Module-4: Electrodynamics

Maxwell's Equations

Q.1. **Explain Gauss's laws for static electric and static magnetic fields in differential and integral forms.** **[5 Marks] [May-2022, May-2023]**

Ans:

UQ. Derive Maxwell's First equation and state its significance. **MU - Q. 4(C), Dec. 22, 4 Marks**

(I) In the differential or point form

UQ. State and derive maxwell's equation which describes how the electric field circulates around the time varying magnetic field (Differential form).
MU - Q. 2(B), May 22, 4 Marks, Q. 4(B), May 23, 5 Marks, Q. 5(A), May 23, 5 Marks

(i) Consider an arbitrary surface S bounding an arbitrary volume V in a dielectric medium. We know that for any dielectric medium, the total charge density is the sum of free charge density ρ_v and polarized charge density ρ_p.

(ii) According to the Gauss's law of electrostatics, the total electric flux ψ crossing the closed surface = the total charge enclosed by that surface i.e.

$$\psi = \iint \vec{E}\, dS = \oint_S \vec{E}\, dS = \frac{1}{\epsilon_0} \oint_v (\rho_v + \rho_p)\, dV$$

Where, $q = \rho_v$ and $\rho_p = -\nabla \overline{P}$

$$\therefore \oint_S \epsilon_0 \vec{E}\, dS = \int_v (\rho_v - \nabla \vec{P})\, dV$$

(iii) Using the divergence theorem, convert surface integral to volume integral,

$$\oint_S \epsilon_0 E\, dS = \oint_v \nabla (\epsilon_0 \vec{E})\, dV = \oint (\rho_v - \nabla \vec{P})\, dV$$

$$\therefore \oint_v \nabla (\epsilon_0 \vec{E} + P)\, dV = \oint \rho_v\, dV$$

(iv) Since, electrical displacement vector $\vec{D} = \epsilon_0 \vec{E} + P$

$$\therefore \oint_v \nabla \cdot \vec{D}\, dV = \oint \rho_v\, dV$$

$$\text{Or} \quad \oint (\nabla \cdot \vec{D} - \rho_v)\, dV = 0$$

$$\therefore \nabla \cdot \vec{D} - \rho_v = 0 \quad \text{or} \quad \nabla \cdot \vec{D} = \rho_v$$

This is the Maxwell's first equation in the differential or point form.

(II) In the integral form

(i) the Maxwell's first equation in the differential or point form i.e.

$$\nabla \cdot \vec{D} = \rho_v$$

$\therefore$ integrating both the sides w.r.t. V, we get,

$$\int_V \nabla \cdot \vec{D}\, dV = \int_V \rho_v\, dV \quad \text{...(4.6.1)}$$

(ii) from the Gauss's theorem we know that,

$$\int_V \nabla \cdot \vec{D}\, dV = \oint_S \overline{D} \cdot dS \quad \text{...(4.6.2)}$$

$\therefore$ from equations (4.6.1) and (4.6.2), we get

$$\oint_S \vec{D} \cdot dS = \int_v \rho_v\, dV \quad \text{...(4.6.3)}$$

(iii) But $\int_V \rho_v\, dV = q$

$$\therefore \oint_S \vec{D} \cdot dS = \int_v \rho_v\, dV$$

This is the Maxwell's first equation in the integral form as required.

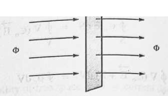
Significance of Maxwell's first equations :

It signifies that 'the total electric displacement through the surface enclosing a volume, is equal to the total charge within the volume.'

(2) Maxwell's second equation

(I) In the differential or point form

(i) We know that the number of magnetic lines flux entering any surface normally is exactly the same as the number of magnetic lines of flux leaving that surface as shown in the Fig. 4.6.1.



(2C24)**Fig. 4.6.1 : Magnetic lines of flux entering and leaving a surface**

$$\therefore \oint_V \overline{B}\, dS = 0$$

(ii) Using Gauss's divergence theorem, we get,

$$\oint_S \overline{B}\, dS = \oint_V \overline{\nabla} \cdot \overline{B}\, dV = 0$$

$$\therefore \nabla \cdot \overline{B} = 0$$

This is the Maxwell's second equation in the differential or point form as required.

(II) In the integral form

(i) From the Maxwell's second equation in the differential or point form i.e.

$$\nabla \cdot \overline{B} = 0 \quad \therefore \int_V \nabla \cdot \overline{B}\, dV = 0$$

(ii) Using Gauss's divergence theorem, we get

$$\oint_S \overline{B} \cdot d\overline{S} = 0$$

This the Maxwell's second equation in the integral form as required.

