Multidimensional optimization

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Victor Barbarich

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CHAPTER

ONE

CONCEPTUAL MODEL

1.1 Imagine situation:

Factory's management has a function that describes the income from the sale of products. The function depends on many variables, e.g. the salary by each employee, the cost of each product and so on.

The management gave us the function $g(\mathbf{x})$, the running costs, the gradient of the functions in analytical form (Their mathematicians did a good job). But it will better if our solution automatically calculates gradient.

And our goal is to ensure the **best cost distribution**.

* For unifying let's introduce $f(\mathbf{x}) = -g(\mathbf{x})$.

CHAPTER TWO

MATHEMATICAL MODEL

We have analyzed their problem. The solution is to use the gradient methods.

We have chosen 2 methods:

1. Gradient Descent
2. Nonlinear conjugate gradient method

2.1 Problem statement

1. $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$
2. $\mathbf{x} \in X \subseteq \mathbb{R}^n$
3. $f \longrightarrow \min\limits_{\mathbf{x} \in X}$
4. The f is defined and differentiable on the X
5. Convergence to a local minimum can be guaranteed. When the function f is convex, gradient descent can converge to the global minima.

2.2 Gradient descent

2.2.1 Equations

1. Function argument:

$$\mathbf{x} = \begin{bmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \dots & \mathrm{x}_n \end{bmatrix}^\top \tag{2.1}$$

2. Gradient:

$$\nabla f = \left[\frac{\partial f}{\partial \mathrm{x}_1} \; \frac{\partial f}{\partial \mathrm{x}_2} \; \dots \; \frac{\partial f}{\partial \mathrm{x}_n} \right]^\top \tag{2.2}$$

3. Gradient step:

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \gamma_i \cdot \nabla f(\mathbf{x}_i) \tag{2.3}$$

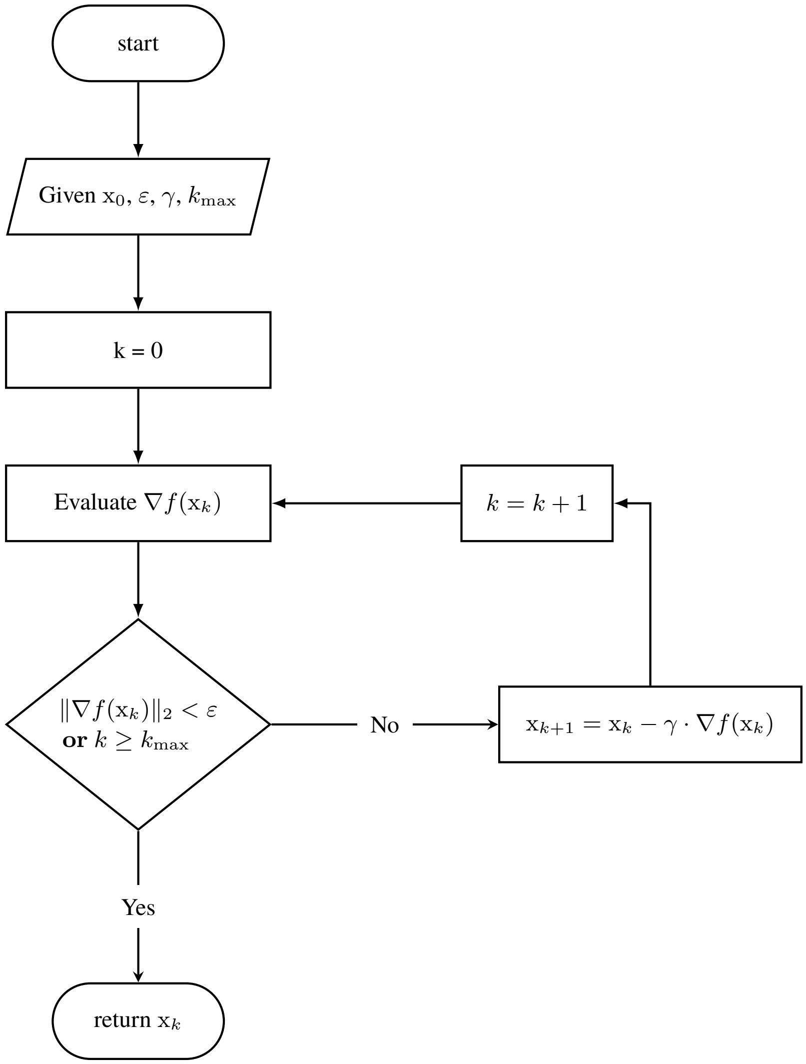
4. Terminate condition:

