# Homework 14: Connectedness

*Assignments should be **stapled**. Problems 4 is optional.*

1. §6.1, #6.5, 6.8.

2. (a) Prove that a topological space $Z$ is connected if and only if there does not a continuous, surjective map $f : Z \to \{0, 1\}$, where $\{0, 1\}$ is given the discrete topology.

   (b) Use (a) to give a new proof of Theorem 6.6 in the texbook.

3. Let $A$ be a subset of a topological space $X$. Suppose that $D \subseteq X$ is connected and intersects both $A$ and $A^c$. Prove that $D$ intersects $\partial A$.

   Hint: Recall that $X = A^\circ \cup \partial A \cup (A^c)^\circ$.

4. (Real Analysis) Let $X$ denote the subset of $\mathbb{R}^\infty$ consisting of sequences $(x_1, x_2, x_3, \ldots)$ such that $\sum x_i^2$ converges. You may use without proof any standard facts about infinite series.

   (a) Prove that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges.

   Hint: Use problem 5(b) from Homework 13 to show that the partial sums are bounded.

   (b) Let $c \in \mathbb{R}$. Prove that if $\mathbf{x}, \mathbf{y} \in X$, then $\mathbf{x} + \mathbf{y}, c\mathbf{x} \in X$.

   (c) Prove that

   $$d(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

   is a well-defined metric on $X$. It is called the $\ell^2$ metric.