Q.2. **State and derive Maxwell's equation in differential form which describes how the electric field circulates around the time-varying magnetic field.** **[5 Marks] [Dec-2018, May-2022, Dec-2022]**

Ans:

(3) Maxwell's Third Equation

UQ. Derive Maxwell's third equation. **MU - Q. 5(b), Dec. 18, 5 Marks**

(I) In the differential form

UQ. How will you state Faraday's law in differential (in point) form explain with appropriate derivation. **MU - Q. 6(c), Dec. 22, Q. 1(4), May 22, 5 Marks**

(i) According to Faraday's second law of electromagnetic induction we know with the usual notation that e.m.f. induced in a closed loop is given by $e = -\frac{d\phi}{dt}$

(ii) Total magnetic flux over any arbitrary surface is given by $\phi = \oint_S \bar{B}\, dS$

$$\therefore \quad e = -\frac{d}{dt}\left(\oint_S \bar{B}\cdot dS\right) = -\oint_S \left(\frac{\partial \bar{B}}{\partial t}\right) dS$$

(iii) The emf is the work done in carrying a unit charge around the closed path.

$$\therefore \quad e = \oint_l \bar{E}\, dl \qquad \therefore \quad \oint_l \bar{E}\, dl = -\oint_S \left(\frac{\partial \bar{B}}{\partial t}\right) dS$$

(iv) Now using Stoke's theorem to convert the line integral into the surface integral, we get

$$\oint_l \bar{E}\cdot dl = \oint_S (\bar{\nabla}\times\bar{E})\, dS \qquad \therefore \quad \oint_S (\bar{\nabla}\times\bar{E})\, dS = -\oint_S \frac{\partial \bar{B}}{\partial t}\, dS$$

$$\therefore \quad \oint_S \left(\bar{\nabla}\times\bar{E} + \frac{\partial \bar{B}}{\partial t}\right) dS = 0 \quad \text{or} \quad \bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

This is the Maxwell's third equation in the differential form. It can also be called as the statement of Faraday;s law in the differential form.

(II) In the integral form

(i) Form the Maxwell's third equation in the differential form i.e.

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

(ii) $\therefore \oint_S (\bar{\nabla}\times\bar{E})\, dS = -\int_S \frac{\partial \bar{B}}{\partial t}\, dS$

(iii) Using Stoke's theorem, we get $\oint_l \bar{E}\cdot \bar{dl} = -\int_S \frac{\partial \bar{B}}{\partial t}\, dS$

This is the Maxwell's third equation in the integral form

Q.3. **Obtain Ampere's circuital law for a static magnetic field in differential and integral forms. [5 Marks] Dec-2018, May-2022, Dec-2022]**

Ans:

(4) Maxwell's Fourth Equation

Q. Derive Maxwell's fourth equation.

(I) In the differential form

(i) According to Ampere's circuital law, $\oint_l \bar{B} \cdot d\bar{l} = \mu_0 I$ ...(4.6.4)

(ii) According to stoke's theorem, $\oint_l \bar{B} \cdot d\bar{l} = \oint_S (\bar{\nabla} \times \bar{B})\, d\bar{S}$ ...(4.6.5)

(iii) $\therefore$ from Equations (4.6.4) and (4.6.5) above we get, $\oint_S (\bar{\nabla} \times \bar{B})\, d\bar{S} = \mu_0 I$

(iv) But we know that,

$$I = \oint_S \bar{J} \cdot d\bar{S}$$

$$\therefore \oint_S (\bar{\nabla} \times \bar{B})\, dS = \mu_0 \oint_S \bar{J} \cdot d\bar{S}$$

Or $\oint_S \left(\bar{\nabla} \times \bar{B} - \mu_0 \bar{J} \right) \cdot d\bar{S} = 0$ $\quad \therefore \bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$ $\quad$ but $\bar{B} = \mu_0 \bar{H}$

$$\therefore \bar{\nabla} \times \mu_0 \bar{H} = \mu_0 \bar{J}, \quad \therefore \bar{\nabla} \times \bar{H} = \bar{J} \qquad ...(4.6.6)$$

(v) But the above Equation (4.6.5) is true for static fields only. Therefore, for removing this discrepancy and to make the Equation (4.6.6) generalized, Maxwell introduced the concept of the displacement current by replacing $\bar{J}$ by $\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)$ where $\frac{\partial \bar{D}}{\partial t}$ represents the displacement current density.

(vi) The term $\frac{\partial \bar{D}}{\partial t}$ is known as Maxwell's correction.

$$\therefore \bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

This is the Maxwell's fourth equation in the differential form.

(II) In the integral form

(i) From the Maxwell's fourth equation in the differential form i.e.

