EQUIVARIANT K-THEORY AND TANGENT SPACES TO SCHUBERT VARIETIES

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Abstract. Tangent spaces to Schubert varieties of type A were characterized by Lakshmibai and Seshadri [LS84]. This result was extended to the other classical types by Lakshmibai [Lak95], [Lak00b], and [Lak00a]. We give a uniform characterization of tangent spaces to Schubert varieties in cominuscule G/P. Our results extend beyond cominuscule G/P; they describe the tangent space to any Schubert variety in G/B at a point xB, where x is a cominuscule Weyl group element in the sense of Peterson. Our results also give partial information about the tangent space to any Schubert variety at any point. Our method is to describe the tangent spaces of Kazhdan-Lusztig varieties, and then recover results for Schubert varieties. Our proof uses a relationship between weights of the tangent space of a variety with torus action, and factors of the class of the variety in torus equivariant K-theory. The proof relies on a formula for Schubert classes in equivariant K-theory due to Graham [Gra02] and Willems [Wil06], as well as a theorem on subword complexes due to Knutson and Miller [KM04], [KM05].

Contents

1. Introduction

One goal in the study of Schubert varieties is to understand their singularities. A related goal is to understand their Zariski tangent spaces, or equivalently, the weights of their Zariski tangent spaces at fixed points of the action of a maximal torus. A

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description of these tangent spaces in type A was given by Lakshmibai and Seshadri [LS84]. In [Lak95], [Lak00b], and [Lak00a], Lakshmibai extended this result to all classical types (see also [BL00, Chapter 5]). We give a different description of tangent spaces to Schubert varieties, which is uniform across all types. Our description, however, recovers only part of the tangent space, except in certain cases, such as Schubert varieties in cominuscule G/P, in which it recovers the entire tangent space. Our results hold for an algebraically closed ground field k of characteristic 0. The results of [LS84], [Lak95], [Lak00b], and [Lak00a] hold in arbitrary characteristic.

Rather than studying Schubert varieties directly, we focus on the smaller Kazhdan-Lusztig varieties, which differ locally only by a well-prescribed affine space. We study general Kazhdan-Lusztig varieties, but we do not attempt to recover all weights of the tangent space. Rather, we restrict our attention to those weights of the tangent space which are integrally indecomposable in an ambient space V, which is to say that they cannot be written as the sum of other weights of V. We characterize such weights. When all weights of V are integrally indecomposable in V, our characterization captures all weights of the tangent space. This occurs, for example, for Kazhdan-Lusztig varieties in cominuscule G/P, or more generally, for any Kazhdan-Lusztig variety at a T-fixed point (i.e., point of tangency) which is a cominuscule Weyl group element.

1.1. **Statement of results.** Let G be a semisimple algebraic group defined over an algebraically cloesd field k of characteristic 0. Let $P \supseteq B \supseteq T$ be a parabolic subgroup, Borel subgroup, and maximal torus of G respectively. We denote the set of weights of a representation E of T by $\Phi(E)$. Let W be the Weyl group of (G, T), and S the set of simple reflections in W relative to B.

Fix $w \leq x \in W$. Let X^w be the Schubert variety $\overline{B^-wB}$, and Y^w_x the Kazhdan-Lusztig variety $BxB \cap \overline{B^-wB}$, in G/B. The Kazhdan-Lusztig variety Y^w_x (and thus its tangent space at x, $T_xY^w_x$) is an affine subvariety of an ambient space V in G/B with weights $\Phi(V) = I(x^{-1})$, the inversion set of x^{-1}. If $\mathbf{s} = (s_1, \ldots, s_l)$, $s_i \in S$, is a reduced expression for x, then the elements of $I(x^{-1})$ are given explicitly by the formula $\gamma_i = s_1 \cdots s_{i-1}(\alpha_i)$, $i = 1, \ldots, l$, where α_i is the simple root corresponding to s_i.

Our main result is the following theorem (see Theorem 5.8):

Theorem A. *Suppose γ_j is integrally indecomposable in $I(x^{-1})$. Then the following are equivalent:*

(i) $\gamma_j \in \Phi(T_xY^w_x)$.
(ii) *There exists a reduced subexpression of $(s_1, \ldots, \widehat{s}_j, \ldots, s_l)$ for w.*
(iii) *The Demazure product of $(s_1, \ldots, \widehat{s}_j, \ldots, s_l)$ is greater than or equal to w.*

This theorem, which applies to Kazhdan-Lusztig varieties in G/B, extends to Schubert varieties and to G/P. Moreover, when x is a cominuscule Weyl group element of

W, all γ_j are integrally indecomposable in $I(x^{-1})$, so Theorem A recovers all weights of the tangent space.

Remark 1.1. If γ_j is not integrally indecomposable in $I(x^{-1})$, then (ii) and (iii) of Theorem A are still equivalent, but (i) is no longer equivalent to (ii) and (iii) in general.

Remark 1.2. Let us denote by $TE_xY_x^w$ the span of the tangent lines to T-invariant curves through x in Y_x^w; then $TE_xY_x^w \subseteq T_xY_x^w$. It is known that condition (iii) of Theorem A, with the Demazure product of $(s_1, \dots, \widehat{s}_j, \dots, s_l)$ replaced by the ordinary product $s_1 \cdots \widehat{s}_j \cdots s_l$, gives a characterization of all weights of $TE_xY_x^w$ (and not just the integrally indecomposable weights) [Car95] [CK03]. Thus, Theorem A can be viewed as a characterization of the integrally indecomposable weights of $T_xY_x^w$ which is similar to this known characterization of all weights of the smaller space $TE_xY_x^w$.

Remark 1.3. The paper [GK20] proves that in simply-laced types, the Demazure product of $(s_1, \dots, \widehat{s}_j, \dots, s_l)$ of Theorem A (iii) is equal to the ordinary product $s_1 \cdots \widehat{s}_j \cdots s_l$, provided that γ_j is integrally indecomposable in $I(x^{-1})$. As a corollary, it is proved that in simply-laced types, when x is a cominuscule Weyl group element, $\Phi(T_xX^w) = \Phi(TE_xX^w)$.

1.2. **Outline of proof.** Our proof of Theorem A uses equivariant K-theory. Let us fix notation and give some basic definitions and properties. If T acts on a smooth scheme M, the Grothendieck group of T-equivariant coherent sheaves (or vector bundles) on M is denoted by $K_T(M)$. If N is a T-stable subscheme of M, then the class in $K_T(M)$ of the pushfoward of the structure sheaf $\mathcal{O}_N$ of N is denoted by $[\mathcal{O}_N]_M$, or sometimes just $[\mathcal{O}_N]$. A T-equivariant vector bundle on a point is a representation of T, so $K_T(\{\text{point}\})$ can be identified with $R(T)$, the representation ring of T. The inclusion $i_m : \{m\} \to M$ of a T-fixed point induces a pullback $i_m^* : K_T(M) \to K_T(\{m\}) = R(T)$.

Consider for the moment a more general situation than that of the previous subsection: V a representation of T such that all weights of V lie in an open half-space and have multiplicity one, $Y \subseteq V$ a T-stable subscheme, and $x \in Y$ a T-fixed point. The structure sheaf $\mathcal{O}_Y$ defines a class $[\mathcal{O}_Y] \in K_T(V)$. We show that the factors of $i_x^*[\mathcal{O}_Y] \in R(T)$ contain information about the tangent space T_xY. Let us say that $1 - e^{-\theta}$ is a *simple factor* of $i_x^*[\mathcal{O}_Y]$ if $i_x^*[\mathcal{O}_Y] = (1 - e^{-\theta})Q$ for some $Q \in R(T)$ which is a polynomial in $e^{-\lambda}$, $\lambda \in \Phi(V) \setminus \{\theta\}$. We prove (see Proposition 3.5)

Proposition B. *Suppose θ is integrally indecomposable in $\Phi(V)$. Then $\theta \in \Phi(T_xY)$ if and only if $1 - e^{-\theta}$ is not a simple factor of $i_x^*[\mathcal{O}_Y]$.*

Now set V and x as in Subsection 1.1 and set Y to be the Kazhdan-Lusztig variety Y_x^w. For $\theta \in \Phi(V) = I(x^{-1})$, we have $\theta = \gamma_j$ for some j. Proposition B then becomes

Proposition C. *Suppose γ_j is integrally indecomposable in $I(x^{-1})$. Then $\gamma_j \in \Phi(T_xY_x^w)$ if and only if $1 - e^{-\gamma_j}$ is not a simple factor of $i_x^*[\mathcal{O}_{Y_x^w}]$.*

This characterization of $\Phi(T_x Y_x^w)$ would appear to suffer from a computational difficulty: determining whether $1 - e^{-\gamma_j}$ is a factor of $i_x^*[\mathcal{O}_{Y_x^w}]$, let alone whether it is a simple factor, seems to be a nontrivial problem. It requires some sort of division algorithm in $R(T)$. One approach would be to search for an expression for $i_x^*[\mathcal{O}_{Y_x^w}]$ as a sum of terms in which $1 - e^{-\gamma_j}$ appears explicitly as a factor of each summand. To rule out the possibility that $1 - e^{-\gamma_j}$ is a factor of $i_x^*[\mathcal{O}_{Y_x^w}]$, then, one would need to show that no such expression exists. This would presumably require knowledge of all possible expressions for $i_x^*[\mathcal{O}_{Y_x^w}]$.

