Recursively Defined Wave Functions

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Abstract

This is the abstract.

1 Introduction

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2 Compact Convergence

(I'll say more before this at some point.) Consider the double angle formula for cosine, $\cos 2x = 2\cos^2 x - 1$. This can be rewritten as $\cos x = 2\cos^2 \frac{x}{2} - 1$, so $\cos x$ can be thought of as a fixed point of the mapping $f(x) \mapsto 2f(\frac{x}{2})^2 - 1$. This raises the question, can we get a sequence which converges to cosine by iterating this mapping? As it turns out, we can. Recall that a sequence of functions converges compactly if it converges uniformly on any compact subset of their domain.

Theorem 1. *Let $f_0(x) = x$ and $f_{n+1}(x) = 2f_n(\frac{x}{2})^2 - 1$. Then $f_n(x) \to \cos x$ compactly as $n \to \infty$.*

Proof. We claim that $f_n(x) = \cos(2^n \cos^{-1}(\frac{x}{2^n}))$ for all $|x| \leq 2^n$ and proceed by induction. For $n = 0$, clearly $f_0(x) = \cos(\cos^{-1} x)$ for $|x| \leq 1$, which proves the base case. Suppose $f_n(x) = \cos(2^n \cos^{-1}(\frac{x}{2^n}))$ for all $|x| \leq 2^n$ for some n. Then if $|x| \leq 2^{n+1}$ then $|\frac{x}{2}| \leq 2^n$, so $f_{n+1}(x) = 2f_n(\frac{x}{2})^2 - 1 = 2\cos^2(2^n \cos^{-1}(\frac{x}{2^{n+1}})) - 1 = \cos(2^{n+1}\cos^{-1}(\frac{x}{2^{n+1}}))$. This completes the induction step. It remains to show that this sequence converges uniformly on every $[-r, r]$ to $\cos x$. Observe that since $\cos^{-1} x$ is smooth, $\cos^{-1} x = \frac{\pi}{2} - x + x^2 g(x)$ for some smooth $g(x)$. Therefore,

$$\begin{aligned} f_n(x) &= \cos\left(2^n\left(\frac{\pi}{2} - \frac{x}{2^n} + \frac{x^2}{2^{2n}} g\left(\frac{x}{2^n}\right)\right)\right) \\ &= \cos\left(2^{n-1}\pi - x + \frac{x^2}{2^n} g\left(\frac{x}{2^n}\right)\right) \\ &= \cos\left(x - \frac{x^2}{2^n} g\left(\frac{x}{2^n}\right)\right) \end{aligned}$$

for $n \geq 2$. Since $g(x)$ is smooth and therefore continuous, $\frac{x^2}{2^n} g(\frac{x}{2^n}) \to 0$ uniformly on every closed interval $[-r, r]$ as $n \to \infty$. Therefore, since $\cos x$ is uniformly continuous, $\cos(x - \frac{x^2}{2^n} g(\frac{x}{2^n})) \to \cos x$ uniformly on every $[-r, r]$ as $n \to \infty$, and the result follows. □

Now let's consider this more general functional equation for real numbers $c > 0$ and integers $p \geq 2$:

$$f(x) = 2f(cx)^p - 1. \tag{1}$$

Our goal is to prove that analytic solutions exist and can be obtained by iterating this equations. We will also examine the extent to which these solutions are unique, and the properties that they have. For now we will treat only the case when p is even. Define the sequence of functions $f_0(x), f_1(x), \ldots$ as follows:

$$f_0(x) = x \tag{2}$$

$$f_n(x) = 2f_{n-1}(cx)^p - 1, n \geq 0. \tag{3}$$

If this sequence converges compactly, its limit will be a solution to this equation. As we will see, when $c < \frac{1}{\sqrt[p]{2p}}$ the sequence simply converges to 1, and when $c > \frac{1}{\sqrt[p]{2p}}$ the limit does not exist, but when $c = \frac{1}{\sqrt[p]{2p}}$ it converges to an analytic function whose power series begins with $1 - \frac{1}{p}x^p$. Note that when $p = 2$, $c = \frac{1}{2}$, and the first two terms of the power series are $1 - \frac{1}{2}x^2$, as we would expect.

