Bezout's Theorem follows by comparing the degree of each side in the above equation. □

Example 9. [continued] In Example 9, we had two cubic curves which intersected in the points $(0, 0, 1)$, $(1, 1, 1)$ and $(4/7, -8/7, 1)$ of multiplicity 5, 3 and 1 respectively. These add up to $9 = 3 \cdot 3$, as desired. If you look back at Example 9, you'll see why we needed to change coordinates in order to compute intersection multiplicities. In the original coordinates, $\text{Res}(f, g, z) = -2y(x - y)^3(2x + y)$, which would give multiplicities 1, 3 and 1. Even without computing the correct multiplicities, we know these can't be right since they don't add up to 9!

Finally, we show that the intersection multiplicities in Definition 8 are well-defined.

Lemma 11. *In Definition 8, all coordinate change matrices satisfying* (3) *give the same intersection multiplicities $I_p(C, D)$ for $p \in C \cap D$.*

Proof. Although this result holds over any algebraically closed field, our proof will use continuity arguments and hence is special to $\mathbb{C}$. We begin by describing carefully the coordinate changes we will use. As in Example 9, pick a point

$$r \notin C \cup D \cup \bigcup_{p \neq q \text{ in } C \cap D} L_{pq}$$

and a matrix $A \in \text{GL}(3, \mathbb{C})$ such that $A(r) = (0, 0, 1)$. This means $A^{-1}(0, 0, 1) = r$, so that the condition on A is

$$A^{-1}(0, 0, 1) \notin C \cup D \cup \bigcup_{p \neq q \text{ in } C \cap D} L_{pq}.$$

Let $l_{pq} = 0$ be the equation of the line L_{pq}, and set

$$h = f \cdot g \cdot \prod_{p \neq q \text{ in } C \cap D} \ell_{pq}.$$

The condition on A is thus $A^{-1}(0, 0, 1) \notin \mathbf{V}(h)$, i.e., $h(A^{-1}(0, 0, 1)) \neq 0$.

We can formulate this problem without using matrix inverses as follows. Consider matrices $B \in M_{3\times 3}(\mathbb{C})$, where $M_{3\times 3}(\mathbb{C})$ is the set of all 3×3 matrices with entries in $\mathbb{C}$, and define the function $H : M_{3\times 3}(\mathbb{C}) \to \mathbb{C}$ by

$$H(B) = \det(B) \cdot h(B(0, 0, 1)).$$

If $B = (b_{ij})$, note that $H(B)$ is a polynomial in the b_{ij}. Since a matrix is invertible if and only if its determinant is nonzero, we have

$$H(B) \neq 0 \iff B \quad \text{is invertible and} \quad h(B(0, 0, 1)) \neq 0.$$

Hence the coordinate changes we want are given by $A = B^{-1}$ for $B \in M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$.

Let $C \cap D = \{p_1, \dots, p_s\}$, and for each $B \in M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$, let $B^{-1}(p_i) = (u_{i,B}, v_{i,B}, w_{i,B})$. Then, by the argument given in Theorem 10, we can write

(4) $$\mathrm{Res}(f \circ B, g \circ B, z) = c_B (v_{1,B}x - u_{1,B}y)^{m_{1,B}} \cdots (v_{s,B}x - u_{s,B}y)^{m_{s,B}},$$

where $c_B \neq 0$. This means $I_{p_i}(C, D) = m_{i,B}$ in the coordinate change given by $A = B^{-1}$. Thus, to prove the lemma, we need to show that $m_{i,B}$ takes the same value for *all* $B \in M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$.

To study the exponents $m_{i,B}$, we consider what happens in general when we have a factorization
$$G(x, y) = (vx - uy)^m H(x, y)$$
where G and H are homogeneous and $(u, v) \neq (0, 0)$. Here, one calculates that

(5) $$\frac{\partial^{i+j} G}{\partial x^i \partial y^j}(u, v) = \begin{cases} 0 & \text{if } 0 \le i + j < m \\ m!v^i(-u)^j H(u, v) & \text{if } i + j = m, \end{cases}$$

(see Exercise 9). In particular, if $H(u, v) \neq 0$, then $(u, v) \neq (0, 0)$ implies that some mth partial of G doesn't vanish at (u, v).

