LINEAR ALGEBRA (MATH 301) DEFINITIONS AND THEOREMS
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These notes contain a list of the main definitions, lemmas, propositions, and theorems discussed in this course.

1. Subspace, Linear Combination, Span, Linear Independence, Basis

Definition 1.1. *A **vector space** is ...*

Definition 1.2. *A **subspace** of a vector space V is a subset W of V which is a vector space (under the addition and scalar multiplication of V).*

Definition 1.3. *A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a vector of the form $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r$, where $k_1, \ldots, k_r$ are scalars.*

Definition 1.4. *The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_r$, denoted $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.*

Definition 1.5. *Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is **linearly independent** if $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ implies $k_1 = \cdots = k_r = 0$.*

Definition 1.6. *A **basis** for a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that*

(i) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, *and*
(ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is linearly independent.*

Definition 1.7. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a vector space V, and let $\mathbf{u}$ be a vector in V. Then $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for unique scalars $c_1, \ldots, c_n$, by Theorem 1.6. The scalars $c_1, \ldots, c_n$ are called the **coordinates** of $\mathbf{u}$ relative to $\mathcal{B}$, and the vector*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

*denoted by $[\mathbf{u}]_{\mathcal{B}}$, is called the **coordinate vector** of $\mathbf{u}$ relative to $\mathcal{B}$.*

Definition 1.8. *The **dimension** of a nonzero vector space V is the number of vectors in any basis for V.*

Definition 1.9. *Let $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ be an $m \times n$ matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$. Then $A\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.*

Definition 1.10. *Let A be an $m \times n$ matrix. The **nullspace** of A, denoted by $\operatorname{Nul} A$, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$.*

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Definition 1.11. *Let A be an $m \times n$ matrix. The **column space** of A, denoted by* $\operatorname{Col} A$, *is the span of the column vectors of A.*

Theorem 1.1. *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar. Then*

(i) $0\mathbf{u} = \mathbf{0}$
(ii) $k\mathbf{0} = \mathbf{0}$
(iii) $(-1)\mathbf{u} = -\mathbf{u}$

Theorem 1.2. *Let V be a vector space and let W be a subset of V. Then W is a subspace of V if:*

(i) *for every $\mathbf{u}$ and $\mathbf{v}$ in W, $\mathbf{u} + \mathbf{v}$ is in W (i.e., W is closed under vector addition); and*
(ii) *for every $\mathbf{u}$ in W and scalar k, $k\mathbf{u}$ is in W (i.e., W is closed under scalar multiplication); and*
(iii) *the zero vector of V lies in W.*

Theorem 1.3. *Let V be a vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in V. Then* $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ *is a subspace of V.*

Theorem 1.4. *A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.*

Theorem 1.5. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.*

(i) *If any vector of V is added to $\mathcal{B}$, then $\mathcal{B}$ is no longer linearly independent.*
(ii) *If any vector is removed from $\mathcal{B}$, then $\mathcal{B}$ no longer spans V.*

Theorem 1.6. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a basis of a vector space V. Then every $\mathbf{u}$ in V can be written in exactly one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, that is, can be expressed as*

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n,$$

for unique scalars $c_1, \ldots, c_n$.

Theorem 1.7. *All bases of a vector space V have the same number of elements.*

Theorem 1.8. *In $\mathbb{R}^n$, the following have the same solutions:*

(i) *The vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{u}$.*
(ii) *The linear system of equations with augmented matrix $[\mathbf{v}_1 \cdots \mathbf{v}_p \mid \mathbf{u}]$.*
(iii) *The matrix equation $[\mathbf{v}_1 \ \cdots \mathbf{v}_p]\,\mathbf{x} = \mathbf{u}$.*

Lemma 1.1. *Let A be an $m \times n$ matrix, let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then*

(i) *$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and*
(ii) *$A(c\mathbf{u}) = c(A\mathbf{u})$.*

Theorem 1.9. *Let A be an $m \times n$ matrix. Then* $\operatorname{Nul} A$ *is a subspace of $\mathbb{R}^n$.*

