TANGENT SPACES AND T-INVARIANT CURVES OF SCHUBERT VARIETIES

WILLIAM GRAHAM AND VICTOR KREIMAN

Abstract. The set of T-invariant curves in a Schubert variety through a T-fixed point is relatively easy to characterize in terms of its weights, but the tangent space is more difficult. We prove that the weights of the tangent space are contained in the rational cone generated by the weights of the T-invariant curves. In simply laced types, this remains true if "rational" is replaced by "integral". We also obtain conditions under which every weight of the tangent space is the weight of a T-invariant curve, as well as a smoothness criterion. The results rely on equivariant K-theory, as well as the study of different notions of decomposability of roots.

Contents

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1. Introduction

Let G be a semisimple algebraic group over an algebraically closed field of characteristic zero. Let $B \supset T$ be a Borel subgroup and maximal torus of G, respectively, and let P be a parabolic subroup containing B. Let X be a Schubert variety in the generalized flag variety G/P, by which we mean the closure of an orbit of the Borel subgroup B^- opposite to B, and let x be a T-fixed point of X. In this paper we study two spaces: the tangent space to X at x, denoted T_xX, and the span of tangent lines to T-invariant curves through x, denoted TE_xX. These spaces are multiplicity-free representations of T, and so are characterized by their sets of weights, which we denote by $\Phi_{\tan}$ and Φ_{cur} respectively.

In [LS84], Lakshmibai and Seshadri obtained a formula for $\Phi_{\tan}$ in type A and gave criteria based on this formula for X to be smooth at x. Lakshmibai then gave detailed formulas for $\Phi_{\tan}$ in all classical types [Lak95], [Lak00a], and [Lak00b]. Polo gave a representation-theoretic description of $\Phi_{\tan}$ in [Pol94]. See also [BL00, Chapter 5]. In [Car94], Carrell and Peterson gave a formula for Φ_{cur} and discovered a test for determining whether X is rationally smooth at x.

The Carrell-Peterson formula for Φ_{cur} has certain advantages: it holds in all types, is type-independent and relatively simple, and has clear connections to combinatorics. The purpose of this paper is to study the relationship between $\Phi_{\tan}$ and Φ_{cur}, with an eye toward replacing $\Phi_{\tan}$ by the computationally simpler Φ_{cur} in certain applications. It is known that $\Phi_{\text{cur}} \subseteq \Phi_{\tan}$, with equality in type A. Our main result is the following:

Theorem 1.1. $\Phi_{\tan} \subseteq \operatorname{Cone}_A \Phi_{\text{cur}}$.

In this theorem and throughout this section $A = \mathbb{Q}$; in simply laced types all results hold for $A = \mathbb{Z}$ as well. By $\operatorname{Cone}_A \Phi_{\text{cur}}$, we mean the set of all nonnegative A-linear combinations of elements of Φ_{cur}. An alternative proof of this result for $A = \mathbb{Q}$, suggested by the referee (who attributed it to "folklore"), is given in Remark 9.5.

For several special cases, Theorem 1.1 is known. These are discussed in the paragraph following Theorem 1.2. For the remaining cases, however, Theorem 1.1 is new, even in classical types (besides type A). Indeed, Lakshmibai's formulas for the tangent spaces are very complicated, and it is not obvious how to use them to deduce Theorem 1.1.

Theorem 1.1 is equivalent to the assertion that $\Phi_{\tan}$ and Φ_{cur} generate the same cone C over A. This in turn is equivalent to the assertion that $\Phi_{\tan}$ and Φ_{cur} have the same A-indecomposable elements, where an element of a set is A-indecomposable if it cannot be written as a positive A-linear combination of other elements of the set. In the case $A = \mathbb{Q}$, such indecomposable elements correspond to the edges of C. More precisely, each indecomposable element lies on one edge, and each edge contains one indecomposable element.

Under certain conditions, Theorem 1.1 can be strengthened. The T-fixed point x can be represented by a Weyl group element, which, by abuse of notation, we denote by x as well.

Theorem 1.2. *Suppose that G is simply laced and that (i) G/P is cominuscule; or (ii) x is a cominuscule Weyl group element in the sense of Peterson; or (iii) G is of type D and x is fully commutative. Then $\Phi_{\tan} = \Phi_{\mathrm{cur}}$.*

For minuscule G/P, the conclusion of Theorem 1.2 follows from [Pol94, Corollary 4.3] (attributed there to [LS84]). Note that the list of minuscule and cominuscule G/P differ only in types B_n and C_n (see [BL00, 2.11.15, 9.0.14]). By [Car97, Corollary 2], the conclusion of the theorem also holds in type C_n when $x = w_0$.

Part (iii) of Theorem 1.2 is a generalization in type D of (ii), since cominuscule Weyl group elements are fully commutative, but Stembridge (see [Ste01]) provides an example in type D of a fully commutative element which is not cominuscule. See Remark 7.5 below.

As an application of Theorem 1.2, a smoothness criterion can be deduced. The Schubert variety X is defined by a Weyl group element w such that $w \leq x$ in the Bruhat-Chevalley order. Thus, any reduced expression $\mathbf{s}$ for x contains a reduced subexpression for w.

Theorem 1.3. *Suppose that G is simply laced and any of the three conditions of Theorem 1.2 are satisfied. Let $\mathbf{s}$ be any reduced decomposition for x. Then X is smooth at x if and only if $\mathbf{s}$ contains a unique reduced subexpression for w.*

It can be deduced from [GK15, Corollary 2.11] that this criterion for smoothness in fact holds whenever G/P is cominuscule, even without the simply laced requirement. Further discussion of the criterion appears in [GK24].

In [Car97], Carrell obtained results which are related to those of this paper. Let us denote the linear span of the reduced tangent cone to X at x by TR_xX. Then

$$TE_xX \subseteq TR_xX \subseteq T_xX. \tag{1.1}$$

Clearly, in all cases discussed above for which $\Phi_{\tan} = \Phi_{\mathrm{cur}}$, the three spaces of (1.1) are equal. The space TR_xX, like TC_xX and T_xX, is a representation of T, and so is characterized by its set of weights. In [Car97, Theorem 2], Carrell proved that this set of weights is contained in the set of roots in the real convex hull of Φ_{cur}, with equality in simply laced types.

In addition to the results by Lakshmibai-Seshadri and Carroll-Peterson discussed above, a number of other papers have studied smoothness and rational smoothness of Schubert varieties. These include Lakshmibai-Sandhya [LS90], Kumar [Kum96], Billey [Bil98], Lakshmibai-Littelmann-Magyar [LLM98], Brion [Bri99], Billey-Warrington

[BW03], Boe-Graham [BG03], Carrell-Kuttler [CK03], Gaussent [Gau03], and Kassel-Lascoux-Reutenauer [KLR03]. We refer the reader to [BL00, Chapters 6 and 8] for a detailed discussion of this topic.

Another application of Theorem 1.1 appears in [GK24], which studies multiplicities of singular points of Schubert varieties.

1.1. **Outline of the proof of Theorem 1.1.** This theorem builds on the work of [GK22] connecting Demazure products with weights of tangent spaces. Here, that connection appears as Theorem 1.4, which is proved using the methods of [GK22]. The theorem of Carrell and Peterson describing Φ_{cur} ([Car94]) also plays a key role. The proof of Theorem 1.1 also requires a detailed study of decomposability of roots. An important new ingredient is the notion of iso-decomposability, which, as shown by Theorem 4.11, appears naturally in the study of inversion sets of Weyl group elements.

In order to prove Theorem 1.1, which concerns the Schubert variety X, we first prove an analogous but stronger result for the Kazhdan-Lusztig variety $Y \subseteq X$ with T-fixed point x. The reason we work with Y rather than X is that the tangent space to Y at x lives in an ambient space with weights $I(x^{-1})$, the inversion set of x^{-1}, and algebraic properties related to $I(x^{-1})$ are essential to our proof.

Fix a reduced expression $\mathbf{s} = (s_1, \ldots, s_l)$ for x. It is known that the roots of $I(x^{-1})$ can be enumerated explicitly by the formula $\gamma_i = s_1 \cdots s_{i-1}(\alpha_i)$, $i = 1, \ldots, l$, where α_i is the simple root corresponding to s_i. Based on this same expression $\mathbf{s}$, define x_i and z_i to be the ordinary and Demazure products respectively of $(s_1, \ldots, \widehat{s}_i, \ldots, s_l)$ (see Section 4.1 for the formal definition of z_i). Denote the weights of the tangent space to Y at x by $\Phi_{\text{tan}}^{\text{KL}}$, and the weights of the tangent lines to T-invariant curves of Y through x by $\Phi_{\text{cur}}^{\text{KL}}$. It follows easily from a result of Carrell and Peterson ([Car94]) that $\text{Cone}_A\, \Phi_{\text{cur}}^{\text{KL}} = \text{Cone}_A\{\gamma_i \mid x_i \geq w\}$ (indeed, this result holds before taking cones). For $\Phi_{\text{tan}}^{\text{KL}}$, we have the following result, which follows from the methods of [GK22].

Theorem 1.4. $\Phi_{\text{tan}}^{\text{KL}} \subseteq \text{Cone}_A\{\gamma_i \mid z_i \geq w\}$.

See Proposition 9.1. The main result of [GK22] is that if γ_i is an integrally indecomposable element of $I(x^{-1})$, then γ_i is in $\Phi_{\text{tan}}^{\text{KL}}$ if and only if $z_i \geq w$. Theorem 1.4 is a related result which applies to all elements of $\Phi_{\text{tan}}^{\text{KL}}$.

The following equality of cones, which appears in the body of the paper as Corollary 5.5, is our main technical result.

Theorem 1.5. $\text{Cone}_A\{\gamma_i \mid z_i \geq w\} = \text{Cone}_A\{\gamma_i \mid x_i \geq w\}$.

The proof of Theorem 1.5 is taken up in Sections 3 - 5. Before discussing this proof, we point out that Theorems 1.4 and 1.5 together clearly imply $\Phi_{\text{tan}}^{\text{KL}} \subseteq \text{Cone}_A\, \Phi_{\text{cur}}^{\text{KL}}$. This statement is stronger than Theorem 1.1. Indeed, a neighborhood of x in X is isomorphic to the product Y by the tangent space to the B-orbit at x, which is a representation of T. Denoting the weights of this representation by Φ', we have $\Phi_{\tan} = \Phi_{\tan}^{\mathrm{KL}} \sqcup \Phi'$ and $\Phi_{\mathrm{cur}} = \Phi_{\mathrm{cur}}^{\mathrm{KL}} \sqcup \Phi'$. Thus $\Phi_{\tan}^{\mathrm{KL}} \subseteq \mathrm{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}}$ implies $\Phi_{\tan} \subseteq \mathrm{Cone}_A \Phi_{\mathrm{cur}}$.

The proof of Theorem 1.5 entails establishing various relationships among cones, indecomposability, 0-Hecke algebras, and Weyl groups. The inclusion $\mathrm{Cone}_A\{\gamma_i \mid x_i \geq w\} \subseteq \mathrm{Cone}_A\{\gamma_i \mid z_i \geq w\}$ follows from the fact that $x_i \leq z_i$ for all i. In order to prove the other inclusion, we introduce two types of indecomposable elements: *increasing A-indecomposable* and *iso-indecomposable*. Their definitions are deferred to Section 2. In Sections 3, 5, and 4 respectively, we prove:

(i) Every element of $\{\gamma_i \mid z_i \geq w\}$ is a positive A-linear combination of increasing A-indecomposable elements which lie in $\{\gamma_i \mid z_i \geq w\}$ (Corollary 3.8).
(ii) Increasing A-indecomposable elements are iso-indecomposable (Proposition 5.3).
(iii) Iso-indecomposable elements γ_i satisfy $z_i = x_i$ (Corollary 4.12).

Together these statements imply that every element of $\{\gamma_i \mid z_i \geq w\}$ is a positive A-linear combination of elements which lie in $\{\gamma_i \mid x_i \geq w\}$. Thus $\{\gamma_i \mid z_i \geq w\} \subseteq \mathrm{Cone}_A\{\gamma_i \mid x_i \geq w\}$, completing the proof of Theorem 1.5.

The proof of Theorem 1.1 does not rely on Sections 6 and 7. The main result of these sections, Theorem 6.1, is that for inversion sets in classical and type G_2 root systems, the notions of rational indecomposability and iso-indecomposability are equivalent, and in simply laced types are equivalent to integral indecomposability. This result, which we view as of independent interest, has a number of consequences, and is used in the proofs of Theorems 1.2 and 1.3.

1.2. **Organization of the paper.** In Section 2 we define various notions of decomposition and indecomposability in *root sets*, where a root set is a generalization of the set of positive roots of a root system. Sections 3 - 7 mainly address general root sets and the root set $I(x^{-1})$. The purpose of Sections 3, 4, and 5, as discussed above, is to prove Theorem 1.5, the main technical result needed for the proof of Theorem 1.1. The purpose of Sections 6 and 7 is to establish further properties of indecomposability in $I(x^{-1})$ needed in later sections to prove Theorems 1.2 and 1.3 respectively. Specifically, in Section 6, we show that various types of indecomposability in $I(x^{-1})$ are equivalent in classical types and type G_2; in Section 7, we examine conditions under which all elements of a root set are indecomposable.

In Sections 8 - 10, we narrow our focus to the root sets $\Phi_{\mathrm{cur}}, \Phi_{\tan} \subseteq I(x^{-1})$. In Section 8, we review known properties of these two roots sets together with some known facts about Schubert varieties, Kazhdan-Lusztig varieties, and T-invariant curves. In Section 9, Theorem 1.1 is proved, and in Section 10, Theorems 1.2 and 1.3 are proved. Finally, Section 11 contains examples: we apply Theorem 1.1 to study tangent spaces of singular three-dimensional Schubert varieties, and verify Theorem 1.1 by direct calculation for a family of examples in type D_n.

1.3. **Notation.** We introduce some notation related to algebraic groups, which will be used throughout the paper. Let G be a semisimple Lie group defined over an algebraically closed field of characteristic 0, and let $B \supset T$ denote a Borel subgroup and maximal torus of G, respectively. If V is a representation of T, we write $\Phi(V)$ for the set of T-weights of V. If N is a unipotent subgroup of G normalized by T, then as a T-space, N is isomorphic to its Lie algebra $\mathfrak{n}$, which is a representation of T. We write $\Phi(N)$ for $\Phi(\mathfrak{n})$, and refer to this set as the weights of N.

Let Φ be the set of roots of G relative to T. Let Φ^+ and Φ^- be the sets of positive and negative roots, chosen so that root spaces corresponding to positive roots are contained in Lie(B). Let B^- be the opposite Borel subgroup to B.

Let $W = N_G(T)/T$, the Weyl group of G. We will use the same letter to denote an element of W and a lift of this element to G. We denote by s_α the reflection in W corresponding to $\alpha \in \Phi$. Let S' be the set of simple reflections in W relative to B. The length of $w \in W$ is denoted by $\ell(w)$; the longest element in W is w_0.

Let P be a parabolic subgroup containing B, and let P^- be the opposite parabolic subgroup to P. Let $L = P \cap P^-$ be the Levi subgroup of P containing T, and let $W_P = N_L(T)/T$, the Weyl group of L. Each coset uW_P in W/W_P contains a unique representative of minimal length; denote the set of minimal length coset representatives by $W^P \subseteq W$. When $P = B$, we have $L = T$, and $W^P = W$.

The torus acts on the generalized flag variety G/P by left translation. The T-fixed points are the cosets wP for $w \in W^P$. The Schubert varieties we consider are closures of B^--orbits: X_0^w is the Schubert cell $B^- \cdot wP$, and its closure is the Schubert variety X^w. The closures of B-orbits will be referred to as opposite Schubert varieties; in particular, $X_w^0 = B \cdot wP$ denotes the opposite Schubert cell at wP, and X_w its closure. The relation between the two types of Schubert varieties is that

$$w_0 X_{w_0 w} = X^w \text{ and } w_0 \Phi(T_{w_0 x} X_{w_0 w}) = \Phi(T_x X^w). \tag{1.2}$$

This relation can be used to translate results from one type of Schubert variety to the other.

We have $X^x \subseteq X^w$ if and only if $x \geq w$ in the Bruhat-Chevalley order (see [BL00, Section 2.7]). In particular, $X^e = G/P$ and X^{w_0} is the B-fixed point. The T-fixed points of X^w are the xP with $x \geq w$ in the Bruhat order.

Let U, U^-, and U_P^- be the unipotent radicals of B, B^-, and P^- respectively. Define $U^-(x) = xU^-x^{-1}$ and $U_P^-(x) = xU_P^-x^{-1}$. The subgroups U, U^-, U_P^-, $U^-(x)$, and $U_P^-(x)$ are unipotent. The weights of the first three of these under the T action are denoted by Φ^+, Φ^-, and Φ_P^- respectively; the weights of the last two are then $x\Phi^-$ and $x\Phi_P^-$ respectively.