We also need a method for measuring the distance between matrices $B, C \in M_{3\times 3}(\mathbb{C})$. If $B = (b_{ij})$ and $C = (c_{ij})$, then the distance between B and C is defined to be
$$d(B, C) = \sqrt{\sum\nolimits_{i,j=1}^{3} |b_{ij} - c_{ij}|^2},$$
where for a complex number $z = a + ib$, $|z| = \sqrt{a^2 + b^2}$. A crucial fact is that any polynomial function $F : M_{3\times 3}(\mathbb{C}) \to \mathbb{C}$ is continuous. This means that given $B_0 \in M_{3\times 3}(\mathbb{C})$, we can get $F(B)$ arbitrarily close to $F(B_0)$ by taking B sufficiently close to B_0 (as measured by the above distance function). In particular, if $F(B_0) \neq 0$, it follows that $F(B) \neq 0$ for B sufficiently close to B_0.

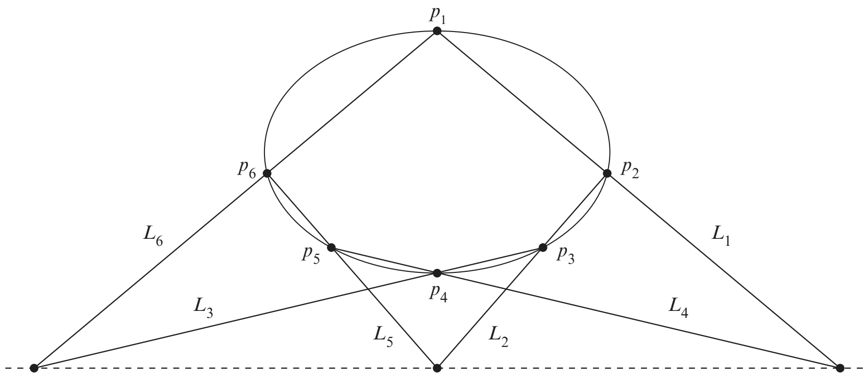
Now consider the exponent $m = m_{i,B_0}$ for fixed B_0 and i. We claim that $m_{i,B} \le m$ if B is sufficiently close to B_0. To see this, first note that (4) and (5) imply that some mth partial of $\mathrm{Res}(f \circ B_0, g \circ B_0, z)$ is nonzero at (u_{i,B_0}, v_{i,B_0}). If we write out $(u_{i,B}, v_{i,B})$ and this partial derivative of $\mathrm{Res}(f \circ B, g \circ B, z)$ explicitly, we get formulas which are rational functions with numerators that are polynomials in the entries of B and denominators that are powers of $\det(B)$. Thus this m-th partial of $\mathrm{Res}(f \circ B, g \circ B, z)$, when evaluated at $(u_{i,B}, v_{i,B})$, is a rational function of the same form. Since it is nonzero at B_0, the continuity argument from the previous paragraph shows that this m-th partial of $\mathrm{Res}(f \circ B, g \circ B, z)$ is nonzero at $(u_{i,B}, v_{i,B})$, once B is sufficiently close to B_0. But then, applying (4) and (5) to $\mathrm{Res}(f \circ B, g \circ B, z)$, we conclude that $m_{i,B} \le m$ [since $m_{i,B} > m$ would imply that all m-th partials would vanish at $(u_{i,B}, v_{i,B})$].

However, if we sum the inequalities $m_{i,B} \le m = m_{i,B_0}$ for $i = 1, \dots, s$, we obtain
$$mn = \sum_{i=1}^{s} m_{i,B} \le \sum_{i=1}^{s} m_{i,B_0} = mn.$$
This implies that we must have term-by-term equalities, so that $m_{i,B} = m_{i,B_0}$ when B is sufficiently close to B_0.

This proves that the function sending B to $m_{i,B}$ is *locally constant*, i.e., its value at a given point is the same as the values at nearby points. In order for us to conclude that the function is actually constant on all of $M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$, we need to prove that $M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$ is *path connected.* This will be done in Exercise 10, which also gives a precise definition of path connectedness. Since the Intermediate Value Theorem from calculus implies that a locally constant function on a path connected set is constant (see Exercise 10), we conclude that $m_{i,B}$ takes the same value for all $B \in M_{3\times 3}(\mathbb{C}) - \mathbf{V}(H)$. Thus the intersection multiplicities of Definition 8 are well-defined. □

The intersection multiplicities $I_p(C, D)$ have many properties which make them easier to compute. For example, one can show that $I_p(C, D) = 1$ if and only if p is a nonsingular point of C and D and the curves have distinct tangent lines at p. A discussion of the properties of multiplicities can be found in Chapter 3 of KIRWAN (1992). We should also point out that using resultants to define multiplicities is unsatisfactory in the following sense. Namely, an intersection multiplicity $I_p(C, D)$ is clearly a *local* object—it depends only on the part of the curves C and D near p—while the resultant is a *global* object, since it uses the equations for all of C and D. Local methods for computing multiplicities are available, though they require slightly more sophisticated mathematics. The local point of view is discussed in Chapter 3 of FULTON (1969) and Chapter IV of WALKER(1950).