We show a way around this apparent computational difficulty. When γ_j is integrally indecomposable in $I(x^{-1})$, the question of whether or not $1 - e^{-\gamma_j}$ is a simple factor of $i_x^*[\mathcal{O}_{Y_x^w}]$ can be answered by using a single expression for $i_x^*[\mathcal{O}_{Y_x^w}]$ due to Graham [Gra02] and Willems [Wil06]:

$$i_x^*[\mathcal{O}_{Y_x^w}] = \sum_{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}} (-1)^{e(\mathbf{t})} \prod_{i \in \mathbf{t}} (1 - e^{-\gamma_i}) \in R(T), \tag{1.1}$$

where $\mathcal{T}_{w,\mathbf{s}}$ is the set of sequences $\mathbf{t} = (i_1, \ldots, i_m)$, $1 \le i_1 < \cdots < i_m \le l$, such that $H_{s_{i_1}} \cdots H_{s_{i_m}} = H_w$ in the 0-Hecke algebra, and $e(\mathbf{t}) = m - \ell(w)$. More precisely, we prove (see Theorem 5.6):

Theorem D. *Suppose γ_j is integrally indecomposable in $I(x^{-1})$. Then $1 - e^{-\gamma_j}$ is a simple factor of $i_x^*[\mathcal{O}_{Y_x^w}]$ if and only if $1 - e^{-\gamma_j}$ occurs explicitly as a factor of every summand of $\sum_{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}} (-1)^{e(\mathbf{t})} \prod_{i \in \mathbf{t}} (1 - e^{-\gamma_i})$, i.e., if and only if j belongs to every $\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}$.*

We note that one direction of this theorem follows immediately from (1.1). Combining Proposition C and Theorem D yields

Theorem E. *Suppose that γ_j is integrally indecomposable in $I(x^{-1})$. Then $\gamma_j \in \Phi(T_x Y_x^w)$ if and only if j does not belong to every $\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}$.*

Now the equivalence of (i) and (ii) of Theorem A is essentially a reformulation of Theorem E, using some properties of 0-Hecke algebras. The equivalence of (ii) and (iii) is due to Knutson-Miller [KM04, Lemma 3.4 (1)].

The paper is organized as follows. In Section 2, we recall definitions and properties of equivariant K-theory and weights of tangent spaces to schemes with T-actions. In Section 3 we prove Proposition B. In Section 4, we give a corollary to a theorem by Knutson-Miller on subword complexes [KM04], [KM05]. Our proof of Theorem D relies on this corollary. In Section 5, we apply the material of the previous sections to Kazhdan-Lusztig varieties in order to prove Proposition C and Theorems D and E. In Section 6, we show how to extend these results to G/P and discuss the case of cominuscule Weyl group elements and cominuscule G/P.

The related paper [GK20] examines *rationally indecomposable* weights of the ambient space V. Rational indecomposability is a stricter condition than integral indecomposability, so the set of rationally indecomposable weights is contained in the set of integrally indecomposable weights. For this smaller set of weights, [GK20] obtains stronger results. For example, it is shown that the elements of $\Phi(T_xY_x^w)$ which are rationally indecomposable in $I(x^{-1})$ lie in $\Phi(TE_xY_x^w)$. Some of the results of [GK20] rely on those of this paper.

2. Preliminaries

In this section, we collect information concerning equivariant K-theory and tangent spaces and tangent cones. We include proofs for the convenience of the reader.

2.1. **The pullback to a fixed point in T-equivariant K-theory.** Let $T = (k^*)^n$ be a torus, and let $\widehat{T} = \mathrm{Hom}(T, k^*)$ be the character group of T. The mapping $\lambda \mapsto d\lambda$ from a character to its differential at $1 \in T$ embeds $\widehat{T}$ in the dual $\mathfrak{t}^*$ of the Lie algebra of T. We will usually view $\widehat{T}$ as a subset of $\mathfrak{t}^*$ under this embedding and express the group operation additively. If λ denotes an element of $\widehat{T}$ viewed as an element of $\mathfrak{t}^*$, then the corresponding homomorphism $T \to k^*$ is written as e^λ. The representation ring $R(T)$ is defined to be the free $\mathbb{Z}$-module with basis e^λ, $\lambda \in \widehat{T}$, with multiplication given by $e^\lambda e^\mu = e^{\lambda+\mu}$.

Let V be a representation of T such that all weights of V have multiplicity one and lie in an open half-space in the real span of the characters of T. Denote the set of weights of T on V by $\Phi(V)$, the set of nonnegative integer linear combinations of elements of $\Phi(V)$ in $\mathfrak{t}^*$ by $\mathrm{Cone}_{\mathbb{Z}}\,\Phi(V)$, and $\prod_{\lambda\in\Phi(V)}(1 - e^{-\lambda})$ by $\lambda_{-1}(V^*)$. For $\Phi_A \subseteq \Phi(V)$, let $\mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_A]$ denote the subring of $R(T)$ generated over $\mathbb{Z}$ by $e^{-\lambda}$, $\lambda \in \Phi_A$. The dual representation V^* has weights $-\Phi(V)$. The coordinate ring $k[V]$ of V is the symmetric algebra $\mathrm{Sym}(V^*)$ and has weights $-\mathrm{Cone}_{\mathbb{Z}}\,\Phi(V)$.

Lemma 2.1. *If $\nu \in \mathrm{Cone}_{\mathbb{Z}}\,\Phi(V)$, then $e^{-\nu}$ can be expressed as a monomial in $e^{-\lambda}$, $\lambda \in \Phi(V)$.*

Proof. Write $\nu = c_1\lambda_1 + \cdots + c_t\lambda_t$, where $\lambda_i \in \Phi(V)$ and c_i are nonnegative integers. Then $e^{-\nu} = e^{\sum -c_i\lambda_i} = \prod(e^{-\lambda_i})^{c_i}$. □

The map $i_0^* : K_T(V) \to R(T)$ is an isomorphism, which we denote by i_V^*.

Lemma 2.2. *Let Y be a T-stable closed subvariety of V. Then $i_V^*[\mathcal{O}_Y]_V \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi(V)]$.*

Proof. We adapt a method appearing in [Ros89]. Denote $k[V]$ by R. All modules in this proof will be T-stable R-modules, and all maps T-equivariant R-homomorphisms.

Consider the projection

$$R \to k[Y] \to 0 \tag{2.1}$$

The kernel is a T-stable ideal I of R which is generated by a finite number of weight vectors $r_{1,1}, \dots, r_{1,n_1}$. Let $\nu_{1,j}$ be the weight of $r_{1,j}$; $k_{\nu_{1,j}}$ the T-representation of weight $\nu_{1,j}$; and $F_1 = \oplus_{j=0}^{n_1} R \otimes k_{\nu_{1,j}}$. Note that R acts on the first factor of F_1 and T on both, and that $\Phi(F_1) \subseteq \Phi(R)$. There exists a map $f_1 : F_1 \to R$ such that

$$F_1 \to R \to k[Y] \to 0 \tag{2.2}$$

is exact (f_1 maps $1 \otimes 1$ from the j-th summand of F_1 to $r_{1,j}$).