2. Indecomposability in root sets: definitions and basic properties

As discussed briefly in Section 1, there is a connection between cones and indecomposability. This connection is explored in Section 3. In this section we focus on indecomposability. We introduce various types of indecomposability and prove some of their basic properties. In order to study indecomposability in a general framework, we introduce the notion of a *root set*.

2.1. Indecomposability definitions.

Let M be a lattice which is isomorphic to $\mathbb{Z}^n$. Let $V_{\mathbb{R}}$ be the associated real vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $V_{\mathbb{Q}}$ the rational subspace $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall that a subset S of $V_{\mathbb{R}}$ is contained in an open half-space of $V_{\mathbb{R}}$ if there is a positive definite inner product $\langle\, , \rangle$ on $V_{\mathbb{R}}$ and an element δ in $V_{\mathbb{R}}$ such that $\langle \delta, \alpha \rangle > 0$ for all $\alpha \in S$. We define a **root set** to be a finite subset S of M such that S is contained in an open half-space of $V_{\mathbb{R}}$ and does not contain both α and $c\alpha$ for a scalar $c \neq 1$. An element of a root set is often referred to as a **root**. A **weight** on a root set is a map $z\colon S \to W$, where W is a partially ordered set. If W consists of a single element and z is the constant map, then the weight is said to be **trivial**. A root set with a weight is called a **weighted root set**. Any subset of a root set is a root set, and any subset of a weighted root set is a weighted root set. In our applications, M will be the root lattice of a semisimple Lie algebra over an algebraically closed field of characteristic 0, S a set of positive roots, and W the Weyl group of the Lie algebra, equipped with the Bruhat-Chevalley order.

Definition 2.1. Let S be a root set. Let $A \subseteq \mathbb{R}$ and $E \subseteq S$. A linear combination $\alpha = \sum c_i \alpha_i$, with $c_i \in \mathbb{R}_{\geq 0}$ and $\alpha, \alpha_i \in S$, is said to be

- an **A-linear combination** if $c_i \in A$ for all i,
- **in E** or **by elements of E** if $\alpha_i \in E$ for all i,
- a **decomposition** if $\alpha_i \neq \alpha$ for all i,
- an **A-decomposition** if $c_i \in A$ and $\alpha_i \neq \alpha$ for all i,
- an **iso-decomposition** if it is a $\mathbb{Q}$-decomposition of the form $\alpha = c\alpha_1 + c\alpha_2$ with $\|\alpha_1\| = \|\alpha_2\|$,
- **increasing** if S is weighted and $z(\alpha_i) \geq z(\alpha)$ for all i.

Elements of S for which there exists no decomposition are said to be **indecomposable**. Elements $\alpha \in S$ for which there exists no A-decomposition (resp. iso-decomposition, increasing A decomposition) are said to be ***A*-indecomposable** (resp. **iso-indecomposable**, **increasing *A*-indecomposable**); the set of all such α is denoted by S^A (resp. $S^{\ddagger}$, $S^{\uparrow A}$).

When referring to A-linear combinations, A-decompositions, or A-indecomposability, the term **rational** or **integral** is often substituted for A when $A = \mathbb{Q}$ or $A = \mathbb{Z}$ respectively.

$$\begin{tikzcd}
 & S^{\uparrow\mathbb{Q}} \arrow[dr] & \\
S^{\mathbb{Q}} \arrow[ur] \arrow[r] \arrow[dr] & S^{\ddagger} & S^{\uparrow\mathbb{Z}} \\
 & S^{\mathbb{Z}} \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. Relationships among five types of indecomposability. Arrows represent inclusions.

Let S be a root set and $A \subseteq \mathbb{R}$. Define

$$\operatorname{Cone}_A S = \left\{ \sum c_i\alpha_i : c_i \in A_{\geq 0}, \alpha_i \in S \right\} \subseteq V_{\mathbb{R}}.$$

The set S is said to generate $\operatorname{Cone}_A S$.

2.2. **Basic properties of A-indecomposability.**

Lemma 2.2. *Let E and F be subsets of a root set S, and let $A \subseteq \mathbb{R}$. Then*

(i) $F \cap S^A \subseteq F^A$.
(ii) $F \cap S^A = F^A \cap S^A$.
(iii) *If* $F^A = E^A$, *then* $F \cap S^A = E \cap S^A$.
(iv) $F \cap S^A = (F \cap S^A)^A$.
(v) *If* $F^A = F$ *and* $E \subseteq F$, *then* $E^A = E$.

Proof. (i) If $\alpha \in F$ is A-indecomposable in S, then α is A-indecomposable in the smaller set F.

(ii) Since $F^A \subseteq F$, $F^A \cap S^A \subseteq F \cap S^A$. By (i), $F \cap S^A = (F \cap S^A) \cap S^A \subseteq F^A \cap S^A$.

(iii) By (ii), $F \cap S^A = F^A \cap S^A = E^A \cap S^A = E \cap S^A$.

(iv) Applying (i) twice, $F \cap S^A = (F \cap S^A) \cap F^A \subseteq (F \cap S^A)^A$. The other inclusion is clear.

(v) By (i), $E = E \cap F = E \cap F^A \subseteq E^A$, and the other inclusion is clear. $\square$

Definition 2.3. If S is a root set with weight $z\colon S \to W$, then we define $S_{z \geq w} = \{s \in S \mid z(s) \geq w\}$.

Lemma 2.4. *Let S be a root set with weights $z, x\colon S \to W$, and let $A \subseteq \mathbb{R}$. Then*

(i) $(S^A)_{z\geq w} \subseteq (S_{z\geq w})^A$.
(ii) $(S^A)_{z\geq w} = (S_{z\geq w})^A \cap S^A$.

(iii) *If* $(S_{z\geq w})^A = (S_{x\geq w})^A$, *then* $(S^A)_{z\geq w} = (S^A)_{x\geq w}$.
(iv) $(S^A)_{z\geq w} = ((S^A)_{z\geq w})^A$.

Proof. Noting that $S_{z\geq w} \cap S^A = (S^A)_{z\geq w}$, one obtains (i) - (iv) of this lemma from (i) - (iv) respectively of Lemma 2.2 by setting $E = S_{x\geq w}$ and $F = S_{z\geq w}$. □

Remark 2.5. In this paper, $(S_{z\geq w})^A$ is more important than $(S^A)_{z\geq w}$. The reason is that the elements of $(S_{z\geq w})^A$ are used in decomposing elements of $S_{z\geq w}$. To be precise, in Section 3, we will show that an element of $S_{z\geq w}$ can be written in terms of indecomposable elements of $S_{z\geq w}$—that is, in terms of elements of $(S_{z\geq w})^A$. In Corollary 6.7, we show that if S is an inversion set in a root system of classical type or of type G_2, then $(S^A)_{z\geq w} = (S_{z\geq w})^A$.

3. Decomposing into indecomposables

The base $\Delta \subseteq \Phi^+$ consists of the roots $\gamma \in \Phi^+$ which cannot be expressed as a sum $\gamma = \alpha + \beta$, where $\alpha, \beta \in \Phi^+$. In the literature, such roots γ are said to be *indecomposable*. (This traditional usage differs from our definition of indecomposable in Section 2.) A fundamental property of root systems is that every element of Φ^+ is a positive integer linear combination of roots in Δ. It is well-known that a similar property holds for $S^A \subseteq S$, where $A = \mathbb{Z}$ or $\mathbb{Q}$: every element of S is a positive A-linear combination of roots in S^A (for $A = \mathbb{Q}$, see Remark 3.12).

In this section we show that an analogous property holds for $S^{\uparrow A} \subseteq S$, where S is a weighted root set and $A = \mathbb{Z}$ or $\mathbb{Q}$: every element $\alpha \in S$ is an increasing A-linear combination of roots $\alpha_i \in S^{\uparrow A}$. If $z(\alpha) \geq w$, then, since the linear combination is increasing, $z(\alpha_i) \geq w$ for all i. This proves $S_{z\geq w} \subseteq \mathrm{Cone}_A(S^{\uparrow A} \cap S_{z\geq w})$, the main result we will need from this section.

We first study increasing $\mathbb{Z}$-linear combinations and then the more difficult case of increasing $\mathbb{Q}$-linear combinations. Recall that δ is chosen so that $\langle \delta, \alpha \rangle > 0$ for all $\alpha \in S$.

Proposition 3.1. *Let S be a weighted root set. Then every element of $S \setminus S^{\uparrow\mathbb{Z}}$ has an increasing $\mathbb{Z}$-decomposition by elements of $S^{\uparrow\mathbb{Z}}$.*

Proof. Let $\alpha \in S \setminus S^{\uparrow\mathbb{Z}}$. We can write $\alpha = \sum_i c_i\alpha_i$, where $\alpha_i \in S$ satisfy $z(\alpha_i) \geq z(\alpha)$, and the c_i are nonnegative integers, at least two of which are nonzero. For all i such that $c_i \neq 0$, $\langle \delta, \alpha_i \rangle < \langle \delta, \alpha \rangle$. By induction on $\langle \delta, \cdot \rangle$, if $\alpha_i \notin S^{\uparrow\mathbb{Z}}$, then α_i has an increasing $\mathbb{Z}$ decomposition by elements of $S^{\uparrow\mathbb{Z}}$. We conclude that α has an increasing $\mathbb{Z}$-decomposition by elements of $S^{\uparrow\mathbb{Z}}$. □

The inductive proof above does not extend to the case of increasing $\mathbb{Q}$-decompositions. This is because $\langle \delta, \alpha_i \rangle$ may not be strictly less than $\langle \delta, \alpha \rangle$. Hence the inductive iteration may not terminate, as seen in the following example.

Example 3.2. Suppose Φ is of type B_2, $S = \Phi^+ = \{\epsilon_1, \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\}$, and z is the trivial weight. The element ϵ_1 has a $\mathbb{Q}$-decomposition

$$\epsilon_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2). \tag{3.1}$$

The long root $\epsilon_1 + \epsilon_2$ has a $\mathbb{Q}$-decomposition as a sum of the short roots ϵ_1 and ϵ_2: write this decomposition as $\epsilon_1 + \epsilon_2 = (\epsilon_1) + (\epsilon_2)$. But now we can decompose the summand ϵ_1 as in (3.1). Substituting into (3.1), we obtain

$$\epsilon_1 = \frac{1}{2}\Big(\frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2) + \epsilon_2\Big) + \frac{1}{2}(\epsilon_1 - \epsilon_2).$$

This process can be repeated indefinitely without terminating. Note that in this example, $S^{\uparrow\mathbb{Q}} = \{\epsilon_1 - \epsilon_2, \epsilon_2\}$, and $\epsilon_1 = (\epsilon_1 - \epsilon_2) + \epsilon_2$ is the desired $\mathbb{Q}$-decomposition of ϵ_1 by elements of $S^{\uparrow\mathbb{Q}}$.

Thus, a more complicated approach is required in order to extend Proposition 3.1 to increasing $\mathbb{Q}$-decompositions.

Lemma 3.3. *Let $\alpha_1, \ldots, \alpha_n$ be distinct elements of a weighted root set S. Suppose that $\alpha_k = \sum_{i=1}^n c_i\alpha_i$ with c_i nonnegative rational numbers, $z(\alpha_i) \geq z(\alpha_k)$ whenever $c_i \neq 0$, and $c_j > 0$ for some $j \neq k$. Then there exists an increasing $\mathbb{Q}$-decomposition $\alpha_k = \sum_{i=1}^n d_i\alpha_i$ (with $d_k = 0$).*

Proof. We have

$$(1 - c_k)\alpha_k = \sum_{i \neq k} c_i\alpha_i. \tag{3.2}$$

The right side of (3.2) has a positive inner product with δ; hence so does the left side. This implies that $c_k < 1$. Set $d_i = c_i/(1 - c_k)$ for $i \neq k$, and $d_k = 0$. If $d_i \neq 0$, then $c_i \neq 0$, so $z(\alpha_i) \geq z(\alpha_k)$. Hence $\alpha_k = \sum_{i=1}^n d_i\alpha_i$ is our desired increasing $\mathbb{Q}$-decomposition. $\square$

Lemma 3.4. *Let S be a weighted root set. Suppose $E_1 = E \cup \{\alpha\} \subseteq S$, where $\alpha \notin E$, and suppose that α has an increasing $\mathbb{Q}$-decomposition in E. Then any element of S which has an increasing $\mathbb{Q}$-decomposition in E_1 has an increasing $\mathbb{Q}$-decomposition in E.*

Proof. Since α has an increasing $\mathbb{Q}$-decomposition in E, we see that $|E| \geq 2$. Write $E = \{\alpha_1, \ldots, \alpha_{n-1}\}$, with $n \geq 3$, and let $\alpha_n = \alpha$. Let γ be an element of S with an increasing $\mathbb{Q}$-decomposition in E_1, and let

$$\gamma = \sum_{i=1}^n c_i\alpha_i \tag{3.3}$$

be such an increasing $\mathbb{Q}$-decomposition. If $c_n = 0$ we are done; thus assume $c_n \neq 0$. By hypothesis, there is an increasing $\mathbb{Q}$-decomposition of α_n in E:

$$\alpha_n = \sum_{i=1}^{n-1} d_i \alpha_i \tag{3.4}$$

where we may assume that $d_j \neq 0$ for some j such that $\alpha_j \neq \gamma$. Let $e_i = c_i + c_n d_i$ for $i < n$; then $e_i \geq 0$ and

$$\gamma = \sum_{i=1}^{n-1} e_i \alpha_i. \tag{3.5}$$

Since $c_n \neq 0$ and $d_j \neq 0$, $e_j \neq 0$. We claim that if $e_i > 0$ then $z(\alpha_i) \geq z(\gamma)$. Indeed, if $e_i > 0$, then either $c_i > 0$, in which case the claim follows since (3.3) is a increasing $\mathbb{Q}$-decomposition; or $c_n d_i > 0$, in which case, since both (3.4) and (3.3) are increasing $\mathbb{Q}$-decompositions, we have $z(\alpha_i) \geq z(\alpha_n) \geq z(\gamma)$. This proves the claim. If $\gamma \notin E$, then (3.5) is an increasing $\mathbb{Q}$-decomposition of γ in E and we are done. Therefore, suppose $\gamma \in E$. We claim that if one of the α_i occurring in the summation of (3.5) with $e_i > 0$ is equal to γ, then this summation must have at least three terms with nonzero coefficients. Indeed, suppose that $\alpha_i = \gamma$ and $e_i > 0$. Then $c_i = 0$ (since γ cannot have been one of the terms with nonzero coefficient in the summation of (3.3), by definition of decomposition). Since (3.3) is a decomposition of γ, there must be exist at least two positive c_k in the summation (3.3). At least one of these must satisfy $k < n$, and this yields $e_k > 0$. So there are at least two nonzero terms in the summation (3.5), namely $e_i\alpha_i$ and $e_k\alpha_k$. These cannot be the only two nonzero terms, since $\gamma = \alpha_i$ but α_k is not a multiple of α_i (by the definition of root set). Therefore there must be at least three nonzero terms in (3.5), proving the claim. Now apply Lemma 3.3 to obtain an increasing $\mathbb{Q}$-decomposition of γ in E. □

Lemma 3.5. *If S is a nonempty weighted root set, then $S^{\uparrow\mathbb{Q}}$ is nonempty.*

Proof. We prove the result by induction on $|S|$. If S has one element, then $S = S^{\uparrow\mathbb{Q}}$ and the result holds. For the inductive step, suppose that $S = E \cup \{\alpha\}$, where $|E| \geq 1$ and $\alpha \notin E$. Our inductive hypothesis is that $E^{\uparrow\mathbb{Q}}$ is nonempty. If α is increasing $\mathbb{Q}$-indecomposable in S, then $S^{\uparrow\mathbb{Q}}$ contains α and we are done, so assume that α is increasing $\mathbb{Q}$-decomposable in S. We will show that $S^{\uparrow\mathbb{Q}} = E^{\uparrow\mathbb{Q}}$; this suffices.

Observe that $S^{\uparrow\mathbb{Q}} \subseteq E^{\uparrow\mathbb{Q}}$. This holds because any element of E which is increasing $\mathbb{Q}$-indecomposable in S remains increasing $\mathbb{Q}$-indecomposable in the smaller set E; moreover, α is increasing $\mathbb{Q}$-decomposable in S. For the reverse inclusion $S^{\uparrow\mathbb{Q}} \supseteq E^{\uparrow\mathbb{Q}}$, we require that if $\gamma \in E$ does not have an increasing $\mathbb{Q}$-decomposition in E, then it does not have an increasing $\mathbb{Q}$-decomposition in S. This follows from Lemma 3.4, with $E_1 = S$. □

Lemma 3.6. *Let S be a weighted root set, let E and F be disjoint subsets of S, and let $E_1 = E \cup F$. Suppose that any $\beta \in F$ has an increasing $\mathbb{Q}$-decomposition in E_1.*

Then any $\gamma \in S$ that has an increasing $\mathbb{Q}$-decomposition in E_1 has an increasing $\mathbb{Q}$-decomposition in E.