Lemma 2. *Let p be a positive even integer and $c = \frac{1}{\sqrt[p]{2p}}$. Then $|f_n(x)| \leq 1$ for every n and $|x| \leq (2p)^{n/p}$.*

Proof. Observe that if $|y| \leq 1$ then $|2y^p - 1| \leq 1$. Therefore, if $|f_n(x)| \leq 1$ for every $|x| \leq (2p)^{n/p}$, $|f_{n+1}(x)| \leq 1$ for every $|x| \leq (2p)^{(n+1)/p}$ by the recursive formula for $f_n(x)$. Since obviously $|f_0(x)| \leq 1$ for all $|x| \leq 1$, the result follows by induction. □

Theorem 3. *Let p be a positive even integer and $c = \frac{1}{\sqrt[p]{2p}}$. Then $f_n(x)$ converges compactly.*

Proof. First observe that $f_1(x) = \frac{1}{p}x^p - 1$. However, since p is even, $f_2(x)$ will be the same whether $f_1(x) = \frac{1}{p}x^p - 1$ or $f_1(x) = 1 - \frac{1}{p}x^p$. Finally, shifting the entire sequence by 1, we can replace $f_0(x) = x$ with $f_0(x) = 1 - \frac{1}{p}x^p$. When $|x| \leq \sqrt[p]{p}$, $f_0(x) \geq 0$, and $f_1(x) = 2f_0(\frac{x}{\sqrt[p]{2p}})^p - 1 = 2(1 - \frac{1}{2p^2}x^p)^p - 1 \geq 2(1 - \frac{1}{2p}x^p) - 1 = 1 - \frac{1}{p}x^p = f_0(x)$ by Bernoulli's inequality.

We now claim that $0 \leq f_n(x) \leq f_{n+1}(x)$ for all $|x| \leq \sqrt[p]{p}$ and proceed by induction. Suppose for some n, $0 \leq f_{n-1}(x) \leq f_n(x)$ for all $|x| \leq \sqrt[p]{p}$. If $|x| \leq \sqrt[p]{p}$ then $|\frac{x}{\sqrt[p]{2p}}| \leq \frac{1}{\sqrt[p]{2}} < \sqrt[p]{p}$ so $0 \leq f_{n-1}(\frac{x}{\sqrt[p]{2p}}) \leq f_n(\frac{x}{\sqrt[p]{2p}})$. Since both of these numbers are non-negative $f_{n-1}(\frac{x}{\sqrt[p]{2p}})^p \leq f_n(\frac{x}{\sqrt[p]{2p}})^p$, and so $2f_{n-1}(\frac{x}{\sqrt[p]{2p}})^p - 1 \leq 2f_n(\frac{x}{\sqrt[p]{2p}})^p - 1$. But this is just $f_n(x) \leq f_{n+1}(x)$. We already know that $f_n(x) \geq 0$ from the induction hypothesis, so $0 \leq f_n(x) \leq f_{n+1}(x)$. Now, since $(2p)^{n/p} > \sqrt[p]{p}$ for all n, $|f_n(x)| \leq 1$ for all $|x| \leq \sqrt[p]{p}$ and all n by Lemma

2. It follows that $f_n(x)$ converges pointwise on $[-\sqrt[p]{p}, \sqrt[p]{p}]$ by MCT, and thus converges uniformly on this interval by Dini's theorem.