2. Introduction to Linear Transformations

Definition 2.1. *Let V and W be vector spaces. A transformation (or mapping) $T : V \to W$ is **linear** if it satisfies the following conditions:*

(i) *For every $\mathbf{u}, \mathbf{v}$ in V, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.*
(ii) *For every $\mathbf{u}$ in V and scalar c, $T(c\mathbf{u}) = cT(\mathbf{u})$.*

Theorem 2.1. *Let $T : V \to W$ be linear. Then*

(i) $T(\mathbf{0}) = \mathbf{0}$.
(ii) $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$, *for any scalars $c_1, \ldots, c_p$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in V.*

Definition 2.2. *A **matrix transformation** is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, for some fixed $m \times n$ matrix A.*

Theorem 2.2. *A matrix transformation is linear.*

Definition 2.3. *Let $T : V \to W$ be linear. Then*

(i) *The **kernel** of T, denoted $\ker(T)$, is the set of vectors in V which T maps to $\mathbf{0}$.*
(ii) *The **range** of T, denoted $R(T)$, is the set of vectors in W which have at least one vector in V mapping to them.*

Theorem 2.3. *Let $T : V \to W$ be linear. Then $\ker(T)$ is a subspace of V and $R(T)$ is a subspace of W.*

Theorem 2.4. *Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Then $\ker(T) = \operatorname{Nul} A$ and $R(T) = \operatorname{Col} A$.*

Theorem 2.5. *Let $T : V \to W$ be linear. Then $\dim(\ker T) + \dim(R(T)) = \dim V$.*

Theorem 2.6. *Let $T : V \to W$ be linear. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.*

Theorem 2.7. *Let W be a subspace of V. If $\dim W = \dim V$, then $W = V$.*

Theorem 2.8. *Let $T : V \to W$ be linear, and suppose that $\dim V = \dim W$. Then T is one-to-one if and only if T is onto.*

Definition 2.4. *Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the **composition** of S with T, denoted $S \circ T$, is the map from U to W defined by $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$ for $\mathbf{u} \in U$.*

Theorem 2.9. *Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the composition $S \circ T : U \to W$ is a linear transformation.*

Definition 2.5. *For any vector space V, the **identity transformation** $I : V \to V$ is defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V.*

Theorem 2.10. *Let $T : V \to W$ be a linear transformation. Then $T \circ I = I \circ T = T$.*

Definition 2.6. *Let $T : V \to W$ be one-to-one. Then there exists an **inverse transformation** $T^{-1} : R(T) \to V$ such that $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V.*

Theorem 2.11. *Let $T : V \to W$ be one-to-one. Then $T^{-1} \circ T = I$.*

Definition 2.7. *An **isomorphism** is a bijective linear transformation.*

Definition 2.8. *If $T : V \to W$ is an isomorphism, then V and W are said to **isomorphic**.*

Theorem 2.12. *If $T : V \to W$ is an isomorphism, then* $\dim V = \dim W$.

Theorem 2.13. *Suppose that V is a vector space and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V. Then the mapping $T : V \to \mathbb{R}^n$ given by $T(\mathbf{u}) = [\mathbf{u}]_B$ is an isomorphism.*

3. The Matrix of a Linear Transformation

Theorem 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then for $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$. The matrix $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ is called the **standard matrix** for T.*

Theorem 3.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then T is a linear transformation if and only if T is a matrix transformation.*

Theorem 3.3. *Suppose that the standard matrix for S is A and the standard matrix for T is B. Then the standard matrix for $S \circ T$ is AB.*

Definition 3.1. *Let A be an $n \times n$ matrix. Then A is said to be **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called the **inverse** of A, and we write $B = A^{-1}$.*

Theorem 3.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let A be the standard matrix for T. Then T is an isomorphism if and only if A is invertible. In this case, the standard matrix for T^{-1} is A^{-1}.*

Theorem 3.5. *Let A be an $n \times n$ matrix. Then A is invertible if and only if A can be row reduced to I_n.*

Theorem 3.6. *Let A be an $n \times n$ matrix, and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