Proof. We may assume that F is nonempty, since otherwise the lemma is trivial. Thus, by Lemma 3.5, $(E_1)^{\uparrow\mathbb{Q}}$ is nonempty. Since F does not intersect $(E_1)^{\uparrow\mathbb{Q}}$, we must have that $E \supseteq (E_1)^{\uparrow\mathbb{Q}}$, and thus E is nonempty. Let $E = \{\alpha_1, \ldots, \alpha_r\}$ and $F = \{\alpha_{r+1}, \ldots, \alpha_n\}$, where $r \geq 1$ and $n \geq r+1$. Let $C(j)$ be the assertion that α_j has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_{j-1}\}$. We will show that $C(j)$ holds for $j \in \{r+1, \ldots, n\}$.

The assertion $C(n)$ holds by hypothesis. Suppose that $r+1 \leq j < n$ and that $C(j+1), \ldots, C(n)$ hold. We show by contradiction that $C(j)$ holds as well. Assume that it does not. Since $j \geq r+1$, α_j has an increasing decomposition in E_1. Let

$$\alpha_j = \sum_{i=1}^{m} c_i \alpha_i \tag{3.6}$$

be an increasing $\mathbb{Q}$-decomposition for α_j in E_1 with m minimal. Since $C(j)$ does not hold, $m \geq j+1$. Since α_j has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_m\}$ and, by $C(m)$, α_m has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_{m-1}\}$, Lemma 3.4 implies that α_j has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_{m-1}\}$. This contradicts the minimality of m and proves $C(j)$. By induction, $C(j)$ holds for $j \in \{r+1, \ldots, n\}$.

We now complete the proof of the lemma. Suppose that $\gamma \in S$ has an increasing $\mathbb{Q}$-decomposition in $E_1 = \{\alpha_1, \ldots, \alpha_n\}$. Let m be the smallest integer such that γ has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_m\}$. We must show $m \leq r$. If not, then $m \geq r+1$, so by $C(m)$, α_m has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_{m-1}\}$. Lemma 3.4 then implies that γ has an increasing $\mathbb{Q}$-decomposition in $\{\alpha_1, \ldots, \alpha_{m-1}\}$. This contradicts the minimality of m. We conclude that $m \leq r$, as desired. □

Theorem 3.7. *Let S be a weighted root set. Then every element of $S \setminus S^{\uparrow\mathbb{Q}}$ has an increasing $\mathbb{Q}$-decomposition by elements of $S^{\uparrow\mathbb{Q}}$.*

Proof. Every $\gamma \in S \setminus S^{\uparrow\mathbb{Q}}$ has an increasing $\mathbb{Q}$-decomposition in S. Thus, by Lemma 3.6 with $E = S^{\uparrow\mathbb{Q}}$ and $F = S \setminus S^{\uparrow\mathbb{Q}}$, every such γ has an increasing $\mathbb{Q}$ decomposition in $S^{\uparrow\mathbb{Q}}$. □

Corollary 3.8. *Let S be a weighted root set, and let $A = \mathbb{Q}$ or $\mathbb{Z}$. Then $S_{z \geq w} \subseteq \mathrm{Cone}_A((S^{\uparrow A})_{z \geq w})$.*

Proof. Let $\alpha \in S_{z \geq w}$. By Theorem 3.7 and Proposition 3.1, α is a positive A-linear combination of elements $\alpha_i \in S^{\uparrow A}$ such that $z(\alpha_i) \geq z(\alpha)$. Since $z(\alpha) \geq w$, $z(\alpha_i) \geq w$ for all i, and thus each α_i lies in $S_{z \geq w}$. □

Remark 3.9. The reason that we introduce inceasing linear combinations in this paper is that they preserve $S_{z \geq w}$, in the sense that if $\alpha = \sum c_i \alpha_i$ is increasing and $\alpha \in S_{z \geq w}$, then $\alpha_i \in S_{z \geq w}$ for all i. This property is used to prove both the above corollary and Lemma 5.2.

Corollary 3.10. *Let S be a root set, and let $A = \mathbb{Q}$ or $\mathbb{Z}$. Then $S \subseteq \mathrm{Cone}_A(S^A)$, and thus $\mathrm{Cone}_A(S) = \mathrm{Cone}_A(S^A)$.*

Proof. Take z to be the trivial weight in Corollary 3.8. □

Corollary 3.11. *Let E, F be subsets of a root set S. Let $A = \mathbb{Q}$ or $\mathbb{Z}$. Then $\mathrm{Cone}_A(E) = \mathrm{Cone}_A(F)$ if and only if $E^A = F^A$.*

Proof. Assume $\mathrm{Cone}_A(E) = \mathrm{Cone}_A(F)$. Suppose that there exists $\alpha \in E^A \setminus F^A$. Then $\alpha \in \mathrm{Cone}_A(E) = \mathrm{Cone}_A(F) = \mathrm{Cone}_A(F^A)$. Thus $\alpha = \sum_i c_i\alpha_i$, where $\alpha_i \in F^A$, and $c_i \in A_{\geq 0}$, at least two of which are nonzero (otherwise $\alpha \in F^A$). But since $\alpha_i \in F^A \subseteq \mathrm{Cone}_A(E)$, $\alpha \notin E^A$, a contradiction. Thus $E^A \subseteq F^A$, and similarly $F^A \subseteq E^A$.

Conversely, if $E^A = F^A$, then $\mathrm{Cone}_A(E) = \mathrm{Cone}_A(E^A) = \mathrm{Cone}_A(F^A) = \mathrm{Cone}_A(F)$. □

Remark 3.12. Proofs of Corollaries 3.10 and 3.11 for the case $A = \mathbb{R}$ can be obtained by using the fact that if S is a root set, then $\mathrm{Cone}_{\mathbb{R}} S$ is a convex polyhedral cone, and $S^{\mathbb{R}}$ is equal to the set of elements of S which lie on the one-dimensional faces of $\mathrm{Cone}_{\mathbb{R}} S$ (see [Ful93, Section 1.2]). One can then use the density of $\mathbb{Q}$ in $\mathbb{R}$ to obtain alternative proofs of these two corollaries for the case $A = \mathbb{Q}$.

4. Iso-indecomposability in inversion sets

In this section we introduce the notion of iso-decomposability. The main results of this section, Theorem 4.11, and its corollaries, play a major role in this paper.

4.1. **Preliminaries and notation.**

We keep the notation of Section 1.3. For the remainder of this paper we limit attention to root sets $S \subseteq \Phi$; our convention will be that any statement involving A holds for $A = \mathbb{Q}$, and if Φ is simply laced, it holds for $A = \mathbb{Z}$ as well.

Recall that W denotes the Weyl group and S' the set of simple reflections. The 0-Hecke algebra $\mathcal{H}$ associated to (W, S') over a commutative ring R is the associative R-algebra generated by H_u, $u \in W$, and subject to the following relations: H_1 is the identity element, and if $u \in W$ and $s \in S'$, then $H_uH_s = H_{us}$ if $\ell(us) > \ell(u)$ and $H_uH_s = H_u$ if $\ell(us) < \ell(u)$.

Throughout the paper, we will assume that we have chosen $w \leq x \in W$ and a reduced expression $\mathbf{s} = (s_1, \ldots, s_l)$, $s_i \in S'$, for x. Then x_i, z_i, and γ_i will have the following meaning. For $i \in \{1, \ldots, l\}$, define

- $x_i = s_1 \cdots \widehat{s_i} \cdots s_l = s_{\gamma_i} x \in W$.

- $z_i \in W$ by the equation $H_{z_i} = H_{s_1} \cdots \widehat{H}_{s_i} \cdots H_{s_l}$.
- $\gamma_i = s_1 \cdots s_{i-1}(\alpha_i) \in \Phi$, where α_i is the simple root corresponding to s_i.

The equality $x_i = s_{\gamma_i} x$ is well-known; it follows from the equation $s_{\gamma_i} = (s_1 \cdots s_{i-1}) s_i (s_1 \cdots s_{i-1})^{-1}$. It is known that the elements $\gamma_1, \ldots, \gamma_l$ enumerate $I(x^{-1}) = \{\alpha \in \Phi^+ \mid x^{-1}(\alpha) \in \Phi^-\}$, the inversion set of x^{-1} (see [Hum90, Exercise 5.6.1]).

Let $w \in W$, $w \leq x$. For any root set $S \subseteq I(x^{-1})$, define the **Coxeter weight** $x \colon S \to W$ by $x(\gamma_i) = x_i$, and the **Demazure weight** $z \colon S \to W$ by $z(\gamma_i) = z_i$. Then $S_{x \geq w} = \{\gamma_i \in S \mid x_i \geq w\}$ and $S_{z \geq w} = \{\gamma_i \in S \mid z_i \geq w\}$ (see Definition 2.3).

The main result of Section 4 is that for $S = I(x^{-1})$, γ_i is iso-indecomposable in S if and only if $(s_1, \ldots, \widehat{s}_i, \ldots, s_l)$ is reduced; in this case, $z_i = x_i$. Consequently, $(S^{\ddagger})_{z \geq w} = (S^{\ddagger})_{x \geq w}$.

Remark 4.1. Since $x_i = s_{\gamma_i} x$, the Coxeter weight $x : I(x^{-1}) \to W$, $\gamma_i \mapsto x_i$, is independent of the reduced expression $\mathbf{s}$ for x. On the other hand, the Demazure weight $z : I(x^{-1}) \to W$, $\gamma_i \mapsto z_i$, is not. (For example, let $x = \sigma_1\sigma_2\sigma_1$ in type A_2. For $\mathbf{s} = (\sigma_1, \sigma_2, \sigma_1)$, one checks that $\gamma_2 = \alpha_1 + \alpha_2$ and $z_2 = \sigma_1$; for $\mathbf{s} = (\sigma_2, \sigma_1, \sigma_2)$, we again have $\gamma_2 = \alpha_1 + \alpha_2$, but now $z_2 = \sigma_2$.) The dependence of the Demazure weight on $\mathbf{s}$ can be removed by restricting the domain to the set of iso-indecomposable elements (since $z_i = x_i$ if γ_i is iso-indecomposable).

4.2. **Demazure products.** If $\mathbf{q} = (r_1, \ldots, r_k)$ is any (not necessarily reduced) sequence of simple reflections in S, define the Demazure product[1] $z_{\mathbf{q}} \in W$ by the equation $H_{z_{\mathbf{q}}} = H_{r_1} \cdots H_{r_k}$, and define $x_{\mathbf{q}} = r_1 \cdots r_k$. It is well known that if $\mathbf{q}$ is reduced, then $\mathbf{q}$ contains a subexpression which multiplies to u if and only if $x_{\mathbf{q}} \geq u$ (see Theorem 5.10 of [Hum90]). A generalization of this result in which $\mathbf{q}$ is not required to be reduced is given by [KM04, Lemma 3.4(1)]:

Lemma 4.2. $\mathbf{q}$ *contains a subexpression which multiplies to* $u \Leftrightarrow z_{\mathbf{q}} \geq u$.

Corollary 4.3. *There exists a subexpression of* $\mathbf{q}$ *which is a reduced expression for* $z_{\mathbf{q}}$.

Proof. By Lemma 4.2 with $u = z_{\mathbf{q}}$, $\mathbf{q}$ contains a subexpression which multiplies to $z_{\mathbf{q}}$, and hence it contains a reduced subexpression which multiplies to $z_{\mathbf{q}}$. ☐

Corollary 4.4. *We have*

(i) $z_{\mathbf{q}} \geq x_{\mathbf{q}}$, *with equality if* $\mathbf{q}$ *is reduced.*
(ii) $z_{\mathbf{q}} \geq z_{\mathbf{p}}$ *if* $\mathbf{p}$ *is a subexpression of* $\mathbf{q}$.

Proof. (i) If $\mathbf{q}$ is reduced, then $z_{\mathbf{q}} = x_{\mathbf{q}}$ by definition. The inequality is due to Lemma 4.2 with $u = x_{\mathbf{q}}$.

[1] In [KM04] and [GK22], the Demazure product $z_{\mathbf{q}}$ is instead denoted by $\delta(\mathbf{q})$.

(ii) By Corollary 4.3, there exists a subexpression $\mathbf{p}'$ of $\mathbf{p}$ which is a reduced expression for $z_{\mathbf{p}}$. By (i), $x_{\mathbf{p}'} = z_{\mathbf{p}}$. Since $\mathbf{p}'$ is a subexpression of $\mathbf{p}$, it is a subexpression of $\mathbf{q}$, so by Lemma 4.2, $z_{\mathbf{q}} \geq x_{\mathbf{p}'}$. Hence $z_{\mathbf{q}} \geq z_{\mathbf{p}}$, as required. □

Remark 4.5. In Corollary 4.4(i), equality can occur even if $\mathbf{q}$ is not reduced. For example, if σ_1 and σ_2 denote the transpositions $(1,2)$ and $(2,3)$ respectively in type A_2, then for $\mathbf{q} = (\sigma_1, \sigma_1, \sigma_2, \sigma_1, \sigma_2)$, $z_{\mathbf{q}} = x_{\mathbf{q}} = \sigma_1\sigma_2\sigma_1$, although $\mathbf{q}$ is not reduced.

Corollary 4.6. *$z_i \geq x_i$ for $i \in \{1, \ldots, l\}$, with equality if $(s_1, \ldots, \widehat{s_i}, \ldots, s_l)$ is reduced.*

Proof. This is a special case of Corollary 4.4(i). □

4.3. **Iso-indecomposability in $I(x^{-1})$.**

Lemma 4.7. *Let $\alpha, \beta, \gamma \in \Phi^+$. If $c\alpha = \beta + \gamma$ and $\|\beta\| = \|\gamma\|$, then $c = \langle \beta, \alpha^\vee \rangle = \langle \gamma, \alpha^\vee \rangle > 0$.*

Proof. Since $\alpha, \beta, \gamma > 0$, we must have $c > 0$. By applying $\langle \cdot, \alpha^\vee \rangle$ to both sides of the equation $c\alpha = \beta + \gamma$, we find that $c = (1/2)\langle \beta, \alpha^\vee \rangle + (1/2)\langle \gamma, \alpha^\vee \rangle$. Since $\|\beta\| = \|\gamma\|$,

$$(\beta, c\alpha) = (\beta, \beta + \gamma) = \|\beta\|^2 + (\beta, \gamma) = \|\gamma\|^2 + (\beta, \gamma) = (\gamma, \beta + \gamma) = (\gamma, c\alpha),$$

implying $\langle \beta, \alpha^\vee \rangle = \langle \gamma, \alpha^\vee \rangle$, as desired. □

As a consequence, we obtain the following well-known fact.

Lemma 4.8. *Let $\alpha, \beta, \gamma \in \Phi^+$, where Φ is simply laced. If $c\alpha = \beta + \gamma$ and $\beta \neq \alpha$, then $c = 1$.*

Proof. This follows from Lemma 4.7. □

Proposition 4.9. *If $S \subseteq \Phi^+$ is a root set and Φ is simply laced, then $S^{\mathbb{Z}} \subseteq S^{\ddagger}$.*

Proof. Suppose α is iso-decomposable in S. Then $c\alpha = \beta + \gamma$, for some $\beta, \gamma \in S$. By Lemma 4.8, $c = 1$. Thus α is integrally decomposable in S. □

Lemma 4.10. *Let $i, j, k \in [l]$.*

(i) *$\gamma_i \neq \gamma_j$ if $i \neq j$.*
(ii) *If $j < i$ then $s_j \cdots s_1(\gamma_i) > 0$; otherwise $s_j \cdots s_1(\gamma_i) < 0$.*
(iii) *If $c\gamma_i = \gamma_j + \gamma_k$, $j < k$, then $j < i < k$.*

Proof. (i), (ii) See [Hum90, Section 1.7].

(iii) By Lemma 4.7, $c > 0$. Assume that $i < j$ and $i < k$, and let $y = s_i \cdots s_1$. Then $cy\gamma_i = y\gamma_j + y\gamma_k$. By (ii), $cy\gamma_i < 0$ and $y\gamma_j + y\gamma_k > 0$, contradiction. A similar argument eliminates the possibility that $i > j$ and $i > k$. (This proof also appears in the proof of [Ste01, Theorem 5.3].) □

The following theorem plays a central role in this paper: it is required for all subsequent results of this section and the next. The theorem also illustrates how iso-indecomposability arises naturally in the study of Coxeter groups and inversion sets. Indeed, the problem of finding some property of γ_i which is equivalent to reducedness of $s_1 \cdots \widehat{s}_i \cdots s_l$ initially led us to this theorem and the definition of iso-indecomposability.

