Homework 2: Functions, More Supremums (Due 2/11/2019)

Assignments should be ***stapled*** *and written clearly and legibly. Problem 5 is optional..*

1. §2.3, #5(a), (b), 7(b), (c), (e), 8.

2. Let a and b be elements of an ordered field $\mathbb{F}$. Prove that if $a < b + \epsilon$ for every $\epsilon \in \mathbb{F}$ such that $\epsilon > 0$, then $a \leq b$.

3. Suppose that A and B are bounded sets in $\mathbb{R}$ and that $\sup A < \sup B$. Prove that there exists $b \in B$ that is an upper bound for A. Then show by example that this is not always the case if we only assume $\sup A \leq \sup B$.

4. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Prove that $\sup T = b$. You may use any of the theorems we've proved in class.

5. A real number $x \in \mathbb{R}$ is said to be **algebraic** if there exist integers $a_0, a_1, a_2, \ldots, a_n$, not all 0, such that
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{1}$$
Real numbers that are not algebraic are said to be **transcendental**.
   (a) Show that every rational number is algebraic.
   (b) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2} + \sqrt{3}$ are algebraic.
   (c) For fixed $n \in \mathbb{N}$, let A_n denote the set of algebraic numbers which are roots of polynomials of degree n with integer coefficients. Prove that A_n is countable. (You may assume that every polynomial has a finite number of roots.)
   (d) Prove that the set of all algebraic numbers is countable.
   (e) What can we conclude about the set of transcendental numbers?