As an application of what we've done so far in this section, we will prove the following result of Pascal. Suppose we have six distinct points $p_1, \ldots, p_6$ on an irreducible conic in $\mathbb{P}^2$. By Bezout's Theorem, a line meets the conic in at most 2 points (see Exercise 11). Hence, we get six distinct lines by connecting p_1 to p_2, p_2 to $p_3, \ldots,$ and p_6 to p_1. If we label these lines $L_1, \ldots, L_6$, then we get the following picture:



We say that lines L_1, L_4 are *opposite*, and similarly the pairs L_2, L_5 and L_3, L_6 are opposite. The portions of the lines lying inside the conic form a hexagon, and opposite lines correspond to opposite sides of the hexagon.

In the above picture, the intersections of the opposite pairs of lines appear to lie on the same line. The following theorem reveals that this is no accident.

Theorem 12 (Pascal's Mystic Hexagon). *Given six points on an irreducible conic, connected by six lines as above, the points of intersection of the three pairs of opposite lines are collinear.*

Proof. Let the conic be C. As above, we have six points $p_1, \dots, p_6$ and three pairs of opposite lines $\{L_1, L_4\}$, $\{L_2, L_5\}$, and $\{L_3, L_6\}$. Now consider the curves $C_1 = L_1 \cup L_3 \cup L_5$ and $C_2 = L_2 \cup L_4 \cup L_6$. These curves are defined by cubic equations, so that by Bezout's Theorem, the number of points in $C_1 \cap C_2$ is 9 (counting multiplicities). However, note that $C_1 \cap C_2$ contains the six original points $p_1, \dots, p_6$ and the three points of intersection of opposite pairs of lines (you should check this carefully). Thus, these are all of the points of intersection, and all of the multiplicities are one.

Suppose that $C = \mathbf{V}(f)$, $C_1 = \mathbf{V}(g_1)$ and $C_2 = \mathbf{V}(g_2)$, where f has total degree 2 and g_1 and g_2 have total degree 3. Now pick a point $p \in C$ distinct from $p_1, \dots, p_6$. Thus, $g_1(p)$ and $g_2(p)$ are nonzero (do you see why?), so that $g = g_2(p)g_1 - g_1(p)g_2$ is a cubic polynomial which vanishes at $p, p_1, \dots, p_6$. Furthermore, g is nonzero since otherwise g_1 would be a multiple of g_2 (or vice versa). Hence, the cubic $\mathbf{V}(g)$ meets the conic C in at least seven points, so that the hypotheses for Bezout's Theorem are not satisfied. Thus, either g is not reduced or $\mathbf{V}(g)$ and C have a common irreducible component. The first of these can't occur, since if g weren't reduced, the curve $\mathbf{V}(g)$ would be defined by an equation of degree at most 2 and $\mathbf{V}(g) \cap C$ would have at most 4 points by Bezout's Theorem. Hence, $\mathbf{V}(g)$ and C must have a common irreducible component. But C is irreducible, which implies that $C = \mathbf{V}(f)$ is a component of $\mathbf{V}(g)$. By Proposition 4, it follows that f must divide g.

Hence, we get a factorization $g = f \cdot l$, where l has total degree 1. Since g vanishes where the opposite lines meet and f doesn't, it follows that l vanishes at these points. Since $\mathbf{V}(l)$ is a projective line, the theorem is proved. □

Bezout's Theorem serves as a nice introduction to the study of curves in $\mathbb{P}^2$. This part of algebraic geometry is traditionally called *algebraic curves* and includes many interesting topics we have omitted (inflection points, dual curves, elliptic curves, etc.). Fortunately, there are several excellent texts on this subject. In addition to FULTON (1969), KIRWAN (1992) and WALKER (1950) already mentioned, we also warmly recommend CLEMENS (1980) and BRIESKORN and KNÖRRER (1986). For students with a background in complex analysis and topology, we also suggest GRIFFITHS (1989).

