Homework 12: Compositions, Inverses

1. §1.5, #9, 11(a), 15, 20(a).

2. §4.10, # 12.

3. In $\mathbb{R}^3$, let T be counterclockwise rotation by θ about the z-axis, and let S be counterclockwise rotation by ψ about the y-axis. (Here counterclockwise means as viewed from the positive axis.) Let R be counterclockwise rotation by θ about the z-axis followed by counterclockwise rotation by ψ in $\mathbb{R}^3$ about the y-axis. Find standard matrices for T, S, and R.

   Hint: The standard matrix for R is obtained by multiplying the other two matrices (in the correct order).

4. In this problem you will prove the following trigonometric identities:

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \end{aligned} \tag{1}$$

   Use the following strategy. Let T be counterclockwise rotation by x and S counterclockwise rotation by y (both in $\mathbb{R}^2$). Find the standard matrix for $S \circ T$ in two ways: (i) by multiplying the standard matrices for S and T, and (ii) by observing that $S \circ T$ is rotation by $x + y$. The matrices obtained in (i) and (ii) must be equal, so their entries must be equal.

5. Use the trigonometric identities (1) to find formulas for $\cos(2x)$ and $\sin(2x)$. (Disclaimer: this is not really a linear algebra problem.)

6. Let $T : V \to W$ be linear. A **left inverse** of T is a linear transformation $L : W \to V$ such that $L \circ T = I_V$, and a **right inverse** of T is a linear transformation $R : W \to V$ such that $T \circ R = I_W$.

   (a) Prove that if T has a left inverse, then T is one-to-one.

   (b) Prove that if T has a right inverse, then T is onto.

   (c) Prove that if T has a left inverse L and a right inverse R, then T is an isomorphism and $L = R$.

   (d) Give an example of a linear transformation T that has a left inverse but not a right inverse.

   (e) Give an example of a linear transformation T that has a right inverse but not a left inverse.