Homework 13: Metric Spaces

*Assignments should be **stapled**. Problem 5 is optional.*

1. §5.1, #5.6, 5.8.
2. §5.3, #5.25.
3. §6.1, #6.2.
4. (Linear Algebra) In this problem you will prove that the Euclidean metric d on $\mathbb{R}^n$ is a metric. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define $\mathbf{x} + \mathbf{y}$, $c\mathbf{x}$, and $\mathbf{x} \cdot \mathbf{y}$ to be the standard vector addition, scalar multiplication, and dot product respectively.
   (a) Prove that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.
   (b) Prove the Cauchy-Schwartz inequality: $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.
   Hint: If $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\|^2 \geq 0$.
   (c) Prove that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
   Hint: Compute $\|\mathbf{x} + \mathbf{y}\|^2$ and apply (b).
   (d) Verify that d is a metric.
5. (Challenge) §5.3, #5.23.