UNIVERSITY OF MANNHEIM
School of Social Sciences

Quantitative Methods in Political Science: Linear Regression I

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Quiz

Suppose we draw a random sample and find that the mean government popularity (measured on a $0 - 100$ scale) reported by the respondents ($\hat{\mu}$) is 45, with a standard error of 2.5. With a large sample, a 95% confidence interval for the "true" mean level of government popularity in the population (μ) is between 40 and 50.
Which of the following statements are correct?

1. The probability that μ lies within the confidence interval is either 0 or 1.
2. We can be 95% sure that μ lies between 40 and 50.
3. There is a .05 probability that μ lies outside the confidence interval.
4. 95% of the confidence intervals one would draw in repeated samples will include μ.
5. Given a sample size of 1600, the standard deviation ($\hat{\sigma}$) of the sample mean is 100.

The Course

Roadmap

- Understand and model stochastic processes
- Understand statistical inference
- Implement it mathematically and learn how to estimate it
  - OLS
  - Maximum Likelihood
- Implement it using software
  - R
  - Basic programming skills

Overview: Week 4

- The Linear Regression Model
- Estimation
    - The OLS Approach
    - Deriving the OLS Estimator
- OLS Regression in Practice
- Regression Diagnostics
- Transformation and Nonlinearity
- Statistical Inference for Linear Models
    - Assumptions
    - Standard Errors and Confidence Intervals

The Linear Regression Model

Regression Analysis

- Regression analysis examines the relationship between a dependent variable, Y, and one or more independent variables, $X_1, \ldots, X_k$.
- The dependent variable is the quantity we want to explain.
    - Examples: vote choice, level of corruption, income, democratization.
- The independent variable is the quantity that we use to explain variation in the dependent variable.
    - Examples: Economic, political, institutional, or demographic variables.

The Linear Model

- Linear model: $Y = a + bX$.
    - a is the intercept: value of Y when X is zero.
    - b is the slope: change in Y for a one-unit increase in X.
- This model implies a perfect linear relationship.
- In actual research, this is never the case, so we need to add an error term:

$$Y = \beta_0 + \beta_1 X + \epsilon$$

- The error term (or disturbance), ϵ, represents unobserved factors other than X that affect Y.

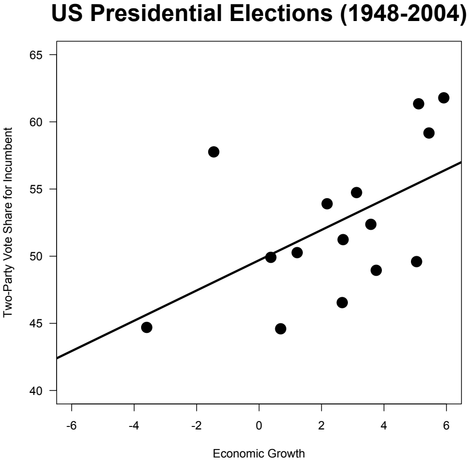
Example

- Let's re-examine the relationship between economic growth *prior* to an election and the vote share received by the incumbent presidential party in the US.
- Let $i = 1, ..., n$ denote observations with total sample size n.
- For each observation i, we can write:

$$VoteShare_i = \beta_0 + \beta_1 Growth_i + \epsilon_i$$

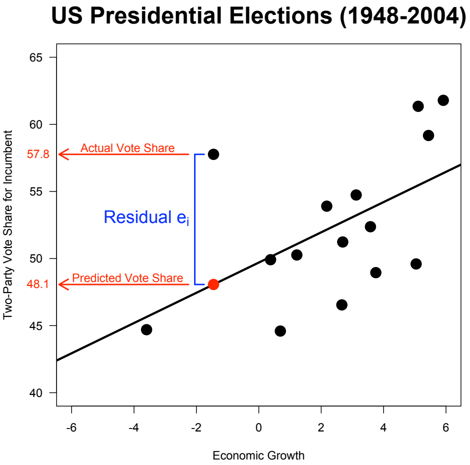
What is the relationship between *X* and *Y*?

$$VoteShare_i = \beta_0 + \beta_1 Growth_i + \epsilon_i$$



Residuals: How bad is our model prediction?

RESIDUAL = ACTUAL - PREDICTED



Estimation

How to Pick the Best Line? The OLS Approach

- OLS is short for "Ordinary Least Squares".
- The best line is the line that minimizes the sum of squared residuals (SSR).
- Residuals are vertical deviations from the line (the observed fitting errors):

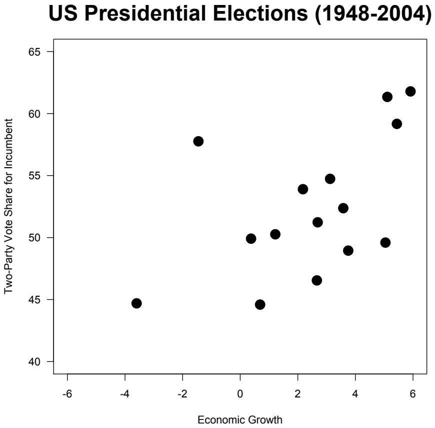
$$e_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$$
$$e_i = y_i - \hat{y}_i$$

- Conceptually, we minimize $e_i^2 = \sum_{i=1}^{n} (ACTUAL_i - PREDICTED_i)^2 = \sum_{i=1}^{n} (RESIDUAL_i)^2$.
- Mathematically, we solve the following optimization problem ("Least Squares"):

$$\min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{n} e_i^2 \quad \Leftrightarrow \quad \min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

How to Pick the Best Line? The OLS Approach

We minimize the sum of squared residuals: $\min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} e_i^2$.



