Applications of Chatterjee's Correlation in MCMC

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Sourav Chatterjee proposed a correlation coefficient in his [add reference]. This coefficient is (a) as simple as the classical ones, (b) is a consistent estimator of some measure of dependence which is 0 iff the variables are independent, and 1 iff one is a measurable function of the other, and (c) has a simple asymptotic theory under the hypothesis of independence, like the classical coefficients.

Let (X, Y) be a pair of random variables, where Y is not a constant (for our purposes, both X and Y are continuous). Let $\{(X_i, Y_i)\}_{i=1}^n$ be i.i.d. pairs following the same distribution as (X, Y).

1. The case when $X_i's$ and $Y_i's$ have no ties. Rearrange the data as $(X_{(1)}, Y_{(1)}), \ldots, (X_{(n)}, Y_{(n)})$ such that $X_{(1)} < \cdots < X_{(n)}$. Let r_i be the rank of $Y_{(i)}$, i.e. the number of j such that $Y_{(j)} \leq Y_{(i)}$. Then the correlation coefficient ξ_n is defined to be

$$\xi_n(X, Y) := 1 - \frac{3 \sum_{i=1}^{n-1} |r_{i+1} - r_i|}{n^2 - 1}$$

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2. In the case of ties. If there are ties in $X_i's$, choose an increasing arrangement as follows and break ties uniformly at random. Let r_i defined as

above, and define l_i to be the number of j such that $Y_{(j)} \geq Y_{(i)}$. Define

$$\xi_n(X, Y) := 1 - \frac{n \sum_{i=1}^{n-1} |r_{i+1} - r_i|}{2 \sum_{i=1}^{n-1} l_i(n - l_i)}$$

. When there are no ties among the $Y_i's, l_1, \ldots, l_n$ is just a permutation of $1, \ldots, n$ and the denominator is just $n(n^2 - 1)/3$, which reduces to the definition in the no ties case.

Theorem 4.1. If Y is not almost surely a constant, then as $n \to \infty$, $\xi_n(X, Y)$ converges almost surely to the deterministic limit

$$\xi(X, Y) := \frac{\int Var(\mathbb{E}(1_{\{Y \geq t\}}|X))d\mu(t)}{\int Var(1_{\{Y \geq t\}})d\mu(t)}$$

where μ is the pdf of Y. This limit belongs to the interval $[0, 1]$. It is 0 iff X and Y are independent, and it is 1 iff there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$ almost surely.

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