Homework 15: The Mean Value Theorem (Due April 27, 2022)

Directions. Assignments should be ***stapled*** *and written clearly and legibly. Problem 7 is optional.*

1. §6.2, #16.

2. Find a twice differentiable function $f(x)$ such that $f'(1) = -1$, $f'(4) = 7$, and $f''(x) > 3$ for all x, or prove that such a function cannot exist.

3. Suppose that $f : [a, b] \to \mathbb{R}$ has continuous derivatives f' and f'', and assume there exists $c \in (a, b)$ such that $f(a) = f(c) = f(b)$. Prove that there exists d in (a, b) such that $f''(d) = 0$.

   Hint. First prove that there exist x_1 and x_2 such that $a < x_1 < x_2 < b$ and $f'(x_1) = f'(x_2) = 0$.

4. Recall that a number c is called a **fixed point** of a function f if $f(c) = c$. Prove that if f is a differentiable function and $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point. (Hint: Consider the function $g(x) = f(x) - x$.)

5. Suppose that f and g are differentiable functions such that $f' = g$ and $g' = -f$. Prove that $h(x) = (f(x))^2 + (g(x))^2$ is a constant function.

6. (a) Prove that if $f : D \to \mathbb{R}$ is uniformly continuous and D is bounded, then $f(D)$ is bounded.

   (b) Give an example of a function $g : (0, 1) \to \mathbb{R}$ which is continuous but unbounded (i.e., $g((0, 1))$ is unbounded).

   (c) Give an example of a function $h : (0, 1) \to \mathbb{R}$ which is continuous, not uniformly continuous, and bounded. You need not prove your answer.

7. Let h be **Thomae's function**:

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ and } x = m/n \text{ in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Prove that h is continuous at every $x \notin \mathbb{Q}$ and discontinuous at every $x \in \mathbb{Q}$.