Theorem 4.11. *For $i \in [l]$, the following are equivalent:*

(i) $s_1 \cdots \widehat{s}_i \cdots s_l$ *is not reduced.*
(ii) *There exist $j < i < k$ such that $\alpha_j = s_{j+1} \cdots \widehat{s}_i \cdots s_{k-1}(\alpha_k)$.*
(iii) *There exist $j < k$ and $c \in \mathbb{Q}$ such that $c\gamma_i = \gamma_j + \gamma_k$ and $\|\gamma_j\| = \|\gamma_k\|$.*
(iv) γ_i *is iso-decomposable in $I(x^{-1})$.*

Moreover, in case one (and thus all) of these statements hold, it must be true that $j < i < k$; and that j, k satisfy (ii) if and only if they satisfy (iii); and that $c = \langle \gamma_k, \gamma_i^\vee \rangle > 0$.

Proof. (i) $\Leftrightarrow$ (ii) See [Hum90, Theorem 1.7]; (iii) $\Leftrightarrow$ (iv) by definition.

Now suppose $j < i < k$. Let $\beta = s_{i+1} \cdots s_{k-1}(\alpha_k)$. Then $\langle \gamma_k, \gamma_i^\vee \rangle = \langle s_1 \cdots s_i \beta, -s_1 \cdots s_i \alpha_i^\vee \rangle = \langle -\beta, \alpha_i^\vee \rangle$. Thus

$$\begin{aligned} \langle \gamma_k, \gamma_i^\vee \rangle \gamma_i &= \langle -\beta, \alpha_i^\vee \rangle \gamma_i \\ &= s_1 \cdots s_{i-1}(\langle -\beta, \alpha_i^\vee \rangle \alpha_i) \\ &= s_1 \cdots s_{i-1}(-\beta + s_i \beta) = -s_1 \cdots \widehat{s}_i \cdots s_{k-1}(\alpha_k) + \gamma_k. \end{aligned} \tag{4.1}$$

(ii) $\Rightarrow$ (iii) Let $j < i < k$ be as in (ii). Substituting α_j for $s_{j+1} \cdots \widehat{s}_i \cdots s_{k-1}(\alpha_k)$ in (4.1) produces $\langle \gamma_k, \gamma_i^\vee \rangle \gamma_i = \gamma_j + \gamma_k$.

(iii) $\Rightarrow$ (ii) Lemma 4.10(iii) forces $j < i < k$. By Lemma 4.7, $c = \langle \gamma_k, \gamma_i^\vee \rangle$, so $\langle \gamma_k, \gamma_i^\vee \rangle \gamma_i = \gamma_j + \gamma_k$. Substituting this into (4.1), we obtain $\gamma_j = -s_1 \cdots \widehat{s}_i \cdots s_{k-1}(\alpha_k)$. On the other hand, by definition, $\gamma_j = s_1 \cdots s_{j-1}(\alpha_j)$. Equating these two expressions for γ_j and simplifying yields $s_{j+1} \cdots \widehat{s}_i \cdots s_{k-1}(\alpha_k) = \alpha_j$, as required. $\square$

Corollary 4.12. *Let $S = I(x^{-1})$. If $\gamma_i \in S^\ddagger$, then $z_i = x_i$.*

Proof. If $\gamma_i \in S^\ddagger$, then $(s_1, \ldots, \widehat{s}_i, \ldots, s_l)$ is reduced by Theorem 4.11, and thus $z_i = x_i$ by Corollary 4.6. $\square$

Corollary 4.13. *Let $S = I(x^{-1})$. Then $(S^\ddagger)_{z \geq w} = (S^\ddagger)_{x \geq w}$ and $(S^A)_{z \geq w} = (S^A)_{x \geq w}$.*

Proof. The first equation is due to Corollary 4.12. Note that the first equation implies that for any $E \subseteq S^\ddagger$, we have $E_{z \geq w} = E_{x \geq w}$. The second equation now follows by observing that $S^{\mathbb{Q}} \subseteq S^\ddagger$, and if Φ is simply laced, then $S^{\mathbb{Z}} \subseteq S^\ddagger$ by Proposition 4.9. $\square$

Remark 4.14. Corollary 4.13 proves the assertions of [GK22, Remark 5.9].

5. Increasing and iso-indecomposability

In this section we show that for $S = I(x^{-1})$ we have that $S^{\uparrow A} \subseteq S^{\ddagger}$, where increasing A-decompositions are relative to the Demazure weight. Using this and results of previous sections, we prove our main technical result: $\mathrm{Cone}_A(S_{z\geq w}) = \mathrm{Cone}_A(S_{x\geq w})$.

Lemma 5.1. *Let $S = I(x^{-1})$, and let $\gamma \in S$. If there exists an iso-decomposition of γ in S, then there exists an increasing iso-decomposition of γ in S.*

Proof. We have that $\gamma = \gamma_i$ for some i. Assume that γ_i is iso-decomposable. By Theorem 4.11, $s_1 \cdots \widehat{s}_i \cdots s_l$ is not reduced. Choose $k > i$ minimal such that $\ell(s_1 \cdots \widehat{s}_i \cdots s_k) < \ell(s_1 \cdots \widehat{s}_i \cdots s_{k-1})$, and then $j < i$ maximal such that $\ell(s_j \cdots \widehat{s}_i \cdots s_k) < \ell(s_{j+1} \cdots \widehat{s}_i \cdots s_k)$. It is shown in the proof of [Hum90, Theorem 1.7] that j, k satisfy Theorem 4.11(ii); thus they satisfy Theorem 4.11(iii), i.e., $c\gamma_i = \gamma_j + \gamma_k$ with $\|\gamma_j\| = \|\gamma_k\|$. We will show that $z_j, z_k \geq z_i$, thus completing the proof.

By choice of k, $s_1 \cdots \widehat{s}_i \cdots s_{k-1}$ is reduced but $s_1 \cdots \widehat{s}_i \cdots s_k$ is not. Thus

$$H_{s_1} \cdots \widehat{H}_{s_i} \cdots H_{s_{k-1}} = H_{s_1 \cdots \widehat{s}_i \cdots s_{k-1}} = H_{s_1 \cdots \widehat{s}_i \cdots s_{k-1}} H_{s_k} = H_{s_1} \cdots \widehat{H}_{s_i} \cdots H_{s_k}$$

If we multiply on the right by $H_{s_{k+1}} \cdots H_{s_l}$, we obtain $H_{s_1} \cdots \widehat{H}_{s_i} \cdots \widehat{H}_{s_k} \cdots H_{s_l} = H_{s_1} \cdots \widehat{H}_{s_i} \cdots H_{s_l}$. Letting $\mathbf{r} = (s_1, \ldots, \widehat{s}_i, \ldots, \widehat{s}_k, \ldots, s_l)$, we have $z_{\mathbf{r}} = z_i$. Since $\mathbf{r}$ is a subexpression of $(s_1, \ldots, \widehat{s}_k, \ldots, s_l)$, Corollary 4.4(ii) implies $z_k \geq z_{\mathbf{r}}$. Therefore $z_k \geq z_i$.

Using the fact that $s_{j+1} \cdots \widehat{s}_i \cdots s_k$ is reduced but $s_j \cdots \widehat{s}_i \cdots s_k$ is not, a similar argument yields $z_j \geq z_i$. $\square$

Lemma 5.2. *Let $S = I(x^{-1})$, and let $\gamma \in S_{z\geq w}$. If there exists an iso-decomposition of γ in S, then there exists an iso-decomposition of γ in $S_{z\geq w}$.*

Proof. This follows from Lemma 5.1 and the fact that increasing linear combinations preserve $S_{z\geq w}$ (see Remark 3.9). $\square$

Proposition 5.3. *Let $S = I(x^{-1})$. Then $S^{\uparrow A} \subseteq S^{\ddagger}$.*

Proof. Suppose $\gamma_i \notin S^{\ddagger}$. By Lemma 5.1, there exist j, k such that $c\gamma_i = \gamma_j + \gamma_k$ is increasing and $c \in \mathbb{Q}$. If Φ is simply laced, then $c = 1$, by Lemma 4.7. Thus $\gamma_i \notin S^{\uparrow A}$. $\square$

Corollary 5.4. *Let $S = I(x^{-1})$. Then $(S_{z\geq w})^{\ddagger} = (S^{\ddagger})_{z\geq w}$.*

Proof. By definition, $(S^{\ddagger})_{z\geq w} = S^{\ddagger} \cap S_{z\geq w}$. The inclusion $(S_{z\geq w})^{\ddagger} \supseteq (S^{\ddagger})_{z\geq w}$ holds because any element of $S_{z\geq w}$ that is iso-indecomposable in S is iso-indecomposable in the smaller set $S_{z\geq w}$. The reverse inclusion is the contrapositive of Lemma 5.2. $\square$

Corollary 5.5. *Let $S = I(x^{-1})$. Then $\mathrm{Cone}_A(S_{z\geq w}) = \mathrm{Cone}_A(S_{x\geq w})$, or equivalently, $(S_{z\geq w})^A = (S_{x\geq w})^A$.*

Proof. Since $x_i \leq z_i$ for all i, $\mathrm{Cone}_A(S_{x\geq w}) \subseteq \mathrm{Cone}_A(S_{z\geq w})$. The other inclusion follows from $S_{z\geq w} \subseteq \mathrm{Cone}_A((S^{\uparrow A})_{z\geq w}) \subseteq \mathrm{Cone}_A((S^{\ddagger})_{z\geq w}) = \mathrm{Cone}_A((S^{\ddagger})_{x\geq w}) \subseteq \mathrm{Cone}_A(S_{x\geq w})$, where the first and second inclusions are due to Propositions 3.8 and 5.3, and the equality is due to Corollary 4.13. The equivalence of the second equality of this corollary is due to Corollary 3.11. □

Remark 5.6. By Lemma 2.4(iii), Corollary 5.5 is a stronger statement than $(S^A)_{z\geq w} = (S^A)_{x\geq w}$, which was proved in Corollary 4.13.

6. Equivalent indecomposabilities

The main theorem of this section is the following.

Theorem 6.1. *Let Φ be of classical type or of type G_2, and let $S = I(x^{-1})$. Let $\alpha \in S$. Then α is rationally indecomposable $\Leftrightarrow$ α is iso-indecomposable. If Φ is simply laced, these conditions are equivalent to the condition that α is integrally indecomposable.*

A more precise statement, Proposition 6.4, is given in Section 6.1. The special case of $S = \Phi^+$ is studied in Section 6.2.

The equivalence of indecomposabilities given by Theorem 6.1, which we view as of independent interest, has two main applications in this paper. First, it is used in this section to prove that in classical types and type G_2, for $S = I(x^{-1})$, we have $(S^A)_{z\geq w} = (S_{z\geq w})^A$ and $(S^A)_{x\geq w} = (S_{x\geq w})^A$. In Section 9, we show that the first of these equalities implies that in classical types and type G_2, $\Phi^{\mathrm{KL}}_{\tan} \cap S^A = (\Phi^{\mathrm{KL}}_{\tan})^A$. Second, it allows us to prove Corollary 7.4: in types A and D, x is fully commutative if and only if all elements of $I(x^{-1})$ are integrally indecomposable. This leads, in Section 10, to smoothness criteria for fully commutative x in types A and D.

6.1. **Indecomposability in closed subsets of Φ^+.**

It is convenient to introduce the following characterization of inversion sets, which is a slight variation of the characterization given by Papi [Pap94]. We shall say that a root set $S \subseteq \Phi^+$ is *closed* if (i) $\alpha, \beta \in S$ and $r\alpha + s\beta \in \Phi$ for positive real numbers r and s imply $r\alpha + s\beta \in S$, and (ii) $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in S$ imply $\alpha \in S$ or $\beta \in S$. The following lemma is a simple consequence of Papi's characterization.

Lemma 6.2. *Let $S \subseteq \Phi^+$. Then S is closed $\Leftrightarrow$ $S = I(x^{-1})$ for some $x \in W$.*

Proof. Consider the condition (i′): $\alpha, \beta \in S$ and $\alpha + s\beta \in \Phi$ s implies $\alpha + \beta \in S$. Papi proved that S satisfies (i′) and (ii) if and only if S is of the form $I(x^{-1})$.

Suppose S is closed. Since S satisfies (i) and (ii), it satisfies (i′) and (ii), so $S = I(x^{-1})$ for some $x \in W$. Conversely, suppose $S = I(x^{-1})$. The set S satisfies (ii) by Papi's result. If $\alpha, \beta \in S$ and $r\alpha + s\beta \in \Phi$ for positive real numbers r and s, then $x^{-1}\alpha$ and $x^{-1}\beta$ are negative roots, so the root $x^{-1}(r\alpha + s\beta)$ must be negative as well (because when written as a sum of simple roots, all coefficients of $x^{-1}(r\alpha + s\beta)$ are negative). Hence $r\alpha + s\beta \in S$, so S satisfies (i). Hence S is closed. $\square$

Theorem 6.1 is an immediate consequence of Proposition 6.4 below. We divide the proof into smaller steps by introducing a new variation of indecomposability.

Definition 6.3. Let S be a root set. A rational decomposition in S of the form $\alpha = c_1\alpha_1 + c_2\alpha_2$ is called a **bi-decomposition** of α. The element α is called **bi-decomposable** if α has a bi-decomposition and **bi-indecomposable** otherwise.

Proposition 6.4. *Let S be a closed subset of Φ^+ and let $\alpha \in S$. Consider the following conditions:*

(i) *α is iso-decomposable.*
(ii) *α is bi-decomposable.*
(iii) *α is rationally decomposable.*
(iv) *α is integrally decomposable.*

We have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftarrow$ (iv). If Φ is of classical type or of type G_2, then (iii) $\Rightarrow$ (ii) and thus (i) – (iii) are equivalent. If Φ is simply laced, then (i) $\Rightarrow$ (iv). Thus in types A and D, (i) - (iv) are equivalent.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftarrow$ (iv) is clear from the definitions; (i) $\Rightarrow$ (iv) if Φ is simply laced, by Proposition 4.9.

(ii) $\Rightarrow$ (i) We first provide several facts about rank 2 root systems, referring the reader to [Ser01, Ch. 5, §3 and §7] for additional background and details. There are four rank 2 root systems: $A_1 \times A_1$, A_2, B_2, and G_2. Every root λ of a rank 2 root system has two "nearest neighbors", one to either side, which we denote by λ' and λ''. One checks that $|\lambda'| = |\lambda''|$, and in types A_2, B_2, and G_2, $\lambda = c\lambda' + c\lambda''$, where $c = 1$ or $c = 1/2$. This gives an iso-decomposition of λ in Φ.

Since α is bi-decomposable, we can write

$$\alpha = r\beta + t\gamma \tag{6.1}$$

where r, t are positive rational numbers, $\beta, \gamma \in S$, and α, β, γ are distinct. Let $X = \{\alpha, \beta, \gamma\}$. Since the three elements of X are distinct, they do not lie on the same line through 0. By (6.1), they lie on the same plane through 0. Thus the $\mathbb{R}$ span of X, which we denote by V_X, is two dimensional. By [Bou02, Ch. VI, §1, no. 1, Prop. 4(ii)], $\Phi_X := \Phi \cap V_X$ is a root system. Its rank is two, and thus it must be of type $A_1 \times A_1$, A_2, B_2, or G_2. Since it contains X, and X contains three elements none of which is the negative of any other, we can rule out type $A_1 \times A_1$.

Let $C = \mathrm{Cone}_{\mathbb{R}}\{\beta, \gamma\} = \{x\beta + y\gamma : x, y \in \mathbb{R}, x, y \geq 0\}$, the convex polyhedral cone generated by $\{\beta, \gamma\}$. Geometrically, C is the locus of points in V_X lying on or between the ray through β and the ray through γ. By (6.1), α is in the interior of C. Since

$\beta, \gamma \in \Phi_X$, it follows that α', α'', the nearest neighbors to α in Φ_X, must also lie in C. Since $\beta, \gamma \in S$ and $\alpha', \alpha'' \in \Phi$, condition (i) of S being closed implies $\alpha', \alpha'' \in S$. As shown above, $\alpha = c\alpha' + c\alpha''$, where $c = 1$ or $c = 1/2$. Hence α is iso-decomposable in S.

(iii) $\Rightarrow$ (ii) if Φ is of classical type. We proceed by contradiction. Suppose that α is rationally decomposable but not bi-decomposable. Then we can write $\alpha = r_1\alpha_1 + \cdots + r_n\alpha_n$, where $n \geq 3$, r_i positive rational numbers, $\alpha_i \in \Phi^+$ distinct, and $\alpha_i \neq \alpha$, but we cannot express α as such a linear combination with $n = 2$. Clearing denominators, we obtain

$$d\alpha = m_1\alpha_1 + \cdots + m_n\alpha_n, \tag{6.2}$$

where $d, m_1, \ldots, m_n$ are positive integers. Among such expressions, consider those with d minimal; among these, consider those with $m_1 + \cdots + m_n$ minimal; among these, choose one with $m_1|\alpha_1|^2 + \cdots + m_n|\alpha_n|^2$ minimal. Assume that d, n, $m_1, \ldots, m_n$, $\alpha_1, \ldots, \alpha_n$ are so chosen. We can rewrite (6.2) as

$$\alpha + \cdots + \alpha = (\alpha_1 + \cdots + \alpha_1) + \cdots + (\alpha_n + \cdots + \alpha_n), \tag{6.3}$$

where α occurs d times and each α_i occurs m_i times. The total number of summands on the right hand side is $m_1 + \cdots + m_n$.

