Homework 16: The Riemann Integral (Due May 2, 2022)

*Directions. Assignments should be **stapled** and written clearly and legibly. Problems 5 and 6 are optional.*

1. Let $f(x) = x^2 - x$ and $P = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$. Find $U(f, P)$ and $L(f, P)$.

2. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all $x \in [a, b]$, and $f(x) > 0$ for at least one value $c \in [a, b]$. Using definitions, prove that $\int_a^b f > 0$. (You may assume that f is integrable.)

3. §5.4, #5.

4. §7.1, #15.

5. (Challenge) Consider the function $f : [0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

   Prove that f is integrable and find $\int_a^b f$.

6. (Challenge) Let h be Thomae's function of Homework 15, Problem 7. In this problem, you will prove that h is integrable.
   (a) Prove that $L(h, P) = 0$ for any partition P of $[0, 2]$.
   (b) Let $\epsilon > 0$. Let $S = \{x \in [0, 2] : h(x) > \epsilon/4\}$. Determine whether S is finite or infinite.
   (c) Explain how to construct a partition P of $[0, 2]$ for which $U(h, P) < \epsilon$. Prove that your partition works.
   (d) Complete the proof.