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

(ii) $\therefore \oint_S (\nabla \times \bar{H})\, dS = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right) dS$

(iii) Using Stoke's theorem, we get $\oint_l \bar{H} \cdot d\bar{l} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right) dS = \bar{I} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\bar{S}$

This is the Maxwell's fourth equation in the integral form.

Q1. Gradient, Divergence and Curl[1]

(a) **What are scalar and vector fields? How is a del operator expressed? [3 Marks] [May-2019, May-2023]**

(b) **If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(-1, -2, 1)$. [3 Marks] [May-2019, May-2023]**

(c) **What is the divergence of a vector field? Find the divergence of a field $\mathbf{F} = xz\hat{i} + y^2z^3\hat{j} - xyz\hat{k}$ at a point $(3, -1, 2)$. Interpret the result you obtain. [3 Marks] [May-2017, May-2022 Dec-2022, Dec-2023]**

(d) **Explain the term 'curl of a vector' and state its significance. Show that the divergence of the curl of a vector is zero. [3 Marks] [Dec-17,May-2023, Dec-2023]**

[1]Similar numericals based on the same concept were asked; however, only one example is presented here. As it is a numerical problem, students are encouraged to practice similar problems for better understanding.

Ans:

Q.1. a) # SCALAR FIELDS:

- A scalar field is mathematical function that assigns a scalar value (a single number) to each point in space.

Ex: Temperature distribution in a room, density of material, electric potential etc.

VECTOR FIELDS:

- A vector field is a mathematical function that assigns a vector to each point in space

Ex: Fluid velocity in pipe, electric field, etc.

Vector fields are represented by function $F(x, y, z) = (F_x, F_y, F_z)$ that takes three spatial coordinates as input and returns a vector with three components

- Whereas, scalar is represented by function $f(x, y, z)$ that takes three spatial coordinates as input and returns a scalar value.

DELL OPERATOR ($\vec{\nabla}$):

Dell operator is expressed as: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

- It acts like a vector and also like a differential eq^n thus it will obey rules relating to vectors as well as differential operator. It is used for finding:

i) Gradient ($\vec{\nabla} \cdot f$)

ii) Divergence ($\vec{\nabla} \cdot F$)

iii) Curl ($\vec{\nabla} \times F$)

b) $\Phi(x, y, z) = 3x^2y - y^3z^2$

To find: $\vec{\nabla} \cdot \phi$

$$\phi(x,y,z) = 3x^2y - y^3z^2$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla}\cdot\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(3x^2y - y^3z^2)$$

$$= \frac{\partial \phi\hat{i}}{\partial x} + \frac{\partial \phi\hat{j}}{\partial y} + \frac{\partial \phi\hat{k}}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} + (-y^3 2z)\hat{k}$$

$$(\vec{\nabla}\cdot\phi)_{(-1,-2,1)} = 6(-1)(-2)\hat{i} + 3(-1)^2 - 3(-2)^2(1)^2\hat{j} - 2(-2)^3(1)\hat{k}$$

$$= 12\hat{i} - 9\hat{j} + 16\hat{k}$$

c) The divergence of vector field $F(x,y,z) = (F_x, F_y, F_z)$ is a scalar value that represents flux of vector field at a given point. It measures how much vector field diverges or converges at that point.

$$\text{div} F = \vec{\nabla}\cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$F = xz\hat{i} + y^2z^3\hat{j} - xyz\hat{k}$ (Given)

$\vec{\nabla}\cdot F$ – To find

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{div}F = \vec{\nabla}\cdot F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xz\hat{i} + y^2z^3\hat{j} - xyz\hat{k}\right)$$

$$= \frac{\partial xz}{\partial x} + \frac{\partial y^2z^3}{\partial y} + \frac{\partial(-xyz)}{\partial z}$$

$$= z + 2yz^3 - xy$$

$$\vec{\nabla}\cdot F_{(3,-1,2)} = 2 + 2(-1)(2)^3 - (3(-1))$$

$$= 2 - 16 + 3 = 2 - 13$$

$$\boxed{\vec{\nabla}\cdot F_{(3,-1,2)} = -11}$$

- The negative divergence indicates that vector field is converging
- It means more vectors are entering the region around this point than leaving it

d) - The curl of vector field $F(x,y,z) = (F_x, F_y, F_z)$ is a vector that measures rotation or circulation of field around a point.
- Denoted by $\vec{\nabla}\times F$ and it signifies rotation, circulation, magnetic field, etc

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$F = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\therefore \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \vec{\nabla}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \vec{\nabla}\cdot\left[\hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]$$

$$= \vec{\nabla}\cdot 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= 0$$

Hence proved, divergence of curl is zero.