We now claim that $f_n(x)$ converges uniformly on $[-(2p)^{k/p}\sqrt[p]{p}, (2p)^{k/p}\sqrt[p]{p}]$ for every integer $k \geq 0$ and proceed by induction. We have already proven this for $k = 0$, so the base case is complete. For the induction step, suppose this is true for some $k \geq 0$, and call the limit of the sequence $f(x)$, which is defined on the interval $[-(2p)^{k/p}\sqrt[p]{p}, (2p)^{k/p}\sqrt[p]{p}]$. Let $\epsilon > 0$ be arbitrary. The function $g(x) = x^p$ is uniformly continuous on $[-1, 1]$, so we may choose δ such that if $x, y \in [-1, 1]$ and $|x - y| < \delta$ then $|x^p - y^p| < \frac{\epsilon}{2}$. Choose N such that if $n \geq N$ then $|f_n(x) - f(x)| < \delta$ for all $|x| \leq (2p)^{k/p}\sqrt[p]{p}$. Let $n \geq N + 1$ and suppose $|x| \leq (2p)^{(k+1)/p}\sqrt[p]{p}$. Then $|\frac{x}{\sqrt[p]{2p}}| \leq (2p)^{k/p}\sqrt[p]{p}$ and so $|f_{n-1}(\frac{x}{\sqrt[p]{2p}}) - f(\frac{x}{\sqrt[p]{2p}})| < \delta$ since $n - 1 \geq N$. By our choice of δ, this implies that $|f_n(x) - (2f(\frac{x}{\sqrt[p]{2p}})^p - 1)| = |(2f_{n-1}(\frac{x}{\sqrt[p]{2p}})^p - 1) - (2f(\frac{x}{\sqrt[p]{2p}})^p - 1)| = 2|f_{n-1}(\frac{x}{\sqrt[p]{2p}})^p - f(\frac{x}{\sqrt[p]{2p}})^p| < \epsilon$. Since ϵ was arbitrary, it follows that $f_n(x)$ converges uniformly on $[-(2p)^{(k+1)/p}\sqrt[p]{p}, (2p)^{(k+1)/p}\sqrt[p]{p}]$. The result follows by choosing k such that $(2p)^{k/p}\sqrt[p]{p} > r$ for any given r. $\square$

Corollary 4. *Assume that p is even. If $c < \frac{1}{\sqrt[p]{2p}}$ then $f_n(x)$ converges compactly to 1, and if $c > \frac{1}{\sqrt[p]{2p}}$, $f_n(x)$ does not converge compactly to anything.*

Proof. Denote $f_n(x)$ for a particular c by $f_n(x, c)$. We claim that $f_n(x, c) = f_n((\sqrt[p]{2p}c)^n x, \frac{1}{\sqrt[p]{2p}})$ for all integers $n \geq 0$ and real numbers $c > 0$ and proceed by induction on n. Certainly $f_0(x, c) = f_0(x, \frac{1}{\sqrt[p]{2p}}) = x$. For the induction step, if $f_{n-1}(x, c) = f_{n-1}((\sqrt[p]{2p}c)^{n-1}x, \frac{1}{\sqrt[p]{2p}})$ then $f_n(x, c) = 2f_{n-1}(cx, c)^p - 1 = 2f_{n-1}((\sqrt[p]{2p}c)^{n-1}cx, \frac{1}{\sqrt[p]{2p}})^p - 1 = 2f_{n-1}(\frac{(\sqrt[p]{2p}c)^n}{\sqrt[p]{2p}}x, \frac{1}{\sqrt[p]{2p}})^p - 1 = f_n((\sqrt[p]{2p}c)^n x, \frac{1}{\sqrt[p]{2p}})$. Now, $f_n(x, \frac{1}{\sqrt[p]{2p}})$ converges compactly to $f(x)$. When $c < \frac{1}{\sqrt[p]{2p}}$, $f_n(x, c) = f_n((\sqrt[p]{2p}c)^n x, \frac{1}{\sqrt[p]{2p}})$ converges compactly to 1 which proves the first part of the result.

For the second part, note that since the power series of $f(x)$ begins $1 - \frac{1}{p}x^p$ and therefore contains at least one non-constant term, so we can choose $x_0 > 0$ such that $f(x_0) \neq 1$. Let $c > \frac{1}{\sqrt[p]{2p}}$ and suppose $f_n(x, c)$ converges compactly to some $F(x)$. The sequence $\frac{x_0}{(\sqrt[p]{2p}c)^n}$ converges to 0, so $f_n(\frac{x_0}{(\sqrt[p]{2p}c)^n}, c)$ must converge to $F(0) = 1$. But $f_n(\frac{x_0}{(\sqrt[p]{2p}c)^n}, c) = f_n(x_0, \frac{1}{\sqrt[p]{2p}})$ which converges to $f(x_0) \neq 1$, a contradiction. $\square$

We have now proven the existence of this limit on the real number line, but what about complex inputs? The importance of Dini's Theorem and the output lying on the interval $[-1, 1]$ makes generalizing this proof difficult. The answer lies in replacing $f_0(x) = x$ with $g_0(x) = 1 + x$ and setting $c = \frac{1}{2p}$. Note that $c < \frac{1}{\sqrt[p]{2p}}$, but since $g_0(x)$ is not x, Corollary 4 does not apply. We next show that this sequence of functions converges compactly. We shall denote the limit by $g(x)$.