We make several preliminary observations about (6.3): for all i, j such that $i \neq j$,

(a) $\alpha_i + \alpha_j$ is not an integer multiple of α.
(b) $\alpha_i + \alpha_j \notin \Phi^+$.
(c) $(\alpha_i, \alpha_j) \geq 0$.
(d) $(\alpha, \alpha_i) > 0$.

To prove (a), note that if $\alpha_i + \alpha_j = e\alpha$ for some positive integer e, then α is bi-decomposable in S, a contradiction. To prove (b), suppose that $\alpha_i + \alpha_j \in \Phi^+$. Then, since S is closed, $\alpha_i + \alpha_j \in S$. Thus two summands α_i, α_j on the right hand side of (6.3) can be replaced by the single summand $\alpha_i + \alpha_j$. This replacement decreases $m_1 + \cdots + m_n$ by 1, contradicting the minimality of this sum. Now (c) follows immediately from (b) and (d) follows from (c), since $(\alpha, \alpha_i) = (1/d) \sum m_j(\alpha_j, \alpha_i) > 0$.

Let us recall the positive roots of classical type:

$$\begin{array}{ll} A_{n-1} & \Phi^+ = \{\epsilon_p - \epsilon_q : 1 \leq p < q \leq n\} \\ B_n & \Phi^+ = \{\epsilon_p \pm \epsilon_q : 1 \leq p < q \leq n\} \cup \{\epsilon_p : 1 \leq p \leq n\} \\ C_n & \Phi^+ = \{\epsilon_p \pm \epsilon_q : 1 \leq p < q \leq n\} \cup \{2\epsilon_p : 1 \leq p \leq n\} \\ D_n & \Phi^+ = \{\epsilon_p \pm \epsilon_q : 1 \leq p < q \leq n\} \end{array}$$

Based on these representations, we can make an additional observation about (6.3):

(e) Suppose that some α_i has a component of ϵ_s but α does not. Then $\alpha_i = \epsilon_r \pm \epsilon_s$ for some $r < s$, and there exists j such that $\alpha_j = \epsilon_r \mp \epsilon_s$. Moreover, α has a component of ϵ_r with positive coefficient.

Indeed, since α_i has a component of ϵ_s but α does not, some α_j must have a component of ϵ_s but with coefficient of opposite sign. This results in $(\alpha_i, \alpha_j) < 0$, unless $\alpha_i = \epsilon_r \pm \epsilon_s$ and $\alpha_j = \epsilon_r \mp \epsilon_s$ for some $r < s$. The final claim of (e) now follows from (d), which tells us that $(\alpha, \alpha_i) > 0$ and $(\alpha, \alpha_j) > 0$.

If $\alpha = \epsilon_p - \epsilon_q$ (in any type), then all terms on the right hand side of (6.3) must be of the form $\epsilon_a - \epsilon_b$ (since the sum of the coefficients of the ϵ_i is 0). There must be one root α_i on the right hand side of (6.3) of the form $\epsilon_r - \epsilon_q$, with $r \neq p$, since the coefficient of ϵ_q is negative. Then since the coefficient of ϵ_r in the sum is 0, there must be a root α_j of the form $\epsilon_a - \epsilon_r$, but then $(\alpha_i, \alpha_j) = -1$, which is impossible by (c). Therefore α is not of the form $\epsilon_p - \epsilon_q$. This completes the proof for type A_{n-1}.

If $\alpha = \epsilon_p$ in type B_n or $\alpha = 2\epsilon_p$ in type C_n, then some α_i must be of the form $\epsilon_p \pm \epsilon_s$, $s \neq p$. By (e), $p < s$ and some α_j must be of the form $\epsilon_p \mp \epsilon_s$. But this contradicts (a). Therefore α is not of the form ϵ_p in type B_n or $2\epsilon_p$ in type C_n.

Hence, $\alpha = \epsilon_p + \epsilon_q$ in type B_n, C_n, or D_n. Suppose that some $\alpha_i = \epsilon_p - \epsilon_q$. Then, since α has a component of ϵ_q with positive coefficient, some α_j must as well. But then $(\alpha_i, \alpha_j) < 0$, contradicting (c). We conclude that no α_i equals $\epsilon_p - \epsilon_q$.

Consider types B_n and C_n. Some α_i must have a component of ϵ_s which α does not. (In type C_n, this is true because the only alternative is $\alpha_i = 2\epsilon_p$, $\alpha_j = 2\epsilon_q$ for some i, j, contradicting (a). In type B_n, a similar argument applies.) Thus, by (e), either α_i is of the form $\epsilon_p \pm \epsilon_s$ and there exists j such that α_j is of the form $\epsilon_p \mp \epsilon_s$, or α_i is of the form $\epsilon_q \pm \epsilon_s$ and there exists j such that α_j is of the form $\epsilon_q \mp \epsilon_s$. Assume the former; the proof for the latter is similar. In type B_n, since S is closed, $\epsilon_p = (1/2)\alpha_i + (1/2)\alpha_j \in S$. Thus α_i, α_j, two summands of the right hand side of (6.3), can be replaced by ϵ_p, ϵ_p. With this replacement, $m_1|\alpha_1|^2 + \cdots + m_n|\alpha_n|^2$ decreases by 2, contradicting the minimality of this quantity. In type C_n, $2\epsilon_p = \alpha_i + \alpha_j \in S$. Now α_i, α_j can be replaced by the single root $2\epsilon_p$. This replacement decreases $m_1 + \cdots + m_n$ by 1, contradicting the minimality of this sum. This completes the proof for types B_n and C_n.

This leaves only the possibility that $\alpha = \epsilon_p + \epsilon_q$ in type D_n. We have seen that no α_i equals $\epsilon_p - \epsilon_q$. By (e), all of the α_i are of the form $\epsilon_p \pm \epsilon_a$ and $\epsilon_q \pm \epsilon_b$, where a, b are not equal to either p or q. Note that we cannot have both $\epsilon_p + \epsilon_a$ and $\epsilon_q - \epsilon_a$ on the right hand side, since the inner product would be -1. Similarly, we cannot have both $\epsilon_p - \epsilon_a$ and $\epsilon_q + \epsilon_a$ on the right hand side. All coefficients except those of ϵ_p and ϵ_q are 0, so we conclude that on the right hand side, $\epsilon_p + \epsilon_a$ occurs iff $\epsilon_p - \epsilon_a$ occurs, and $\epsilon_q + \epsilon_b$ occurs iff $\epsilon_q - \epsilon_b$ occurs. By minimality of the expression (6.3), we conclude that this expression must have the form

$$2(\epsilon_p + \epsilon_q) = (\epsilon_p + \epsilon_a) + (\epsilon_p - \epsilon_a) + (\epsilon_q + \epsilon_b) + (\epsilon_q - \epsilon_b) \tag{6.4}$$

for a, b not equal to p or q, and $a \neq b$. Now, $\epsilon_p + \epsilon_a \in S$, and

$$\epsilon_p + \epsilon_a = (\epsilon_p - \epsilon_q) + (\epsilon_a + \epsilon_q).$$

Since S is closed, at least one of the roots $\epsilon_p - \epsilon_q$ or $\epsilon_a + \epsilon_q$ must be in S. If $\epsilon_p - \epsilon_q \in S$, then $\epsilon_p - \epsilon_b$ is in S as it is the sum $(\epsilon_p - \epsilon_q) + (\epsilon_q - \epsilon_b)$, where both summands are in S; but then

$$\alpha = \epsilon_p + \epsilon_q = (\epsilon_p - \epsilon_b) + (\epsilon_q + \epsilon_b),$$

which contradicts the assumption that α is not bi-decomposable. On the other hand, if $\epsilon_a + \epsilon_q \in S$, then

$$\alpha = \epsilon_p + \epsilon_q = (\epsilon_p - \epsilon_a) + (\epsilon_a + \epsilon_q),$$

again contradicting the assumption that α is not bi-decomposable.

(iii) $\Rightarrow$ (ii) if Φ is of type G_2. The root system G_2 was discussed above in the proof of (ii) $\Rightarrow$ (i). Its six positive roots lie in a half-plane. Suppose that $\alpha \in S$ is rationally decomposable. Then $\alpha = r_1\alpha_1 + \cdots + r_t\alpha_t$ for some $t \geq 2$, $\alpha_i \in S$ distinct, $\alpha_i \neq \alpha$, and $r_i \in \mathbb{Q}_{>0}$. Let β and γ be the leftmost and rightmost roots of $\alpha_1, \ldots, \alpha_t$. Then β and γ lie on the two edges of $\mathrm{Cone}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_t\}$. Thus

$$\alpha \in \mathrm{Cone}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_t\} = \mathrm{Cone}_{\mathbb{Q}}\{\beta, \gamma\},$$

implying α is bi-decomposable. $\square$

The proof of the implication (iii) $\Rightarrow$ (ii) in classical types is clearly the most difficult part of this proof. The question of whether this implication holds in types E_n and F_4 is open.

Remark 6.5. If Φ is not simply laced, then rational decomposability is not equivalent to integral decomposability. For example, in type B_2, one can check that the subset $S = \{\epsilon_1 - \epsilon_2, \epsilon_1, \epsilon_1 + \epsilon_2\} \subseteq \Phi^+$ is closed. In S, the roots $\epsilon_1 - \epsilon_2$ and $\epsilon_1 + \epsilon_2$ are rationally indecomposable, but ϵ_1 is not. However, all three are integrally indecomposable.

Proposition 6.6. *If $S = I(x^{-1})$ and Φ is of classical type or of type G_2, then $(S_{z\geq w})^A = (S_{z\geq w})^{\ddagger}$.*

Proof. The inclusion $\subseteq$ holds because any A-indecomposable element is iso-indecomposable. We prove the reverse inclusion. Suppose $\gamma \in (S_{z\geq w})^{\ddagger}$. If γ is A-decomposable in $S_{z\geq w}$, then γ is A-decomposable in S, so by Theorem 6.1, γ is iso-decomposable in S. By Lemma 5.2, γ is iso-decomposable in $S_{z\geq w}$, a contradiction. Hence $\gamma \in (S_{z\geq w})^A$. $\square$

Suppose that $S = I(x^{-1})$. One sees easily (see Lemma 2.4(i)) that $(S^A)_{z\geq w} \subseteq (S_{z\geq w})^A$ and $(S^A)_{x\geq w} \subseteq (S_{x\geq w})^A$. The following corollary shows that in classical types and type G_2, all four of these sets are equal.

Corollary 6.7. *If $S = I(x^{-1})$ and Φ is of classical type or of type G_2, then $(S^A)_{x\geq w} = (S_{x\geq w})^A = (S^A)_{z\geq w} = (S_{z\geq w})^A$.*

Proof. By Corollary 5.5 and Remark 5.6, $(S^A)_{x\geq w} = (S^A)_{z\geq w}$ and $(S_{x\geq w})^A = (S_{z\geq w})^A$. The proof is completed by observing that

$$(S^A)_{z\geq w} = (S^{\ddagger})_{z\geq w} = (S_{z\geq w})^{\ddagger} = (S_{z\geq w})^A,$$

where the first equality is due to Theorem 6.1, the second to Corollary 5.4, and the third to Proposition 6.6. $\square$

Remark 6.8. This corollary implies that if x is fixed and one wants to calculate $(S_{z\geq w})^A$ for multiple z, it is not necessary to check indecomposability separately for each z. Rather, one can compute the set S^A of A-indecomposable elements in S, and then intersect with $S_{z\geq w}$.

Figure 2 summarizes the relationships we have found among the five main types of indecomposability in $I(x^{-1})$. For general root sets, rational indecomposability implies the other four types (see Figure 1). The other four implications of Figure 2 are proved in Propositions 4.9, 5.3, and 6.4.

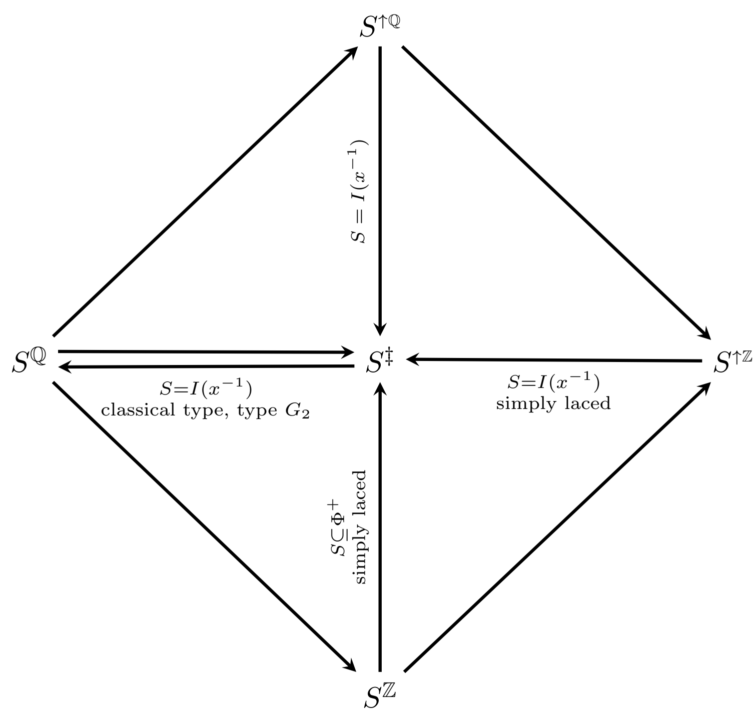

FIGURE 2. Relationships among five types of indecomposability. Arrows represent inclusions. (See also Figure 1 on page 8.)

6.2. **Indecomposability in Φ^+.** When $S = \Phi^+$, several types of indecomposability are easily proved to be equivalent in all types. This is due to the following properties of the base Δ of Φ^+:

(a) Δ is linearly independent.
(b) Each root of Φ^+ is a non-negative integer linear combination of elements of Δ.

(c) Any root of Φ^+ not in Δ can be written as a sum of two positive roots.

Proposition 6.9. *Let $S = \Phi^+$ and $\alpha \in S$. The following conditions are equivalent:*

(i) *α is iso-decomposable.*
(ii) *α is bi-decomposable.*
(iii) *$\alpha \notin \Delta$.*
(iv) *α is integrally decomposable.*
(v) *α is rationally decomposable.*

Proof. (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Leftarrow$ (i) are clear from the definitions, (ii) $\Leftarrow$ (iii) follows from (c), and (i) $\Leftarrow$ (ii) holds by Proposition 6.4.

(iii) $\Leftarrow$ (v) is an immediate consequence of (a) and (b). Indeed, let $\alpha \in \Delta$. If β is an element of $\Phi^+ \setminus \{\alpha\}$, then a (nonnegative) linear combination for β in the elements of Δ must contain at least one term other than α. Thus the same is true if β is a nontrivial nonnegative linear combination of elements of $\Phi^+ \setminus \{\alpha\}$. Therefore a nonnegative linear combination of elements of $\Phi^+ \setminus \{\alpha\}$ cannot equal α, i.e., α is rationally indecomposable. $\square$

Corollary 6.10. $(x\Phi^-)^\ddagger = x\Delta^- = (x\Phi^-)^{\mathbb{Z}} = (x\Phi^-)^{\mathbb{Q}}$.

Proof. This follows from Proposition 6.9, as x is a length-preserving linear automorphism of Φ. $\square$

7. When all elements are indecomposable

In this section we study root sets $S \subseteq \Phi^+$. We are interested in examining conditions under which all elements of S are rationally, integrally, or iso-indecomposable. When this occurs, results in other sections concerning such indecomposable elements of S will of course apply to all elements of S.

We say that S is **coplanar** if there exists $v \in \mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in S$. If $I(x^{-1})$ is coplanar, then x is said to be a **cominuscule** Weyl group element. Cominuscule Weyl group elements were first studied by Peterson.

The Weyl group element x is said to be **fully commutative** if there does not exist a reduced expression for x which contains a subword of the form $s_i s_j s_i \cdots$ of length $m \geq 3$, where m is the order of $s_i s_j$. In [BJS93], Billey, Jockusch, and Stanley showed that in type A, fully commutative Weyl group elements can alternatively be characterized as 321-avoiding permutations, and that their Schubert polynomials are flag skew Schur functions. Full commutativity was studied extensively by Fan and Stembridge in [Fan95], [Fan97], [FS97], [Ste96], [Ste97].

