Homework 3: Vector Spaces

1. §4.1, #2, 24

2. Let V be the set of all matrices of real numbers with one column and two rows, with addition and scalar multiplication defined as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 v_2 \end{bmatrix} \qquad \text{and} \qquad c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ u_2 \end{bmatrix}$$

   Determine whether V is a vector space. If not, identify all vector space axioms which fail, and briefly explain why each of these axioms fail. If axiom 3 holds, then give an explicit zero vector.

3. Let V be a vector space. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in V, and let b and c be scalars. Using only the definition of a vector space, prove
   (a) $c\,\mathbf{0} = \mathbf{0}$
   (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$
   (c) $(b + c)(\mathbf{u} + \mathbf{v}) = (c\mathbf{v} + b\mathbf{u}) + (b\mathbf{v} + c\mathbf{u})$

4. (Challenge) Prove that there does not exist a vector space which contains exactly two elements.

5. (Challenge) Is it possible for a vector space other than the zero vector space to have a finite number of elements? Prove your answer.