Homework 11: Continuity
(Due April 8, 2022)

Directions. Assignments should be ***stapled*** *and written clearly and legibly.*

1. Let $a > 0$. Use the definition of limit to prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

   Hint: Use the inequality $|\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \dfrac{|x-a|}{\sqrt{a}}$

2. Let $f : D \to \mathbb{R}$. Use the definition of continuity to prove that if c is an isolated point of D, then f is continuous at c.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at 0.

4. Suppose that f, g, h are three functions which are defined on (a, b) and continuous at $c \in (a, b)$.
   (a) Prove that if $f(c) \neq 0$, then there exists a neighborhood U of c such that $f(x) \neq 0$ for every $x \in U$.
   (b) Prove that if $g(c) \neq h(c)$, then there exists a neighborhood U of c such that $g(x) \neq h(x)$ for every $x \in U$. (Hint: consider the function $p(x) = g(x) - h(x)$, and apply part (a)).

5. Let D be a subset of $\mathbb{R}$ containing 0, and let $f : D \to \mathbb{R}$ be bounded on D (i.e., $f(D)$ is a bounded subset of $\mathbb{R}$). Define a new function $g : D \to \mathbb{R}$ by $g(x) = xf(x)$.
   (a) Use the definition of continuity to prove that g is continuous at $x = 0$.
   (b) Suppose $c \neq 0$. Prove that g is continuous at c if and only if f is continuous at c. (Hint: Use a theorem on continuity.)