Theorem 7.1. *If x is a cominuscule Weyl group element, then x is fully commutative.*

Proof. This result is due to Stembridge (see [Ste01, Proposition 2.1]). $\square$

Theorem 7.2. *For $S \subseteq \Phi^+$, consider the following statements:*

(i) *S is coplanar.*
(ii) *$S^{\mathbb{Z}} = S$.*
(iii) *S does not contain three roots of the form $\alpha, \beta, \alpha+\beta$.*
(iv) *$S^{\mathbb{Q}} = S$.*
(v) *$S^{\ddagger} = S$.*
(vi) *x is fully commutative.*

We have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii). If Φ is simply laced, then (iii) $\Leftrightarrow$ (v). If Φ is simply laced and $S = I(x^{-1})$, then (i) $\Rightarrow$ (iv) and (iii) $\Leftrightarrow$ (vi). (See Figure 3.)

$$
\begin{tikzcd}
\text{(i) } S \text{ coplanar} \arrow[r] \arrow[d, "{S=I(x^{-1})}" , "\text{simply laced}"'] & \text{(ii) } S^{\mathbb{Z}} = S \arrow[d] & \text{(vi) } x \text{ fully commut.} \arrow[dl, leftrightarrow, "{S=I(x^{-1})}"', "\text{simply laced}"] \\
\text{(iv) } S^{\mathbb{Q}} = S \arrow[ur] & \text{(iii) } \nexists\, \alpha, \beta, \alpha+\beta \in S \arrow[r, leftrightarrow, "\text{simply laced}"] & \text{(v) } S^{\ddagger} = S
\end{tikzcd}
$$

FIGURE 3. To accompany Theorem 7.2. Arrows represent implications. Labelings of an arrow indicate conditions under which the implication or equivalence holds.

Proof. (i) $\Rightarrow$ (ii) is proved in the same manner as [GK22, Proposition 6.4], but with $I(x^{-1})$ replaced by S; (ii) $\Rightarrow$ (iii) is clear from definitions; (iv) $\Rightarrow$ (ii) since $S^{\mathbb{Q}} \subseteq S^{\mathbb{Z}}$; (iii) $\Leftrightarrow$ (v) if Φ is simply laced follows from Lemma 4.8; (iii) $\Leftrightarrow$ (vi) if Φ is simply laced and $S = I(x^{-1})$ by [FS97, Theorem 2.4].

(i) $\Rightarrow$ (iv) if Φ is simply laced and $S = I(x^{-1})$. Assume S is coplanar. Then for all $\alpha, \beta \in S$, $(\alpha, \beta) \geq 0$. Indeed, if this were not true, then we would have $\alpha + \beta \in \Phi^+$. Since $S = I(x^{-1})$ is closed under addition, this would imply $\alpha + \beta \in S$, violating (iii).

Suppose there exists $\beta \in S \setminus S^{\mathbb{Q}}$. Let

$$\beta = \sum r_i \beta_i \tag{7.1}$$

$r_i \in \mathbb{Q}_{>0}$, $\beta_i \in S \setminus \{\beta\}$ be a positive rational decomposition of β. Applying $(\,\cdot\,, \beta_i)$ to both sides of (7.1) yields $(\beta, \beta_i) > 0$. Thus $\langle \beta_i, \beta \rangle = 1$.

Applying $\langle \cdot, \beta \rangle$ to both sides of (7.1), we obtain $2 = \sum r_i$. Let v be such that $\alpha(v) = 1$ for all $\alpha \in S$. Applying v to both sides of (7.1), we find that $1 = \sum r_i$, a contradiction. □

Remark 7.3. Some of the conclusions of Theorem 7.2 (i) - (vi) can be strengthened. For example, (iii) $\Rightarrow$ (vi) does not require S to be simply laced (although the converse does).

Corollary 7.4. *If Φ is of type A or D, then all elements of $I(x^{-1})$ are integrally indecomposable if and only if x if fully commutative.*

Proof. In types A and D, $I(x^{-1})^{\mathbb{Q}} = I(x^{-1})^{\ddagger}$, by Proposition 6.4. Thus, in Theorem 7.2, conditions (ii) through (vi) are equivalent (see Figure 3). □

Remark 7.5. In [GK22, Remark 6.5], the Weyl group element $x = s_2s_1s_3s_4s_2$ of type D_4 is considered. It is observed that all elements of $I(x^{-1})$ are integrally indecomposable, but x is not cominuscule (see [Ste01, Remark 5.4]). Nevertheless, x is fully commutative, as required by Corollary 7.4. Note that Stembridge's results apply since cominuscule Weyl group elements are exactly the minuscule elements for the dual root system (see [GK21, Section 5.2]).

8. Tangent spaces and T-invariant curves

In this section we recall some known results on tangent spaces and T-invariant curves of Schubert and Kazhdan-Lusztig varieties in G/P. We include some proofs for the convenience of the reader. Fix $w \leq x \in W^P$.

8.1. **Schubert and Kazhdan-Lusztig varieties.** We use the notation of Section 1.3. Let U_α denote the root subgroup of G corresponding to $\alpha \in \Phi$. If V is a unipotent subgroup of G normalized by T, then $U_\alpha \subseteq V$ if and only if $\alpha \in \Phi(V)$; otherwise $U_\alpha \cap V$ is the identity. Moreover, $V \cong \prod_{\alpha \in \Phi(V)} U_\alpha$.

Part (i) of the next lemma goes back at least to [Kos61], and part(ii) is an immediate consequence. Part (iii), which also follows easily from (i), is used implicitly in [Knu09, Section 7.3].

Lemma 8.1. *We have*

(i) $x\Phi_P^- \cap \Phi^+ = I(x^{-1})$.
(ii) $x\Phi_P^- = (x\Phi_P^- \cap \Phi^-) \sqcup I(x^{-1})$.
(iii) $U_P^-(x) \cap U = U^-(x) \cap U$.

Proof. (i) By definition, $I(x^{-1}) = x\Phi^- \cap \Phi^+$. The left coset analogue of [Kos61, Remark 5.13] states, in our notation, that $x(\Phi^+ \setminus \Phi_P^+) \subseteq \Phi^+$. Thus $x(\Phi^- \setminus \Phi_P^-) \subseteq \Phi^-$. This implies that $I(x^{-1}) = x\Phi^- \cap \Phi^+ = x\Phi_P^- \cap \Phi^+$, as claimed.

(ii) $x\Phi_P^- = (x\Phi_P^- \cap \Phi^-) \sqcup (x\Phi_P^- \cap \Phi^+) = (x\Phi_P^- \cap \Phi^-) \sqcup I(x^{-1})$, by (i).

(iii) $U_\alpha \subseteq U_P^-(x) \cap U$ if $\alpha \in x\Phi_P^- \cap \Phi^+$, and $U_\alpha \subseteq U^-(x) \cap U$ if $\alpha \in I(x^{-1})$. By (i), $x\Phi_P^- \cap \Phi^+ = I(x^{-1})$. Hence $U_P^-(x) \cap U$ and $U^-(x) \cap U$ contain the same root subgroups, so they are equal. □

Combining this lemma with a well-known fact about the Bruhat order, we obtain the following corollary.

Corollary 8.2. *Let $w \leq x \in W^P$. Then $x\Phi_P^- \cap \Phi^+ = I(x^{-1}) = \{\alpha \in x\Phi_P^- \mid s_\alpha x < x\}$ and $x\Phi_P^- \cap \Phi^- = \{\alpha \in x\Phi_P^- \mid s_\alpha x > x\}$.*

Proof. Let $\alpha \in \Phi$. Observe that $s_\alpha x > x$ if and only if $x^{-1}s_\alpha > x^{-1}$ if and only if α and $x^{-1}\alpha$ have the same sign. Thus, if $\alpha \in x\Phi_P^-$, then $s_\alpha x > x$ implies $\alpha \in x\Phi_P^- \cap \Phi^-$, and $s_\alpha x < x$ implies $\alpha \in x\Phi_P^- \cap \Phi^+ = I(x^{-1})$. □

As noted in Section 1.3, the T-fixed points in G/P are the cosets zP for $z \in W^P$. We will sometimes denote the fixed point zP simply by z, when it is understood that we are considering a point in G/P. The Kazhdan-Lusztig variety Y_x^w is the intersection of the Schubert variety X^w with the opposite Schubert cell $X_x^0 = B \cdot xP$. It is irreducible of dimension $\ell(x) - \ell(w)$ (see [KL80, 1.4]), and is an open subvariety of the Richardson variety $X^w \cap X_x$. The unipotent subgroup $U_P^-(x)$ embeds as an open subset of G/P under the mapping $U_P^-(x) \to U_P^-(x)xP$, and $X_x^0 = BxP = (U_P^-(x) \cap U)xP \subseteq U_P^-(x)xP$. Thus,

$$Y_x^w = X^w \cap (U_P^-(x) \cap U)xP. \tag{8.1}$$

We see that Y_x^w is an affine subvariety of $(U_P^-(x) \cap U)xP \cong U_P^-(x) \cap U$. As observed by Kazhdan and Lusztig (in the setting of the full flag variety G/B), a neighborhood of X^w near xP is isomorphic to the product of Y_x^w with an affine space (see [KL79, Lemma A4]). In the G/P setting, this takes the form of a T-equivariant isomorphism

$$X^w \cap U_P^-(x)xP \cong Y_x^w \times (U_P^-(x) \cap U^-)xP. \tag{8.2}$$

Note that $(U_P^-(x) \cap U^-)xP = B^-xP = X_0^x$. The T-fixed points of X^w are the cosets $zP \in G/P$ such that $z \in W^P$ and $z \geq w$, whereas xP is the unique T-fixed point of Y_x^w.

We remark that although Kazhdan and Lusztig considered the full flag variety G/B, we have followed [Knu09] and used the term Kazhdan-Lusztig variety in the G/P setting. A more detailed discussion of slices such as this may be found in [GK21].

8.2. **T-invariant curves of Schubert and Kazhdan-Lusztig varieties.** In this section we recall some results describing the sets of weights of tangent spaces and of T-invariant curves of Schubert varieties and Kazhdan-Lusztig varieties. These are obtained from the decomposition (8.2), as well as the description by Carrell and Peterson of the T-invariant curves in Schubert varieties (see the proof of Proposition 8.3(i)). They will be used in the proof of our main result in the next section.

If Z is a variety with a T-action, then a T-invariant curve of Z is defined to be an irreducible curve C which is closed in Z and stable under the T-action, i.e., $t \cdot C \subseteq C$ for all $t \in T$. If z is a T-fixed point of Z, then denote the set of all T-invariant curves of Z containing z by $E(Z, z)$. Denote the tangent space to Z at z by $T_z Z$, and $\sum_{C \in E(Z,z)} T_x C \subseteq T_z Z$ by $TE_z Z$.

The T-invariant curves in G/P are the curves of the form $\overline{U_\alpha xP}$, where $x \in W^P$ and $\alpha \in x\Phi_P^-$. Such a curve is isomorphic to $\mathbb{P}^1$, and is equal to $U_\alpha xP \cup \{s_\alpha xP\}$. If Z is a T-invariant subvariety of G/P containing xP, then $E(Z, x)$ consists of the curves $\overline{U_\alpha xP}$ such that $\overline{U_\alpha xP} \cap Z$ has dimension 1. Hence $E(Z, x)$ is determined by the set of weights $\Phi(TE_x Z)$, since $\overline{U_\alpha xP} \cap Z \in E(Z, x)$ if and only if $\alpha \in \Phi(TE_x Z)$.

The description by Carrell and Peterson of T-invariant curves in Schubert varieties yields the following result.

Proposition 8.3. *Let $w \leq x \in W^P$.*

(i) $\Phi(TE_x X^w) = \{\alpha \in x\Phi_P^- \mid s_\alpha x \geq w\}$.
(ii) $\Phi(TE_x Y_x^w) = \{\alpha \in I(x^{-1}) \mid s_\alpha x \geq w\}$.

Proof. (i) By the result of Carrell and Peterson (see [Car94], [CK03]),

$$E(X^w, xP) = \{\overline{U_\alpha xP} \mid \alpha \in x\Phi_P^-, s_\alpha x \geq w\}. \tag{8.3}$$

By the discussion before the proposition, this is equivalent to (i).

(ii) The definition of Y_x^w implies that $E(Y_x^w, x)$ is the intersection of $E(X^w \cap U_P^-(x)xP, x)$ and $E((U_P^-(x) \cap U)xP, x)$. By (i),

$$E(X^w \cap U_P^-(x)xP, x) = \{U_\alpha xP \mid \alpha \in x\Phi_P^-, s_\alpha x \geq w\}. \tag{8.4}$$

Also, we have $U_\alpha xP \subseteq (U_P^-(x) \cap U)xP \Leftrightarrow U_\alpha \subseteq U_P^-(x) \cap U \Leftrightarrow \alpha \in \Phi(U_P^-(x) \cap U) = x\Phi_P^- \cap \Phi^+ = I(x^{-1})$ (here we have used Lemma 8.1). Thus,

$$E((U_P^-(x) \cap U)xP, x) = \{U_\alpha xP \mid \alpha \in I(x^{-1})\}. \tag{8.5}$$

Hence,

$$E(Y_x^w, xP) = \{U_\alpha xP \mid \alpha \in I(x^{-1}), s_\alpha x \geq w\}, \tag{8.6}$$

which implies (ii). □

For later use, we summarize some facts about weights of tangent spaces and T-invariant curves in the following corollary.

Corollary 8.4. *Let $w \leq x \in W^P$.*

(i) $\Phi(T_x X^w) \subseteq x\Phi_P^-$.
(ii) $\Phi(T_x X^w) = (x\Phi_P^- \cap \Phi^-) \sqcup \Phi(T_x Y_x^w) = \{\alpha \in x\Phi_P^- \mid s_\alpha x > x\} \sqcup \Phi(T_x Y_x^w)$.

(iii) $\Phi(T_xY_x^w) = \Phi(T_xX^w) \cap I(x^{-1})$.
(iv) $\Phi(TE_xX^w) = (x\Phi_P^- \cap \Phi^-) \sqcup \Phi(TE_xY_x^w)$.
(v) $\Phi(TE_xY_x^w) = \Phi(TE_xX^w) \cap I(x^{-1}) = \{\alpha \in x\Phi_P^- \mid x > s_\alpha x \geq w\}$.

Proof. (i) $T_xX^w \subseteq T_x(G/P) \cong U_P^-(x)$, which has weights $x\Phi_P^-$.

(ii) The first equality follows from the decomposition (8.2) (it also appears in [GK22, Lemma 6.2(i)]). The second then follows from Corollary 8.2.

(iii) This follows from (i), (ii), and Lemma 8.1(ii).

(iv) By Lemma 8.1(ii) and Proposition 8.3, $\Phi(TE_xX^w) = \{\alpha \in x\Phi_P^- \cap \Phi^- \mid s_\alpha x \geq w\} \sqcup \Phi(TE_xY_x^w)$. But by Corollary 8.2, for all $\alpha \in x\Phi_P^- \cap \Phi^-$, $s_\alpha x \geq w$.

(v) The first equality follows from (i), (iv), and Lemma 8.1(ii). The second can be deduced from Proposition 8.3(i) and Corollary 8.2. □

9. Indecomposable weights of tangent spaces and T-invariant curves

Fix $w \leq x \in W^P$. Define $\Phi_{\tan} = \Phi(T_xX^w)$, $\Phi_{\mathrm{cur}} = \Phi(TE_xX^w)$, $\Phi_{\tan}^{\mathrm{KL}} = \Phi(T_xY_x^w)$, and $\Phi_{\mathrm{cur}}^{\mathrm{KL}} = \Phi(TE_xY_x^w)$. We refer to $\Phi_{\tan}$ and $\Phi_{\tan}^{\mathrm{KL}}$ as sets of tangent weights, and Φ_{cur} and $\Phi_{\mathrm{cur}}^{\mathrm{KL}}$ as sets of curve weights.

In this section we prove the main result of this paper, Theorem 1.1 (see Corollary 9.3): $\Phi_{\tan} \subseteq \mathrm{Cone}_A \Phi_{\mathrm{cur}}$. We deduce this from the stronger statement $\Phi_{\tan}^{\mathrm{KL}} \subseteq \mathrm{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}}$. This result relies on properties of $\Phi_{\tan}^{\mathrm{KL}}$ studied in [GK22], the characterization of $\Phi_{\mathrm{cur}}^{\mathrm{KL}}$ by Carrell-Peterson [Car94], and Corollary 5.5.

Observe that $\Phi_{\mathrm{cur}}^{\mathrm{KL}} \subseteq \Phi_{\tan}^{\mathrm{KL}} \subseteq I(x^{-1})$, where the second inclusion follows from Corollary 8.4(iii). Also,

$$\Phi_{\mathrm{cur}}^{\mathrm{KL}} = I(x^{-1})_{x \geq w}, \tag{9.1}$$

as follows from Proposition 8.3(ii) and the equality $x_i = s_{\gamma_i}x$.