Theorem 5. *Let $p \geq 2$ be a positive integer and $c = \frac{1}{2p}$. Then $g_n(x)$ converges compactly. In particular, the limit is bounded above by e^x for $x \geq -1$.*

Proof. By the same argument used in Theorem 3, we only need to show that the sequence converges on some $[-B, B]$. In particular, consider the interval $[-1, 1]$. On this interval, Bernoulli's inequality gives $g_1(x) = 2(1 + \frac{x}{2p})^p - 1 \geq 2(1 + \frac{x}{2}) - 1 = 1 + x = g_0(x) \geq 0$. The same inductive argument as in Theorem 3 gives $0 \leq g_n(x) \leq g_{n+1}(x)$. The only thing we are missing to complete an analogous argument is an upper bound. We claim that $g_n(x) \leq e^x$ for all $n \geq 0$ and $x \geq -1$. For the base case, $g_0(x) = 1 + x \leq e^x$ is true for all real numbers. For the induction step, suppose $g_n(x) \leq e^x$. Then $g_{n+1}(x) = 2g_n(\frac{x}{2p})^p - 1 \leq 2(e^{\frac{x}{2p}})^p - 1 = 2e^{\frac{x}{2}} - 1$. To see that this final expression is no more than e^x, consider that $(e^{\frac{x}{2}} - 1)^2 \geq 0$ for all real x, so $e^x - 2e^{\frac{x}{2}} + 1 \geq 0$ and finally $e^x \geq 2e^{\frac{x}{2}} - 1$. $\square$

To generalize to the complex case, we will need the following important lemma on the compact convergence of sequences of holomorphic functions. This is also what will allow us to later determine some interesting property of these functions.

Lemma 6. *Let $F_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$ be any sequence of holomorphic functions such that $a_{n,k}$ converges to some a_k as $n \to \infty$. Furthermore, suppose $F(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic, and that there exist real numbers m_k and R such that $|a_{n,k}| \leq m_k$ for all n and $\sum_{k=0}^{\infty} m_k z^k$ is analytic whenever $|z| \leq R$. Then $F_n(z)$ converges compactly to $F(z)$ on the entire complex plane.*

Proof. Let $\epsilon > 0$. Choose M large enough that $|\sum_{k=M}^{\infty} a_k z^k| < \frac{\epsilon}{3}$ and $\sum_{k=M}^{\infty} m_k |z|^k < \frac{\epsilon}{3}$ for all $|z| \leq R$. Then choose N large enough that for all $n \geq N$, $\sum_{k=0}^{M-1} R^k |a_{n,k} - a_k| < \frac{\epsilon}{3}$. Then for all $n \geq N$ and $z \in \mathbb{C}$ such that $|z| \leq R$,

$$\begin{aligned}
|F(z) - F_n(z)| &= \left| \sum_{k=0}^{\infty} (a_k - a_{n,k}) z^k \right| \\
&\leq \left| \sum_{k=0}^{M-1} (a_k - a_{n,k}) z^k \right| + \left| \sum_{k=M}^{\infty} (a_k - a_{n,k}) z^k \right| \\
&\leq \sum_{k=0}^{M-1} R^k |a_{n,k} - a_k| + \left| \sum_{k=M}^{\infty} a_k z^k \right| + \left| \sum_{k=M}^{\infty} a_{n,k} z^k \right| \\
&\leq \sum_{k=0}^{M-1} R^k |a_{n,k} - a_k| + \left| \sum_{k=M}^{\infty} a_k z^k \right| + \sum_{k=M}^{\infty} m_k |z|^k \\
&< \epsilon
\end{aligned}$$