EXERCISES FOR §7

1. This exercise is concerned with the parabola $y = x^2$ and the ellipse $x^2 + 4(y - \lambda)^2 = 4$ from Example 1.
   a. Show that these curves have empty intersection over $\mathbb{R}$ when $\lambda < -1$. Illustrate the cases $\lambda < -1$ and $\lambda = -1$ with a picture.
   b. Find the smallest positive real number λ_0 such that the intersection over $\mathbb{R}$ is empty when $\lambda > \lambda_0$. Illustrate the cases $\lambda > \lambda_0$ and $\lambda = \lambda_0$ with a picture.

c. When $-1 < \lambda < \lambda_0$, describe the possible types of intersections that can occur over $\mathbb{R}$ and illustrate each case with a picture.

d. In the pictures for parts (a), (b), and (c) use the intuitive idea of multiplicity from Example 1 to determine which ones represent intersections with multiplicity > 1.

e. Without using Bezout's Theorem, explain why over $\mathbb{C}$, the number of intersections (counted with multiplicity) adds up to 4 when λ is real. Hint: Use the formulas for x and y given in Example 1.

2. In Example 2, we intersected the parabola $y = x^2$ with a line L in affine space. Assume that L is not vertical.

   a. Over $\mathbb{R}$, show that the number of points of intersection can be 0, 1, or 2. Further, show that you get one point of intersection exactly when L is tangent to $y = x^2$ in the sense of Chapter 3, §4.

   b. Over $\mathbb{C}$, show (without using Bezout's Theorem) that the number of intersections (counted with multiplicity) is exactly 2.

3. In proving Proposition 4, we showed that if $f = gh$ is homogeneous and $g = g_m + \cdots + g_0$, where g_i is homogeneous of total degree i and $g_m \neq 0$, and similarly $h = h_n + \cdots + h_0$, then $f = g_m h_n$. Complete the proof by showing that $g = g_m$ and $h = h_n$. Hint: Let m_0 be the smallest index m_0 such that $g_{m_0} \neq 0$, and define $h_{n_0} \neq 0$ similarly.

4. In this exercise, we sketch an alternate proof of Lemma 5. Given f and g as in the statement of the lemma, let $R(x, y) = \mathrm{Res}(f, g, z)$. It suffices to prove that $R(tx, ty) = t^{mn} R(x, y)$.

   a. Use $a_i(tx, ty) = t^i a_i(x, y)$ and $b_i(tx, ty) = t^i b_i(x, y)$ to show that $R(tx, ty)$ is given by a determinant whose entries are either 0 or $t^i a_i(x, y)$ or $t^i b_i(x, y)$.

   b. In the determinant from part (a), multiply column 2 by t, column 3 by $t^2, \ldots,$ column n by t^{n-1}, column $n + 2$ by t, column $n + 3$ by $t^2, \ldots,$ and column $n + m$ by t^{m-1}. Use this to prove that $t^q R(tx, ty)$, where $q = n(n-1)/2 + m(m-1)/2$, equals a determinant where in each row, t appears to the *same* power.

   c. By pulling out the powers of t from the rows of the determinant from part (b) prove that $t^q R(tx, ty) = t^r R(x, y)$, where $r = (m + n)(m + n - 1)/2$.

   d. Use part (c) to prove that $R(tx, ty) = t^{mn} R(x, y)$, as desired.

5. Complete the proof of Lemma 6 using the hints given in the text. Hint: Use Proposition 7 and Exercise 11 from §2.

6. This exercise is concerned with the proof of Theorem 7.

   a. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a nonzero polynomial. Prove that $\mathbf{V}(f)$ and $\mathbb{C}^n - \mathbf{V}(f)$ are nonempty. Hint: Use the Nullstellensatz and Proposition 5 of Chapter 1, §1.

   b. Use part (a) to prove that you can find $q \notin C \cup D \cup \bigcup_{i<j} L_{ij}$ as claimed in the proof of Theorem 7.

   c. Given $q \in \mathbb{P}^2(\mathbb{C})$, find $A \in \mathrm{GL}(3, \mathbb{C})$ such that $A(q) = (0, 0, 1)$. Hint: Regard q and $(0, 0, 1)$ as nonzero column vectors in $\mathbb{C}^3$ and use linear algebra to find an invertible matrix A such that $A(q) = (0, 0, 1)$.

   d. Prove that the projective line connecting $(0, 0, 1)$ to (u, v, w) intersects the line $z = 0$ in the point $(u, v, 0)$. Hint: Use equation (10) of §6.