Proposition 9.1. *We have* $\Phi_{\tan}^{\mathrm{KL}} \subseteq \mathrm{Cone}_{\mathbb{Z}}(I(x^{-1})_{z \geq w})$.

Proof. We first give definitions of three terms which appear in equation (9.2) below. Recall that $\mathbf{s} = (s_1, \ldots, s_l)$ is a fixed reduced expression for x. Define $\mathcal{T}_{w,\mathbf{s}}$ to be the set of sequences $\mathbf{t} = (i_1, \ldots, i_m)$, $1 \leq i_1 < \cdots < i_m \leq l$, such that $H_{s_{i_1}} \cdots H_{s_{i_m}} = H_w$. For such $\mathbf{t}$, define $e(\mathbf{t}) = m - \ell(w)$. For $\zeta \in \mathrm{Cone}_{\mathbb{Z}}\{\gamma_i : i \notin \mathbf{t}\}$, define n_ζ to be the number of ways to express ζ as a nonnegative integer linear combination of the γ_i, $i \notin \mathbf{t}$.

Let B and C be the coordinate rings of the tangent space and scheme-theoretic tangent cone respectively of Y_x^w at x, and let B_1 and C_1 be the degree one components of these rings. By equation (5.2) of [GK22] (see also [GK22, Theorem 6.1]) and the simplifications following this equation, the character of C for the T-action is

$$\mathrm{Char}\, C = \sum\nolimits_{\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}} \sum\nolimits_{\zeta \in \mathrm{Cone}_{\mathbb{Z}}\{\gamma_i : i \notin \mathbf{t}\}} (-1)^{e(\mathbf{t})} n_\zeta e^{-\zeta}. \tag{9.2}$$

For each γ_i in this summation, $i \notin \mathbf{t}$ for some $\mathbf{t} \in \mathcal{T}_{w,\mathbf{s}}$; by [GK22, Theorem 5.8], $z_i \geq w$. Therefore all weights ζ of C lie in $-\operatorname{Cone}_{\mathbb{Z}}(I(x^{-1})_{z\geq w})$. So too do all weights of B_1, since B_1 and C_1 are canonically identified. But B_1 is the dual space of $T_xY_x^w$. Hence all weights of $T_xY_x^w$ lie in $\operatorname{Cone}_{\mathbb{Z}}(I(x^{-1})_{z\geq w})$. ☐

Theorem 9.2. *We have:*

(i) $\operatorname{Cone}_A \Phi_{\tan}^{\mathrm{KL}} = \operatorname{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}}$, *or equivalently,* $(\Phi_{\tan}^{\mathrm{KL}})^A = (\Phi_{\mathrm{cur}}^{\mathrm{KL}})^A$. *Hence* $\Phi_{\tan}^{\mathrm{KL}} \subseteq \operatorname{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}}$.
(ii) $\Phi_{\tan}^{\mathrm{KL}} \cap I(x^{-1})^A = \Phi_{\mathrm{cur}}^{\mathrm{KL}} \cap I(x^{-1})^A$.
(iii) $\Phi_{\tan}^{\mathrm{KL}} \cap I(x^{-1})^A = \Phi_{\mathrm{cur}}^{\mathrm{KL}} \cap I(x^{-1})^A = (\Phi_{\mathrm{cur}}^{\mathrm{KL}})^A = (\Phi_{\tan}^{\mathrm{KL}})^A$, *if* Φ *is of classical type or of type* G_2.

Proof. (i) For $S = I(x^{-1})$,

$$\Phi_{\tan}^{\mathrm{KL}} \subseteq \operatorname{Cone}_{\mathbb{Z}}(S_{z\geq w}) \subseteq \operatorname{Cone}_A(S_{z\geq w}) = \operatorname{Cone}_A(S_{x\geq w}) = \operatorname{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}},$$

where the inclusions and equalities are due, respectively, to Proposition 9.1, $\mathbb{Z} \subseteq A$, Corollary 5.5, and equation (9.1). It follows that $\operatorname{Cone}_A \Phi_{\tan}^{\mathrm{KL}} \subseteq \operatorname{Cone}_A \Phi_{\mathrm{cur}}^{\mathrm{KL}}$, and the other inclusion is clear. The equivalence of the second equality of (i) is due to Corollary 3.11. Note that in general, we need to take $A = \mathbb{Q}$ to apply Corollary 5.5. However, if G is simply laced, Corollary 5.5 holds for $A = \mathbb{Z}$ as well, so we can take either $A = \mathbb{Q}$ or $A = \mathbb{Z}$.

(ii) follows from (i) and Lemma 2.2(iii).

(iii) The first and third equalities are (ii) and (i) respectively, and the second equality follows from Corollary 6.7 and equation (9.1). ☐

We can deduce from Theorem 9.2 analogous results for Schubert varieties.

Corollary 9.3. *We have:*

(i) $\operatorname{Cone}_A \Phi_{\tan} = \operatorname{Cone}_A \Phi_{\mathrm{cur}}$, *or equivalently,* $(\Phi_{\tan})^A = (\Phi_{\mathrm{cur}})^A$. *Hence* $\Phi_{\tan} \subseteq \operatorname{Cone}_A \Phi_{\mathrm{cur}}$.
(ii) $\Phi_{\tan} \cap (x\Phi_P^-)^A = \Phi_{\mathrm{cur}} \cap (x\Phi_P^-)^A$.
(iii) $\Phi_{\tan} \cap x\Delta^- = \Phi_{\mathrm{cur}} \cap x\Delta^-$, *if* $X^w \subseteq G/B$.

Proof. (i) follows from Theorem 9.2(i) and Corollary 8.4(ii), (iv).

(ii) follows from (i) and Lemma 2.2(iii).

(iii) follows from (ii) and Corollary 6.10. ☐

If we restrict attention to Schubert varieties in G/B and $x = w_0$, the longest element of the Weyl group, some of our results simplify. In this case, since $P = B$, $\Phi_P^- = \Phi^-$ and $x\Phi_P^- = w_0\Phi^- = \Phi^+$. Thus, by Corollary 9.3(i) and Proposition 8.3(i), $\Phi_{\tan} \subseteq$

$\mathrm{Cone}_A \Phi_{\mathrm{cur}} = \mathrm{Cone}_A\{\alpha \in \Phi^+ \mid s_\alpha w_0 \geq w\}$. Moreover, $s_\alpha w_0 \geq w$ is equivalent to $s_\alpha \leq ww_0$ (see [Hum90, Example 5.9.3]). Hence we obtain

$$\Phi_{\tan} \subseteq \mathrm{Cone}_A\{\alpha \in \Phi^+ \mid s_\alpha \leq ww_0\}.$$

The following proposition gives a stronger result for classical types and type G_2.

Proposition 9.4. *Suppose Φ is of classical type or of type G_2, $P = B$, and $x = w_0$. Then $(\Phi_{\mathrm{cur}})^A = \{\alpha \in \Delta \mid s_\alpha \leq ww_0\}$, and thus*

$$\Phi_{\tan} \subseteq \mathrm{Cone}_A\{\alpha \in \Delta \mid s_\alpha \leq ww_0\}. \tag{9.3}$$

Proof. Since $x = w_0$, $S = I(x^{-1}) = \Phi^+$. By Proposition 6.9, $S^A = \Delta$. We have

$$(S_{x \geq w})^A = (S^A)_{x \geq w} = \{\alpha \in \Delta \mid s_\alpha w_0 \geq w\},$$

where the first equality is by Corollary 6.7, and the second is because $S^A = \Delta$. By (9.1), $S_{x\geq w} = \Phi^{\mathrm{KL}}_{\mathrm{cur}}$. Since $x = w_0$, by Corollary 8.4(iv), $\Phi^{\mathrm{KL}}_{\mathrm{cur}} = \Phi_{\mathrm{cur}}$. We conclude that

$$(\Phi_{\mathrm{cur}})^A = \{\alpha \in \Delta \mid s_\alpha w_0 \geq w\}.$$

Since $s_\alpha w_0 \geq w \Leftrightarrow s_\alpha \leq ww_0$, and $\Phi_{\tan} \subseteq \mathrm{Cone}_A((\Phi_{\mathrm{cur}})^A)$ (see Corollary 3.10), the result follows. ☐

We remark that the condition $s_\alpha \leq ww_0$ appearing in Proposition 9.4 is equivalent to the condition that the simple reflection s_α occurs in a reduced expression for ww_0.

Remark 9.5. In Corollary 9.3, we proved that $\Phi(T_xX^w) \subseteq \mathrm{Cone}_{\mathbb{Q}} \Phi(TE_xX^w)$. We sketch an alternative proof of this result provided by the referee (who attributed it to "folklore"), and we thank the referee for the same. To our knowledge, the stronger result for simply laced types, namely that $\Phi(T_xX^w) \subseteq \mathrm{Cone}_{\mathbb{Z}} \Phi(TE_xX^w)$, is not recovered by this proof.

Since $\Phi(T_xX^w)$ is contained in an open half-space, x is said to be an *attractive* fixed point of X^w (see [Bri99, 1.3]). Hence there exists an open affine T-invariant neighborhood U_x of x in X^w together with a T-equivariant closed embedding $\iota : U_x \to T_xX^w$ (see [ByB73, Corollary 2] or [Bri99, Proposition A2]). If $p : T_xX^w \to TE_xX^w$ is the natural T-equivariant projection, then the T-equivariant map $\pi = p \circ \iota : U_x \to TE_xX^w$ is known to be finite (see proof of [Bri98, Theorem 17]).

We shall denote the coordinate ring of an affine scheme V by $k[V]$, where k is the ground field. For $\lambda \in \Phi(T_xX^w)$, there exists $x_\lambda \in k[T_xX^w]$ with weight $-\lambda$. Let $y = \iota^*(x_\lambda) \in k[U_x]$, where ι^* is the map on coordinate rings induced by ι. Then y also has weight $-\lambda$. Since π is finite, y is integral over $k[TE_xX^w]$. Thus there exist $a_0, \ldots, a_m \in k[TE_xX^w]$ such that

$$y^m + a_{m-1}y^{m-1} + \cdots + a_1y + a_0 = 0. \tag{9.4}$$

The coordinate ring $k[TE_xX^w]$ is equal to the polynomial ring $k[z_\mu \colon \mu \in \Phi(TE_xX)]$, where z_μ represents a vector in $(TE_xX^w)^*$ of weight $-\mu$. Since the weight of y^m

is $m(-\lambda)$, equation (9.4) implies that for some $i < m$, a_i must contain a monomial $(z_{\mu_1})^{j_1} \cdots (z_{\mu_t})^{j_t}$ such that the weight of $(z_{\mu_1})^{j_1} \cdots (z_{\mu_t})^{j_t} y^i$ is also $m(-\lambda)$. Thus

$$j_1(-\mu_1) + \cdots + j_t(-\mu_t) = (m - i)(-\lambda).$$

Hence λ is a nonnegative rational linear combination of elements of $\Phi(TE_x X^w)$. We conclude that $\Phi(T_x X^w) \subseteq \mathrm{Cone}_{\mathbb{Q}}\, \Phi(TE_x X^w)$, as desired. □

10. When tangent weights are curve weights

In this section, we use Theorem 9.2 and Corollary 9.3 to study cases where elements of $\Phi_{\tan}$ lie in Φ_{cur}, and thus can be easily characterized. We use this analysis to give smoothness criteria for X^w at certain points x.

10.1. **Characterizations of tangent spaces.** Recall that $\Phi_{\tan} \subseteq x\Phi_P^-$ and $\Phi_{\tan}^{\mathrm{KL}} \subseteq x\Phi_P^- \cap \Phi^+ = I(x^{-1})$.

Theorem 10.1. *(cf.* [GK22, Theorem A]*) Let $\alpha \in I(x^{-1})^A$. Then $\alpha \in \Phi_{\tan}^{\mathrm{KL}}$ if and only if $s_\alpha x \geq w$.*

Proof. For $\alpha \in I(x^{-1})^A$, Theorem 9.2(ii) implies that $\alpha \in \Phi_{\tan}^{\mathrm{KL}}$ if and only if $\alpha \in \Phi_{\mathrm{cur}}^{\mathrm{KL}}$. This occurs if and only if $s_\alpha x \geq w$ by Proposition 8.3(ii), □

Corollary 10.2. *Let $\alpha \in x\Phi_P^-$.*

(i) *If $s_\alpha x > x$, then $\alpha \in \Phi_{\tan}$.*
(ii) *If $s_\alpha x < x$, then $\alpha \in I(x^{-1})$. In this case, if $\alpha \in I(x^{-1})^A$, then $\alpha \in \Phi_{\tan}$ if and only if $s_\alpha x \geq w$.*

Proof. (i) If $s_\alpha x > x$, then $\alpha \in x\Phi_P^- \cap \Phi^-$ by Corollary 8.2, so $\alpha \in \Phi_{\tan}$ by Corollary 8.4(ii). Note that $x\Phi_P^- \cap \Phi^-$ is the set of weights of $X_0^x = B^- \cdot xP$ (cf. (8.2) and the comment after this equation).

(ii) If $s_\alpha x < x$, then $\alpha \in I(x^{-1})$ by Corollary 8.2, and thus, by Corollary 8.4(iii), $\alpha \in \Phi_{\tan}$ if and only if $\alpha \in \Phi_{\tan}^{\mathrm{KL}}$. By Theorem 10.1, if $\alpha \in I(x^{-1})^A$, this occurs if and only if $s_\alpha x \geq w$. □

Corollary 10.3. *Let $X^w \subseteq G/B$ and let $\alpha \in x\Delta^-$. Then $\alpha \in \Phi_{\tan}$ if and only if $s_\alpha x \geq w$.*

Proof. For $\alpha \in x\Delta^-$, Corollary 9.3(iii) implies that $\alpha \in \Phi_{\tan}$ if and only if $\alpha \in \Phi_{\mathrm{cur}}$, which by Proposition 8.3(i) is equivalent to $s_\alpha x \geq w$. □

Recall from Section 7 that a root set $S \subseteq \Phi^+$ is said to be *coplanar* if there exists $v \in \mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in S$; and that x is said to be a *cominuscule* Weyl group element if $I(x^{-1})$ is coplanar. If $\Phi_{\tan}^{\mathrm{KL}}$ is coplanar, then x is called a *KL cominuscule point of X^w*. Since $\Phi_{\tan}^{\mathrm{KL}} \subseteq I(x^{-1})$, if x is a cominuscule Weyl group element, then x

is a KL cominuscule point of X^w. See [GK24] or [GK21] for further discussion of KL cominuscule points. The maximal parabolic subgroup $P \supseteq B$ (or sometimes G/P) is said to be *cominuscule* if the simple root corresponding to P occurs with coefficient 1 when the highest root of G is written as a linear combination of the simple roots.

Theorem 10.4. *Suppose that Φ is simply laced and that any of the following hold:*

(i) *All elements of $\Phi_{\tan}^{\mathrm{KL}}$ are integrally indecomposable in $\Phi_{\tan}^{\mathrm{KL}}$.*
(ii) *All elements of $I(x^{-1})$ are integrally indecomposable in $I(x^{-1})$.*
(iii) *x is fully commutative and Φ is of types A or D.*
(iv) *x is a KL cominuscule point of X^w.*
(v) *x is a cominuscule Weyl group element.*
(vi) *P is cominuscule.*

Then $\Phi_{\tan}^{\mathrm{KL}} = \Phi_{\mathrm{cur}}^{\mathrm{KL}}$ and $\Phi_{\tan} = \Phi_{\mathrm{cur}}$. If $\alpha \in x\Phi_P^-$, then $\alpha \in \Phi_{\tan}^{\mathrm{KL}}$ if and only if $x > s_\alpha x \geq w$, and $\alpha \in \Phi_{\tan}$ if and only if $s_\alpha x \geq w$.

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) by Corollary 7.4 and Lemma 2.2(v) respectively; (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (i) by [GK22, Proposition 6.7], definition, and Theorem 7.2 respectively.

Thus we may assume that (i) holds, i.e., $(\Phi_{\tan}^{\mathrm{KL}})^{\mathbb{Z}} = \Phi_{\tan}^{\mathrm{KL}}$. By Lemma 2.2(v), $(\Phi_{\mathrm{cur}}^{\mathrm{KL}})^{\mathbb{Z}} = \Phi_{\mathrm{cur}}^{\mathrm{KL}}$. Hence Theorem 9.2(i) implies $\Phi_{\tan}^{\mathrm{KL}} = \Phi_{\mathrm{cur}}^{\mathrm{KL}}$. From Corollary 8.4(ii) and (iv) it follows that $\Phi_{\tan} = \Phi_{\mathrm{cur}}$. Let $\alpha \in x\Phi_P^-$. By Corollary 8.4(v), $\alpha \in \Phi_{\tan}^{\mathrm{KL}} = \Phi_{\mathrm{cur}}^{\mathrm{KL}} \Leftrightarrow x > s_\alpha x \geq w$. By Proposition 8.3(i), $\alpha \in \Phi_{\tan} = \Phi_{\mathrm{cur}} \Leftrightarrow s_\alpha x \geq w$. □

The decomposition (8.2) implies that x is a smooth point of X^w if and only if x is a smooth point of Y_x^w. Theorem 10.4 yields the following smoothness criterion.