$\square$

We now have everything we need to generalize to the complex case:

Theorem 7. *For $p \geq 2$ and $c = \frac{1}{2p}$, the sequence $g_n(z)$ converges compactly to some $g(z)$ on the entire complex plane, and the nth coefficient of $g(z)$ is greater than 0 and less than or equal to $\frac{1}{2^{n-1}n!}$.*

Proof. Suppose $h(z) = 1 + z + \sum_{n=2}^{\infty} c_n z^n$ is a holomorphic function. Then $2h(\frac{z}{2p})^p - 1 = 1 + z + \sum_{n=2}^{\infty} P_n(c_2, c_3, \ldots, c_n) z^n$ for polynomials P_n with positive coefficients. Importantly, if $0 < c_n < c'_n$ for all $n \geq 2$ then $P_n(c_2, c_3, \ldots, c_n) < P_n(c'_2, c'_3, \ldots, c'_n)$. Consider $h(z) = e^z$. Then $2h(\frac{z}{2p})^p - 1 = 2e^{\frac{z}{2}} - 1$, so $P_n(\frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}) = \frac{1}{2^{n-1}n!} < \frac{1}{n!}$. Now let $g_n(z)$ be the sequence in theorem 4, and write $g_n(z) = 1 + z + \sum_{k=2}^{\infty} a_{n,k} z^k$. For all $k \geq 2$, $a_{0,k} = 0 < \frac{1}{2^{k-1}k!}$, and if $a_{n,k} < \frac{1}{2^{k-1}k!}$ for all $k \geq 2$ then

$$\begin{aligned} a_{n+1,k} &= P_k(a_{n,2}, a_{n,3}, \ldots, a_{n,k}) \\ &< P_k\left(\frac{1}{2^{2-1}2!}, \frac{1}{2^{3-1}3!}, \ldots, \frac{1}{2^{k-1}k!}\right) \\ &< P_k\left(\frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{k!}\right) \\ &= \frac{1}{2^{k-1}k!} \end{aligned}$$

so $a_{n,k} < \frac{1}{2^{k-1}k!}$ for all $n \geq 0$ and $k \geq 2$ by induction. Now, since $g_n(z)$ converges compactly on the real number line, each sequence $a_{n,k}$ converges to some a_k. We can conclude that $0 < a_k \leq \frac{1}{2^{k-1}k!}$. Convergence then follows from Lemma 6. □

Corollary 8. *$|g(z)| \leq g(|z|) \leq e^{|z|}$ for all $z \in \mathbb{C}$.*

Proof. This follows from the positivity of the coefficients. □

Corollary 9. *For any even p and $c = \frac{1}{\sqrt[p]{2p}}$, the $f_n(z)$ converge compactly to some $f(z)$ on the entire complex plane. In particular, $f(z) = g(-\frac{z^p}{p})$, and so $|f(z)| \leq g(\frac{|z|^p}{p}) \leq e^{\frac{|z|^p}{p}}$ for all $z \in \mathbb{C}$. [NOTE: I spotted a major error: p does need to be even for $f(z)$, just not for $g(z)$.]*

Proof. Recall from Theorem 3 that we can take $f_0(z)$ to be $1 - \frac{1}{p}z^p = g_0(-\frac{z^p}{p})$. If $f_n(z) = g_n(-\frac{z^p}{p})$ then $f_{n+1}(z) = 2f_n(\frac{z}{\sqrt[p]{2p}})^p - 1 = 2g_n(-\frac{z^p}{2p^2})^p - 1 = g_{n+1}(-\frac{z^p}{p})$, so $f_n(z) = g_n(-\frac{z^p}{p})$ for all n by induction. Taking the limit as $n \to \infty$, the result follows. □

Corollary 10. *$f(z)$ is of the form $\sum_{n=0}^{\infty} (-1)^n a_n z^{np}$ for some sequence of real numbers a_n satisfying $0 < a_n \leq \frac{1}{2^{n-1}p^n n!}$.*

