Homework 8: Linear Transformations

1. Show that each of the following transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear by finding a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Describe geometrically what each transformation does.
   (a) $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} -x_1 \\ x_2 \end{smallmatrix}\right]$
   (b) $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x_2 \\ x_1 \end{smallmatrix}\right]$
   (c) $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right]$
   (d) $T(\mathbf{x}) = -\mathbf{x}$
   (e) $T(\mathbf{x}) = \frac{1}{2}\mathbf{x}$

2. Determine whether each of the following transformations T is linear. Prove your answers.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 4x_1 - 2x_2 \\ 3|x_2| \end{smallmatrix}\right]$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x_1 - 2x_2 \\ x_1 + 3x_2 \\ 3x_1 - 4x_2 \end{smallmatrix}\right]$.
   (c) $T : P_2 \to P_2$ defined by $T(a + bx + cx^2) = a + b(x+1) + b(x+1)^2$.
   (d) $T : F(-\infty, \infty) \to \mathbb{R}^2$ defined by $T(f) = \left[\begin{smallmatrix} f(0) \\ f(2) \end{smallmatrix}\right]$.

3. Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors that spans V. Prove that if

   $$T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0},$$

   then $T(\mathbf{u}) = \mathbf{0}$ for all vectors $\mathbf{u}$ in V.

   Hint. Use Theorem 2.1.

4. Let $T : V \to W$ be a linear transformation. Suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly dependent set of vectors in V. Using the definition of linear dependence and Theorem 2.1, prove that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent in W.

   Hint. Begin the proof as follows:

   > *"Since $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent, I can write $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, where $c_1, \dots, c_p$ are scalars which are not all zero."*