

# LINEAR ALGEBRA (MATH 301) DEFINITIONS AND THEOREMS

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These notes contain a list of the main definitions and theorems which will be discussed in this course.

### 1. SUBSPACE, LINEAR COMBINATION, SPAN, LINEAR INDEPENDENCE, BASIS

**Definition 1.1.** A **vector space** is ...

**Definition 1.2.** A **subspace** of a vector space  $V$  is a subset  $W$  of  $V$  which is itself a vector space.

**Definition 1.3.** A **linear combination** of  $\mathbf{v}_1, \dots, \mathbf{v}_r$  is a vector of the form  $k_1\mathbf{v}_1 + \dots + k_r\mathbf{v}_r$ , where  $k_1, \dots, k_r$  are scalars.

**Definition 1.4.** The **span** of  $\mathbf{v}_1, \dots, \mathbf{v}_r$ , denoted  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ , is the set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_r$ .

**Definition 1.5.** Let  $V$  be a vector space and let  $\mathbf{v}_1, \dots, \mathbf{v}_r$  be vectors in  $V$ . Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$  is **linearly independent** if  $k_1\mathbf{v}_1 + \dots + k_r\mathbf{v}_r = \mathbf{0}$  implies  $k_1 = \dots = k_r = 0$ .

**Definition 1.6.** A **basis** for a vector space  $V$  is a set of vectors  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  such that

- (i)  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = V$ , and
- (ii)  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly independent.

**Definition 1.7.** Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of a vector space  $V$ , and let  $\mathbf{u}$  be a vector in  $V$ . Then  $\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$  for unique scalars  $c_1, \dots, c_n$ , by Theorem 1.6. The scalars  $c_1, \dots, c_n$  are called the **coordinates** of  $\mathbf{u}$  relative to  $\mathcal{B}$ , and the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

denoted by  $[\mathbf{u}]_{\mathcal{B}}$ , is called the **coordinate vector** of  $\mathbf{u}$  relative to  $\mathcal{B}$ .

**Definition 1.8.** The **dimension** of a nonzero vector space  $V$  is the number of vectors in a basis for  $V$ .

**Definition 1.9.** Let  $A = [\mathbf{v}_1 \dots \mathbf{v}_n]$  be an  $m \times n$  matrix with column vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , and

let  $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  be a vector in  $\mathbb{R}^n$ . Then  $A\mathbf{x} = x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n$ .

**Definition 1.10.** Let  $A$  be an  $m \times n$  matrix. The **nullspace** of  $A$ , denoted by  $\text{Nul } A$ , is the set of all solutions to  $A\mathbf{x} = \mathbf{0}$ .

**Definition 1.11.** Let  $A$  be an  $m \times n$  matrix. The **column space** of  $A$ , denoted by  $\text{Col } A$ , is the span of the column vectors of  $A$ .

**Theorem 1.1.** Let  $V$  be a vector space,  $\mathbf{u}$  a vector in  $V$ , and  $k$  a scalar. Then

- (i)  $0\mathbf{u} = \mathbf{0}$
- (ii)  $k\mathbf{0} = \mathbf{0}$
- (iii)  $(-1)\mathbf{u} = -\mathbf{u}$

**Theorem 1.2.** Let  $V$  be a vector space and let  $W$  be a subset of  $V$ . Then  $W$  is a subspace of  $V$  if:

- (i) for every  $\mathbf{u}$  and  $\mathbf{v}$  in  $W$ ,  $\mathbf{u} + \mathbf{v}$  is in  $W$  (i.e.,  $W$  is closed under vector addition); and
- (ii) for every  $\mathbf{u}$  in  $W$  and scalar  $k$ ,  $k\mathbf{u}$  is in  $W$  (i.e.,  $W$  is closed under scalar multiplication); and
- (iii) the zero vector of  $V$  lies in  $W$ .

