

LINEAR ALGEBRA (MATH 301) DEFINITIONS AND THEOREMS

UNIVERSITY OF WISCONSIN - PARKSIDE, FALL 2020

These notes contain a list of the main definitions, lemmas, propositions, and theorems discussed in this course.

1. SUBSPACE, LINEAR COMBINATION, SPAN, LINEAR INDEPENDENCE, BASIS

Definition 1.1. A **vector space** is ...

Definition 1.2. A **subspace** of a vector space V is a subset W of V which is a vector space (under the addition and scalar multiplication of V).

Definition 1.3. A **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_r$ is a vector of the form $k_1\mathbf{v}_1 + \dots + k_r\mathbf{v}_r$, where k_1, \dots, k_r are scalars.

Definition 1.4. The **span** of $\mathbf{v}_1, \dots, \mathbf{v}_r$, denoted $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, is the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_r$.

Definition 1.5. Let V be a vector space and let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be vectors in V . Then $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is **linearly independent** if $k_1\mathbf{v}_1 + \dots + k_r\mathbf{v}_r = \mathbf{0}$ implies $k_1 = \dots = k_r = 0$.

Definition 1.6. A **basis** for a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that

- (i) $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = V$, and
- (ii) $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent.

Definition 1.7. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of a vector space V , and let \mathbf{u} be a vector in V . Then $\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ for unique scalars c_1, \dots, c_n , by Theorem 1.6. The scalars c_1, \dots, c_n are called the **coordinates** of \mathbf{u} relative to \mathcal{B} , and the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

denoted by $[\mathbf{u}]_{\mathcal{B}}$, is called the **coordinate vector** of \mathbf{u} relative to \mathcal{B} .

Definition 1.8. The **dimension** of a nonzero vector space V is the number of vectors in any basis for V .

Definition 1.9. Let $A = [\mathbf{v}_1 \dots \mathbf{v}_n]$ be an $m \times n$ matrix with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and

let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in \mathbb{R}^n . Then $A\mathbf{x} = x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n$.

Definition 1.10. Let A be an $m \times n$ matrix. The **nullspace** of A , denoted by $\text{Nul } A$, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$.

Definition 1.11. Let A be an $m \times n$ matrix. The **column space** of A , denoted by $\text{Col } A$, is the span of the column vectors of A .

Theorem 1.1. Let V be a vector space, \mathbf{u} a vector in V , and k a scalar. Then

- (i) $0\mathbf{u} = \mathbf{0}$
- (ii) $k\mathbf{0} = \mathbf{0}$
- (iii) $(-1)\mathbf{u} = -\mathbf{u}$

Theorem 1.2. Let V be a vector space and let W be a subset of V . Then W is a subspace of V if:

- (i) for every \mathbf{u} and \mathbf{v} in W , $\mathbf{u} + \mathbf{v}$ is in W (i.e., W is closed under vector addition); and
- (ii) for every \mathbf{u} in W and scalar k , $k\mathbf{u}$ is in W (i.e., W is closed under scalar multiplication); and
- (iii) the zero vector of V lies in W .

Theorem 1.3. Let V be a vector space and let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be vectors in V . Then $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a subspace of V .

Theorem 1.4. A set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

Theorem 1.5. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for a vector space V .

- (i) If any vector of V is added to \mathcal{B} , then \mathcal{B} is no longer linearly independent.
- (ii) If any vector is removed from \mathcal{B} , then \mathcal{B} no longer spans V .

Theorem 1.6. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of a vector space V . Then every \mathbf{u} in V can be written in exactly one way as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$, that is, can be expressed as

$$\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n,$$

for unique scalars c_1, \dots, c_n .

Theorem 1.7. All bases of a vector space V have the same number of elements.

Theorem 1.8. In \mathbb{R}^n , the following have the same solutions:

- (i) The vector equation $x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{u}$.
- (ii) The linear system of equations with augmented matrix $[\mathbf{v}_1 \dots \mathbf{v}_p \mid \mathbf{u}]$.
- (iii) The matrix equation $[\mathbf{v}_1 \dots \mathbf{v}_p] \mathbf{x} = \mathbf{u}$.

Lemma 1.1. Let A be an $m \times n$ matrix, let \mathbf{u}, \mathbf{v} be vectors in \mathbb{R}^n , and let c be a scalar. Then

- (i) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and
- (ii) $A(c\mathbf{u}) = c(A\mathbf{u})$.

Theorem 1.9. Let A be an $m \times n$ matrix. Then $\text{Nul } A$ is a subspace of \mathbb{R}^n .

2. INTRODUCTION TO LINEAR TRANSFORMATIONS

Definition 2.1. Let V and W be vector spaces. A transformation (or mapping) $T : V \rightarrow W$ is **linear** if it satisfies the following conditions:

- (i) For every \mathbf{u}, \mathbf{v} in V , $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
- (ii) For every \mathbf{u} in V and scalar c , $T(c\mathbf{u}) = cT(\mathbf{u})$.

