Homework 15: Components and Total Disconnectedness

Instructions. Assignments should be ***stapled****. In all proofs, state all hypotheses you are using and fully justify all assertions.*

1. Let X be a topological space. Suppose that A is an open and closed subset of X. Prove that A is the union of components of X.

2. (a) Give all continuous functions $f : \mathbb{R} \to \mathbb{R}_l$. Prove your answer.
   Hint: Consider the components of $\mathbb{R}$ and $\mathbb{R}_l$.

   (b) Use (a) to prove $\mathbb{R} \ncong \mathbb{R}_l$.

3. Let X be a topological space. Define an equivalence relation on X by $x \sim y$ if there exists a connected subset of X containing both x and y. In class we verified that $\sim$ is an equivalence relation and defined the components of X to be the equivalence classes. Prove that $X/\sim$ with quotient topology is totally disconnected.

4. Prove that the complement of a dense open subset of $[0, 1]$ is totally disconnected.

5. Give an explicit homeomorphism $f : \mathring{B}^2 \to \mathbb{R}^2$, and give its inverse function explicitly (see top of page 17 for the definition of $\mathring{B}^2$). You need not prove that f or its inverse function are continuous. However, you must explain why both f and its inverse function are well defined, and verify that they are inverse to eachother. This problem shows that "boundedness" is not a topological property.