**Theorem 1.3.** Let  $V$  be a vector space and let  $\mathbf{v}_1, \dots, \mathbf{v}_r$  be vectors in  $V$ . Then  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$  is a subspace of  $V$ .

**Theorem 1.4.** A set  $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$  of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

**Theorem 1.5.** Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis for a vector space  $V$ .

- (i) If any vector of  $V$  is added to  $\mathcal{B}$ , then  $\mathcal{B}$  is no longer linearly independent.
- (ii) If any vector is removed from  $\mathcal{B}$ , then  $\mathcal{B}$  no longer spans  $V$ .

**Theorem 1.6.** Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of a vector space  $V$ . Then every  $\mathbf{u}$  in  $V$  can be written in exactly one way as a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , that is, can be expressed as

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$

for unique scalars  $c_1, \dots, c_n$ .

**Theorem 1.7.** All bases of a vector space  $V$  have the same number of elements.

**Theorem 1.8.** In  $\mathbb{R}^n$ , the following have the same solutions:

- (i) The vector equation  $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{u}$ .
- (ii) The linear system of equations with augmented matrix  $[\mathbf{v}_1 \cdots \mathbf{v}_p \mid \mathbf{u}]$ .
- (iii) The matrix equation  $[\mathbf{v}_1 \cdots \mathbf{v}_p]\mathbf{x} = \mathbf{u}$ .

**Lemma 1.1.** Let  $A$  be an  $m \times n$  matrix, let  $\mathbf{u}, \mathbf{v}$  be vectors in  $\mathbb{R}^n$ , and let  $c$  be a scalar. Then

- (i)  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ , and
- (ii)  $A(c\mathbf{u}) = c(A\mathbf{u})$ .

**Theorem 1.9.** Let  $A$  be an  $m \times n$  matrix. Then  $\text{Nul } A$  is a subspace of  $\mathbb{R}^n$ .

## 2. INTRODUCTION TO LINEAR TRANSFORMATIONS

**Definition 2.1.** Let  $V$  and  $W$  be vector spaces. A transformation (or mapping)  $T : V \rightarrow W$  is **linear** if it satisfies the following conditions:

- (i) For every  $\mathbf{u}, \mathbf{v}$  in  $V$ ,  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ .
- (ii) For every  $\mathbf{u}$  in  $V$  and scalar  $c$ ,  $T(c\mathbf{u}) = cT(\mathbf{u})$ .

**Theorem 2.1.** Let  $T : V \rightarrow W$  be linear. Then

- (i)  $T(\mathbf{0}) = \mathbf{0}$ .
- (ii)  $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$ , for any scalars  $c_1, \dots, c_p$  and vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  in  $V$ .

**Definition 2.2.** A **matrix transformation** is a mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by  $T(\mathbf{x}) = A\mathbf{x}$ , for some fixed  $m \times n$  matrix  $A$ .

**Theorem 2.2.** A matrix transformation is linear.

**Definition 2.3.** Let  $T : V \rightarrow W$  be linear. Then

- (i) The **kernel** of  $T$ , denoted  $\ker(T)$ , is the set of vectors in  $V$  which  $T$  maps to  $\mathbf{0}$ .
- (ii) The **range** of  $T$ , denoted  $R(T)$ , is the set of vectors in  $W$  which have at least one vector in  $V$  mapping to them.

**Theorem 2.3.** Let  $T : V \rightarrow W$  be linear. Then  $\ker(T)$  is a subspace of  $V$  and  $R(T)$  is a subspace of  $W$ .

**Theorem 2.4.** Let  $A$  be an  $m \times n$  matrix, and let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be the matrix transformation  $T(\mathbf{x}) = A\mathbf{x}$ . Then  $\ker(T) = \text{Nul } A$  and  $R(T) = \text{Col } A$ .

**Theorem 2.5.** Let  $T : V \rightarrow W$  be linear. Then  $\dim(\ker T) + \dim(R(T)) = \dim V$ .

**Theorem 2.6.** Let  $T : V \rightarrow W$  be linear. Then  $T$  is one-to-one if and only if  $\ker T = \{\mathbf{0}\}$ .

