Homework 4: Subspace, Linear Combination, Span

*Assignments should be **stapled** and written clearly and legibly. Problem 5 is optional.*

1. §4.2, #3(b), #5(a)(b), 10(a)(c), 12(a)(b), 14.

   Note: All answers to textbook problems should be justified.

2. Give an example of two nonzero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ such that $\mathbf{u} \neq \mathbf{v}$ and $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is not equal to $\mathbb{R}^2$.

3. Let $\mathbf{u}, \mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ be vectors in a vector space V. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and c is a scalar. Prove that:

   (a) $\mathbf{x} + \mathbf{y}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   (b) $c\mathbf{x}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   Note: For this problem, you may not use any theorems. You should use only definitions.

4. Let $\mathbf{u}, \mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in a vector space V. Prove that if $\mathbf{z}$ is in $\text{span}\{\mathbf{x}, \mathbf{y}\}$, and $\mathbf{x}$ and $\mathbf{y}$ are in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{z}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

5. (Putnam Competition) Let S be a set and let $\circ$ be a binary operation on S satisfying the two laws

$$x \circ x = x \text{ for all } x \text{ in } S, \text{ and}$$
$$(x \circ y) \circ z = (y \circ z) \circ x \text{ for all } x, y, z \text{ in } S.$$

   Show that $\circ$ is associative and commutative.