Theorem 3.7. *Let $T : V \to W$ be linear. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W. Then there exists a matrix $[T]_{\mathcal{B}',\mathcal{B}}$ such that for every $\mathbf{v}$ in V, $[T(\mathbf{v})]_{\mathcal{B}'} = [T]_{\mathcal{B}',\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$.*

Theorem 3.8. *Let $T : V \to W$ be linear. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for V and $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W. Then*

$$[T]_{\mathcal{B}',\mathcal{B}} = \Big[[T(\mathbf{u}_1)]_{\mathcal{B}'} \cdots [T(\mathbf{u}_n)]_{\mathcal{B}'} \Big]$$

Theorem 3.9. *Let $T : U \to V$ and $S : V \to W$ be linear. Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be bases for vector spaces U, V, W respectively. Then $[S \circ T]_{\mathcal{B}'',\mathcal{B}} = [S]_{\mathcal{B}'',\mathcal{B}'} \cdot [T]_{\mathcal{B}',\mathcal{B}}$.*

Definition 3.2. *Let $\mathcal{B}, \mathcal{B}'$ be bases for a vector space V. Then $[I]_{\mathcal{B}',\mathcal{B}}$ is called the **change of coordinates matrix** from $\mathcal{B}$ to $\mathcal{B}'$ coordinates.*

Theorem 3.10. *Let $\mathcal{B}, \mathcal{B}'$ be bases for a vector space V. Then*

(i) *For any $\mathbf{v}$ in V, $[\mathbf{v}]_{\mathcal{B}'} = [I]_{\mathcal{B}',\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$.*
(ii) *$[I]_{\mathcal{B},\mathcal{B}} = I_n$, where $n = \dim V$.*
(iii) *$[I]_{\mathcal{B}',\mathcal{B}}$ is invertible.*
(iv) *$([I]_{\mathcal{B}',\mathcal{B}})^{-1} = [I]_{\mathcal{B},\mathcal{B}'}$.*

Notation 3.1. *$[T]_{\mathcal{B},\mathcal{B}}$ is often denoted by just $[T]_{\mathcal{B}}$.*

Theorem 3.11 (Change of Basis Formula)**.** *Let $T : V \to V$ be a linear operator. Let $\mathcal{B}, \mathcal{B}'$ be bases for V. Then*

$$[T]_{\mathcal{B}'} = [I]_{\mathcal{B}',\mathcal{B}} \cdot [T]_{\mathcal{B}} \cdot [I]_{\mathcal{B},\mathcal{B}'}$$

4. Inner Product Spaces

Definition 4.1. *Let V be a vector space. An* ***inner product*** *on V is a rule which assigns to each pair of vectors $\mathbf{u}, \mathbf{v}$ in V a scalar, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c,*

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
(ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
(iii) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.
(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, *with equality if and only if* $\mathbf{v} = \mathbf{0}$.

A vector space with an inner product is called an ***inner product space****.*

Definition 4.2. *Let V be an inner product space.*

(i) *For $\mathbf{v}$ in V, the* ***norm*** *(or* ***length****) of $\mathbf{v}$ is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.*
(ii) *For $\mathbf{u}, \mathbf{v}$ in V, the* ***distance*** *between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.*
(iii) *A* ***unit vector*** *is a vector of norm 1.*
(iv) *The set of all unit vectors in V is called the* ***unit circle*** *of V.*

Definition 4.3. *Let V be an inner product space. Vectors $\mathbf{u}$ and $\mathbf{v}$ in V are* ***orthogonal*** *if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.*

Definition 4.4. *A set S of two or more vectors in an inner product space is said to be* ***orthogonal*** *if every two distinct vectors in S are orthogonal. The set S is* ***orthonormal*** *if S is orthogonal and consists entirely of unit vectors.*

Definition 4.5. *Let V be an inner product space and let W be a subspace of V. The* ***orthogonal complement*** *of W, denoted $W^{\perp}$, is the set of vectors of V which are orthogonal to all vectors in W.*

