

### Homework 3: Vector Spaces

Instructions. Assignments should be **stapled** and written clearly and legibly. Problem 4 is optional.

1. §4.1, #2, 7, TF (True-False Exercises).
2. Let  $V$  be the set of all matrices of real numbers with one column and two rows, with addition and scalar multiplication defined as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1v_1 \\ u_2 + v_2 \end{bmatrix} \quad \text{and} \quad c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 5cu_2 \\ 5cu_1 \end{bmatrix}$$

With this addition and scalar multiplication,  $V$  is not a vector space. Identify all vector space axioms which fail, and briefly explain why each of these axioms fail (in most cases, a counterexample will be sufficient). If axiom 3 holds, then give an explicit additive identity.

3. Let  $V$  be a vector space. Let  $\mathbf{u}, \mathbf{v}$  and  $\mathbf{w}$  be vectors in  $V$ , and let  $b$  and  $c$  be scalars. Using only the definition of a vector space, prove
  - (a)  $c\mathbf{0} = \mathbf{0}$
  - (b)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$  (Hint: use only the commutative and associative axioms.)
  - (c)  $(b + c)(\mathbf{u} + \mathbf{v}) = (c\mathbf{u} + c\mathbf{v}) + (b\mathbf{u} + b\mathbf{v})$
4. (Challenge) Prove that there does not exist a vector space which contains exactly two elements.