The kernel of f_1 is finitely generated over R (since F_1 is finitely generated and R is Noetherian), and thus is generated by a finite number of weight vectors. Thus the above procedure can be repeated to produce a module F_2 and map $F_2 \to F_1$, which, when appended to (2.2), yields an exact sequence. Moreover, $\Phi(F_2) \subseteq \Phi(F_1) \subseteq \Phi(R)$. When iterated, this procedure must terminate, by the Hilbert Syzygy Theorem. The resulting complex is a resolution of $k[Y]$:

$$0 \to F_d \to \cdots \to F_1 \to R \to k[Y] \to 0 \tag{2.3}$$

where $F_i = \oplus_j R \otimes k_{\nu_{i,j}}$ and $\Phi(F_i) \subseteq \Phi(R)$. Thus $\nu_{i,j} \in \Phi(F_i) \subseteq \Phi(R) = -\operatorname{Cone}_{\mathbb{Z}} \Phi(V)$.

The resolution (2.3) corresponds to a resolution of $\mathcal{O}_Y$ over $\mathcal{O}_V$:

$$0 \to \mathcal{F}_d \to \cdots \to \mathcal{F}_1 \to \mathcal{O}_V \to \mathcal{O}_Y \to 0$$

where $\mathcal{F}_i = \oplus_j \mathcal{O}_V \otimes k_{\nu_{i,j}}$. Since $[\mathcal{O}_V]_V = 1$, we have

$$i_V^*[\mathcal{O}_Y]_V = 1 + \sum_{i,j} (-1)^i e^{\nu_{i,j}} i_V^*[\mathcal{O}_V]_V = 1 + \sum_{i,j} (-1)^i e^{\nu_{i,j}}$$

By Lemma 2.1, this lies in $\mathbb{Z}[e^{-\lambda}, \lambda \in \Phi(V)]$. □

2.2. **Weights of tangent and normal spaces.** The coordinate ring of V is the polynomial ring $k[x_\lambda, \lambda \in \Phi(V)]$, where x_λ denotes a vector of V^* of weight $-\lambda$. Observe that this polynomial ring is graded by the coordinates x_λ and also by the weights of the T action. The weights of any T-stable subspace of V form a subset of $\Phi(V)$, whose corresponding weight vectors span the subspace. Thus there is a bijection between the T-stable subspaces of V and the subsets of $\Phi(V)$. The coordinate ring of a T-stable subspace Z is $\operatorname{Sym}(Z^*) = k[x_\lambda, \lambda \in \Phi(Z)]$.

Let $Y \to V$ be a T-equivariant closed immersion, with T-fixed point $x \in Y$ mapping to $0 \in V$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals in the local rings of Y at x and of V at 0 respectively. Let

$$A = \operatorname{Sym}(\mathfrak{n}/\mathfrak{n}^2), \qquad B = \operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2), \qquad C = \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

The tangent space to V at 0, which can be identified with V, is defined to be $\operatorname{Spec} A$. The tangent space and tangent cone to Y at x, which we denote by T_xY and TC_xY

respectively, are defined to be $\operatorname{Spec} B$ and $\operatorname{Spec} C$ respectively. The degree one components of B and C, which are denoted by B_1 and C_1 respectively, are both equal to $\mathfrak{m}/\mathfrak{m}^2$, and hence are canonically identified. The projections $A \twoheadrightarrow B \twoheadrightarrow C$ induce inclusions $TC_xY \hookrightarrow T_xY \hookrightarrow V$.

All spaces above are T-stable and all maps are T-equivariant. Since T_xY is a T-stable subspace of V, its coordinate ring B is equal to $k[x_\lambda, \lambda \in \Phi(T_xY)]$, with character

$$\operatorname{Char} B = \frac{1}{\prod_{\lambda \in \Phi(T_xY)}(1-e^{-\lambda})} = \frac{1}{\lambda_{-1}(T_xY^*)}.$$

This lives in $\widehat{R}(T)$, the set of expressions of the form $\sum_{\lambda \in \widehat{T}} c_\lambda e^\lambda$. Similarly, we have a formula for the character of C.

Proposition 2.3. $\operatorname{Char} C = \dfrac{i_V^*[\mathcal{O}_{TC_xY}]_V}{\lambda_{-1}(V^*)} = \dfrac{i_{T_xY}^*[\mathcal{O}_{TC_xY}]_{T_xY}}{\lambda_{-1}(T_xY^*)}$.

Proof. The first equality is proved in [GK15, Proposition 2.1] and the second in [GK17, (3.10)]. □

Proposition 2.4. *$[\mathcal{O}_{TC_xY}]_V = [\mathcal{O}_Y]_V$ in $K_T(V)$.*

Proof. See [GK17, Proposition 3.1(2)]. □

3. Factors in equivariant K-theory

We keep the notations, definitions, and assumptions of the previous section. Denote $\Phi(V), \Phi(T_xY)$, and $\Phi(V/T_xY) = \Phi(V) \setminus \Phi(T_xY)$ by $\Phi_{\mathrm{amb}}, \Phi_{\mathrm{tan}}$, and Φ_{nor} respectively. Then $\Phi_{\mathrm{amb}} = \Phi_{\mathrm{tan}} \sqcup \Phi_{\mathrm{nor}}$. If $P \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\mathrm{amb}}]$, then we will say that $1-e^{-\theta}$ is a **simple factor** of P if $P = (1-e^{-\theta})Q$, for $Q \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\mathrm{amb}} \setminus \{\theta\}]$.

Example 3.1. Suppose $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Phi_{\mathrm{amb}}$ are distinct, and $\lambda_3 = \lambda_1 + \lambda_2$. Consider

$$P = (1-e^{-\lambda_1})(1-e^{-\lambda_4}) + (1-e^{-\lambda_2})(1-e^{-\lambda_4}) - (1-e^{-\lambda_1})(1-e^{-\lambda_2})(1-e^{-\lambda_4}),$$

an element of $\mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\mathrm{amb}}]$. Then P can be expressed as $(1-e^{-(\lambda_1+\lambda_2)})(1-e^{-\lambda_4})$. Thus both $1-e^{-(\lambda_1+\lambda_2)}$ and $1-e^{-\lambda_4}$ are simple factors of P.

Remark 3.2. In Section 5.3, we will need to distinguish between the nature of the two factors $1-e^{-(\lambda_1+\lambda_2)}$ and $1-e^{-\lambda_4}$ of P in Example 3.1. While the latter factor appears explicitly as a factor of each summand, and thus is easily identifiable as a factor of P, the former does not. We will refer to $1-e^{-\lambda_4}$ as an *explicit factor* of the expression P.

There are usually many ways to express an element $P \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\mathrm{amb}}]$. Explicit factors depend on the particular expression of P, while (non-explicit) factors do not.

We wish to study whether it is possible to determine whether a weight θ lies in Φ_{nor} or Φ_{tan} based on whether or not $1-e^{-\theta}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$. We begin with the following observation:

Proposition 3.3. *If $\theta \in \Phi_{\mathrm{nor}}$, then $1 - e^{-\theta}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$.*

Proof. By Propositions 2.4 and 2.3,

$$i_V^*[\mathcal{O}_Y]_V = i_V^*[\mathcal{O}_{TC_xY}]_V = \frac{\lambda_{-1}(V^*)}{\lambda_{-1}(T_xY^*)} i_{T_xY}^*[\mathcal{O}_{TC_xY}]_{T_xY}.$$

Now, $\frac{\lambda_{-1}(V^*)}{\lambda_{-1}(T_xY^*)} = \lambda_{-1}((V/T_xY)^*) = \prod_{\lambda \in \Phi_{\mathrm{nor}}}(1 - e^{-\lambda})$, and $1 - e^{-\theta}$ occurs among the terms of this product exactly once. Moreover, since TC_xY is a closed subvariety of T_xY, Lemma 2.2 implies $i_{T_xY}^*[\mathcal{O}_{TC_xY}]_{T_xY} \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi(T_xY)] \subseteq \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\mathrm{amb}} \setminus \{\theta\}]$. □

The converse of this proposition is false, as illustrated by the following example.

