Homework 3: Proofs with Quantifiers (Due 2/17/2023)

*Assignments should be **stapled** and written clearly and legibly. Problems 7, 8, 9, and 10 are optional.*

1. §1.2, #8, 9(d), 10(c), 11(f).

2. §1.4, #11.

3. Prove that for every integer b, there exists a positive integer a such that $|a - |b|| \leq 1$.

4. Prove that for every positive real number e, there exists a positive real number d such that if x is a real number with $|x| < d$, then $2|x| < e$.

5. Prove that for every positive real number ϵ, there exists a natural number N such that if $n > N$, then $\dfrac{1}{n^2+1} < \epsilon$.

6. §3.3, #8.

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions whose ranges are bounded. Justify each of the following inequalities:

$$\begin{aligned}
\inf\{f(x) : x \in \mathbb{R}\} + \inf\{g(x) : x \in \mathbb{R}\} &\leq \inf\{f(x) + g(x) : x \in \mathbb{R}\} \\
&\leq \sup\{f(x) + g(x) : x \in \mathbb{R}\} \\
&\leq \sup\{f(x) : x \in \mathbb{R}\} + \sup\{g(x) : x \in \mathbb{R}\}
\end{aligned}$$

8. Let S be a nonempty set. Prove that the following three assertions are equivalent:
   (a) S is countable.
   (b) There exists an injection $f : S \to \mathbb{N}$.
   (c) There exists a surjection $g : \mathbb{N} \to S$.

9. Give an explicit bijection $f : [0, 1) \to (0, 1)$.

10. Which is greater, π^3 or 3^π?