Homework 2: Basis for a Topology

*Assignments should be **stapled** and written clearly and legibly.*

1. §1.1, #1.4.

2. §1.2, #1.11(a),(b).

3. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases on a set X, and let $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}'}$ be the corresponding topologies generated by the bases. Consider the following two statements:

   (i) *For every $B' \in \mathcal{B}'$ and $x \in B'$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B'$.*
   (ii) *For every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

   (a) Prove that $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$ if and only if (i) holds.

   (b) Prove that $\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}'}$ if and only if (i) and (ii) both hold. (Hint: use part (a).)

   (c) Use the Union Lemma to express (i) in terms of unions.

4. §1.2 #1.12, 1.13.