**Theorem 2.7.** Let  $W$  be a subspace of  $V$ . If  $\dim W = \dim V$ , then  $W = V$ .

**Theorem 2.8.** Let  $T : V \rightarrow W$  be linear, and suppose that  $\dim V = \dim W$ . Then  $T$  is one-to-one if and only if  $T$  is onto.

**Definition 2.4.** Let  $T : U \rightarrow V$  and  $S : V \rightarrow W$  be linear transformations. Then the **composition** of  $S$  with  $T$ , denoted  $S \circ T$ , is the map from  $U$  to  $W$  defined by  $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$  for  $\mathbf{u} \in U$ .

**Theorem 2.9.** Let  $T : U \rightarrow V$  and  $S : V \rightarrow W$  be linear transformations. Then the composition  $S \circ T : U \rightarrow W$  is a linear transformation.

**Definition 2.5.** For any vector space  $V$ , the **identity transformation**  $I : V \rightarrow V$  is defined by  $I(\mathbf{v}) = \mathbf{v}$  for all  $\mathbf{v}$  in  $V$ .

**Theorem 2.10.** Let  $T : V \rightarrow W$  be a linear transformation. Then  $T \circ I = I \circ T = T$ .

**Definition 2.6.** Let  $T : V \rightarrow W$  be one-to-one. Then there exists an **inverse transformation**  $T^{-1} : R(T) \rightarrow V$  such that  $T^{-1}(T(\mathbf{v})) = \mathbf{v}$  for all  $\mathbf{v}$  in  $V$ .

**Theorem 2.11.** Let  $T : V \rightarrow W$  be one-to-one. Then  $T^{-1} \circ T = I$ .

**Definition 2.7.** An **isomorphism** is a bijective linear transformation.

**Definition 2.8.** If  $T : V \rightarrow W$  is an isomorphism, then  $V$  and  $W$  are said to be **isomorphic**.

**Theorem 2.12.** If  $T : V \rightarrow W$  is an isomorphism, then  $\dim V = \dim W$ .

**Theorem 2.13.** Suppose that  $V$  is a vector space and  $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $V$ . Then the mapping  $T : V \rightarrow \mathbb{R}^n$  given by  $T(\mathbf{u}) = [\mathbf{u}]_B$  is an isomorphism.

### 3. THE MATRIX OF A LINEAR TRANSFORMATION

**Theorem 3.1.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then for  $\mathbf{x}$  in  $\mathbb{R}^n$ ,  $T(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is the matrix  $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ . The matrix  $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$  is called the **standard matrix** for  $T$ .

**Theorem 3.2.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping. Then  $T$  is a linear transformation if and only if  $T$  is a matrix transformation.

**Theorem 3.3.** Suppose that the standard matrix for  $S$  is  $A$  and the standard matrix for  $T$  is  $B$ . Then the standard matrix for  $S \circ T$  is  $AB$ .

**Definition 3.1.** Let  $A$  be an  $n \times n$  matrix. Then  $A$  is said to be **invertible** if there exists an  $n \times n$  matrix  $B$  such that  $AB = BA = I_n$ . In this case,  $B$  is called the **inverse** of  $A$ , and we write  $B = A^{-1}$ .

**Theorem 3.4.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation, and let  $A$  be the standard matrix for  $T$ . Then  $T$  is an isomorphism if and only if  $A$  is invertible. In this case, the standard matrix for  $T^{-1}$  is  $A^{-1}$ .

**Theorem 3.5.** Let  $A$  be an  $n \times n$  matrix. Then  $A$  is invertible if and only if  $A$  can be row reduced to  $I_n$ .

**Theorem 3.6.** Let  $A$  be an  $n \times n$  matrix, and let  $\mathbf{b}$  be a vector in  $\mathbb{R}^n$ . If  $A$  is invertible, then  $A\mathbf{x} = \mathbf{b}$  has a unique solution, namely,  $\mathbf{x} = A^{-1}\mathbf{b}$ .