$$\|\nabla f(\mathbf{x}_i)\|_2 < \varepsilon \tag{2.4}$$

2.2.2 Algorithm with constant step

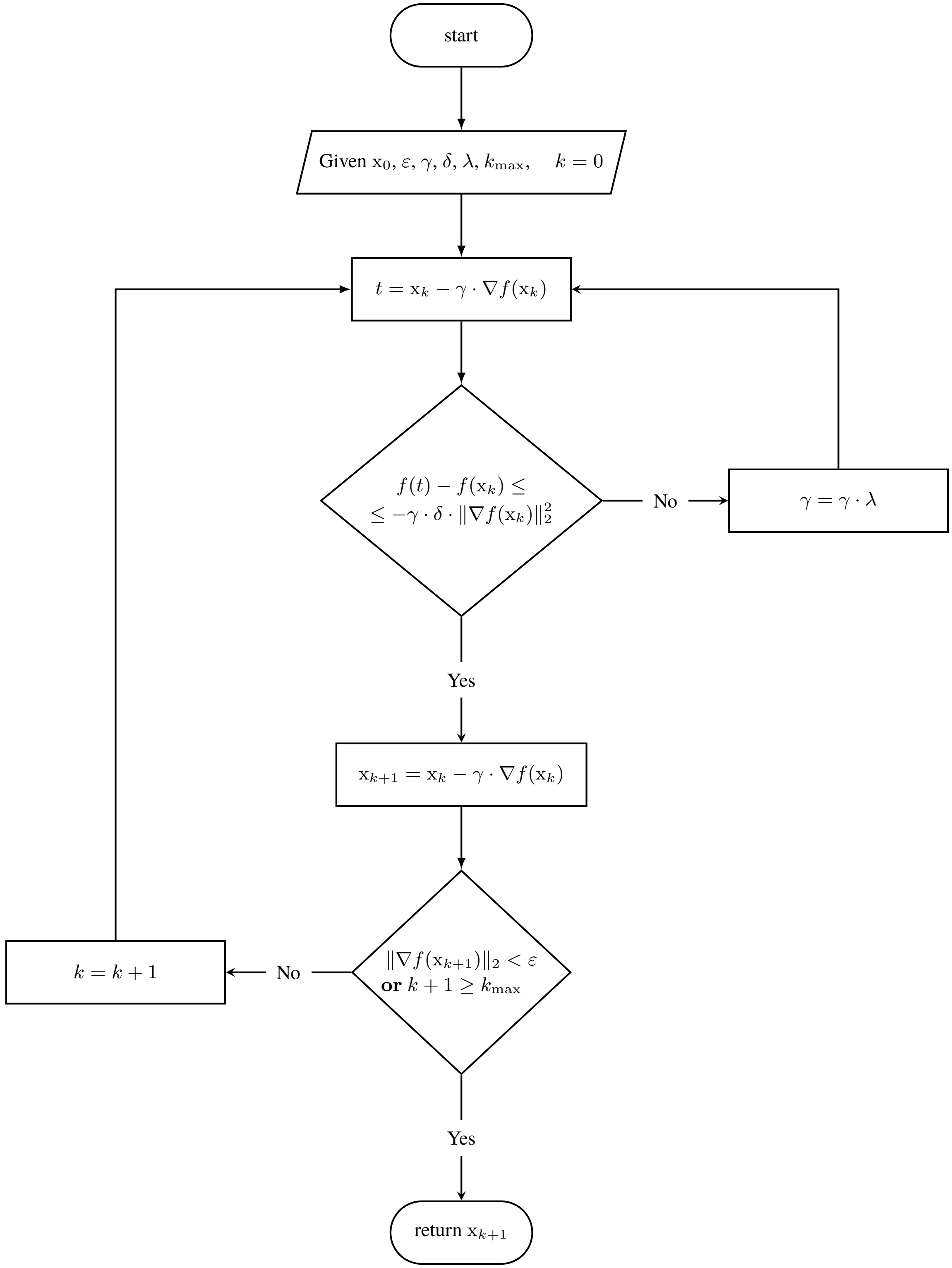
The gradient of the function shows us the direction of increasing the function. The idea is to move in the opposite direction to $\mathbf{x}_{k+1}$ where $f(\mathbf{x}_{k+1}) < f(\mathbf{x}_k)$.

But, if we add a gradient to $\mathbf{x}_k$ without changes, our method will often diverge. So we need to add a gradient with some weight γ.



