Homework 9: Compact Sets in $\mathbb{R}$
(Due March 30, 2022)

Assignments should be ***stapled*** *and written clearly and legibly. You may use the Heine-Borel Theorem only for Question 2. Problems 5 and 6 are optional.*

1. §3.5, #3(a), (c)

2. Prove that every compact set has a maximum. (Hint: use the Heine-Borel Theorem.)

3. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Prove that S is compact using the definition of compactness (and not the Heine-Borel Theorem). In other words, prove directly that every open cover of S has a finite subcover.

4. Use the definition of compactness to prove that the union of a finite collection of compact sets is compact. Show by example that the union of an infinite collection of compact sets may not be compact.

5. (Challenge) Use the definition of compactness to prove that if S is compact, then every infinite subset of S has a limit point in S.

6. Use the definition of compactness to prove that $[0, 1] \cap \mathbb{Q}$ is not compact.

The next three questions pertain to the new material on metric spaces.

7. For $x, y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= |x| + |y| \\ d_2(x,y) &= (x-y)^2 \\ d_3(x,y) &= |x-2y| \\ d_4(x,y) &= |x^2 - y^2| \end{aligned}$$

   Determine whether each of these is a metric on $\mathbb{R}$. Justify your answers.

8. Let X be the set of sequences of real numbers which are bounded. Define a function $d : X \times X \to \mathbb{R}$ by $d((a_n),(b_n)) = \sup\{|a_n - b_n| : n \in \mathbb{N}\}$. Verify that d is a metric.

9. §3.6, #7