Example 3.4. Suppose that T acts on $V = k^3$, and that the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are weight vectors with corresponding weights $\lambda_1, \lambda_2, \lambda_3 = \lambda_1 + \lambda_2$. Letting $x_1, x_2, x_3 \in V^*$ denote the dual of the standard basis, we have, for $t \in T$, $tx_i = e^{-\lambda_i}(t)x_i$, $i = 1, 2, 3$. Let Y be the affine variety in V defined by the ideal $I = (x_1x_2)$, and let x be the origin. Then the ideal of the tangent space and tangent cone of Y at x are $\{0\}$ and (x_1x_2) respectively. The tangent space of Y at x is all of V, so $\Phi_{\mathrm{tan}} = \Phi_{\mathrm{amb}}$ and Φ_{nor} is empty. The tangent cone of Y at x is the union of the x_2x_3-plane and the x_1x_3 plane, so its coordinate ring C has character

$$\operatorname{Char} C = \left(\frac{1}{1 - e^{-\lambda_1}} + \frac{1}{1 - e^{-\lambda_2}} - 1\right) \frac{1}{1 - e^{-(\lambda_1+\lambda_2)}}$$

Additionally, $\lambda_{-1}(V^*) = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2})(1 - e^{-(\lambda_1+\lambda_2)})$, and thus by Proposition 2.3,

$$\begin{aligned} i_V^*[\mathcal{O}_Y]_V = i_V^*[\mathcal{O}_C]_V &= (1 - e^{-\lambda_1}) + (1 - e^{-\lambda_2}) - (1 - e^{-\lambda_1})(1 - e^{-\lambda_2}) \\ &= 1 - e^{-(\lambda_1+\lambda_2)} \end{aligned}$$

Hence $1 - e^{-\lambda_3} = 1 - e^{-(\lambda_1+\lambda_2)}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$, but λ_3 lies in Φ_{tan}.

In this example, the fact that there exists λ_3 in Φ_{tan} such that $1 - e^{-\lambda_3}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$, thus violating the converse of Proposition 3.3, appears to be related to the fact that λ_3 can be expressed as the sum of other weights of Φ_{amb}. This suggests that the converse of Proposition 3.3 may hold if we restrict to weights θ which cannot be expressed as such a sum. This assertion is true, and is proved in the following proposition. Let us say that a weight of Φ_{amb} is **integrally decomposable** if it can be expressed as a positive integer linear combination of other elements of Φ_{amb}, or **integrally indecomposable** otherwise.

Proposition 3.5. *Let θ be an integrally indecomposable element of Φ_{amb}. Then the following are equivalent:*

(i) $\theta \in \Phi_{\mathrm{tan}}$.

(ii) $x_\theta \in B_1$.
(iii) $x_\theta \in C_1$.
(iv) $-\theta$ *is a weight of* C.
(v) $1-e^{-\theta}$ *is not a simple factor of* $i_V^*[\mathcal{O}_Y]_V$.

Proof. (v) $\Rightarrow$ (i) by Proposition 3.3; (i) $\Rightarrow$ (ii) since $B = k[x_\lambda, \lambda \in \Phi_{\text{tan}}]$; (ii) $\Rightarrow$ (iii) since $C_1 = B_1$; and (iii) $\Rightarrow$ (iv) because x_θ has weight $-\theta$.

It remains to prove (iv) $\Rightarrow$ (v). We prove the contrapositive. Thus assume that $1-e^{-\theta}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$. Then $i_V^*[\mathcal{O}_Y]_V = (1-e^{-\theta})Q$, where $Q \in \mathbb{Z}[e^{-\lambda}, \lambda \in \Phi_{\text{amb}} \setminus \{\theta\}]$. By Propositions 2.3 and 2.4,

$$\operatorname{Char} C = \frac{Q}{\prod_{\lambda \in \Phi_{\text{amb}} \setminus \{\theta\}}(1-e^{-\lambda})} = Q \prod_{\lambda \in \Phi_{\text{amb}} \setminus \{\theta\}} (1 + e^{-\lambda} + e^{-2\lambda} + \cdots).$$

Expanding, one obtains an infinite sum of terms $e^{-\nu}$, $\nu \in \operatorname{Cone}_{\mathbb{Z}}(\Phi_{\text{amb}})$. None of these terms is equal to $e^{-\theta}$. (This is because none of the factors in the above product for $\operatorname{Char} C$ contain a term $e^{-\theta}$; since θ is integrally indecomposable in Φ_{amb}, the term $e^{-\theta}$ cannot be obtained by expanding the product.) Thus $-\theta$ is not a weight of C, as required. $\square$

4. Euler characteristics of subword complexes

In Section 5, we will apply the results of the previous section to Schubert varieties and Kazhdan-Lusztig varieties. One main tool for this purpose is Corollary 4.6, whose proof relies on a theorem by Knutson-Miller on subword complexes [KM05], [KM04].

4.1. **The reduced Euler characteristic.** In this subsection we give a brief review of simplicial complexes and their Euler characteristics.

Recall that an (abstract) simplicial complex on a finite set A is a set Δ of subsets of A, called faces, with the property that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. The dimension of a face F is $\#F - 1$, and the dimension of Δ is the maximum dimension of a face. A maximal face of Δ is called a facet. Note that $\Delta = \emptyset$ and $\Delta = \{\emptyset\}$ are distinct simplicial complexes, called the void complex and irrelevant complex respectively. If $\Delta \neq \emptyset$, then $\emptyset$ must be a face of Δ.

The reduced Euler characteristic of Δ is defined to be $\widetilde{\chi}(\Delta) = \sum_{F \in \Delta} (-1)^{\dim F}$. If $\Delta \neq \emptyset$, so that $\emptyset \in \Delta$, then $\emptyset$ contributes a summand of -1 to $\widetilde{\chi}(\Delta)$. From this we see, for example, that $\widetilde{\chi}(\{\emptyset\}) = -1$, but $\widetilde{\chi}(\emptyset) = 0$.

Suppose that $\Delta \neq \emptyset$ or $\{\emptyset\}$. Denoting the elements of A by $x_1, \ldots, x_m$, the set A can be embedded in $\mathbb{R}^m$ by mapping x_i to the ith standard basis vector of $\mathbb{R}^m$. For any face F of Δ, let $|F|$ be the convex hull of its vertices in $\mathbb{R}^m$. The geometric realization of Δ is then defined to be $|\Delta| = \bigcup_{F \in \Delta} |F|$, a topological subspace of $\mathbb{R}^m$. If $|\Delta|$ is homeomorphic to a topological space Y, then Δ is called a triangulation of Y.

In this case, the reduced Euler characteristic of Δ is equal to the topological reduced Euler characteristic of Y. If Y is a manifold with boundary and its boundary ∂Y is nonempty, then there exists a subcomplex of Δ which is a triangulation of ∂Y [Mau80, Proposition 5.4.4]. This subcomplex is called the boundary of Δ and denoted by $\partial\Delta$.

For $m \geq 0$, let $B^m = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$ and $S^m = \{x \in \mathbb{R}^{m+1} : \|x\| = 1\}$, the m-ball and m-sphere respectively. Both can be triangulated. In the sequel, when we refer to an m-ball or m-sphere or their notations, $m \geq 0$, we will mean a triangulation of the object. When we refer to the sphere S^{-1}, we will mean the irrelevant complex $\{\emptyset\}$. With these conventions, for $m \geq 0$, $\widetilde{\chi}(B^m) = 0$, $\widetilde{\chi}(S^{m-1}) = (-1)^{m-1}$, $\partial B^m = S^{m-1}$, and $\partial S^{m-1} = \emptyset$. (Observe that ∂B^0 is the irrelevant complex, but ∂S^{m-1} is the void complex.) For $\Delta = B^m$ or S^m, define $\widetilde{\chi}^\circ(\Delta) = \widetilde{\chi}(\Delta) - \widetilde{\chi}(\partial\Delta)$.

Lemma 4.1. *$\widetilde{\chi}^\circ(B^m) = \widetilde{\chi}^\circ(S^m) = (-1)^m$, for $m \geq 0$.*

Proof. The proof is a calculation: $\widetilde{\chi}^\circ(B^m) = 0 - (-1)^{m-1} = (-1)^m$, and $\widetilde{\chi}^\circ(S^m) = (-1)^m - 0 = (-1)^m$. □

4.2. **0-Hecke algebras.** Let G be a semisimple algebraic group, B a Borel subgroup, B^- the opposite Borel subgroup, and $T = B \cap B^-$ a maximal torus. Let $W = N_G(T)/T$, the Weyl group of G. Let S be the set of simple reflections of W relative to B. The 0-Hecke algebra $\mathcal{H}$ associated to (W, S) over a commutative ring R is the associative R-algebra generated by H_u, $u \in W$, and subject to the following relations: H_1 is the identity element, and if $u \in W$ and $s \in S$, then $H_u H_s = H_{us}$ if $\ell(us) > \ell(u)$ and $H_u H_s = H_u$ if $\ell(us) < \ell(u)$. If $\mathbf{q} = (q_1, \ldots, q_l)$ is any sequence of elements of S, define the Demazure product $\delta(\mathbf{q}) \in W$ by the equation $H_{q_1} \cdots H_{q_l} = H_{\delta(\mathbf{q})}$. Define $\ell(\mathbf{q}) = l$ and $e(\mathbf{q}) = \ell(\mathbf{q}) - \ell(\delta(\mathbf{q}))$.

