Homework 8: Null Space and Column Space of a Matrix

1. In Homework 6, you found that $\{1, \ln(2x), \ln(x^2)\}$ is linearly dependent in $F(0,\infty)$. Let $W = \text{Span}\{1, \ln(2x), \ln(x^2)\}$.
   (a) How do we know that W is a subspace of $F(0,\infty)$?
   (b) Find a basis $\mathcal{B}$ for W, and find $\dim W$.
   (c) Determine whether $\ln(7x^{12})$ is in W. If so, find $[\ln(7x^{12})]_{\mathcal{B}}$.

   You should justify your answers, but proofs are not required.

2. A matrix A and an echelon form of A are given:

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for $\text{Nul}\,A$. What is $\dim(\text{Nul}\,A)$?
   (b) Find a basis for $\text{Col}\,A$. What is $\dim(\text{Col}\,A)$?

3. Without using a calculator or computer, find a nonzero vector in $\text{Nul}\,A$, where

$$A = \begin{bmatrix} 51 & 51 & 58 & 2 & 7 \\ 7 & 2001 & 9 & 1 & 2 \\ 3 & 17 & 5 & 0 & 2 \\ 9 & 2023 & 15 & 3 & 6 \\ 3 & \sqrt{2} & 8 & 37 & 5 \\ 7 & \pi & 23 & 19 & 16 \\ 11 & 3.14 & 14 & 0 & 3 \end{bmatrix}$$

4. For each of the following vector spaces, find a matrix A such that the vector space is equal to $\text{Nul}\,A$. Then find a basis for the vector space.
   (a) The line $y = 5x$ in $\mathbb{R}^2$.
   (b) The plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

5. Find a basis for $\text{Col}\left[\begin{smallmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 4 \end{smallmatrix}\right]$ without doing row reduction.

6. Find a basis for $\text{Col}\left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{smallmatrix}\right]$ without doing row reduction, and then a basis for $\text{Nul}\left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{smallmatrix}\right]$ without doing row reduction.

7. Construct a 2×3 matrix C such that $\operatorname{Nul} C = \operatorname{Col} \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$.

8. Construct a matrix A such that $\operatorname{Col} A = \operatorname{Nul} \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & -2 \\ 2 & 6 & 3 \end{bmatrix}$, and let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the three columns of A.

   (a) How many vectors are in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (b) How many vectors are in $\operatorname{Col} A$?

   (c) How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (d) Give two vectors in $\operatorname{Col} A$ which are not in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (e) Write the vector equation which is equivalent to the matrix equation $A\mathbf{x} = \mathbf{0}$.

   (f) Write the linear system of equations which is equivalent to the matrix equation $A\mathbf{x} = \mathbf{0}$.