Homework 11: The Definition of Derivative

1. §6.1, #9.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying (i) $f(0) = 0$, (ii) $f'(0) = 3$, and (iii) $g(0) = 2$. Prove that fg is differentiable at 0, and find $(fg)'(0)$.

   Note: The product rule for derivatives cannot be used for this problem, since g may not be differentiable at 0. You must use the definition of derivative.

3. Let $f : (-1, 1) \to \mathbb{R}$ be a bounded function. In Homework 9, you proved that the function $g : (-1, 1) \to \mathbb{R}$ defined by $g(x) = xf(x)$ is continuous at $x = 0$. Use this result to prove that the function $h : (-1, 1) \to \mathbb{R}$ defined by $h(x) = x^2 f(x)$ is differentiable at 0, and find $h'(0)$.

   Note: The product rule for derivatives cannot be applied here either.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ exists.

   (a) Prove that $\lim\limits_{x \to 0} f(x)$ exists and find its value.

   (b) Prove that if $f(0) = 0$, then f is differentiable at 0.