4.3. **Subword Complexes.** Let $\mathbf{s} = (s_1, \ldots, s_l)$ be a sequence of elements of S and $w \in W$. The subword complex $\Delta(\mathbf{s}, w)$ is defined to be the set of subsequences $\mathbf{r} = (s_{i_1}, \ldots, s_{i_t})$, $1 \leq i_1 < \cdots < i_t \leq l$, whose complementary subsequence $\mathbf{s} \setminus \mathbf{r}$ contains a reduced expression for w. One checks that $\Delta(\mathbf{s}, w)$ is a simplicial complex. Subword complexes were introduced in [KM04], [KM05]. We will require that $\mathbf{s}$ contains a reduced expression for w.

Remark 4.2. The requirement that $\mathbf{s}$ contains a reduced expression for w implies that the empty sequence $\emptyset$ lies in $\Delta(\mathbf{s}, w)$. In particular, $\Delta(\mathbf{s}, w)$ is not the void complex. It is possible, however, for $\Delta(\mathbf{s}, w)$ to be the irrelevant complex. For example, this occurs when $\mathbf{s} = (s_1)$, $w = s_1$.

The following theorem is [KM04, Theorem 3.7]:

Theorem 4.3. *The subword complex $\Delta(\mathbf{s}, w)$ is either a ball or sphere. A face $\mathbf{r}$ is in the boundary of $\Delta(\mathbf{s}, w)$ if and only if $\delta(\mathbf{s} \setminus \mathbf{r}) \neq w$.*

Corollary 4.4. $\displaystyle\sum_{\substack{\mathbf{q}\subseteq\mathbf{s}\\ \delta(\mathbf{q})=w}}(-1)^{e(\mathbf{q})}=1.$

Proof. If $\mathbf{r}\subseteq\mathbf{s}$, then $\mathbf{r}\in\Delta(\mathbf{s},w)$ if and only if $\mathbf{s}\setminus\mathbf{r}$ contains a reduced expression for w if and only if $\delta(\mathbf{s}\setminus\mathbf{r})\geq w$ [KM04, Lemma 3.4 (1)]. The dimension of face $\mathbf{r}$ is equal to $\ell(\mathbf{r})-1$. Thus $\sum_{\{\mathbf{r}\subseteq\mathbf{s},\delta(\mathbf{s}\setminus\mathbf{r})\geq w\}}(-1)^{l(\mathbf{r})-1}=\widetilde{\chi}(\Delta(\mathbf{s},w))$. By the second statement of Theorem 4.3, $\sum_{\{\mathbf{r}\subseteq\mathbf{s},\delta(\mathbf{s}\setminus\mathbf{r})>w\}}(-1)^{l(\mathbf{r})-1}=\widetilde{\chi}(\partial\Delta(\mathbf{s},w))$. Hence

$$\sum_{\substack{\mathbf{r}\subseteq\mathbf{s}\\ \delta(\mathbf{s}\setminus\mathbf{r})=w}}(-1)^{l(\mathbf{r})-1}=\widetilde{\chi}^{\circ}(\Delta(\mathbf{s},w)).$$

But since $\Delta(\mathbf{s},w)$ is either a ball or sphere, $\widetilde{\chi}^{\circ}(\Delta(\mathbf{s},w))=(-1)^{\dim\Delta(\mathbf{s},w)}$, by Lemma 4.1. To compute $\dim\Delta(\mathbf{s},w)$, observe that if $\mathbf{r}\in\Delta(\mathbf{s},w)$ has maximal length, then $\mathbf{s}\setminus\mathbf{r}$ is a reduced word for w; thus $\ell(\mathbf{s})-\ell(\mathbf{r})=\ell(w)$, so $\ell(\mathbf{r})=\ell(\mathbf{s})-\ell(w)$. Thus $\dim\Delta(\mathbf{s},w)=\ell(\mathbf{s})-\ell(w)-1$. We conclude

$$\sum_{\substack{\mathbf{r}\subseteq\mathbf{s}\\ \delta(\mathbf{s}\setminus\mathbf{r})=w}}(-1)^{l(\mathbf{r})-1}=(-1)^{\ell(\mathbf{s})-\ell(w)-1}.$$

Multiplying both sides of this equation by $(-1)^{\ell(\mathbf{s})-\ell(w)-1}$, we obtain

$$\sum_{\substack{\mathbf{r}\subseteq\mathbf{s}\\ \delta(\mathbf{s}\setminus\mathbf{r})=w}}(-1)^{e(\mathbf{s}\setminus\mathbf{r})}=1,$$

since, if $\delta(\mathbf{s}\setminus\mathbf{r})=w$, then we have $(-1)^{\ell(\mathbf{s})-\ell(w)-1}(-1)^{\ell(\mathbf{r})-1}=(-1)^{\ell(\mathbf{s})-\ell(\mathbf{r})-\ell(w)}=(-1)^{\ell(\mathbf{s}\setminus\mathbf{r})-\ell(\delta(\mathbf{s}\setminus\mathbf{r}))}=(-1)^{e(\mathbf{s}\setminus\mathbf{r})}$. Now the desired equation is obtained by re-indexing this summation. Rather than summing over subsequences $\mathbf{r}$ of $\mathbf{s}$, one sums over their complementary subsequences $\mathbf{q}$. ☐

Definition 4.5. Let $w\in W$ and let $\mathbf{s}=(s_1,\ldots,s_p)$ be a sequence of simple reflections in S. Define $\mathcal{T}_{w,\mathbf{s}}$ to be the set of sequences $\mathbf{t}=(i_1,\ldots,i_m)$, $1\leq i_1<\cdots<i_m\leq p$, such that $H_{s_{i_1}}\cdots H_{s_{i_m}}=H_w$. Then $\ell(\mathbf{t})=m$ and $e(\mathbf{t})=\ell(\mathbf{t})-\ell(w)$.

Corollary 4.6. $\sum_{\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}}(-1)^{e(\mathbf{t})}=1$, *if* $\mathcal{T}_{w,\mathbf{s}}\neq\emptyset$.

Proof. We have

$$\sum_{\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}}(-1)^{e(\mathbf{t})}=\sum_{\substack{\mathbf{q}\subseteq\mathbf{s}\\ \delta(\mathbf{q})=w}}(-1)^{e(\mathbf{q})}=1,$$

where the first equality is obtained by re-indexing and the second equality is Corollary 4.4. Note that the hypothesis $T_{w,\mathbf{s}}\neq\emptyset$ assures us that $\mathbf{s}$ contains a reduced expression for w, a requirement for Corollary 4.4. ☐

We remark that the expression $\sum_{\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}}(-1)^{e(\mathbf{t})}$ appearing in Corollary 4.6 has elements in common with the expression for $i_x^*[\mathcal{O}_{X^w}]_{G/B}$ given by (5.1). The similarity between these expressions is critical to our proof of Theorem 5.6.

5. Applications to Kazhdan-Lusztig and Schubert varieties

In Section 3 we saw that the pullback $i_V^*[\mathcal{O}_Y]_V$ can be used to determine whether an integrally indecomposable weight $\alpha \in \Phi_{\mathrm{amb}}$ lies in Φ_{nor} or Φ_{tan}. Specifically, α lies in Φ_{nor} if and only if $1 - e^{-\theta}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$. Computationally, however, an algorithm which utilizes this idea would seem to present difficulties, since it is often possible to express $i_V^*[\mathcal{O}_Y]_V$ in many different ways. Determining whether $1 - e^{-\theta}$ is a factor of $i_V^*[\mathcal{O}_Y]_V$ is nontrivial in general.

In this section we show that when Y is the Kazhdan-Lusztig variety Y_x^w in an appropriate space $V \subseteq G/B$, then a particular expression $P_{w,\mathbf{s}}$ for $i_V^*[\mathcal{O}_Y]_V$ due to Graham and Willems has the property that, if we assume that θ is integrally indecomposable in $\Phi_{\mathrm{amb}} = \Phi(V)$, then whenever $1-e^{-\theta}$ is a simple factor of $i_V^*[\mathcal{O}_Y]_V$, it is a factor of $P_{w,\mathbf{s}}$ in a trivial fashion (see Theorem 5.6). Thus the expression $P_{w,\mathbf{s}}$ allows us to detect simple factors $1 - e^{-\theta}$ of $i_V^*[\mathcal{O}_Y]_V$, θ integrally indecomposable, in a computationally simple manner.