2.2.3 Algorithm with descent step

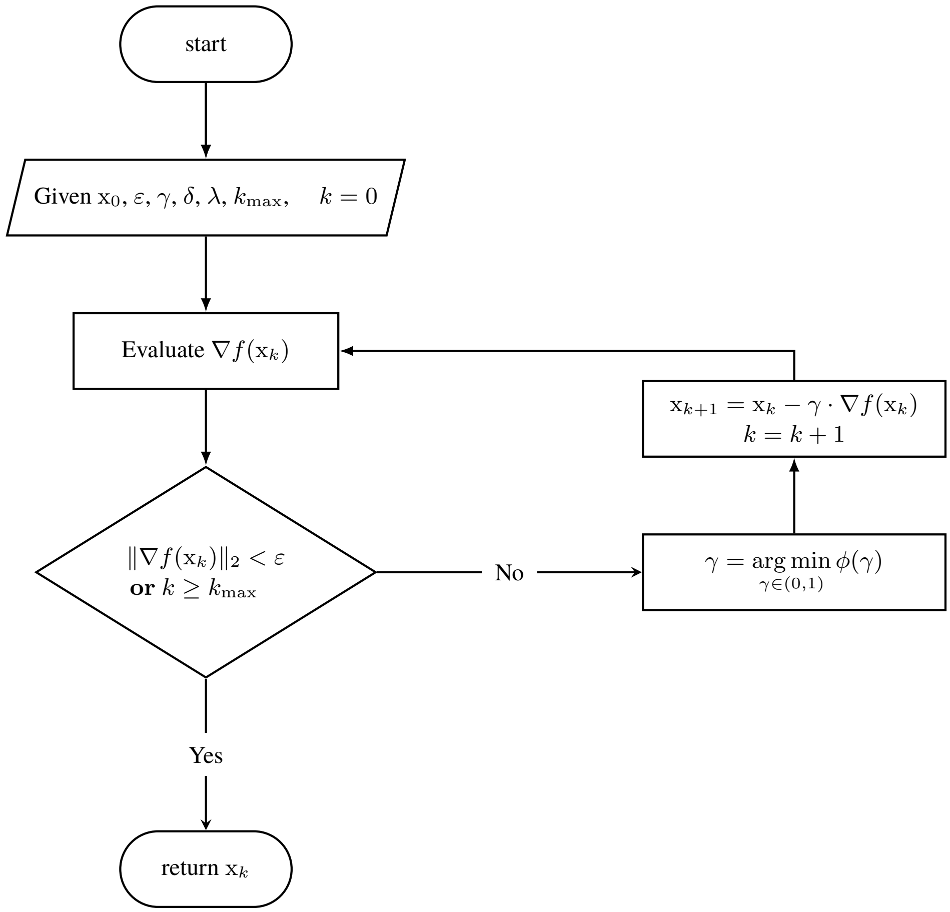
Requirements: $0 < \lambda < 1$ is the step multiplier, $0 < \delta < 1$.



2.2.4 Algorithm with optimal step size

Another good idea is to find a γ that minimizes $\phi(\gamma) = f(\mathrm{x}_k + \gamma \cdot \nabla f(\mathrm{x}_k))$

So we have a task to find the $\gamma_{\min} = \underset{\gamma \in (0,1)}{\arg\min}\, \phi(\gamma)$. We will use Brent's algorithm to search $\gamma_{\min}$.



2.3 Strong Wolfe conditions

The conditions necessary to minimize $\phi(\gamma) = f(\mathrm{x}_k + \gamma p_k)$ and find $\gamma_k = \underset{\gamma}{\arg\min}\, \phi$

$$f(\mathrm{x}_k + \gamma_k p_k) \leq f(\mathrm{x}_k) + c_1 \gamma_k \nabla f_k^\top p_k \tag{2.5}$$

$$|\nabla f(\mathrm{x}_k + \gamma_k p_k)^\top p_k| \leq -c_2 \nabla f_k^\top p_k \tag{2.6}$$

2.4 Nonlinear conjugate gradient method

The Fletcher–Reeves method. p_k is the direction to evaluate x_{k+1}.

