Applications of Chatterjee's Correlation in MCMC
(UGP, 21-22 Even Semester)

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April 17, 2022

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Markov Chain Monte Carlo

Definition

A Markov chain is a discrete time stochastic process $X_1, X_2, \ldots$ such that the next state of the process depends solely on the present state, i.e.

$$\Pr(X_{n+1}|X_1, \ldots, X_n) = \Pr(X_{n+1}|X_n).$$

- A Markov chain can be specified by two things
  1. The initial distribution, i.e. the marginal of X_1.
  2. The transition probabilites, i.e. the conditional distribution of $X_{n+1}|X_n$

Markov chain Monte Carlo

Definition

A Markov chain is said to be **stationary** if the marginal of X_n is independent of n. This invariant distribution is called the stationary distribution.

Definition

A Markov chain is **ergodic** if the distribution of X_n converges to the invariant distribution.

Definition

A stationary Markov chain is **time-reversible** w.r.t. the stationary distribution if X_n and X_{n+1} are exchangable.

Pearson's Correlation Coefficient

- Pearson's Correlation Coefficient measures the linear correlation between two sets of data.

Definition

Given random variables X, Y, Pearson Correlation ρ is defined as

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}.$$

- Pearson Correlation is very powerful in detecting monotone relations and has a well developed asymptotic theory.

Problems with Pearson Correlation

- There are some problems with the Pearson Correlation.

- We would like the correlation to be close to its maximum value iff one variable is a function of the other. In Pearson's case, it is equal to ± 1 iff the variables are linearly dependent.

- We would also like the correlation to be 0 iff the variables are independent. If the variables are independent, Pearson correlation is indeed 0, but the converse is not always true.

- In order to solve these problems, Chatterjee came up with a new measure of dependence in [?]. It overcomes the above mentioned drawbacks, and has a computationally efficient and consistent estimator.

Definition

Given random variables X, Y, where is Y is not a constant, Chatterjee correlation ξ is defined as

$$\xi(X, Y) = \frac{\int Var(\mathbb{E}(1_{\{Y \geq t\}}|X))d\mu(t)}{\int Var(1_{\{Y \geq t\}})d\mu(t)},$$

where μ is the law of Y.

Consistent Estimator of ξ

- Let $\{(X_i, Y_i)\}_{i=1}^{n}$ be i.i.d. pairs following the same distribution as (X, Y). Rearrange the data as $(X_{(1)}, Y_{(1)}), \ldots, (X_{(n)}, Y_{(n)})$ such that $X_{(1)} < \cdots < X_{(n)}$. Let r_i be the rank of $Y_{(i)}$, i.e. the number of j such that $Y_{(j)} \leq Y_{(i)}$. Then the correlation coefficient ξ_n is defined to be

$$\xi_n(X, Y) := 1 - \frac{3 \sum_{i=1}^{n-1} |r_{i+1} - r_i|}{n^2 - 1}.$$

Theorem

If Y is not almost surely a constant, then as $n \to \infty$, $\xi_n(X, Y)$ converges almost surely to $\xi(X, Y)$.

Properties of ξ

- $\xi(X, Y) \in [0, 1]$
- $\xi(X, Y) = 0$ if and only if X and Y are independent.
- $\xi(X, Y) = 1$ if and only if atleast one of X and Y is a measurable function of the other.
- ξ is not symmetric in X, Y. This is intentional and useful as we might want to study if Y is a measurable function of X, or X is a measurable function of Y. To get a symmetric coefficient, it suffices to consider $\max(\xi(X, Y), \xi(Y, X))$.
- ξ_n is based on ranks, and for the same reason, it can be computed in $O(n \log n)$.

Chatterjee's Autocorrelation Function

- We present a new autocorrelation function (ACF) using Chatterjee's correlation coefficient.

Definition

Let $X_1, X_2, \ldots$ be a stationary, time homogeneous Markov chain with stationary distribution π. We define the new lag-k ACF as follows

$$\gamma'_{k,n} := \xi(X_n, X_{n+k}).$$

$\gamma'_{k,n}$ is independent of n

Theorem

$\gamma_k = Cov(X_n, X_{n+k})$ is independent of n.

Theorem

$\gamma'_{k,n} = \xi(X_n, X_{n+k})$ is independent of n, where n and k are in $\mathbb{N}$.

- From now on, we'll denote $\gamma'_{k,n}$ by γ'_k

Symmetricity of γ'_k

Theorem

$\xi(X_n, X_{n+k}) = \xi(X_{n+k}, X_n)$ *for time reversal Markov chains for any* $n, k \in \mathbb{N}$.

Convergence of γ'_k

Theorem

$\lim_{n\to\infty} \xi(X_1, X_n) = 0$ *for an Ergodic Markov chain.*

Proof of consistency of estimator

- Chatterjee presented the proof of consistency of the estimator in [**?**], where the samples drawn from (X, Y) are i.i.d.
- In our case of stationary Markov chains, we have correlated but identically distributed draws.
- We aim to prove the consistency of the estimator for our case as well.

Proof of consistency of estimator

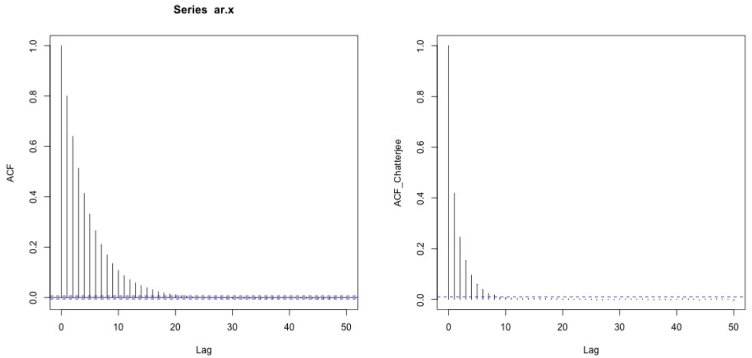
Theorem

Let $X_1, X_2, \dots$ be a stationary time-homogeneous Markov chain with stationary distribution μ. Then $\xi_n(X, Y)$ estimated using the draws from the Markov chain converge to $\xi(X, Y)$ as $n \to \infty$, where X, Y are any two time points in the chain.

- We presented some ideas and a pathway for proving it but could not complete it and is left as future work.

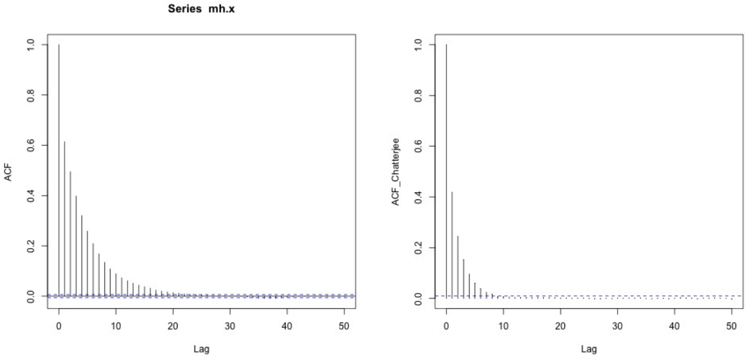
Simulations

- AR(1) process with $\rho = 0.8$.



Simulations

- Metropolis-Hastings algorithm with initial distribution as Exp(0.01) and target distribution $\mathcal{N}(0, 1)$.



Thank you