

Homework 7: Topology of \mathbb{R} (Due March 28, 2023)

Assignments should be **stapled** and written clearly and legibly. Problem 6 is optional.

1. Define the following (as in class): open set, closed set, limit point, A' , \bar{A} . In the problems below, you may use only these definitions. You may not use any theorems from the sections we've covered on topology.
2. §3.4, #14, 15.
3. Let $A \subseteq B$ be two sets of real numbers. We say that A is **dense** in B if $\bar{A} = B$. Prove that \mathbb{Q} is dense in \mathbb{R} . (In other words, prove $\bar{\mathbb{Q}} = \mathbb{R}$.)
4. Let A be a set of real numbers. Prove the following:
 - (a) A' is closed.
 - (b) $(\bar{A})' = A'$.
 - (c) \bar{A} is closed.
5. Let A be a set of real numbers. Prove that if B is a closed set and $B \supseteq A$, then $B \supseteq \bar{A}$.
Note. Problems 4(c) and 5 tell us that \bar{A} is the smallest closed set containing A .
6. (Challenge) For sets $A, B \subseteq \mathbb{R}$, the **Minkowski sum** of A and B is defined to be

$$A + B = \{a + b : a \in A, b \in B\}.$$

- (a) Prove that if A and B are open, then $A + B$ is open.
- (b) Disprove the following by giving a counterexample: If A and B are closed, then $A + B$ is closed. (Hint: for any such counterexample, neither A nor B can be bounded.)