Theorem 2.1. Let $T : V \rightarrow W$ be linear. Then

- (i) $T(\mathbf{0}) = \mathbf{0}$.
- (ii) $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$, for any scalars c_1, \dots, c_p and vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in V .

Definition 2.2. A **matrix transformation** is a mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, for some fixed $m \times n$ matrix A .

Theorem 2.2. A matrix transformation is linear.

Definition 2.3. Let $T : V \rightarrow W$ be linear. Then

- (i) The **kernel** of T , denoted $\ker(T)$, is the set of vectors in V which T maps to $\mathbf{0}$.
- (ii) The **range** of T , denoted $R(T)$, is the set of vectors in W which have at least one vector in V mapping to them.

Theorem 2.3. Let $T : V \rightarrow W$ be linear. Then $\ker(T)$ is a subspace of V and $R(T)$ is a subspace of W .

Theorem 2.4. Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T) = \text{Nul } A$ and $R(T) = \text{Col } A$.

Theorem 2.5. Let $T : V \rightarrow W$ be linear. Then $\dim(\ker T) + \dim(R(T)) = \dim V$.

Theorem 2.6. Let $T : V \rightarrow W$ be linear. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Theorem 2.7. Let W be a subspace of V . If $\dim W = \dim V$, then $W = V$.

Theorem 2.8. Let $T : V \rightarrow W$ be linear, and suppose that $\dim V = \dim W$. Then T is one-to-one if and only if T is onto.

Definition 2.4. Let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear transformations. Then the **composition** of S with T , denoted $S \circ T$, is the map from U to W defined by $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$ for $\mathbf{u} \in U$.

Theorem 2.9. Let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear transformations. Then the composition $S \circ T : U \rightarrow W$ is a linear transformation.

Definition 2.5. For any vector space V , the **identity transformation** $I : V \rightarrow V$ is defined by $I(\mathbf{v}) = \mathbf{v}$ for all \mathbf{v} in V .

Theorem 2.10. Let $T : V \rightarrow W$ be a linear transformation. Then $T \circ I = I \circ T = T$.

Definition 2.6. Let $T : V \rightarrow W$ be one-to-one. Then there exists an **inverse transformation** $T^{-1} : R(T) \rightarrow V$ such that $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all \mathbf{v} in V .

Theorem 2.11. Let $T : V \rightarrow W$ be one-to-one. Then $T^{-1} \circ T = I$.

Definition 2.7. An **isomorphism** is a bijective linear transformation.

Definition 2.8. If $T : V \rightarrow W$ is an isomorphism, then V and W are said to be **isomorphic**.

Theorem 2.12. If $T : V \rightarrow W$ is an isomorphism, then $\dim V = \dim W$.

Theorem 2.13. Suppose that V is a vector space and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V . Then the mapping $T : V \rightarrow \mathbb{R}^n$ given by $T(\mathbf{u}) = [\mathbf{u}]_B$ is an isomorphism.

3. THE MATRIX OF A LINEAR TRANSFORMATION

Theorem 3.1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then for \mathbf{x} in \mathbb{R}^n , $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$. The matrix $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ is called the **standard matrix** for T .

Theorem 3.2. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a mapping. Then T is a linear transformation if and only if T is a matrix transformation.

Theorem 3.3. Suppose that the standard matrix for S is A and the standard matrix for T is B . Then the standard matrix for $S \circ T$ is AB .

Definition 3.1. Let A be an $n \times n$ matrix. Then A is said to be **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called the **inverse** of A , and we write $B = A^{-1}$.

Theorem 3.4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation, and let A be the standard matrix for T . Then T is an isomorphism if and only if A is invertible. In this case, the standard matrix for T^{-1} is A^{-1} .

Theorem 3.5. Let A be an $n \times n$ matrix. Then A is invertible if and only if A can be row reduced to I_n .

Theorem 3.6. Let A be an $n \times n$ matrix, and let \mathbf{b} be a vector in \mathbb{R}^n . If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 3.7. Let $T : V \rightarrow W$ be linear. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W . Then there exists a matrix $[T]_{\mathcal{B}', \mathcal{B}}$ such that for every \mathbf{v} in V , $[T(\mathbf{v})]_{\mathcal{B}'} = [T]_{\mathcal{B}', \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$.

Theorem 3.8. Let $T : V \rightarrow W$ be linear. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W . Then

$$[T]_{\mathcal{B}', \mathcal{B}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}'} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}'} \end{bmatrix}$$

Theorem 3.9. Let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear. Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be bases for vector spaces U, V, W respectively. Then $[S \circ T]_{\mathcal{B}'', \mathcal{B}} = [S]_{\mathcal{B}'', \mathcal{B}'} \cdot [T]_{\mathcal{B}', \mathcal{B}}$.

