# Homework 3: Proofs with Quantifiers
# (Due 2/16/2022)

*Assignments should be* ***stapled*** *and written clearly and legibly. Problems 6, 7, 8, and 9 are optional.*

1. §1.2, #8, 9(c), (d), 10, 11.

2. §1.4, #11.

3. Prove that for every integer $b$, there exists a positive integer $a$ such that $|a - |b|| \leq 1$.

4. Prove that for every positive real number $e$, there exists a positive real number $d$ such that if $x$ is a real number with $|x| < d$, then $2|x| < e$.

5. Prove that for every positive real number $\epsilon$, there exists a natural number $N$ such that if $n > N$, then $\dfrac{1}{n^2+1} < \epsilon$.

6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions whose ranges are bounded. Justify each of the following inequalities:

$$\begin{aligned}\inf\{f(x) : x \in \mathbb{R}\} + \inf\{g(x) : x \in \mathbb{R}\} &\leq \inf\{f(x) + g(x) : x \in \mathbb{R}\} \\ &\leq \sup\{f(x) + g(x) : x \in \mathbb{R}\} \\ &\leq \sup\{f(x) : x \in \mathbb{R}\} + \sup\{g(x) : x \in \mathbb{R}\}\end{aligned}$$

7. Let $S$ be a nonempty set. Prove that the following three assertions are equivalent:
   (a) $S$ is countable.
   (b) There exists an injection $f : S \to \mathbb{N}$.
   (c) There exists a surjection $g : \mathbb{N} \to S$.

8. Give an explicit bijection $f : [0, 1) \to (0, 1)$.

9. Which is greater, $\pi^3$ or $3^\pi$?