Homework 13: The Intermediate Value Theorem (Due April 18, 2022)

*Assignments should be **stapled** and written clearly and legibly. Problems 4, 5, and 6 are optional.*

1. §5.3, #5, 7, 10.

2. Let f be continuous on $[0, 1]$ with $f(0) = f(1)$. Prove that there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

3. Prove that there exists a real number x such that
$$x^{177} + \frac{165}{1 + x^8 + \sin^2 x} = 125.$$

4. The purpose of this exercise is to prove the **Banach Fixed-Point Theorem**. Let $f : \mathbb{R} \to \mathbb{R}$, and suppose there exists $C < 1$ such that
$$|f(x) - f(y)| \leq C|x - y|$$
for all $x, y \in \mathbb{R}$. (Such a function f is said to be **Lipschitz**.)
   (a) Prove that f is continuous.
   (b) For any $x_1 \in \mathbb{R}$, define a sequence $(x_1, x_2, x_3, \ldots)$ recursively by the formula $x_n = f(x_{n-1})$, $n \geq 2$. Prove that this sequence converges. (Hint: Show that it is Cauchy).
   (c) Let $x = \lim_{n\to\infty} x_n$. Prove that x is a fixed point of f, i.e., $f(x) = x$.
   (d) Prove that x is the only fixed point of f.

5. Prove that if $f : [a, b] \to \mathbb{R}$ is injective and continuous, then the inverse function f^{-1} is also continuous.

6. (Putnam Exam) Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1 - x} + \frac{a_1}{1 - x^2} + \cdots + \frac{a_n}{1 - x^{n+1}} = 0.$$
Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$