

## Homework 13: Metric Spaces

*Assignments should be **stapled**. Problems 4 and 5 are optional.*

1. §5.1, #5.8.
2. §5.3, #5.25, 5.29(a).
3. §6.1, #6.2.
4. (Challenge) §5.2, #5.23.
5. (Linear Algebra) In this problem you will prove that the Euclidean metric  $d$  on  $\mathbb{R}^n$  is a metric. For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , define  $\mathbf{x} + \mathbf{y}$ ,  $c\mathbf{x}$ , and  $\mathbf{x} \cdot \mathbf{y}$  to be the standard vector addition, scalar multiplication, and dot product respectively.
  - (a) Prove that  $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$ .
  - (b) Prove the Cauchy-Schwartz inequality:  $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ .  
Hint: If  $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ , let  $a = 1/\|\mathbf{x}\|$  and  $b = 1/\|\mathbf{y}\|$ , and use the fact that  $\|a\mathbf{x} + b\mathbf{y}\|^2 \geq 0$ .
  - (c) Prove that  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ .  
Hint: Compute  $\|\mathbf{x} + \mathbf{y}\|^2$  and apply (b).
  - (d) Verify that  $d$  is a metric.