Homework 12: Compositions, Inverse Transformations, Isomorphisms

*Assignments should be **stapled** and written clearly and legibly. Problem 6 is optional.*

1. §8.2, #20, 26.

2. §8.3, #2, 4, 10(b). (Please provide justifications.)

3. Consider $T : P_2 \to P_2$ defined by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative of $p(x)$. Prove that T is linear. Then determine $\ker(T)$ and $R(T)$. Verify that Theorem 2.5 holds for T.

4. Consider $T : P_2 \to P_2$ given by $T(p(x)) = p(x) + p'(x)$.
   (a) Prove that T is linear.
   (b) Prove that T is an isomorphism.
   (c) Give $\ker(T)$ and $R(T)$.

5. Give an example of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T) = R(T)$.

6. (Optional) Let $T : V \to W$ be a linear transformation, and let Y be a subspace of W. The **inverse image** of Y, denoted $T^{-1}(Y)$, is defined to be

$$T^{-1}(Y) = \{\mathbf{v} \in V : T(\mathbf{v}) \in Y\}$$

   Prove that $T^{-1}(Y)$ is a subspace of V.

   Note: The symbol T^{-1} is used to represent both the inverse image, as defined above, and the inverse transformation, as defined in class on Monday. Although the same symbol is used for both, they are different concepts.