We begin with two subsections reviewing properties of Kazhdan-Lusztig and Schubert varieties in G/B.

5.1. **Unipotent subgroups and affine spaces in G/B.** Let G be a semisimple algebraic group defined over a field k of characteristic 0, B a Borel subgroup, B^- the opposite Borel subgroup, and $T = B \cap B^-$, a maximal torus. Let $W = N_G(T)/T$, the Weyl group of G, and let S be the set of simple reflections of W relative to B.

Unipotent subgroups of G are isomorphic to their Lie algebras. In particular, they are isomorphic to affine spaces, with which we will often identify them. If H is a unipotent subgroup of G and $x \in W$, then xHx^{-1}, which we denote by $H(x)$, is unipotent as well. If H is stable under conjugation by T, then so is $H(x)$. Letting $\Phi(H)$ denote the weights of H, we have $\Phi(H(x)) = x\Phi(H)$.

Two unipotent subgroups of interest are U and U^-, the unipotent radicals of B and B^- respectively. Their weights, $\Phi(U)$ and $\Phi(U^-)$, are by definition the positive and negative roots Φ^+ and Φ^- respectively. The groups $U^-(x)$ and $U^-(x) \cap U$ are also unipotent, with weights $\Phi(U^-(x)) = x\Phi^-$ and $\Phi(U^-(x) \cap U) = x\Phi^- \cap \Phi^+$, the inversion set $I(x^{-1})$ of x^{-1}. This is the set of positive roots which x^{-1} takes to negative roots.

The variety G/B is called the full flag variety. The T-fixed points of G/B are of the form uB, $u \in W$. Under the mapping $\zeta : U^-(x) \to G/B$, $y \mapsto y \cdot xB$, the unipotent subgroup $U^-(x)$ embeds as a T-stable affine space in G/B containing xB. We denote this affine space by C_x. The unipotent subgroup $U^-(x) \cap U$ embeds as an affine subspace, which we denote by V. We note that the weight spaces of C_x are one dimensional and that the weights lie in an open half-space; thus the same is true of V.

5.2. **Schubert and Kazhdan-Lusztig varieties in G/B.** The Schubert variety $X^w \subseteq G/B$ is defined to be $\overline{B^- wB}$, the Zariski closure of the B^- orbit through wB. The Kazhdan-Lusztig variety Y^w_x is defined to be $V \cap X^w$. As the following lemma shows, locally, the two varieties differ only by an affine space with well-prescribed weights.

Lemma 5.1. *Let $w \leq x \in W$. Then*

(i) $C_x \cong (U^-(x) \cap U^-) \times V$.
(ii) $X^w \cap C_x \cong (U^-(x) \cap U^-) \times Y^w_x$.
(iii) $T_x X^w \cong (U^-(x) \cap U^-) \times T_x Y^w_x$.
(iv) $\Phi(T_x X^w) = -(\Phi^+ \setminus I(x^{-1})) \sqcup \Phi(T_x Y^w_x)$.

Proof. (i) [GK17, (4.6)].

(ii) is an application of [GK17, Lemma 4.6], with $H = (U^-(x) \cap U^-)$, $Y = (U^-(x) \cap U) \cdot xB$, and $Z = X^w \cap C_x$.

(iii) follows from (ii) and the fact that the tangent space of a product is isomorphic to the product of the tangent spaces (see [GW10, Proposition 6.9]).

(iv) Note that $(U^-(x) \cap U) \times (U(x) \cap U) \cong U$. Since $\Phi(U^-(x) \cap U) = I(x^{-1})$ and $\Phi(U) = \Phi^+$, we must have $\Phi(U(x) \cap U) = \Phi^+ \setminus I(x^{-1})$, and thus $\Phi(U^-(x) \cap U^-) = -(\Phi^+ \setminus I(x^{-1}))$. Combined with (iii), this yields the desired result. □

Lemma 5.1(iv) shows how to produce $\Phi(T_x X^w)$ from $\Phi(T_x Y^w_x)$. In Section 3, we saw that information about the latter can be obtained from $i^*_V[\mathcal{O}_{Y^w_x}]_V$. The next proposition asserts that this pullback is equal to $i^*_x[\mathcal{O}_{X^w}]_{G/B}$, for which there are known formulas, in particular (5.1) below.

Lemma 5.2. *Let $w \leq x \in W$. Then $i^*_V[\mathcal{O}_{Y^w_x}]_V = i^*_x[\mathcal{O}_{X^w}]_{G/B}$.*

Proof. By Lemma 5.1(i) and (ii), we can apply [GK17, Lemma 2.1] to obtain $i^*_V[\mathcal{O}_{Y^w_x}]_V = i^*_{C_x}[\mathcal{O}_{X^w \cap C_x}]_{C_x}$. Since pullbacks in equivariant K-theory are defined locally, $i^*_{C_x}[\mathcal{O}_{X^w \cap C_x}]_{C_x} = i^*_x[\mathcal{O}_{X^w}]_{G/B}$. □

5.3. **Formulas for weights of the tangent space.** Fix a reduced expression $\mathbf{s} = (s_1, \ldots, s_l)$ for x. The elements of the inversion set $I(x^{-1}) = \Phi^+ \cap x\Phi^-$ are given explicitly by the formula $\gamma_i = s_1 \cdots s_i(\alpha_i)$, $i = 1, \ldots, l$, where α_i is the simple root corresponding to s_i [Hum90]. The following result is due to Graham [Gra02] and Willems [Wil06]:

Theorem 5.3. *Let $w \leq x \in W$, and let $\mathbf{s} = (s_1, \ldots, s_l)$ be a reduced sequence of simple reflections for x. Then*

$$i^*_x[\mathcal{O}_{X^w}]_{G/B} = \sum_{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}} (-1)^{e(\mathbf{t})} \prod_{i \in \mathbf{t}} (1 - e^{-\gamma_i}) \in R(T) \tag{5.1}$$

Denote the expression $\sum_{\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}}(-1)^{e(\mathbf{t})}\prod_{i\in\mathbf{t}}(1-e^{-\gamma_i})$ by $P_{w,\mathbf{s}}$. By Lemma 5.2, we have

Corollary 5.4. *Let $w \le x \in W$. Then $i_V^*[\mathcal{O}_{Y_x^w}]_V = P_{w,\mathbf{s}}$.*

Remark 5.5. In general, there exist numerous expressions for $i_V^*[\mathcal{O}_{Y_x^w}]_V$. Lemma 2.2 assures us that there exists an expression as a polynomial in $1-e^{-\gamma}$, $\gamma\in\Phi(V)=I(x^{-1})$; $P_{w,\mathbf{s}}$ is such an expression.

We shall say that $1-e^{-\gamma_j}$ is an **explicit factor** of $P_{w,\mathbf{s}}$ if $1-e^{-\gamma_j}$ occurs among the factors of every summand $\prod_{i\in\mathbf{t}}(1-e^{-\gamma_i})$ of $P_{w,\mathbf{s}}$, or equivalently, if j belongs to every $\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}$ (see Remark 3.2). Since all of the γ_j, $j=1,\dots,l$, are distinct, every explicit factor of $P_{w,\mathbf{s}}$ is a simple factor of $i_V^*[\mathcal{O}_{Y_x^w}]_V$. The following theorem tells us that when γ_j is integrally indecomposable in $I(x^{-1})$, the converse is true as well.

Theorem 5.6. *Let $w\le x\in W$, and let γ_j be integrally indecomposable in $I(x^{-1})$. If $1-e^{-\gamma_j}$ is a simple factor of $i_V^*[\mathcal{O}_{Y_x^w}]_V$, then it is an explicit factor of $P_{w,\mathbf{s}}$.*

Proof. Let C be the coordinate ring of the tangent cone to Y_x^w at x. We will assume that $1-e^{-\gamma_j}$ is not an explicit factor of $P_{w,\mathbf{s}}$ and show that $-\gamma_j$ is a weight of C (of multiplicity 1). Proposition 3.5 then implies that $1-e^{-\gamma_j}$ is not a simple factor of $i_V^*[\mathcal{O}_{Y_x^w}]_V$, completing the proof. Let $[l]$ denote $\{1,\dots,l\}$.

