LINEAR ALGEBRA (MATH 301) DEFINITIONS AND THEOREMS UNIVERSITY OF WISCONSIN - PARKSIDE, FALL 2020

These notes contain a list of the main definitions, lemmas, propositions, and theorems discussed in this course.

1. Subspace, Linear Combination, Span, Linear Independence, Basis

Definition 1.1. *A **vector space** is ...*

Definition 1.2. *A **subspace** of a vector space V is a subset W of V which is a vector space (under the addition and scalar multiplication of V).*

Definition 1.3. *A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a vector of the form $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r$, where $k_1, \ldots, k_r$ are scalars.*

Definition 1.4. *The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$, denoted $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.*

Definition 1.5. *Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is **linearly independent** if $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ implies $k_1 = \cdots = k_r = 0$.*

Definition 1.6. *A **basis** for a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that*
- (i) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, *and*
- (ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is linearly independent.*

Definition 1.7. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a vector space V, and let $\mathbf{u}$ be a vector in V. Then $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for unique scalars $c_1, \ldots, c_n$, by Theorem 1.6. The scalars $c_1, \ldots, c_n$ are called the **coordinates** of $\mathbf{u}$ relative to $\mathcal{B}$, and the vector*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

*denoted by $[\mathbf{u}]_{\mathcal{B}}$, is called the **coordinate vector** of $\mathbf{u}$ relative to $\mathcal{B}$.*

Definition 1.8. *The **dimension** of a nonzero vector space V is the number of vectors in any basis for V.*

Definition 1.9. *Let $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ be an $m \times n$ matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$. Then $A\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.*

Definition 1.10. *Let A be an $m \times n$ matrix. The **nullspace** of A, denoted by $\operatorname{Nul} A$, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$.*

Date: October 30, 2020.

Definition 1.11. *Let A be an $m \times n$ matrix. The **column space** of A, denoted by* $\operatorname{Col} A$, *is the span of the column vectors of A.*

Theorem 1.1. *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar. Then*

(i) $0\mathbf{u} = \mathbf{0}$
(ii) $k\mathbf{0} = \mathbf{0}$
(iii) $(-1)\mathbf{u} = -\mathbf{u}$

Theorem 1.2. *Let V be a vector space and let W be a subset of V. Then W is a subspace of V if:*

(i) *for every $\mathbf{u}$ and $\mathbf{v}$ in W, $\mathbf{u} + \mathbf{v}$ is in W (i.e., W is closed under vector addition); and*
(ii) *for every $\mathbf{u}$ in W and scalar k, $k\mathbf{u}$ is in W (i.e., W is closed under scalar multiplication); and*
(iii) *the zero vector of V lies in W.*

Theorem 1.3. *Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then* $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ *is a subspace of V.*

Theorem 1.4. *A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.*

Theorem 1.5. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.*

(i) *If any vector of V is added to $\mathcal{B}$, then $\mathcal{B}$ is no longer linearly independent.*
(ii) *If any vector is removed from $\mathcal{B}$, then $\mathcal{B}$ no longer spans V.*

Theorem 1.6. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a basis of a vector space V. Then every $\mathbf{u}$ in V can be written in exactly one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, that is, can be expressed as*

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n,$$

for unique scalars $c_1, \ldots, c_n$.

Theorem 1.7. *All bases of a vector space V have the same number of elements.*

Theorem 1.8. *In $\mathbb{R}^n$, the following have the same solutions:*

(i) *The vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{u}$.*
(ii) *The linear system of equations with augmented matrix $[\mathbf{v}_1 \cdots \mathbf{v}_p \mid \mathbf{u}]$.*
(iii) *The matrix equation $[\mathbf{v}_1\ \cdots \mathbf{v}_p]\, \mathbf{x} = \mathbf{u}$.*

Lemma 1.1. *Let A be an $m \times n$ matrix, let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then*

(i) *$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and*
(ii) *$A(c\mathbf{u}) = c(A\mathbf{u})$.*

Theorem 1.9. *Let A be an $m \times n$ matrix. Then* $\operatorname{Nul} A$ *is a subspace of $\mathbb{R}^n$.*

2. Introduction to Linear Transformations

Definition 2.1. *Let V and W be vector spaces. A transformation (or mapping) $T : V \to W$ is* ***linear*** *if it satisfies the following conditions:*

(i) *For every $\mathbf{u}, \mathbf{v}$ in V, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.*
(ii) *For every $\mathbf{u}$ in V and scalar c, $T(c\mathbf{u}) = cT(\mathbf{u})$.*

Theorem 2.1. *Let $T : V \to W$ be linear. Then*

(i) $T(\mathbf{0}) = \mathbf{0}$.
(ii) $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$, *for any scalars $c_1, \ldots, c_p$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in V.*

Definition 2.2. *A* ***matrix transformation*** *is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, for some fixed $m \times n$ matrix A.*

Theorem 2.2. *A matrix transformation is linear.*

Definition 2.3. *Let $T : V \to W$ be linear. Then*

(i) *The* ***kernel*** *of T, denoted $\ker(T)$, is the set of vectors in V which T maps to $\mathbf{0}$.*
(ii) *The* ***range*** *of T, denoted $R(T)$, is the set of vectors in W which have at least one vector in V mapping to them.*

Theorem 2.3. *Let $T : V \to W$ be linear. Then $\ker(T)$ is a subspace of V and $R(T)$ is a subspace of W.*

Theorem 2.4. *Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T) = \operatorname{Nul} A$ and $R(T) = \operatorname{Col} A$.*

Theorem 2.5. *Let $T : V \to W$ be linear. Then $\dim(\ker T) + \dim(R(T)) = \dim V$.*

Theorem 2.6. *Let $T : V \to W$ be linear. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.*

Theorem 2.7. *Let W be a subspace of V. If $\dim W = \dim V$, then $W = V$.*

Theorem 2.8. *Let $T : V \to W$ be linear, and suppose that $\dim V = \dim W$. Then T is one-to-one if and only if T is onto.*

Definition 2.4. *Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the* ***composition*** *of S with T, denoted $S \circ T$, is the map from U to W defined by $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$ for $\mathbf{u} \in U$.*

Theorem 2.9. *Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the composition $S \circ T : U \to W$ is a linear transformation.*

Definition 2.5. *For any vector space V, the* ***identity transformation*** *$I : V \to V$ is defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V.*

Theorem 2.10. *Let $T : V \to W$ be a linear transformation. Then $T \circ I = I \circ T = T$.*

Definition 2.6. *Let $T : V \to W$ be one-to-one. Then there exists an* ***inverse transformation*** *$T^{-1} : R(T) \to V$ such that $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V.*

Theorem 2.11. *Let $T : V \to W$ be one-to-one. Then $T^{-1} \circ T = I$.*

Definition 2.7. *An* ***isomorphism*** *is a bijective linear transformation.*

Definition 2.8. *If $T : V \to W$ is an isomorphism, then V and W are said to* ***isomorphic***.

Theorem 2.12. *If $T : V \to W$ is an isomorphism, then $\dim V = \dim W$.*

Theorem 2.13. *Suppose that V is a vector space and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Then the mapping $T : V \to \mathbb{R}^n$ given by $T(\mathbf{u}) = [\mathbf{u}]_B$ is an isomorphism.*

3. The Matrix of a Linear Transformation

Theorem 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then for $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$. The matrix $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ is called the* ***standard matrix*** *for T.*

Theorem 3.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then T is a linear transformation if and only if T is a matrix transformation.*