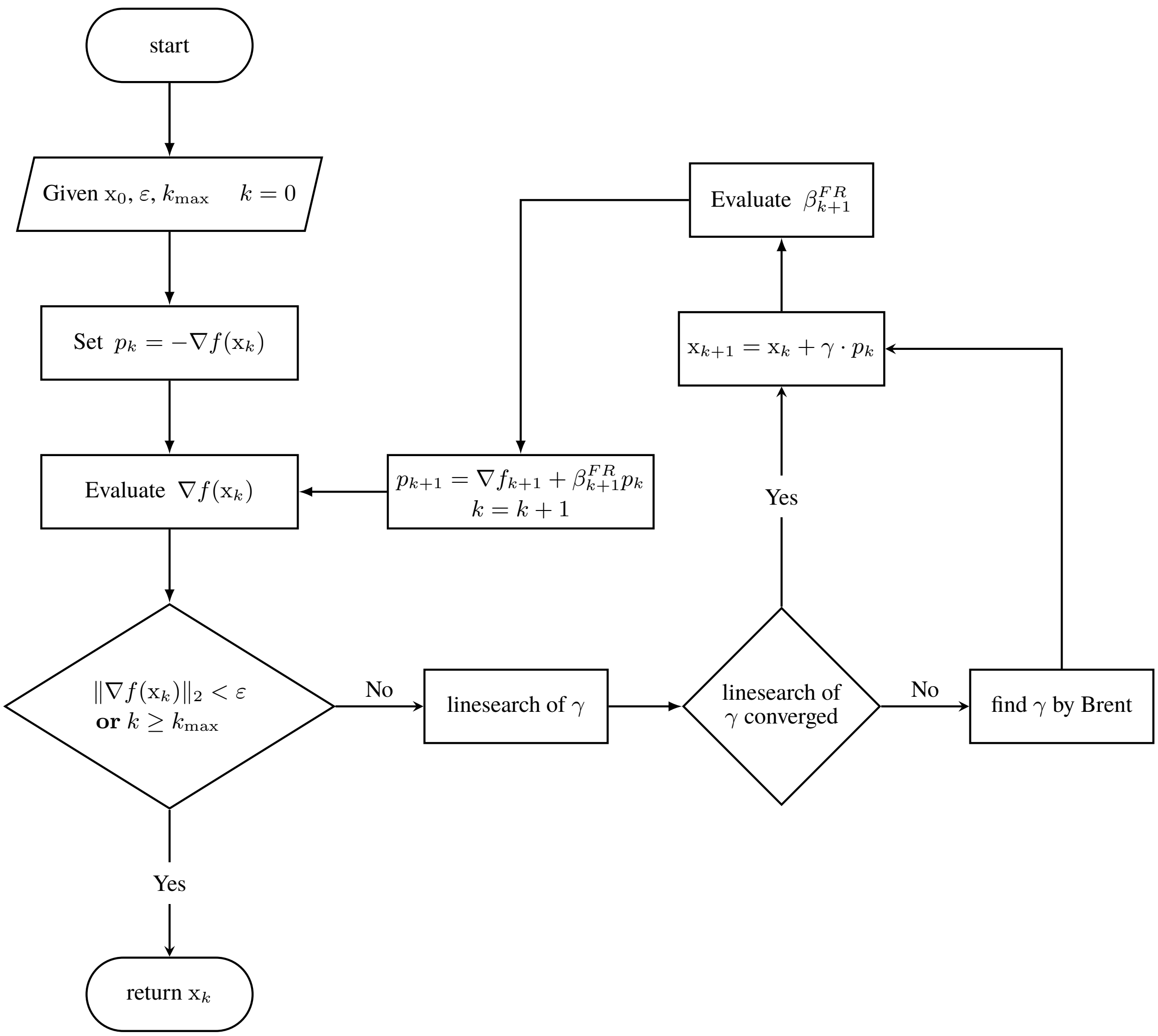
1. $p_0 = -\nabla f_0$
2. $p_{k+1} = \nabla f_{k+1} + \beta^{FR}_{k+1} p_k$

In the RF method, γ is searched using Line Search (Nocedal, Wright (2006) *Numerical Optimization* pp.60-61)

Our modification is that if Line Search does not converge, use Brent's algorithm to search for $\gamma_{\min} = \arg\min_{\gamma} \phi$

$$\beta^{FR}_{k+1} = \frac{\nabla f^\top_{k+1} \nabla f_{k+1}}{\nabla f^\top_k \nabla f_k} \tag{2.7}$$

2.4.1 Algorithm Nonlinear conjugate gradient method



CHAPTER

THREE

REQUIREMENTS FOR WORKING WITH USERS

3.1 Required input fields

1. Field with function input $f(\mathrm{x})$
2. Field with the analytical gradient flag. If analytical gradient flag is True then request the gradient in analytical form. $\nabla f(\mathrm{x})$
3. Field with the start point (The default value is random numbers from a uniform distribution on (-1, 1))
4. Search precision ε