Corollary 10.5. *Suppose that Φ is simply laced. If any of the conditions of Theorem 10.4 are satisfied, then the following are equivalent:*

(i) *x is a smooth point of X^w and Y_x^w.*
(ii) *$|\{\alpha \in x\Phi_P^- : s_\alpha x \geq w\}| = \dim X^w$.*
(iii) *$|\{\alpha \in I(x^{-1}) : s_\alpha x \geq w\}| = \dim Y_x^w$.*

Proof. If any of the conditions of Theorem 10.4 is satisfied, then

$$|\Phi_{\tan}| = |\Phi_{\mathrm{cur}}| = |\{\alpha \in x\Phi_P^- : s_\alpha x \geq w\}|.$$

Thus, (ii) holds if and only if $|\Phi_{\tan}| = \dim X^w$, which by definition is equivalent to smoothness of X^w at x. Hence (i) $\Leftrightarrow$ (ii). The equivalence of (i) and (iii) is proved similarly. □

10.2. **Smoothness and reduced expressions.** Recall that $\mathbf{s} = (s_1, \ldots, s_l)$ is a reduced expression for x. Since $w \leq x$, $\mathbf{s}$ contains a reduced subexpression for w. Let $M = \{i \in [l] : (s_1, \ldots, \widehat{s}_i, \ldots, s_l)$ contains a reduced subexpression for $w\}$.

Lemma 10.6. *$|M| = \ell(x) - \ell(w)$ if and only if* $\mathbf{s}$ *contains a unique reduced subexpression for w.*

Proof. Let $1 \le i_1 < \cdots < i_t \le l$ be such that $(s_1, \ldots, \widehat{s}_{i_1}, \ldots, \widehat{s}_{i_t}, \ldots, s_l)$ is a reduced expression for w. If it is the unique reduced subexpression of $\mathbf{s}$ for w, then $M = \{i_1, \ldots, i_t\}$, and $|M| = l - \ell(w) = \ell(x) - \ell(w)$. If there exists another reduced subexpression of $\mathbf{s}$ for w, then $M \supsetneq \{i_1, \ldots, i_t\}$, and $|M| > \ell(x) - \ell(w)$. □

Theorem 10.7. *Suppose that Φ is simply laced. If any of the conditions (ii), (iii), (v), or (vi) of Theorem 10.4 are satisfied, then x is a smooth point of X^w if and only if* $\mathbf{s}$ *contains a unique reduced subexpression for w.*

Proof. First, we claim that since Φ is simply laced, x is a smooth point of X^w if and only if

$$|\{i \in [l] : x_i \ge w\}| = \ell(x) - \ell(w). \tag{10.1}$$

Indeed, for any $\alpha \in I(x^{-1})$, $\alpha = \gamma_i$ for some $i \in [l]$, and $s_\alpha x = x_i$. Hence the left side of (10.1) equals $|\{\alpha \in I(x^{-1}) : s_\alpha x \ge w\}|$. The right side of (10.1) equals $\dim Y^w_x$. Hence the claim follows from Corollary 10.5.

Conditions (iii), (v), and (vi) of Theorem 10.4 all imply condition (ii) of the same theorem; hence we may assume that $I(x^{-1})^{\mathbb{Z}} = I(x^{-1})$. By Theorem 7.2, $I(x^{-1})^{\ddagger} = I(x^{-1})$; hence $x_i = z_i$ for all i. Consequently,

$$\{i \in [l] : x_i \ge w\} = \{i \in [l] : z_i \ge w\}.$$

By Lemma 4.2, $z_i \ge w$ if and only if $(s_1, \ldots, \widehat{s}_i, \ldots, s_l)$ contains a reduced subexpression for w. Therefore $\{i \in [l] : z_i \ge w\} = M$. Therefore, x is a smooth point of X^w if and only if $|M| = \ell(x) - \ell(w)$. The result now follows from Lemma 10.6. □

For P cominuscule, the smoothness criterion of Theorem 10.7 applies even when Φ is not simply laced. This can be deduced from [GK15, Corollary 2.11], which states that the multiplicity of $x \in X^w$ when P is cominuscule is equal to the number of reduced subexpressions of $\mathbf{s}$ for w. Of course, x is a smooth point of X^w precisely when its multiplicity equals 1. Thus the result is obtained.

Further discussion of the smoothness criterion of Theorem 10.7 appears in [GK24].

11. Examples

In this section we consider examples. In Section 11.1, we determine the tangent space at w_0 to singular 3-dimensional Schubert varieties for an irreducible root system not of type G_2. Section 11.2 focuses on type D. We define a family of elements $w_{ab} \in W$, for $1 \le a < b < n-1$, and let $\Phi_{\tan}$ and Φ_{cur} be defined taking $x = w_0$ and $w = w_{ab}$. We show that $\Phi_{\tan}$ properly contains Φ_{cur}, and verify by direct calculation that $\mathrm{Cone}_A\, \Phi_{\mathrm{cur}} \supseteq \Phi_{\tan}$, as guaranteed by our main result Theorem 1.1.

Throughout this section, we take $P = B$ so $W^P = W$. In this section, $\Phi_{\tan}$ and Φ_{cur} always refer to the sets of curve and tangent weights at the point $x = w_0$.

11.1. **Three-dimensional Schubert varieties.** Schubert varieties of dimension 3 are of the form X^w, where either:

(1) $w = w_0 s_\alpha s_\beta s_\alpha$ for nonorthogonal simple roots α, β.

(2) $w = w_0 s_\alpha s_\beta s_\gamma$, where α, β, γ are distinct simple roots.

For w as in (1), it is shown in [Bri98] that if $\langle \beta, \alpha^\vee \rangle \leq -2$, then X^w is rationally smooth but not smooth at w_0. Using equivariant multiplicities, one can show that for w as in type (2) that X^w is smooth at w_0. The equivariant multiplicity at w_0 can be calculated using a formula for equivariant multiplicities due to Arabia and Rossmann ([Ara89, Prop. 3.3.1] and [Ros89]; see [Bri98, Section 4]); then a result from [Kum96] (see [Bri98, Cor. 19]) may be applied to deduce smoothness.

In light of this discussion, the following proposition describes the tangent space at w_0 to a singular 3-dimensional Schubert variety for an irreducible root system Φ not of type G_2.

Proposition 11.1. *Assume Φ is irreducible. Suppose α and β are simple roots with $\langle \beta, \alpha^\vee \rangle = -2$. Let $\Phi_{\tan}$ and Φ_{cur} correspond to $w = w_0 s_\alpha s_\beta s_\alpha$ and $x = w_0$. Then*

$$\Phi_{\mathrm{cur}} = \{\alpha, \beta, 2\alpha + \beta\} \text{ and } \Phi_{\tan} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}.$$

Proof. Since there are two root lengths in Φ, the action of w_0 on Φ is by multiplication by -1 (cf. [Hum72, Sec. 13, Ex. 5]), so w_0 is in the center of W.

Taking $x = w_0$, we have $\Phi_{\mathrm{cur}} = \{\gamma \in \Phi^+ \mid s_\gamma w_0 \geq w\}$. Equivalently, $\Phi_{\mathrm{cur}} = \{\gamma \in \Phi^+ \mid s_\gamma \leq s_\alpha s_\beta s_\alpha\}$. (The statements are equivalent because $s_\gamma w_0 \geq w$ is equivalent to $w_0 s_\gamma w_0 \leq w_0 w$; since w_0 is in the center of W, this is equivalent to $s_\gamma \leq s_\alpha s_\beta s_\alpha$.) Since the length of s_γ is odd, the only possibilities for s_γ are s_α, s_β, or $s_\alpha s_\beta s_\alpha = s_{s_\alpha \beta}$. We conclude that $\Phi_{\mathrm{cur}} = \{\alpha, \beta, s_\alpha \beta\} = \{\alpha, \beta, 2\alpha + \beta\}$.

We have

$$\Phi_{\mathrm{cur}} \subsetneq \Phi_{\tan} \subseteq \mathrm{Cone}_A \Phi_{\mathrm{cur}}. \tag{11.1}$$

The first inclusion is proper because X^w is singular at w_0, so $\dim T_{w_0} X^w > \dim X^w = 3 = |\Phi_{\mathrm{cur}}|$. The second inclusion follows from our main result. In this example, α and β span a root system of type B_2, with long root β. Thus, there are 4 roots in the cone spanned (over $\mathbb{Q}$ or $\mathbb{Z}$) by α, β, namely, $\alpha, \beta, s_\alpha \beta = 2\alpha + \beta, s_\beta \alpha = \alpha + \beta$, the first three of which are in Φ_{cur}. From (11.1), we conclude that $\Phi_{\tan}$ must consist of all 4 of these roots. $\square$

Note that for Φ of type G_2 (which occurs exactly when $\langle \beta, \alpha^\vee \rangle = -3$), there are more than 4 positive roots, so the arguments above do not suffice to determine $\Phi_{\tan}$.

11.2. **Examples in type D_n.** In this section we assume Φ is of type D_n. We define a family of elements w_{ab} $(a < b < n-1)$, and consider the tangent spaces $T_{w_0}X^{w_{ab}}$ at $x = w_0$. We write $\Phi_{\text{oth}} = \Phi_{\tan} \setminus \Phi_{\text{cur}}$ for the set of "other" roots in the tangent space of $X^{w_{ab}}$ at w_0. (Of course, $\Phi_{\tan}, \Phi_{\text{cur}}$, and Φ_{oth} all depend on the choice of $x = w_0$ and of $w_{ab} \in W$, but we omit this from the notation.) We will calculate Φ_{oth}, and identify enough elements of Φ_{cur} to show that each root in Φ_{oth} is a sum of two roots in Φ_{cur}. Thus, in this example, we can verify our main result that $\Phi_{\tan} \subseteq \text{Cone}_A \Phi_{\text{cur}}$ by direct calculation.

We have chosen $x = w_0$ in this section because the description of the tangent spaces $T_x(X^w)$ in [Lak00a] and [BL00] is much simpler for $x = w_0$ than for arbitrary x.

We use the standard realization of the root system of D_n as in Section 6. Let $w \in W$. We have $\Phi_{\text{cur}} = \{\gamma \in \Phi^+ \mid s_\gamma w_0 \geq w\}$. Applying [Lak00a, Theorem 6.8] or [BL00, Theorem 5.3.1], we see that $\gamma \in \Phi_{\text{oth}}$ if and only if $\gamma = \epsilon_i + \epsilon_j$, with $1 \leq i < j < n-1$, and

(1) $s_\gamma w_0 \not\geq w$, and

(2) $s_{\epsilon_i - \epsilon_n} s_{\epsilon_i + \epsilon_n} s_{\epsilon_j + \epsilon_{n-1}} w_0 \geq w$.

(In translating the result from [Lak00a] we have used the fact that w_0 commutes with $s_{\epsilon_i - \epsilon_n} s_{\epsilon_i + \epsilon_n} s_{\epsilon_j + \epsilon_{n-1}}$, as can easily be seen by writing the elements of W as signed permutations (cf. [BG03].)

Motivated by condition (2), for $1 \leq a < b < n-1$, we define $u_{ab} = s_{\epsilon_a - \epsilon_n} s_{\epsilon_a + \epsilon_n} s_{\epsilon_b + \epsilon_{n-1}}$, and set $w_{ab} = u_{ab} w_0$. If $w = w_{ab}$, then we claim that (1) and (2) are equivalent to the conditions:

(1′) $s_\gamma \not\leq u_{ab}$, and

(2′) $u_{ij} < u_{ab}$.

Indeed, condition (1) states that $s_\gamma w_0 \not\geq w_{ab} = u_{ab} w_0$, which is equivalent to (1′). Condition (2) states that $u_{ij} w_0 \geq w_{ab} = u_{ab} w_0$, which is equivalent to (2′). This verifies the claim.

Proposition 11.2. *Suppose Φ is of type D_n for $n \geq 4$. Fix a, b satisfying $1 \leq a < b < n-1$. Let $\Phi_{\tan}$, Φ_{cur}, and Φ_{oth} be defined as above, corresponding to $x = w_0$ and $w = w_{ab}$. Assume below that i, j denote integers satisfying $1 \leq i < j < n-1$. Then:*

(*a*) $\Phi_{\text{oth}} = \{\epsilon_i + \epsilon_j \mid i \geq a, j \geq b\}$.

(*b*) *Suppose $i \geq a$ and $j \geq b$. Then the roots $\epsilon_i \pm \epsilon_{n-1}$ and $\epsilon_j \pm \epsilon_{n-1}$ are in Φ_{cur}.*

(*c*) $\Phi_{\tan} \subseteq \text{Cone}_A \Phi_{\text{cur}}$.

Proof. We sketch the proof, but omit most details, which involve calculations in the Bruhat order in type D_n. We adopt the conventions and notation of [BG03]. The Weyl group of type D_n can be realized as the group of signed permutations of $1, \ldots, n$, with an even number of negative signs. If $u \in W$, write $u(i) = u_i$; then u can be represented by the sequence $u_1 u_2 \dots u_n$. Here $u_i \in \{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$, where $\overline{a}$ denotes $-a$. Write $\gamma = \epsilon_i + \epsilon_j$.

The proof uses two key facts about the Bruhat order. The first is the fact that if $u \in W$ and β is a positive root with $u\beta > 0$, then $u < us_\beta$. The second is a characterization of the Bruhat order in type D_n in terms of the sequences representing elements of W as signed permutations. This characterization is due to Proctor [Pro82]; the statement may also be found in [BG03, Prop. 2.10].

We first consider (a). We know that $\gamma \in \Phi_{\text{oth}}$ if and only if conditions $(1')$ and $(2')$ are satisfied. We claim first that $(2')$ is satisfied $\Leftrightarrow a \leq i$ and $b \leq j$. The implication $(\Rightarrow)$ is proved by considering the contrapositive. If either of $i \geq a$ or $j \geq b$ is false, then applying Proctor's condition to the signed permutations corresponding u_{ij} and u_{ab}, we see that $u_{ij} \not\leq u_{ab}$. The calculation is made easier because the parity condition in this characterization is not needed, and one can restrict attention to the places where the expressions for u_{ij} and u_{ab} differ from the identity permutation $12 \dots n$ (cf. [BG03, Lemma 3.4]). To prove $(\Leftarrow)$, we suppose $i \geq a$ and $j \geq b$. In this case, we can exhibit a sequence of positive roots $\beta_1, \dots, \beta_r$ such that

$$u_{ij} < u_{ij} s_{\beta_1} < \cdots < u_{ij} s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} = u_{ab}. \tag{11.2}$$

This calculation is facilitated by considering elements of W as signed permutations and using the description of multiplication by reflections in [BG03, (2.1)]. We omit further details. This proves the claim.

To complete the proof of (a), we need to verify that if $\gamma = \epsilon_i + \epsilon_j$, with $a \leq i$ and $b \leq j$, then $s_\gamma \not\leq u_{ab}$. This can be verified by writing down the expressions for s_γ and u_{ab} as signed permutations, and again using Proctor's characterization of the Bruhat order. We omit further details. This completes the proof of (a).

We next consider (b). We know a root ζ is in Φ_{cur} if and only if $s_\zeta w_0 \geq w_{ab} = u_{ab} w_0$, or equivalently, $s_\zeta < u_{ab}$. Thus, we want to show that if ζ is one of the four roots listed in the statement of (b), then $s_\zeta < u_{ab}$. Since $s_{\epsilon_j - \epsilon_{n-1}} < s_{\epsilon_i - \epsilon_{n-1}}$ and $s_{\epsilon_j + \epsilon_{n-1}} < s_{\epsilon_i + \epsilon_{n-1}}$, we only need to prove this for ζ equal to $\epsilon_i - \epsilon_{n-1}$ or $\epsilon_i + \epsilon_{n-1}$. We prove that $s_\zeta < u_{ab}$ in the same way the statement $u_{ij} \leq u_{ab}$ was proved in part (a), by exhibiting sequences analogous to (11.2). We omit further details.

Finally, part (c) follows from parts (a) and (b). Indeed, since

$$\epsilon_i + \epsilon_j = (\epsilon_i - \epsilon_{n-1}) + (\epsilon_j + \epsilon_{n-1}) = (\epsilon_i + \epsilon_{n-1}) + (\epsilon_j - \epsilon_{n-1}),$$

parts (a) and (b) imply that every root in Φ_{oth} is in $\text{Cone}_A \Phi_{\text{cur}}$. $\square$

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Department of Mathematics, University of Georgia, Boyd Graduate Studies Research Center, Athens, GA 30602

Email address: wag@uga.edu

Department of Mathematics and Physics, University of Wisconsin - Parkside, Kenosha, WI 53140

Email address: kreiman@uwp.edu