Proof. Write $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $f(z) = g(-\frac{z^p}{p}) = \sum_{n=0}^{\infty} \frac{(-1)^n b_n}{p^n} z^{np} = \sum_{n=0}^{\infty} (-1)^n a_n z^{np}$ where $a_n = \frac{b_n}{p^n}$. The result follows from our bound on the coefficients of $g(z)$. □

We will now consider the much more general case of a sequence $h_n(z)$ where $h_0(z)$ can be any analytic function such that $h_0(0) = 1$, and c will be chosen based on the specific function $h_0(z)$. Counterintuitively, $h_0(z)$ does not even need to be defined everywhere for the limit of the sequence to be defined everywhere. This is because of the scaling effect that we have already taken advantage of a number of times: the values of $h_n(z)$ on $|z| \le \frac{r}{c}$ are dependent only on the values of $h_{n-1}(z)$ on $|z| \le r$. [Note: I might want to bring this up earlier on.] This theorem will not only greatly generalize our previous results, but will also allow us to demonstrate an interesting behavior of the $f(z)$.

Theorem 11. *Let p be a positive integer, and*

$$h_0(z) = 1 + \sum_{k=k_0}^{\infty} a_k z^k, \qquad h_n(z) = 2h_{n-1}\left(\frac{z}{\sqrt[k_0]{2p}}\right)^p - 1 \tag{4}$$

where $k_0 \ge 1$ and $h_0(z)$ is analytic on some closed disc around the origin. Then $h_n(z)$ converges compactly to $g(a_{k_0} z^{k_0})$.

Proof. Write $h_n(z) = 1+\sum_{k=k_0}^{\infty} a_{n,k} z^k$. The first key observation is that $a_{n+1,k}$ will be of the form $\frac{a_{n,k}}{(2p)^{k/k_0-1}} + P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1})$ where P_k is a polynomial with positive coefficients. Specifically, $a_{n,k_0} = a_{k_0}$ for all n, and therefore a_{n,k_0} converges to a_{k_0}. Suppose that $k > k_0$ and for all $k_0 \le i < k$, $a_{n,i}$ converges to some A_i as $n \to \infty$. Then $P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1})$ also converges. Write

$$\begin{aligned} a_{n+1,k} &= \frac{a_{n,k}}{(2p)^{k/k_0-1}} + P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1}) \\ &= \frac{1}{(2p)^{k/k_0-1}}\left(a_{n,k} - \frac{(2p)^{k/k_0-1}P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1})}{(2p)^{k/k_0-1}-1}\right) \\ &\quad + \frac{(2p)^{k/k_0-1}P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1})}{(2p)^{k/k_0-1}-1} \end{aligned}$$

Define $b_{n,k} = a_{n,k} - \frac{(2p)^{k/k_0-1}P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1})}{(2p)^{k/k_0-1}-1}$. Then $b_{n+1,k} = \frac{b_{n,k}}{(2p)^{k/k_0-1}} + \frac{(2p)^{k/k_0-1}}{(2p)^{k/k_0-1}-1}(P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1}) - P_k(a_{n+1,k_0}, a_{n+1,k_0+1}, \ldots, a_{n+1,k-1}))$. The second expression on the right-hand side will simply go to 0, and so since $(2p)^{k/k_0-1} > 1$, so does $b_{n,k}$. Therefore, from the definition of the $b_{n,k}$, $a_{n,k}$ converges to $\frac{(2p)^{k/k_0-1}P_k(A_{k_0}, A_{k_0+1}, \ldots, A_{k-1})}{(2p)^{k/k_0-1}-1}$. Note that A_k is uniquely determined by the previous A_i, and therefore are uniquely determined by $A_{k_0} = a_{k_0}$. When $h_0(z)$ is simply $1 + a_{k_0} z^{k_0}$, $1 + a_{k_0} z^{k_0} + \sum_{k=k_0+1}^{\infty} A_k z^k = g(a_{k_0} z^{k_0})$. Since the A_k are uniquely determined by a_{k_0}, this holds for any h_0. It follows that the coefficients of the $h_n(z)$ converge to the coefficients of $g(a_{k_0} z^{k_0})$.

Now define $m_{k_0} = |a_{k_0}|$. In general, define $m_k = \max\{|a_k|, \frac{(2p)^{k/k_0-1}P_k(m_{k_0}, m_{k_0+1}, \ldots, m_{k-1})}{(2p)^{k/k_0-1}-1}\}$.