**Theorem 3.7.** Let  $T : V \rightarrow W$  be linear. Let  $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  be a basis for  $V$  and  $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$  a basis for  $W$ . Then there exists a matrix  $[T]_{\mathcal{B}', \mathcal{B}}$  such that for every  $\mathbf{v}$  in  $V$ ,  $[T(\mathbf{v})]_{\mathcal{B}'} = [T]_{\mathcal{B}', \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$ .

**Theorem 3.8.** Let  $T : V \rightarrow W$  be linear. Let  $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  be a basis for  $V$  and  $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$  a basis for  $W$ . Then

$$[T]_{\mathcal{B}', \mathcal{B}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}'} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}'} \end{bmatrix}$$

**Theorem 3.9.** Let  $T : U \rightarrow V$  and  $S : V \rightarrow W$  be linear. Let  $\mathcal{B}, \mathcal{B}', \mathcal{B}''$  be bases for vector spaces  $U, V, W$  respectively. Then  $[S \circ T]_{\mathcal{B}'', \mathcal{B}} = [S]_{\mathcal{B}'', \mathcal{B}'} \cdot [T]_{\mathcal{B}', \mathcal{B}}$ .

**Definition 3.2.** Let  $\mathcal{B}, \mathcal{B}'$  be bases for a vector space  $V$ . Then  $[I]_{\mathcal{B}', \mathcal{B}}$  is called the **change of coordinates matrix** from  $\mathcal{B}$  to  $\mathcal{B}'$  coordinates.

**Theorem 3.10.** Let  $\mathcal{B}, \mathcal{B}'$  be bases for a vector space  $V$ . Then

- (i) For any  $\mathbf{v}$  in  $V$ ,  $[\mathbf{v}]_{\mathcal{B}'} = [I]_{\mathcal{B}', \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$ .
- (ii)  $[I]_{\mathcal{B}, \mathcal{B}} = I_n$ , where  $n = \dim V$ .
- (iii)  $[I]_{\mathcal{B}', \mathcal{B}}$  is invertible.
- (iv)  $([I]_{\mathcal{B}', \mathcal{B}})^{-1} = [I]_{\mathcal{B}, \mathcal{B}'}$ .

**Notation 3.1.**  $[T]_{\mathcal{B},\mathcal{B}}$  is often denoted by just  $[T]_{\mathcal{B}}$ .

**Theorem 3.11** (Change of Basis Formula). *Let  $T : V \rightarrow V$  be a linear operator. Let  $\mathcal{B}, \mathcal{B}'$  be bases for  $V$ . Then*

$$[T]_{\mathcal{B}'} = [I]_{\mathcal{B}',\mathcal{B}} \cdot [T]_{\mathcal{B}} \cdot [I]_{\mathcal{B},\mathcal{B}'}$$

#### 4. INNER PRODUCT SPACES

**Definition 4.1.** *Let  $V$  be a vector space. An **inner product** on  $V$  is a rule which assigns to each pair of vectors  $\mathbf{u}, \mathbf{v}$  in  $V$  a scalar, denoted  $\langle \mathbf{u}, \mathbf{v} \rangle$ , such that for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$  and all scalars  $c$ ,*

- (i)  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ .
- (ii)  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ .
- (iii)  $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ .
- (iv)  $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ , with equality if and only if  $\mathbf{v} = \mathbf{0}$ .

