Homework 14: The Matrix of a Linear Transformation

1. §8.4, #8.

2. Let $T : V \to W$ be linear, and let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ a basis for W. Suppose that $T(\mathbf{u}_1) = 3\mathbf{w}_1 + 5\mathbf{w}_2 - 7\mathbf{w}_3$ and $T(\mathbf{u}_2) = 2\mathbf{w}_1 + 4\mathbf{w}_3$.
   (a) Find $[T]_{\mathcal{B}',\mathcal{B}}$.
   (b) If $[\mathbf{x}]_{\mathcal{B}} = \left[\begin{smallmatrix} 7 \\ -2 \end{smallmatrix}\right]$, then what is $[T(\mathbf{x})]_{\mathcal{B}'}$?

3. Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$, let $\mathcal{C}$ be the basis of $\mathbb{R}^3$ consisting of the three vectors $\mathbf{u}_1 = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right], \mathbf{u}_2 = \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right], \mathbf{u}_3 = \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right]$, and let $\mathcal{D}$ be the standard basis of $\mathbb{R}^3$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\right) = a\mathbf{u}_1 + b\mathbf{u}_2 + (a+b)\mathbf{u}_3$. Find $[T]_{\mathcal{C},\mathcal{B}}$ and $[T]_{\mathcal{D},\mathcal{B}}$.

4. §8.5, #10, 14.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \frac{1}{3}x$. Let $\mathcal{B} = \left\{\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right]\right\}$ and $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2\}$, two bases of $\mathbb{R}^2$.
   (a) Find $[T]_{\mathcal{B}}$.
   (b) Use (a) and the change of basis formula to find $[T]_{\mathcal{C}}$, the standard matrix for T.
   (c) Find the reflection of $\left[\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right]$ across the line $y = \frac{1}{3}x$, i.e., find $T\left(\left[\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right]\right)$.