Certainly $|a_{0,k}| \le m_k$. If $|a_{n,k}| \le m_k$ for some n and all $k \ge k_0$ then

$$\begin{aligned}
|a_{n+1,k}| &= \left| \frac{a_{n,k}}{(2p)^{k/k_0-1}} + P_k(a_{n,k_0}, a_{n,k_0+1}, \ldots, a_{n,k-1}) \right| \\
&\le \frac{|a_{n,k}|}{(2p)^{k/k_0-1}} + P_k(|a_{n,k_0}|, |a_{n,k_0+1}|, \ldots, |a_{n,k-1}|) \\
&\le \frac{m_k}{(2p)^{k/k_0-1}} + P_k(m_{k_0}, m_{k_0+1}, \ldots, m_{k-1}) \\
&\le \frac{m_k}{(2p)^{k/k_0-1}} + \frac{((2p)^{k/k_0-1} - 1)m_k}{(2p)^{k/k_0-1}} \\
&= m_k
\end{aligned}$$

so by induction, $|a_{n,k}| \le m_k$ for all n and k. Now, if there exists some $R > 0$ such that $\sum_{k=0}^{\infty} m_k z^k$ is analytic for $|z| \le R$ then $h_n(z)$ converges uniformly to $g(a_{k_0} z^{k_0})$ on this disc by Lemma 6. If $h_n(z)$ converges to $g(a_{k_0} z^{k_0})$ whenever $|z| \le r$ for some $r > 0$ then it also converges to $g(a_{k_0} z^{k_0})$ whenever $|z| \le \sqrt[k_0]{2p}r$, and thus converges to $g(a_{k_0} z^{k_0})$ compactly on the entire complex plane by the same argument as in the final step of the proof of Theorem 3.

The proof will therefore be complete if we can show that there is some $R > 0$ such that $\sum_{k=0}^{\infty} m_k z^k$ is analytic for $|z| \le R$. To prove this, it suffices to show that $m_k < M^k$ for some $M > 0$ and all $k \ge k_0$. Choose $K_0 \ge k_0$ large enough that whenever $k \ge K_0$, $|a_k| < M_0^k$ for some fixed $M_0 > 0$ and $2^{k/k_0-1}p^{k/k_0} \ge \frac{2p}{2p-1}\binom{k+p-1}{p-1}$, where this inequality holds for sufficiently large k because the left-hand side is exponential in k, whereas the right-hand side is a polynomial. Then choose $M \ge \max\{M_0, 1\}$ [Note: I can't figure out why M needed to be greater than or equal to 1.] such that $m_k < M^k$ for all $k_0 \le k < K_0$. Now, consider $1 + \frac{M^{k_0} z^{k_0}}{1-Mz} = 1 + \sum_{k=k_0}^{\infty} M^k z^k$. We have

$$\begin{aligned}
1 + \sum_{k=k_0}^{\infty} &\left(\frac{M^k}{(2p)^{k/k_0-1}} + P_k(M^{k_0}, M^{k_0+1}, \ldots, M^{k-1}) \right) z^k \\
&= 1 + \frac{M^{k_0} z^{k_0}}{p(1 - M(2p)^{-1/k_0} z)^p} \\
&= 1 + \frac{M^{k_0} z^{k_0}}{p} \sum_{k=0}^{\infty} \binom{k+p-1}{p-1} \left(\frac{Mz}{\sqrt[k_0]{2p}} \right)^k \\
&= 1 + \frac{1}{p} \sum_{k=0}^{\infty} \binom{k+p-1}{p-1} \frac{M^{k+k_0} z^{k+k_0}}{(2p)^{k/k_0}} \\
&= 1 + \sum_{k=k_0}^{\infty} \binom{k+p-1}{p-1} \frac{M^k z^k}{2^{k/k_0-1} p^{k/k_0}}
\end{aligned}$$

so when $k \geq K_0$,

$$\begin{aligned} P_k(M^{k_0}, M^{k_0+1}, \ldots, M^{k-1}) &= \binom{k+p-1}{p-1} \frac{M^k}{2^{k/k_0-1} p^{k/k_0}} - \frac{M^k}{(2p)^{k/k_0-1}} \\ &< \binom{k+p-1}{p-1} \frac{M^k}{2^{k/k_0-1} p^{k/k_0}} \\ &\leq \frac{2p-1}{2p} M^k \end{aligned}$$

