Homework 6: Monotonic and Bounded Sequences (Due March 10, 2023)

Assignments should be ***stapled*** *and written clearly and legibly. Problems 4(c), 5, and 6 are optional.*

1. (a) Give an example of a sequence with subsequences converging to 1, 2, and 3.
   (b) Give an example of a sequence with subsequences converging to every integer.
   (c) Show that there exists a sequence with subsequences converging to every rational number.

2. Prove that the sequence (a_n) converges, where $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

3. In this problem, we give an algorithm for computing $\sqrt{2}$. Let $a_1 = 2$, and define
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \text{ for } n \geq 1. \tag{1}$$
   (a) Prove that $a_n^2 \geq 2$ for all n. (Use proof by induction.)
   (b) Use part (a) and equation (1) to prove that $a_n - a_{n+1} \geq 0$ for all n.
   (c) Conclude that the sequence (a_n) converges.
   (d) Prove that $\lim_{n\to\infty} a_n = \sqrt{2}$.
   (e) Modify the sequence (a_n) so that it converges to $\sqrt{c}$. No formal proof is required for this part, but you should give a brief justification.

4. (a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.
   (b) Use part (a) to prove that the sequence
$$\left(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots\right)$$
   converges.
   (c) (Challenge) Find the limit.

5. Consider the sequence
$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \ldots\right)$$
   For which numbers x does this sequence have a subsequence converging to x? Prove your answer.

6. (Challenge) Let $a_0 = 2\sqrt{3}$ and $b_0 = 3$. Define two sequences recursively by

$$a_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1} + b_{n-1}} \quad \text{and} \quad b_n = \sqrt{a_n b_{n-1}}$$

Prove that (a_n) is decreasing and convergent, and prove that (b_n) is increasing and convergent. Then prove that they both converge to π.