Theorem 4.1. *Let V be an inner product space. Then*

(i) $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ *and* $\langle \mathbf{0}, \mathbf{v} \rangle = 0$, *for every $\mathbf{v}$ in V.*
(ii) $\langle c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \mathbf{w} \rangle = c_1\langle \mathbf{v}_1, \mathbf{w} \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{w} \rangle$, *for all scalars $c_1, \ldots, c_n$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}$.*

Theorem 4.2. *Let V be an inner product space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in V and let c be a scalar. Then*

(i) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
(ii) $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

Theorem 4.3. *Let V be an inner product space. Let $\mathbf{v}$ be in V and let c be a scalar. Then*

(i) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.
(ii) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ *is a unit vector, if* $\mathbf{v} \neq \mathbf{0}$.

Theorem 4.4. *If* $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ *is an orthogonal set of nonzero vectors in an inner product space, then* S *is linearly independent.*

Theorem 4.5. *Let* V *be an inner product space, and let* $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *be an orthogonal basis for* V*. Then for* $\mathbf{u}$ *in* V*,* $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$*, where*

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \quad \text{for } i = 1, \dots, n.$$

Theorem 4.6. *Let* W *be a subspace of an inner product space* V*. Then*

(i) $W^\perp$ *is a subspace of* V*; and*
(ii) $W \cap W^\perp = \{\mathbf{0}\}$.

Theorem 4.7. *Let* $\mathbf{u}, \mathbf{v}$ *be vectors in an inner product space* V *with* $\mathbf{v} \neq \mathbf{0}$*. Let* $L = \operatorname{span}\{\mathbf{v}\}$*, a one-dimensional subspace of* V*. The we can uniquely write* $\mathbf{u} = \mathbf{y} + \mathbf{z}$*, with* $\mathbf{y}$ *in* L *and* $\mathbf{z}$ *in* $L^\perp$*. Explicitly,*

$$\mathbf{y} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}, \quad \text{and} \quad \mathbf{z} = \mathbf{u} - \mathbf{y}.$$

The vector $\mathbf{y}$ *is called the* ***orthogonal projection of*** $\mathbf{u}$ ***onto*** L *and denoted by* $\operatorname{proj}_L \mathbf{u}$ *or* $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Theorem 4.8. *Let* $\mathbf{u}$ *be a nonzero vector in an inner product space* V*, and let* W *be a finite dimensional subspace of* V*. Then we can uniquely write* $\mathbf{u} = \mathbf{y} + \mathbf{z}$*, with* $\mathbf{y}$ *in* W *and* $\mathbf{z}$ *in* $W^\perp$*. The vector* $\mathbf{y}$ *is called the* ***orthogonal projection of*** $\mathbf{u}$ ***onto*** W *and denoted by* $\operatorname{proj}_W \mathbf{u}$*, and* $\mathbf{z}$ *is called the* ***component of*** $\mathbf{u}$ ***orthogonal to*** W*.*

5. Determinants

Theorem 5.1. *Let* A *be a square matrix.*

(i) *If two rows of* A *are interchanged to produce a matrix* B*, then* $\det B = -\det A$.
(ii) *If one row of* A *is multiplied by a constant* k *to produce* B*, then* $\det B = k \det A$.
(iii) *If a multiple of one row of* A *is added to another row to produce* B*, then* $\det B = \det A$.

Theorem 5.2. *Let* A *be a square matrix. Then* A *is invertible if and only if* $\det A \neq 0$.

6. Eigenvectors and Eigenvalues

Definition 6.1. *Let* A *be an* $n \times n$ *matrix. An* ***eigenvector*** *of* A *is a nonzero vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$ *for some scalar* λ*. The scalar* λ *is called the* ***eigenvalue*** *corresponding to* $\mathbf{v}$*.*

Theorem 6.1. *Let* A *be an* $n \times n$ *matrix. Then* λ *is an eigenvalue of* A *if and only if* $\det(\lambda I_n - A) = 0$.

Definition 6.2. *Let* A *be an* $n \times n$ *matrix. The* ***characteristic polynomial*** *of* A *is* $\det(\lambda I_n - A)$.