Now suppose $K \geq K_0$ and $m_k < M^k$ for all $k < K$. Then $|a_K| < M_0^K \leq M^K$, and

$$\begin{aligned} \frac{(2p)^{K/k_0-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1})}{(2p)^{K/k_0-1} - 1} &= \left(\frac{(2p)^{K/k_0-1} - 1}{(2p)^{K/k_0-1}} \right)^{-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1}) \\ &= \left(1 - \frac{1}{(2p)^{K/k_0-1}} \right)^{-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1}) \\ &\leq \left(1 - \frac{1}{2p} \right)^{-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1}) \\ &= \left(\frac{2p-1}{2p} \right)^{-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1}) \\ &= \frac{2p}{2p-1} P_K(m_{k_0}, m_{k_0+1}, \ldots, m_{K-1}) \\ &\leq \frac{2p}{2p-1} P_K(M^{k_0}, M^{k_0+1}, \ldots, M^{K-1}) \\ &< \frac{2p}{2p-1} \left(\frac{2p-1}{2p} M^K \right) \\ &= M^K \end{aligned}$$

so $m_K < M^K$. It follows that $m_k < M^k$ for all $k \geq k_0$ by induction, which completes the proof. □

Corollary 12. *Let p be a positive even integer, and*

$$h_0(z) = \sum_{k=k_0}^{\infty} a_k z^k, \qquad h_n(z) = 2h_{n-1}\left(\frac{z}{\sqrt[k_0 p]{2p}} \right)^p - 1 \tag{5}$$

where $k_0 \geq 1$ and $h_0(z)$ is analytic on some closed disc around the origin. Then $h_n(z)$ converges compactly to $f(a_{k_0} z^{k_0})$.

Proof. The lowest order term of $2h_0(\frac{z}{\sqrt[k_0 p]{2p}})^p$ is $2(a_{k_0}(\frac{z}{\sqrt[k_0 p]{2p}})^{k_0})^p = \frac{a_{k_0}^p}{p} z^{k_0 p}$, so the first two terms of the series for $h_1(z)$ are $-1 + \frac{a_{k_0}^p}{p} z^{k_0 p}$. As in the proof of Theorem 3, we can consider $-h_1(z)$ instead, which can be written as $1 + \sum_{k=k_0 p}^{\infty} b_k z^k$ where $b_{k_0 p} = -\frac{a_{k_0}^p}{p}$. By Theorem 11, $h_n(z)$ converges to $g(b_{k_0 p} z^{k_0 p}) = g(-\frac{a_{k_0}^p}{p} z^{k_0 p}) = f(a_{k_0} z^{k_0})$. □

Recall that in the case of $p = 2$ we were able to determine that $f(z) = \cos z$ by taking advantage of the periodicity of $\cos z$. As it turns out, the $f(z)$ are not period more generally, but the have a related, somewhat more interesting property which can be seen clearly when the functions are plotted on the complex plane. Their graphs appear fractal-like because they are approximately self-similar.

Theorem 13. *Let $f(z)$ be the limit in Theorem 3. Then for any zero of $f(z)$, a, $f(z + (2p)^{n/p}a)$ converges compactly to $f(f'(a)z)$.*

Proof. Observe that $f(z + a) = f'(a)z + \sum_{k=2}^{\infty} a_k z^k$ for some sequence a_k and that $f(z + (2p)^{(n+1)/p}) = f(\sqrt[p]{2p}(\frac{z}{\sqrt[p]{2p}} + (2p)^{n/p})) = 2f(\frac{z}{\sqrt[p]{2p}} + (2p)^{n/p})^p - 1$. The result follows by Corollary 12. $\square$

Note that the scale factor $f'(a)$ is equal to either 1 or -1 in the case of $p = 2$, as we would expect considering that the cosine function is exactly periodic.