*A vector space with an inner product is called an **inner product space**.*

**Definition 4.2.** *Let  $V$  be an inner product space.*

- (i) *For  $\mathbf{v}$  in  $V$ , the **norm** (or **length**) of  $\mathbf{v}$  is defined by  $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ .*
- (ii) *For  $\mathbf{u}, \mathbf{v}$  in  $V$ , the **distance** between  $\mathbf{u}$  and  $\mathbf{v}$  is  $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$ .*
- (iii) *A **unit vector** is a vector of norm 1.*
- (iv) *The set of all unit vectors in  $V$  is called the **unit circle** of  $V$ .*

**Definition 4.3.** *Let  $V$  be an inner product space. Vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V$  are **orthogonal** if  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .*

**Definition 4.4.** *A set  $S$  of two or more vectors in an inner product space is said to be **orthogonal** if every two distinct vectors in  $S$  are orthogonal. The set  $S$  is **orthonormal** if  $S$  is orthogonal and consists entirely of unit vectors.*

**Definition 4.5.** *Let  $V$  be an inner product space and let  $W$  be a subspace of  $V$ . The **orthogonal complement** of  $W$ , denoted  $W^\perp$ , is the set of vectors of  $V$  which are orthogonal to all vectors in  $W$ .*

**Theorem 4.1.** *Let  $V$  be an inner product space. Then*

- (i)  $\langle \mathbf{v}, \mathbf{0} \rangle = 0$  and  $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ , for every  $\mathbf{v}$  in  $V$ .
- (ii)  $\langle c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \mathbf{w} \rangle = c_1\langle \mathbf{v}_1, \mathbf{w} \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{w} \rangle$ , for all scalars  $c_1, \dots, c_n$  and vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}$ .

**Theorem 4.2.** *Let  $V$  be an inner product space. Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be in  $V$  and let  $c$  be a scalar. Then*

- (i)  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ .
- (ii)  $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ .

**Theorem 4.3.** *Let  $V$  be an inner product space. Let  $\mathbf{v}$  be in  $V$  and let  $c$  be a scalar. Then*

- (i)  $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ .
- (ii)  $\frac{\mathbf{v}}{\|\mathbf{v}\|}$  is a unit vector, if  $\mathbf{v} \neq \mathbf{0}$ .

**Theorem 4.4.** If  $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  is an orthogonal set of nonzero vectors in an inner product space, then  $S$  is linearly independent.

**Theorem 4.5.** Let  $V$  be an inner product space, and let  $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthogonal basis for  $V$ . Then for  $\mathbf{u}$  in  $V$ ,  $\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ , where

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \quad \text{for } i = 1, \dots, n.$$

**Theorem 4.6.** Let  $W$  be a subspace of an inner product space  $V$ . Then

- (i)  $W^\perp$  is a subspace of  $V$ ; and
- (ii)  $W \cap W^\perp = \{\mathbf{0}\}$ .

**Theorem 4.7.** Let  $\mathbf{u}, \mathbf{v}$  be vectors in an inner product space  $V$  with  $\mathbf{v} \neq \mathbf{0}$ . Let  $L = \text{span}\{\mathbf{v}\}$ , a one-dimensional subspace of  $V$ . Then we can uniquely write  $\mathbf{u} = \mathbf{y} + \mathbf{z}$ , with  $\mathbf{y}$  in  $L$  and  $\mathbf{z}$  in  $L^\perp$ . Explicitly,

$$\mathbf{y} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}, \quad \text{and } \mathbf{z} = \mathbf{u} - \mathbf{y}.$$

The vector  $\mathbf{y}$  is called the **orthogonal projection of  $\mathbf{u}$  onto  $L$**  and denoted by  $\text{proj}_L \mathbf{u}$  or  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .

**Theorem 4.8.** Let  $\mathbf{u}$  be a nonzero vector in an inner product space  $V$ , and let  $W$  be a finite dimensional subspace of  $V$ . Then we can uniquely write  $\mathbf{u} = \mathbf{y} + \mathbf{z}$ , with  $\mathbf{y}$  in  $W$  and  $\mathbf{z}$  in  $W^\perp$ . The vector  $\mathbf{y}$  is called the **orthogonal projection of  $\mathbf{u}$  onto  $W$**  and denoted by  $\text{proj}_W \mathbf{u}$ , and  $\mathbf{z}$  is called the **component of  $\mathbf{u}$  orthogonal to  $W$** .