By Proposition 2.3, we have

$$\operatorname{Char} C = \frac{i_V^*[\mathcal{O}_{Y_x^w}]_V}{\lambda_{-1}(V^*)} = \frac{\sum_{\mathbf{t}\in\mathcal{T}_{w,\mathbf{s}}}(-1)^{e(\mathbf{t})}\prod_{i\in\mathbf{t}}(1-e^{-\gamma_i})}{\prod_{i\in[l]}(1-e^{-\gamma_i})} \tag{5.2}$$

Each summand of (5.2) can be simplified:

$$\begin{aligned}\frac{(-1)^{e(\mathbf{t})}\prod_{i\in\mathbf{t}}(1-e^{-\gamma_i})}{\prod_{i\in[l]}(1-e^{-\gamma_i})} &= (-1)^{e(\mathbf{t})}\frac{1}{\prod_{i\notin\mathbf{t}}(1-e^{-\gamma_i})}\\ &= (-1)^{e(\mathbf{t})}\prod_{i\notin\mathbf{t}}(1+e^{-\gamma_i}+e^{-2\gamma_i}+\cdots)\\ &= (-1)^{e(\mathbf{t})}\sum_{\zeta\in\operatorname{Cone}_{\mathbb{Z}}\{\gamma_i : i\notin\mathbf{t}\}} n_\zeta e^{-\zeta}\\ &= (-1)^{e(\mathbf{t})} n_{\gamma_j} e^{-\gamma_j} + \text{ other terms}\end{aligned}$$

where n_ζ is the number of ways to express ζ as a nonnegative integer linear combination of the γ_i, $i\notin\mathbf{t}$, and "other terms" refers to an infinite linear combination of characters with no $e^{-\gamma_j}$ term. Since γ_j is integrally indecomposable in $I(x^{-1})$, $n_{\gamma_j}=1$ if $j\notin\mathbf{t}$ and $n_{\gamma_j}=0$ if $j\in\mathbf{t}$. Thus

$$\frac{(-1)^{e(\mathbf{t})}\prod_{i\in\mathbf{t}}(1-e^{-\gamma_i})}{\prod_{i\in[l]}(1-e^{-\gamma_i})} = \begin{cases}(-1)^{e(\mathbf{t})}e^{-\gamma_j} + \text{ other terms}, & \text{if } j\notin\mathbf{t}\\ \text{other terms}, & \text{if } j\in\mathbf{t}\end{cases} \tag{5.3}$$

According to (5.2), $\operatorname{Char} C$ is the sum of fractions as in (5.3), one for each $\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}$. Therefore the coefficient of $e^{-\gamma_j}$ in $\operatorname{Char} C$ is

$$N = \sum_{\{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}} : j \notin \mathbf{t}\}} (-1)^{e(\mathbf{t})}$$

Setting $\mathbf{s}_j = (s_1, \dots, \hat{s}_j, \dots, s_l)$, we have

$$N = \sum_{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}_j}} (-1)^{e(\mathbf{t})}$$

The assumption that $1 - e^{-\gamma_j}$ is not an explicit factor of $P_{w,\mathbf{s}}$ assures us that $\mathcal{T}_{w,\mathbf{s}_j} \neq \emptyset$, and thus this sum equals 1 by Corollary 4.6. Since $N \neq 0$, $-\gamma_j$ is a weight of C. $\square$

Denote $\Phi(T_x Y_x^w)$ and $\Phi(V/T_x Y_x^w)$ by $\Phi_{\tan}$ and Φ_{nor} respectively, so $\Phi_{\tan} \sqcup \Phi_{\mathrm{nor}} = \Phi(V) = I(x^{-1})$.

Corollary 5.7. *Let $w \leq x \in W$, and suppose that γ is an integrally indecomposable element of $I(x^{-1})$. If $\gamma \in \Phi_{\mathrm{nor}}$, then $1 - e^{-\gamma}$ is an explicit factor of $P_{w,\mathbf{s}}$.*

Proof. Since $\gamma \in \Phi_{\mathrm{nor}}$, $1 - e^{-\gamma}$ is a simple factor of $i_V^*[\mathcal{O}_{Y_x^w}]_V$, by Proposition 3.3. Thus $1 - e^{-\gamma}$ is an explicit factor of $P_{w,\mathbf{s}}$, by Theorem 5.6. $\square$

Let $m = \ell(w)$, and define

$$\mathcal{RT}_{w,\mathbf{s}} = \{\mathbf{t} = (t_1, \dots, t_m) \subseteq [l] \mid s_{t_1} \cdots s_{t_m} = w\}.$$

Parts (i) - (iii) of the following theorem summarize the main findings of this section thus far. Parts (iv) and (v) provide a computationally simpler method of determining whether γ_j lies in $\Phi_{\tan}$, by allowing us to substitute $\mathcal{RT}_{w,\mathbf{s}}$ for $\mathcal{T}_{w,\mathbf{s}}$, and thus to perform calculations in the Weyl group rather than the 0-Hecke algebra. Part (vi) gives an alternative characterization of (v) in terms of Demazure products.

Theorem 5.8. *Let $w \leq x \in W$, and let $\mathbf{s} = (s_1, \dots, s_l)$ be a reduced expression for x. If γ_j is an integrally indecomposable element of $I(x^{-1})$, then the following are equivalent:*

- (i) $\gamma_j \in \Phi_{\tan}$.
- (ii) $1 - e^{-\gamma_j}$ *is not an explicit factor of* $P_{w,\mathbf{s}}$.
- (iii) *There exists* $\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}$ *not containing* j.
- (iv) *There exists* $\mathbf{t} \in \mathcal{RT}_{w,\mathbf{s}}$ *not containing* j.
- (v) *There exists a reduced subexpression of* $(s_1, \dots, \widehat{s}_j, \dots, s_l)$ *for* w.
- (vi) $\delta((s_1, \dots, \widehat{s}_j, \dots, s_l)) \geq w$.

Proof. (i) $\Leftrightarrow$ (ii) by Proposition 3.5 and Corollary 5.7; (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) are due to definitions of $P_{w,\mathbf{s}}$, $\mathcal{T}_{w,\mathbf{s}}$, and $\mathcal{RT}_{w,\mathbf{s}}$. The proof of (iii) $\Leftrightarrow$ (iv) follows from $\mathcal{T}_{w,\mathbf{s}} \supseteq \mathcal{RT}_{w,\mathbf{s}}$ and the fact that every element of $\mathcal{T}_{w,\mathbf{s}}$ contains an element of $\mathcal{RT}_{w,\mathbf{s}}$.

(v) $\Leftrightarrow$ (vi) There exists a reduced subexpression of $\mathbf{s}$ for w not containing s_j if and only if there exists a subexpression of $(s_1, \dots, \widehat{s}_j, \dots, s_l)$ for w if and only if

$\delta((s_1, \ldots, \widehat{s}_j, \ldots, s_l)) \geq w$, where the last equivalence is due to [KM04, Lemma 3.4 (1)]. □

Remark 5.9. For $\gamma_j \in I(x^{-1})$, it is known that in type A, $\gamma_j \in \Phi_{\tan}$ if and only if $s_1 \cdots \widehat{s}_j \cdots s_l \geq w$ [LS84]. Theorem 5.8 states that if γ_j is integrally indecomposable in $I(x^{-1})$, then $\gamma_j \in \Phi_{\tan}$ if and only if $\delta((s_1, \ldots, \widehat{s}_j, \ldots, s_l)) \geq w$. These two statements imply that in type A, if γ_j is integrally indecomposable in $I(x^{-1})$, then $\delta((s_1, \ldots, \widehat{s}_j, \ldots, s_l)) \geq w$ if and only if $s_1 \cdots \widehat{s}_j \cdots s_l \geq w$. That this holds for all $w \leq x$ would seem to imply that $\delta((s_1, \ldots, \widehat{s}_j, \ldots, s_l)) = s_1 \cdots \widehat{s}_j \cdots s_l$. This is indeed true, and the above argument can be made rigorous. In [GK20], it is shown that the statement extends to all simply-laced types. It is also shown that if γ_j is *rationally indecomposable* in $I(x^{-1})$, then $\delta((s_1, \ldots, \widehat{s}_j, \ldots, s_l)) = s_1 \cdots \widehat{s}_j \cdots s_l$ in all types.

