Homework 4: Subspace, Linear Combination, Span

*Assignments should be **stapled** and written clearly and legibly.*

1. §4.2, #3(b), #5(a)(b), 10(b)(c), 12(b)(d), 14(a)(b)(c).

   Note: All answers to textbook problems should be justified.

2. Give an example of two nonzero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ such that $\mathbf{u} \neq \mathbf{v}$ and $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is not equal to $\mathbb{R}^2$.

3. Let $\mathbf{u}, \mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ be vectors in a vector space V. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and c is a scalar. Prove that:

   (a) $\mathbf{x} + \mathbf{y}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   (b) $c\mathbf{x}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   Note: For this problem and the next one, you may not use any theorems. You should use only definitions.

4. Let $\mathbf{u}, \mathbf{v}$, $\mathbf{w}$, be vectors in a vector space V.

   (a) Prove that $\text{span}\{\mathbf{u}, \mathbf{v}\} \subseteq \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   Hint. Begin the proof as follows:

   "Suppose that $\mathbf{z}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}\}$. I must show that $\mathbf{z}$ is also in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$."

   (b) Supppose that $\mathbf{w}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}\}$. Prove that $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq \text{span}\{\mathbf{u}, \mathbf{v}\}$.

   Hint. Begin the proof as follows:

   "Suppose that $\mathbf{z}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. I must show that $\mathbf{z}$ is also in $\text{span}\{\mathbf{u}, \mathbf{v}\}$."

   (c) Use parts (a) and (b) to prove that if $\mathbf{w}$ is in $\text{span}\{\mathbf{u}, \mathbf{v}\}$, then $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{span}\{\mathbf{u}, \mathbf{v}\}$.

   For part (c) you should use the definition of what it means for two sets S and T to be equal: $S = T$ if $S \subseteq T$ and $T \subseteq S$.

5. (Putnam Competition) Let S be a set and let $\circ$ be a binary operation on S satisfying the two laws

$$x \circ x = x \text{ for all } x \text{ in } S, \text{ and}$$
$$(x \circ y) \circ z = (y \circ z) \circ x \text{ for all } x, y, z \text{ in } S.$$

   Show that $\circ$ is associative and commutative.