Homework 8: Null Space and Column Space of a Matrix

1. Let H be the set of all vectors of the form $\begin{bmatrix} a \\ a \\ a \\ b \end{bmatrix}$. Show that H is the span of vectors, and use this to prove that H is a subspace of $\mathbb{R}^4$. Find a basis for H. Prove that your basis is indeed a basis.

2. In Homework 6, you found that $\{1, \ln(2x), \ln(x^2)\}$ is linearly dependent in $F(0, \infty)$. Let $W = \text{Span}\{1, \ln(2x), \ln(x^2)\}$.
   (a) How do we know that W is a subspace of $F(0, \infty)$?
   (b) Find a basis $\mathcal{B}$ for W, and find $\dim W$.
   (c) Determine whether $\ln(5x^3)$ is in W. If so, find $[\ln(5x^3)]_{\mathcal{B}}$.

   You should justify your answers, but proofs are not required.

3. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a vector space V. Using the definition of linear independence (and no theorems on linear dependence/independence), prove that if $\mathbf{v}$ is a vector in V which is not in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is still linearly independent.

4. Without using a calculator or computer, find a nonzero vector in $\text{Nul}\, A$, where

$$A = \begin{bmatrix} 51 & 51 & 58 & 2 & 7 \\ 7 & 10 & 9 & 1 & 2 \\ 3 & 17 & 5 & 0 & 2 \\ 9 & 2022 & 15 & 3 & 6 \\ 3 & \sqrt{2} & 8 & 37 & 5 \\ 7 & \pi & 23 & 19 & 16 \\ 11 & 3.14 & 14 & 0 & 3 \end{bmatrix}$$

5. A matrix A and an echelon form of A are given:

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for $\operatorname{Nul} A$. What is $\dim(\operatorname{Nul} A)$?

   (b) Find a basis for $\operatorname{Col} A$. What is $\dim(\operatorname{Col} A)$?

6. For each of the following vector spaces, find a matrix A such that the vector space is equal to $\operatorname{Nul} A$. Then find a basis for the vector space.

   (a) The line $y = 5x$ in $\mathbb{R}^2$.

   (b) The plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

7. Find a basis for $\operatorname{Col} \left[\begin{smallmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 4 \end{smallmatrix}\right]$ without doing **any** calculations.