Remark 5.10. Suppose that γ_j is not integrally indecomposable in Φ_{amb}. Then statements (ii) – (vi) of Theorem 5.8 are still equivalent, but statement (i) is no longer equivalent to the other five in general. The following example shows that (vi) $\Rightarrow$ (i) can fail. In type A_2, let $\mathbf{s} = (\sigma_1, \sigma_2, \sigma_1)$, where σ_i is the simple transposition which exchanges i and $i+1$. Let $w = \sigma_1$ and $j = 2$. Then $\delta((\sigma_1, \widehat{\sigma}_2, \sigma_1)) = \delta((\sigma_1, \sigma_1)) = \sigma_1 \geq w$, so (vi) holds. However, $\sigma_1 \widehat{\sigma}_2 \sigma_1 = e \not\geq w$. Thus $\gamma_2 \notin \Phi_{\tan}$ (see Remark 5.9), so (i) fails.

We note that γ_j is required to be integrally indecomposable in $I(x^{-1})$ for our proofs of both implications of Theorem 5.8 (i) $\Leftrightarrow$ (ii).

6. Partial flag varieties and cominuscule elements

Let P be a parabolic subgroup containing B. In Section 6.1 we show that Lemma 5.1 and Theorem 5.8 extend from G/B to G/P with no changes other than notation. In Section 6.2 we apply the results to cominuscule elements of W and cominuscule G/P.

6.1. **Extending results to G/P.** Let P be a parabolic subgroup containing B. Let L be the Levi subgroup of P containing T, and $W_P = N_L(T)/T$, the Weyl group of L. Each coset uW_P in W/W_P contains a unique representative of minimal length; denote the set of minimal length coset representatives by $W^P \subseteq W$. Unless stated otherwise, in this subsection we assume that all Weyl group elements lie in W^P. The T-fixed points of G/P are of the form uP, $u \in W^P$.

Let P^- the opposite parabolic subgroup to P, and let U_P^- be the the unipotent radical of P^-. Under the mapping $\zeta : U_P^-(x) \to G/P$, $y \mapsto y \cdot xP$, the unipotent subgroup $U_P^-(x)$ embeds as a T-stable affine space in G/P containing xP. The unipotent subgroup $U_P^-(x) \cap U$ embeds as an affine subspace, which we denote by V_P.

The Schubert variety $X_P^w \subseteq G/P$ is defined to be $\overline{B^- wP}$, the Zariski closure of the B^- orbit through wP. The Kazhdan-Lusztig variety $Y_{x,P}^w$ is defined to be $V_P \cap X_P^w$.

The following result appears in [Knu09, Section 7.3]:

Theorem 6.1. *Let $w \le x \in W^P$. Then $V_P \cong V$ and $Y^w_{x,P} \cong Y^w_x$.*

The next theorem extends the main results of Section 5 to G/P.

Theorem 6.2. *Let $w \le x \in W^P$.*

(i) $\Phi(T_x X^w_P) = \Phi(U^-_P(x) \cap U^-) \sqcup \Phi(T_x Y^w_{x,P})$.

(ii) *Let γ_j be an integrally indecomposable element of $I(x^{-1})$. Then $\gamma_j \in \Phi(T_x Y^w_{x,P})$ if and only if $\delta((s_1, \dots, \hat{s}_j, \dots, s_l)) \ge w$.*

Proof. (i) Lemma 5.1(i)-(iii) remain valid if all quantities are replaced by their analogs in G/P. In particular, $T_x X^w_P \cong (U^-_P(x) \cap U^-) \times T_x Y^w_{x,P}$.

(ii) Since $Y^w_{x,P} \cong Y^w_x$, $T_x Y^w_{x,P} \cong T_x Y^w_x$. Thus all parts of Theorem 5.8 remain valid if $\Phi_{\tan} = \Phi(T_x Y^w_x)$ in Theorem 5.8(i) is replaced by $\Phi(T_x Y^w_{x,P})$. ☐

6.2. **Application to cominuscule Weyl group elements and cominuscule G/P.** In this subsection we discuss conditions on x under which all elements of $I(x^{-1})$ are integrally indecomposable, and thus, for any Kazhdan-Lusztig variety containing x, Theorems 5.8 and 6.2(ii) recover all weights of the tangent space at x. In particular, we show that our results completely describe the tangent spaces of Schubert varieties in cominuscule G/P.

Definition 6.3. The element $x \in W$ is said to be *cominuscule* if there exists $v \in \mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in I(x^{-1})$.

This notion was introduced and studied by Peterson (see [GK17, Section 5.2] or [Ste01] for discussion). In type A, the cominuscule Weyl group elements are precisely the 321-avoiding permutations [Knu09, p. 25]. As noted in [GK17], the equality $I(x) = -x^{-1}I(x^{-1})$ implies that x is cominuscule if and only if x^{-1} is.

Proposition 6.4. *If $x \in W$ is cominuscule, then all elements of $I(x^{-1})$ are integrally indecomposable.*

Proof. If x is cominuscule, then there exists $v \in \mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in I(x^{-1})$. Assume that some $\beta \in I(x^{-1})$ is integrally decomposable. Then $\beta = \sum_{i=1}^m \beta_i$, where $m \ge 2$, $\beta_i \in I(x^{-1})$. Since $\beta(v) = -1$ and $\beta_i(v) = -1$ for all i, this leads to a contradiction. ☐

Remark 6.5. The converse of the above proposition is false: there exist non-cominuscule elements x such that every element of $I(x^{-1})$ is integrally indecomposable. The following example is a variation and extension of [Ste01, Remark 5.4]. In type D_4, with the conventions of [Hum90], consider the element $x = s_2 s_1 s_3 s_4 s_2$. The inversion set $I(x^{-1})$ is equal to $\{\epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_2 + \epsilon_4\}$. Every element of $I(x^{-1})$ is integrally indecomposable, but the element x is not cominuscule (cf. [Ste01, Remark 5.4]). Note that [Ste01] uses a different numbering of the vertices of the Dynkin diagram in which node 3 has degree 3 (see [Ste01, Remark 2.7]), so he writes the element x as $s_3 s_1 s_2 s_4 s_3$.

Definition 6.6. The maximal parabolic subgroup $P \supseteq B$ is said to be *cominuscule* if the simple root α_i corresponding to P occurs with coefficient 1 when the highest root of G is written as a linear combination of the simple roots.

If P is cominuscule, then the corresponding flag variety G/P is said to be cominuscule as well. We refer the reader to [BL00, Chapter 9], [Bou02, Chapter VI, §1, Exercise 24], [GK15] for more on cominuscule G/P. The following proposition gives an important class of cominuscule Weyl group elements.

Proposition 6.7. *If $x \in W^P$, where P is cominuscule, then x is a cominuscule element of W.*

Proof. If $x \in W^P$, then $U^-(x) \cap U = U_P^-(x) \cap U$ (see the discussion before Lemma 4.1 in [GK15], cf. [Knu09]). Hence

$$I(x^{-1}) = \Phi((U^-(x) \cap U) = \Phi(U_P^-(x) \cap U) \subset x\Phi(U_P^-).$$

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots of G; these form a basis for $\mathfrak{t}^*$. Denote the dual basis of $\mathfrak{t}$ by $\xi_1, \ldots, \xi_r$. Assume that P corresponds to the simple root α_i. Since P is cominuscule, [GK15, Lemma 2.8] implies α_i must occur with coefficient -1 in all $\alpha \in \Phi(U_P^-)$ (when α is written as a linear combination of the simple roots), so for all such α, we have $\alpha(\xi_i) = -1$. It follows that $v = x\xi_i$ satisfies $\alpha(v) = -1$ for all $\alpha \in x\Phi(U_P^-)$. Hence $\alpha(v) = -1$ for all $\alpha \in I(x^{-1})$, so x is a cominuscule element of W. □

Remark 6.8. The results of this subsection imply that if P is cominuscule and $w \leq x \in W^P$, then Theorem 6.2 characterizes all weights of $T_xY_{x,P}^w$ and $T_xX_P^w$. More generally, suppose $x \in W$ is any cominuscule element (or more generally any element such that each element of $I(x^{-1})$ is integrally indecomposable). Then Lemma 5.1 and Theorem 5.8 characterize all weights of $T_xY_x^w$ and T_xX^w. If in addition $P \supset B$ is a parabolic subgroup such that $w, x \in W^P$, then Theorem 6.2 characterizes all weights of $T_xY_{x,P}^w$ and $T_xX_P^w$.

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