

Homework 14: Connectedness

1. §6.1, #6.5, 6.8.
2. (a) Prove that a topological space Z is connected if and only if there does not exist a continuous, surjective map $f : Z \rightarrow \{0, 1\}$, where $\{0, 1\}$ is given the discrete topology.
(b) Use (a) to give a new proof of Theorem 6.6 in the textbook.
3. Let A be a subset of a topological space X . Suppose that $D \subseteq X$ is connected and intersects both A and A^c . Prove that D intersects ∂A .
Hint: Recall that $X = A^\circ \dot{\cup} \partial A \dot{\cup} (A^c)^\circ$ (the disjoint union of the three sets).
4. (Real Analysis) Let X denote the subset of \mathbb{R}^∞ consisting of sequences $\mathbf{x} = (x_1, x_2, x_3, \dots)$ such that $\sum x_i^2$ converges. You may use without proof any standard facts about infinite series.
 - (a) Prove that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges.
Hint: Use Problem 4(b) from Homework 13 to show that the partial sums are bounded.
 - (b) Let $c \in \mathbb{R}$. Prove that if $\mathbf{x}, \mathbf{y} \in X$, then $\mathbf{x} + \mathbf{y}, c\mathbf{x} \in X$.
 - (c) Prove that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on X . It is called the ℓ^2 metric.