## 5. DETERMINANTS

**Theorem 5.1.** Let  $A$  be a square matrix.

- (i) If two rows of  $A$  are interchanged to produce a matrix  $B$ , then  $\det B = -\det A$ .
- (ii) If one row of  $A$  is multiplied by a constant  $k$  to produce  $B$ , then  $\det B = k \det A$ .
- (iii) If a multiple of one row of  $A$  is added to another row to produce  $B$ , then  $\det B = \det A$ .

**Theorem 5.2.** Let  $A$  be a square matrix. Then  $A$  is invertible if and only if  $\det A \neq 0$ .

## 6. EIGENVECTORS AND EIGENVALUES

**Definition 6.1.** Let  $A$  be an  $n \times n$  matrix. An **eigenvector** of  $A$  is a nonzero vector  $\mathbf{x}$  such that  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . The scalar  $\lambda$  is called the **eigenvalue** corresponding to  $\mathbf{v}$ .

**Theorem 6.1.** Let  $A$  be an  $n \times n$  matrix. Then  $\lambda$  is an eigenvalue of  $A$  if and only if  $\det(\lambda I_n - A) = 0$ .

**Definition 6.2.** Let  $A$  be an  $n \times n$  matrix. The **characteristic polynomial** of  $A$  is  $\det(\lambda I_n - A)$ .

**Theorem 6.2.** Let  $A$  be an  $n \times n$  matrix, and let  $\lambda$  be an eigenvalue of  $A$ . Then  $\mathbf{x}$  is an eigenvector of  $A$  corresponding to  $\lambda$  if and only if  $\mathbf{x} \neq \mathbf{0}$  and  $\mathbf{x}$  is in  $\text{Nul}(\lambda I_n - A)$ .

**Definition 6.3.** Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be an eigenvalue of  $A$ . Then  $\text{Nul}(\lambda I_n - A)$  is called the **eigenspace** of  $A$  corresponding to  $\lambda$  (or sometimes just the  $\lambda$ -**eigenspace** of  $A$ ).

**Theorem 6.3.** Let  $A$  be an  $n \times n$  matrix, and let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the matrix transformation  $T(\mathbf{x}) = A\mathbf{x}$ . Suppose that  $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $\mathbb{R}^n$  consisting of eigenvectors for  $A$  (i.e., an **eigenbasis** for  $A$ ). Suppose that the eigenvalues of  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are  $\lambda_1, \dots, \lambda_n$ . Then  $[T]_B$  is the following diagonal matrix:

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

If  $B'$  is the standard basis for  $\mathbb{R}^n$ , then by the change of basis theorem,  $[T]_{B'} = [I]_{B',B}[T]_B[I]_{B,B'}$ . This is often written  $A = PDP^{-1}$ .

**Definition 6.4.** An  $n \times n$  matrix is said to be **diagonalizable** if it has an eigenbasis, i.e., a basis for  $\mathbb{R}^n$  consisting of eigenvectors for  $A$ .

**Theorem 6.4.** Let  $A$  be an  $n \times n$  matrix. If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$ , and if  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are corresponding eigenvectors, then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is linearly independent.

**Definition 6.5.** Let  $\lambda$  be an eigenvalue of  $A$ .

- (i) The **algebraic multiplicity** of  $\lambda$  is the multiplicity of  $A$  as a zero of the characteristic polynomial of  $A$ .
- (ii) The **geometric multiplicity** of  $\lambda$  is the dimension of the  $\lambda$  eigenspace of  $A$ .

**Theorem 6.5.** Let  $A$  be an  $n \times n$  matrix, and let  $\lambda_1, \dots, \lambda_k$  be the distinct eigenvalues of  $A$ .

- (i) The geometric multiplicity of any eigenvalue is less than or equal to its algebraic multiplicity.
- (ii)  $A$  is diagonalizable if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.