Homework 12: Isomorphisms, The Standard Matrix

*Assignments should be **stapled** and written clearly and legibly. Problem 7 is optional.*

1. §8.3, #2, 4. (For these two, provide brief justifications.)
2. Find the standard matrix for each of the following linear transformations.
   (a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating 2θ **clockwise** about the origin.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting across the line $y = -x$.
   (c) $R : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating $180°$ about the line passing through points $(-1, 0, -1)$ and $(1, 0, 1)$.
3. Find the result of rotating the vector $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ by $180°$ about the line through the points $(-1, 0, -1)$ and $(1, 0, 1)$.
4. Consider $T : P_2 \to P_2$ defined by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative of $p(x)$. Prove that T is linear. Then determine $\ker(T)$ and $R(T)$. Verify that Theorem 2.5 holds for T.
5. Consider $T : P_2 \to P_2$ given by $T(p(x)) = p(x) + p'(x)$.
   (a) Prove that T is linear.
   (b) Prove that T is an isomorphism.
   (c) Give $\ker(T)$ and $R(T)$.
6. Give an example of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T) = R(T)$.
7. (Bonus) Let $T : V \to W$ be a linear transformation, and let Y be a subspace of W. The **inverse image** of Y, denoted $T^{-1}(Y)$, is defined to be

$$T^{-1}(Y) = \{\mathbf{v} \in V : T(\mathbf{v}) \in Y\}$$

   Prove that $T^{-1}(Y)$ is a subspace of V.

   Note: The symbol T^{-1} is used to represent both the inverse image, as defined above, and the inverse function, as defined in class last week. Although the same symbol is used for both, they are different concepts.