Homework 17: The Fundamental Theorem of Calculus (Due May 13, 2022)

Assignments should be ***stapled*** *and written clearly and legibly. Problems 4(f) and 5 are optional.*

1. §7.2, #6, 13.

2. §7.3, #7, 12. Make sure to fully justify your answer to #7.

3. Determine whether each of the following is true or false. Justify each answer.
   (a) If $f = F'$ for some F on $[a, b]$, then f is continuous on $[a, b]$.
   (b) If f is continuous on $[a, b]$, then $f = F'$ for some F on $[a, b]$.
   (c) If $A(x) = \int_a^x f$ is differentiable at some $c \in [a, b]$, then f is continuous at c.

4. Let f be integrable on $[a, b]$. Recall from class that the **average value** of f on $[a, b]$ is defined to be
$$\operatorname{avg}(f) = \frac{1}{b-a} \int_a^b f$$
   (a) Suppose that F is an antiderivative of f. Prove that $\operatorname{avg}(f)$ is the average rate of change of F over $[a, b]$.
   (b) Prove that $\int_a^b \operatorname{avg}(f) = \int_a^b f$.
   (c) Prove that if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m \leq \operatorname{avg}(f) \leq M$.
   (d) Prove that if f is continuous on $[a, b]$, then f assumes its average value for some $c \in [a, b]$; in other words, there exists $c \in [a, b]$ for which $f(c) = \operatorname{avg}(f)$.
   (e) Give an example showing that the conclusion of (d) may fail if f is not continuous on $[a, b]$.
   (f) Part (d) is called the **mean value theorem for integrals**. Which is stronger: the mean value theorem, or the mean value theorem for integrals?

5. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Show that there exists $c \in [0, 1]$ such that
$$f(c) = \int_0^1 f(x) 3x^2 \, dx.$$