Homework 10: One-to-One, Onto

1. §8.2, #1(b), 6, 7, 19(a)(c)(d), 26.

2. Let $T : V \to W$ be a one-to-one linear transformation, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be linearly independent in V. Prove that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly independent in W.

   Hint. Begin the proof as follows:

   "Suppose $c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}$. *I must show that* $c_1 = \cdots = c_p = 0$."

3. Let $T : V \to W$ be a linear transformation, with $\dim V = n$, $\dim W = m$. Prove the following:

   (a) $\dim(R(T)) \leq n$.

   (b) $\dim(R(T)) = n$ if and only if T is one-to-one.

   (c) $\dim(R(T)) = m$ if and only if T is onto.

4. (Challenge) Let $T : V \to V$ be a linear transformation such that $T \circ T = T$, and let $\mathbf{u}$ be a vector in V. Prove that $\{\mathbf{u}, T(\mathbf{u})\}$ is linearly dependent if and only if $T(\mathbf{u}) = \mathbf{u}$ or $T(\mathbf{u}) = \mathbf{0}$.