Definition 3.2. Let $\mathcal{B}, \mathcal{B}'$ be bases for a vector space V . Then $[I]_{\mathcal{B}', \mathcal{B}}$ is called the **change of coordinates matrix** from \mathcal{B} to \mathcal{B}' coordinates.

Theorem 3.10. Let $\mathcal{B}, \mathcal{B}'$ be bases for a vector space V . Then

- (i) For any \mathbf{v} in V , $[\mathbf{v}]_{\mathcal{B}'} = [I]_{\mathcal{B}', \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$.
- (ii) $[I]_{\mathcal{B}, \mathcal{B}} = I_n$, where $n = \dim V$.
- (iii) $[I]_{\mathcal{B}', \mathcal{B}}$ is invertible.
- (iv) $([I]_{\mathcal{B}', \mathcal{B}})^{-1} = [I]_{\mathcal{B}, \mathcal{B}'}$.

Notation 3.1. $[T]_{\mathcal{B},\mathcal{B}}$ is often denoted by just $[T]_{\mathcal{B}}$.

Theorem 3.11 (Change of Basis Formula). *Let $T : V \rightarrow V$ be a linear operator. Let $\mathcal{B}, \mathcal{B}'$ be bases for V . Then*

$$[T]_{\mathcal{B}'} = [I]_{\mathcal{B}',\mathcal{B}} \cdot [T]_{\mathcal{B}} \cdot [I]_{\mathcal{B},\mathcal{B}'}$$

4. INNER PRODUCT SPACES

Definition 4.1. *Let V be a vector space. An **inner product** on V is a rule which assigns to each pair of vectors \mathbf{u}, \mathbf{v} in V a scalar, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c ,*

- (i) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
- (ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
- (iii) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.
- (iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$.

*A vector space with an inner product is called an **inner product space**.*

Definition 4.2. *Let V be an inner product space.*

- (i) *For \mathbf{v} in V , the **norm** (or **length**) of \mathbf{v} is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.*
- (ii) *For \mathbf{u}, \mathbf{v} in V , the **distance** between \mathbf{u} and \mathbf{v} is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.*
- (iii) *A **unit vector** is a vector of norm 1.*
- (iv) *The set of all unit vectors in V is called the **unit circle** of V .*

Definition 4.3. *Let V be an inner product space. Vectors \mathbf{u} and \mathbf{v} in V are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.*

Definition 4.4. *A set S of two or more vectors in an inner product space is said to be **orthogonal** if every two distinct vectors in S are orthogonal. The set S is **orthonormal** if S is orthogonal and consists entirely of unit vectors.*

Definition 4.5. *Let V be an inner product space and let W be a subspace of V . The **orthogonal complement** of W , denoted W^\perp , is the set of vectors of V which are orthogonal to all vectors in W .*

Theorem 4.1. *Let V be an inner product space. Then*

- (i) $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ and $\langle \mathbf{0}, \mathbf{v} \rangle = 0$, for every \mathbf{v} in V .
- (ii) $\langle c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \mathbf{w} \rangle = c_1\langle \mathbf{v}_1, \mathbf{w} \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{w} \rangle$, for all scalars c_1, \dots, c_n and vectors $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}$.

Theorem 4.2. *Let V be an inner product space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in V and let c be a scalar. Then*

- (i) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
- (ii) $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

Theorem 4.3. *Let V be an inner product space. Let \mathbf{v} be in V and let c be a scalar. Then*

- (i) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.
- (ii) $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector, if $\mathbf{v} \neq \mathbf{0}$.

Theorem 4.4. If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Theorem 4.5. Let V be an inner product space, and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal basis for V . Then for \mathbf{u} in V , $\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$, where

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \quad \text{for } i = 1, \dots, n.$$

Theorem 4.6. Let W be a subspace of an inner product space V . Then

- (i) W^\perp is a subspace of V ; and
- (ii) $W \cap W^\perp = \{\mathbf{0}\}$.

Theorem 4.7. Let \mathbf{u}, \mathbf{v} be vectors in an inner product space V with $\mathbf{v} \neq \mathbf{0}$. Let $L = \text{span}\{\mathbf{v}\}$, a one-dimensional subspace of V . Then we can uniquely write $\mathbf{u} = \mathbf{y} + \mathbf{z}$, with \mathbf{y} in L and \mathbf{z} in L^\perp . Explicitly,

$$\mathbf{y} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}, \quad \text{and} \quad \mathbf{z} = \mathbf{u} - \mathbf{y}.$$

The vector \mathbf{y} is called the **orthogonal projection of \mathbf{u} onto L** and denoted by $\text{proj}_L \mathbf{u}$ or $\text{proj}_{\mathbf{v}} \mathbf{u}$.