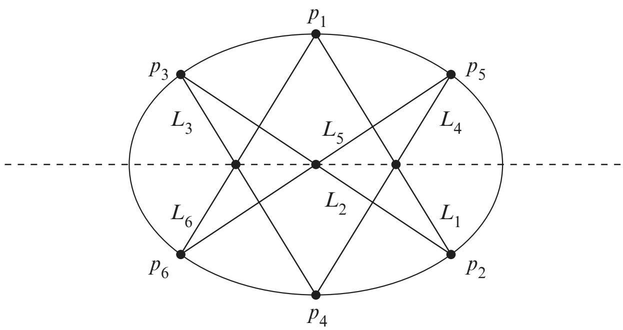
7. In Example 9, we considered the curves $C = \mathbf{V}(f)$ and $D = \mathbf{V}(g)$, where f and g are given in the text.

   a. Verify carefully that $p = (0, 0, 1), q = (1, 1, 1)$ and $r = (4/7, -8/7, 1)$ are the only points of intersection of the curves C and D. Hint: Once you have $\mathrm{Res}(f, g, z)$, you can do the rest by hand.

   b. Show that f and g are reduced. Hint: Use a computer.

   c. Show that $(0, 1, 0) \notin C \cup D \cup L_{pq} \cup L_{pr} \cup L_{qr}$.

8. For each of the following pairs of curves, find the points of intersection and compute the intersection multiplicities.
   a. $C = \mathbf{V}(yz - x^2)$ and $D = \mathbf{V}(x^2 + 4(y-z)^2 - 4z^2)$. This is the projective version of Example 1 when $\lambda = 1$. Hint: Show that the coordinate change given by $A(x, y, z) = (x, y + z, z)$ has the desired properties.
   b. $C = \mathbf{V}(x^2y^3 - 2xy^2z^2 + yz^4 + z^5)$ and $D = \mathbf{V}(x^2y^2 - xz^3 - z^4)$. Hint: There are four solutions, two real and two complex. When finding the complex solutions, computing the GCD of two complex polynomials may help.
9. Prove (5). Hint: Use induction on m, and apply the inductive hypothesis to $\partial G/\partial x$ and $\partial G/\partial y$.
10. (Requires advanced calculus.) An open set $U \subset \mathbb{C}^n$ is *path connected* if for every two points $a, b \in U$, there is a continuous function $\gamma : [0, 1] \to U$ such that $\gamma(0) = a$ and $\gamma(1) = b$.
    a. Suppose that $F : U \to \mathbb{Z}$ is locally constant (as in the text, this means that the value of F at a point of U equals its value at all nearby points). Use the Intermediate Value Theorem from calculus to show that F is constant when U is path connected. Hint: If we regard F as a function $F : U \to \mathbb{R}$, explain why F is continuous. Then note that $F \circ \gamma : [0, 1] \to \mathbb{R}$ is also continuous.
    b. Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Prove that $\mathbb{C} - \mathbf{V}(f)$ is path connected.
    c. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is nonzero, prove that $\mathbb{C} - \mathbf{V}(f)$ is path connected. Hint: Given $a, b \in \mathbb{C}^n - \mathbf{V}(f)$, consider the complex line $\{ta + (1-t)b : t \in \mathbb{C}\}$ determined by a and b. Explain why $f(ta + (1-t)b)$ is a nonzero polynomial in t and use part (b).
    d. Give an example of $f \in \mathbb{R}[x, y]$ such that $\mathbb{R}^2 - \mathbf{V}(f)$ is not path connected. Further, find a locally constant function $F : \mathbb{R}^2 - \mathbf{V}(f) \to \mathbb{Z}$ which is not constant. Thus, it is essential that we work over $\mathbb{C}$.
11. Let C be an irreducible conic in $\mathbb{P}^2(\mathbb{C})$. Use Bezout's Theorem to explain why a line L meets C in at most two points. What happens when C is reducible? What about when C is a curve defined by an irreducible polynomial of total degree n?
12. In the picture drawn in the text for Pascal's Mystic Hexagon, the six points went clockwise around the conic. If we change the order of the points, we can still form a "hexagon," though opposite lines might intersect inside the conic. For example, the picture could be as follows:

