Homework 3: Vector Spaces

Instructions. Assignments should be **stapled** and written clearly and legibly. Problem 4 is optional.

1. §4.1, #2, TF (True-False Exercises).

2. Let V be the set of all matrices of real numbers with one column and two rows, with addition and scalar multiplication defined as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 3u_2v_2 \end{bmatrix} \qquad \text{and} \qquad c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$$

   Determine whether V is a vector space. If not, identify all vector space axioms which fail, and briefly explain why each of these axioms fail (in most cases, a counterexample will be sufficient). If axiom 3 holds, then give an explicit additive identity.

3. Let V be a vector space. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in V, and let b and c be scalars. Using only the definition of a vector space, prove

   (a) $c\,\mathbf{0} = \mathbf{0}$

   (b) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{v}+(\mathbf{w}+\mathbf{u})$ (Hint: use only the commutative and associative axioms.)

   (c) $(b + c)(\mathbf{u} + \mathbf{v}) = (c\mathbf{u} + c\mathbf{v}) + (b\mathbf{u} + b\mathbf{v})$

4. (Challenge) Prove that there does not exist a vector space which contains exactly two elements.