Homework 8: Equivalence Relations and Quotient Sets

1. Let X be a nonempty set. A relation R on X is said to be **circular** if, for all $x, y, z \in X$, $x\,R\,y$ and $y\,R\,z$ implies $z\,R\,x$. Prove that a relation R on X is an equivalence relation if and only if R is reflexive and circular.

2. Let X be the set of directed line segments PQ (with initial point P and terminal point Q) in $\mathbb{R}^2$. Give an equivalence relation $\sim$ for which $X/\sim$ is the usual set of vectors in $\mathbb{R}^2$.

3. Consider $\mathbb{R}$ with relation $Q = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.
   (a) Verify that Q is an equivalence relation.
   (b) Find the equivalence classes.
   (c) Identify $\mathbb{R}/Q$ with an interval in $\mathbb{R}$.
   (d) Identify $\mathbb{R}/Q$ with S^1.

4. Consider $\mathbb{R}^2$ with relation $(a, b) \sim (c, d)$ if $a + 2d = c + 2b$.
   (a) Verify that $\sim$ is an equivalence relation.
   (b) What are the equivalence classes?

5. Let $X = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0\}$. Define relation $\sim$ on X by $(a, b) \sim (c, d)$ if $ad = bc$.
   (a) Verify that $\sim$ is an equivalence relation.
   (b) Give three elements in $[(1, 2)]$.
   (c) $X/\sim$ is sometimes used to define a set with which you are familiar. Give the set.

6. (Number Theory) Consider $\mathbb{Z}$ with relation $a \sim b$ if $5 \mid a - b$.
   (a) Verify that $\sim$ is an equivalence relation.
   (b) Find the equivalence classes.
   (c) What is the cardinality of $\mathbb{Z}/\sim$?

7. (Linear Algebra) Let V be a vector space and W a subspace. Consider the relation $\sim$ on V defined by $\mathbf{u} \sim \mathbf{v}$ if $\mathbf{u} - \mathbf{v} \in W$.
   (a) Verify that $\sim$ is an equivalence relation. The corresponding quotient $V/\sim$ is usually denoted by V/W.
   (b) Let $V = \mathbb{R}^2$ and $W = \text{Span}[\begin{smallmatrix}1\\1\end{smallmatrix}]$, a subspace of V. What are the equivalence classes? What is V/W?