

Homework 14: The Matrix of a Linear Transformation

Assignments should be **stapled** and written clearly and legibly.

1. §1.5, #9, 11(a), 16.
2. In this problem you will prove the following trigonometric identities:

$$\begin{aligned}\cos(x + y) &= \cos x \cos y - \sin x \sin y \\ \sin(x + y) &= \sin x \cos y + \cos x \sin y\end{aligned}\tag{1}$$

Use the following strategy. Let T be counterclockwise rotation by x and S counterclockwise rotation by y (both in \mathbb{R}^2). Find the standard matrix for $S \circ T$ in two ways: (i) by multiplying the standard matrices for S and T , and (ii) by observing that $S \circ T$ is rotation by $x + y$. The matrices obtained in (i) and (ii) must be equal, so their entries must be equal.

3. Use the trigonometric identities (1) to find formulas for $\cos(2x)$ and $\sin(2x)$.
4. In an earlier homework problem, you considered the linear transformation $T : P_2 \rightarrow \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(-1) \\ p(1) \end{bmatrix}$. Determine whether T is onto. **Prove** your answer.
5. Construct the following:
 - (a) A matrix A such that the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one but not onto.
 - (b) A matrix B such that the matrix transformation $T(\mathbf{x}) = B\mathbf{x}$ is onto but not one-to-one.
 - (c) A matrix C such that the matrix transformation $T(\mathbf{x}) = C\mathbf{x}$ is one-to-one and onto.Briefly explain why your given matrices satisfy the required properties.
6. §8.4, #8.
7. Let $T : V \rightarrow W$ be linear, and let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ a basis for W . Suppose that $T(\mathbf{u}_1) = 3\mathbf{w}_1 + 5\mathbf{w}_2 - 7\mathbf{w}_3$ and $T(\mathbf{u}_2) = 2\mathbf{w}_1 + 4\mathbf{w}_3$.
 - (a) Find $[T]_{\mathcal{B}', \mathcal{B}}$.
 - (b) If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then what is $[T(\mathbf{x})]_{\mathcal{B}'}$?
8. Let \mathcal{B} be the standard basis of \mathbb{R}^2 , let \mathcal{C} be the basis of \mathbb{R}^3 consisting of the three vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and let \mathcal{D} be the standard basis of \mathbb{R}^3 . Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\mathbf{u}_1 + b\mathbf{u}_2 + (a+b)\mathbf{u}_3$. Find $[T]_{\mathcal{C}, \mathcal{B}}$ and $[T]_{\mathcal{D}, \mathcal{B}}$.