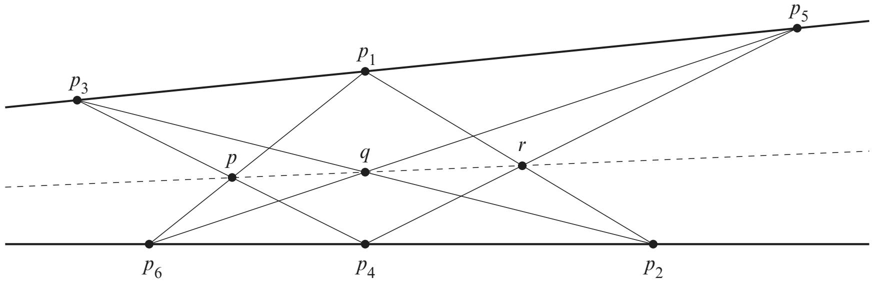


    Explain why the theorem remains true in this case.
13. In Pascal's Mystic Hexagon, suppose that the conic is a circle and the six lines come from a regular hexagon inscribed inside the circle. Where do the opposite lines meet and on what line do their intersections lie?

14. Pappus's Theorem from Exercise 8 of Chapter 6, §4, states that if p_3, p_1, p_5 and p_6, p_4, p_2 are two collinear triples of points and we set

$$\begin{aligned} p &= \overline{p_3p_4} \cap \overline{p_6p_1} \\ q &= \overline{p_2p_3} \cap \overline{p_5p_6} \\ r &= \overline{p_4p_5} \cap \overline{p_1p_2}. \end{aligned}$$

then p, q, r are also collinear. The picture is as follows:



The union of the lines $\overline{p_3p_1}$ and $\overline{p_6p_4}$ is a reducible conic C'. Explain why Pappus's Theorem can be regarded as a "degenerate" case of Pascal's Mystic Hexagon. Hint: See Exercise 12. Note that unlike the irreducible case, we can't choose any six points on C': we must avoid the singular point of C', and each component of C' must contain three of the points.

15. The argument used to prove Theorem 12 applies in much more general situations. Suppose that we have curves C and D defined by reduced equations of total degree n such that $C \cap D$ consists of exactly n^2 points. Furthermore, suppose there is an irreducible curve E with a reduced equation of total degree $m < n$ which contains exactly mn of these n^2 points. Then adapt the argument of Theorem 12 to show that there is a curve F with a reduced equation of total degree $n - m$ which contains the remaining $n(n - m)$ points of $C \cap D$.

16. Let C and D be curves in $\mathbb{P}^2(\mathbb{C})$.
    a. Prove that $C \cap D$ must be nonempty.
    b. Suppose that C is nonsingular in the sense of part (a) of Exercise 9 of §6 [if $C = \mathbf{V}(f)$, this means the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ don't vanish simultaneously on $\mathbb{P}^2(\mathbb{C})$]. Prove that C is irreducible. Hint: Suppose that $C = C_1 \cup C_2$, which implies $f = f_1 f_2$. How do the partials of f behave at a point of $C_1 \cap C_2$?

17. This exercise will explore an informal proof of Bezout's Theorem. The argument is not rigorous but does give an intuitive explanation of why the number of intersection points is mn.
    a. In $\mathbb{P}^2(\mathbb{C})$, show that a line L meets a curve C of degree n in n points, counting multiplicity. Hint: Choose coordinates so that all of the intersections take place in $\mathbb{C}^2$, and write L parametrically as $x = a + ct$, $y = b + dt$.
    b. If a curve C of degree n meets a union of m lines, use part (a) to predict how many points of intersection there are.
    c. When two curves C and D meet, give an intuitive argument (based on pictures) that the number of intersections (counting multiplicity) doesn't change if one of the curves moves a bit. Your pictures should include instances of tangency and the example of the intersection of the x-axis with the cubic $y = x^3$.

d. Use the constancy principle from part (c) to argue that if the m lines in part (b) all coincide (giving what is called a line of *multiplicity* m), the number of intersections (counted with multiplicity) is still as predicted.
e. Using the constancy principle from part (c) argue that Bezout's Theorem holds for general curves C and D by moving D to a line of multiplicity m [as in part (d)]. Hint: If D is defined by $f = 0$, you can "move" D letting all but one coefficient of f go to zero.

In technical terms, this is a *degeneration* proof of Bezout's Theorem. A rigorous version of this argument can be found in BRIESKORN and KNÖRRER (1986). Degeneration arguments play an important role in algebraic geometry.