Homework 9: Kernel, Range

1. §8.1, #10, 11, 24.

2. Let V and W be vector spaces, and let $T : V \to W$ be linear. Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ spans V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ spans $R(T)$.

3. (Bonus) Let $T : V \to W$ be a linear transformation. Let w be an element of W, and let v_0 be an element of V which T maps to w. Prove that the set of all vectors which T maps to w is equal to $\{v_0 + u : u \in \ker T\}$. (Note that this is not, in general, a subspace of V (why not?).)

4. (Bonus – Differential Equations) Let $V = W = C^\infty(-\infty, \infty)$ be the vector space of infinitely differentiable functions. Let $T : V \to W$ be the linear transformation

$$T(f) = a_m f^{(m)} + a_{m-1} f^{(m-1)} + \cdots + a_1 f,$$

where $a_m, \dots, a_1$ are constants.

   (a) Recall that $\ker T = \{f \in V : T(f) = 0\}$. What does $\ker T$ represent, from the perspective of differential equations?

   (b) Let $g \in W$, and let $H = \{h \in V : T(h) = g\}$. What does H represent, from the perspective of differential equations?

   (c) Let h_0 be any element of H. Use Problem 3 to prove that $H = \{h_0 + f : f \in \ker T\}$.