Homework 6: Monotonic and Bounded Sequences

Assignments should be ***stapled*** *and written clearly and legibly.*

1. Let (a_n) be a convergent sequence. Suppose that $\lim a_n > 0$. Use the definition of a limit to prove that there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$.

2. (a) Give an example of a sequence with subsequences converging to 1, 2, and 3.
   (b) Give an example of a sequence with subsequences converging to every integer.
   (c) Show that there exists a sequence with subsequences converging to every rational number.

3. Prove that the sequence (a_n) converges, where $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

4. In this problem, we give an algorithm for computing $\sqrt{2}$. Let $a_1 = 2$, and define

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \text{ for } n \geq 1. \tag{1}$$

   (a) Prove that $a_n^2 \geq 2$ for all n. (Use proof by induction.)
   (b) Use part (a) and equation (1) to prove that $a_n - a_{n+1} \geq 0$ for all n.
   (c) Conclude that the sequence (a_n) converges.
   (d) Prove that $\lim_{n\to\infty} a_n = \sqrt{2}$.
   (e) Modify the sequence (a_n) so that it converges to $\sqrt{c}$. No formal proof is required for this part, but you should give a brief justification.