

## Homework 13: Compositions, Inverses

Assignments should be **stapled** and written clearly and legibly. Problem 7 is optional.

1. §1.5, #9, 11(a), 15, 20(a).
2. §4.10, #9, 12.
3. §8.2, #20.
4. In  $\mathbb{R}^3$ , let  $T$  be counterclockwise rotation by  $\theta$  about the  $z$ -axis, and let  $S$  be counterclockwise rotation by  $\psi$  about the  $y$ -axis. (Here counterclockwise means as viewed from the positive axis.) Let  $R$  be counterclockwise rotation by  $\theta$  about the  $z$ -axis followed by counterclockwise rotation by  $\psi$  in  $\mathbb{R}^3$  about the  $y$ -axis. Find standard matrices for  $T$ ,  $S$ , and  $R$ .

Hint: The standard matrix for  $R$  is obtained by multiplying the other two matrices (in the correct order).

5. In this problem you will prove the following trigonometric identities:

$$\begin{aligned}\cos(x + y) &= \cos x \cos y - \sin x \sin y \\ \sin(x + y) &= \sin x \cos y + \cos x \sin y\end{aligned}\tag{1}$$

Use the following strategy. Let  $T$  be counterclockwise rotation by  $x$  and  $S$  counterclockwise rotation by  $y$  (both in  $\mathbb{R}^2$ ). Find the standard matrix for  $S \circ T$  in two ways: (i) by multiplying the standard matrices for  $S$  and  $T$ , and (ii) by observing that  $S \circ T$  is rotation by  $x + y$ . The matrices obtained in (i) and (ii) must be equal, so their entries must be equal.

6. Use the trigonometric identities (1) to find formulas for  $\cos(2x)$  and  $\sin(2x)$ .
7. (Bonus) Let  $T : V \rightarrow W$  be linear. A **left inverse** of  $T$  is a linear transformation  $L : W \rightarrow V$  such that  $L \circ T = I_V$ , and a **right inverse** of  $T$  is a linear transformation  $R : W \rightarrow V$  such that  $T \circ R = I_W$ .
  - (a) Prove that if  $T$  has a left inverse, then  $T$  is one-to-one.
  - (b) Prove that if  $T$  has a right inverse, then  $T$  is onto.
  - (c) Prove that if  $T$  has a left inverse  $L$  and a right inverse  $R$ , then  $T$  is an isomorphism and  $L = R$ .
  - (d) Give an example of a linear transformation  $T$  that has a left inverse but not a right inverse.
  - (e) Give an example of a linear transformation  $T$  that has a right inverse but not a left inverse.