Homework 8: Topology of $\mathbb{R}$
(Due March 14, 2022)

*Assignments should be **stapled** and written clearly and legibly. Problem 6 is optional.*

1. Define the following (as in class): open set, closed set, limit point, A', $\overline{A}$. In the problems below, you may use only these definitions. You may not use any theorems from the sections we've covered on topology.

2. §3.4, #14, 15.

3. Let $A \subseteq B$ be two sets of real numbers. We say that A is **dense** in B if $\overline{A} = B$. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. (In other words, prove $\overline{\mathbb{Q}} = \mathbb{R}$.)

4. Let A be a set of real numbers. Prove the following:
   (a) A' is closed.
   (b) $(\overline{A})' = A'$.
   (c) $\overline{A}$ is closed.

5. Let A be a set of real numbers. Prove that if B is a closed set and $B \supseteq A$, then $B \supseteq \overline{A}$.
   Note. Problems 4(c) and 5 tell us that $\overline{A}$ is the smallest closed set containing A.

6. (Challenge) For sets $A, B \subseteq \mathbb{R}$, the **Minkowski sum** of A and B is defined to be

$$A + B = \{a + b : a \in A, b \in B\}.$$

   (a) Prove that if A and B are open, then $A + B$ is open.
   (b) Disprove the following by giving a counterexample: If A and B are closed, then $A + B$ is closed. (Hint: for any such counterexample, neither A nor B can be bounded.)