Homework 11: Metric Spaces, Limits of Functions (Due April 4, 2022)

*Assignments should be **stapled** and written clearly and legibly. For problems 4 and 6, you must use the $\epsilon - \delta$ definition of a limit. Problems 3 and 7 are optional.*

1. Let X be a nonempty set and let d be the discrete metric on X. Prove that every subset of X is both open and closed.

2. Consider $\mathbb{R}$ with the discrete metric. Prove that $E = [0, 1]$ is closed and bounded in $\mathbb{R}$, but not compact. (Note that closed, bounded, and compact are in reference to the discrete metric.)

3. (GRE Mathematics Subject Test. This question was answered correctly by 19% of examinees.) Let d be a metric on a set X. Which of the following is also a metric on X?
   (a) $4 + d$
   (b) $e^d - 1$
   (c) $d - |d|$
   (d) d^2
   (e) $\sqrt{d}$

4. §5.1, #7(a), #13 (the Squeeze Theorem).

5. Use the Squeeze Theorem to prove that $\lim_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right) = 0$. Make sure to state what D is.

6. Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, and let a be a limit point of D. Suppose that $\lim_{x\to a} f(x) > 0$. Prove that there exists a deleted neighborhood $N^*_\delta(a)$ of a such that $f(x) > 0$ for all $x \in N^*_\delta(a) \cap D$.

7. Consider $\mathbb{Q}$, viewed as a metric subspace of $\mathbb{R}$ with Euclidean metric. Let $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$. Prove that in this metric subspace, E is closed and bounded, but not compact.