How to Pick the Best Line? The OLS Approach

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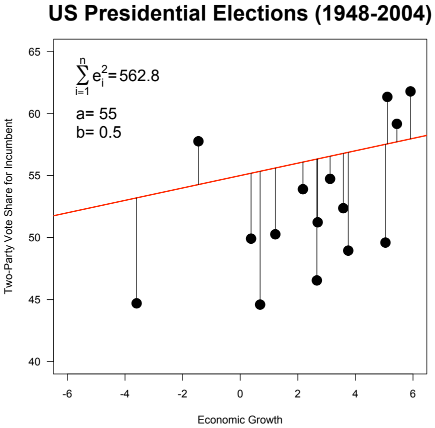

How to Pick the Best Line? The OLS Approach

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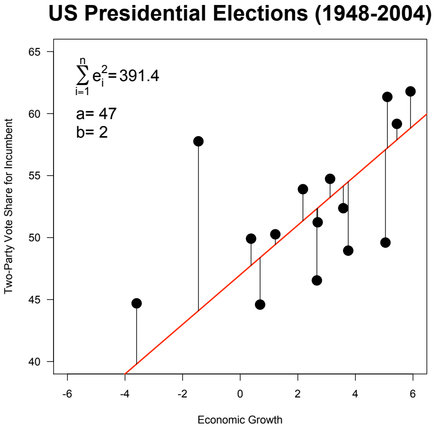
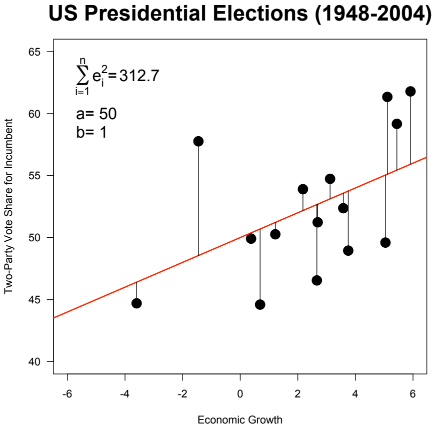


How to Pick the Best Line? The OLS Approach

We minimize the sum of squared residuals: $\min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} e_i^2$.



How to Pick the Best Line? The OLS Approach

- How to pick the best line? Get the best slope and best intercept using differential calculus.
- For $Var(x) \neq 0$, the slope coefficient $\hat{\beta}_1$ is given by

$$\hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{Cov(x, y)}{Var(x)}.$$

- The intercept coefficient $\hat{\beta}_0$ is given by

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} \quad \text{where } \bar{y} = \textstyle\sum_{i=1}^{n} \frac{y_i}{n} \text{ and } \bar{x} = \textstyle\sum_{i=1}^{n} \frac{x_i}{n}.$$

- An estimator is unbiased if its expected value, $E(\hat{\theta})$, is identical to the population value, θ.
- The estimator is best in the sense that it has the lowest variance across all unbiased estimators.
- The OLS estimator is said to be the Best Linear Unbiased Estimator (BLUE).

How to Pick the Best Line? The OLS Approach

- The residual variance (aka *error variance*) $\hat{\sigma}^2$ can be calculated as:

$$\hat{\sigma}^2 = \frac{\sum e_i^2}{n-2}$$

- Thus, the residual standard error is given as:

$$\hat{\sigma} = \sqrt{\frac{\sum e_i^2}{n-2}}$$

- To get an unbiased estimate of the residual variance, we divide by $n-2$ degrees of freedom, rather than the sample size n.
- Degrees of freedom reduce by two because we have two optimization conditions. This generalizes to $n-k$, where k is the number of *parameters*.
- In some literature you will find instead $n-k-1$. Huh? What’s the difference? In this case k denotes the number of *independent variables* (*excluding* the constant).

Digression: Getting the Coefficients

- Let there be the following minimization problem:

$$\min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

- Then, we have the following two partial derivatives:

$$\frac{\partial \sum e_i^2}{\partial \hat{\beta}_0} = 2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1)$$

$$\frac{\partial \sum e_i^2}{\partial \hat{\beta}_1} = 2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i)$$

Digression: Getting the Coefficients ($\hat{\beta}_0$)

- With the above partial derivative, the first-order condition for the first equation is given as:

$$(-2)\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

- This allows us to derive an expression for $\hat{\beta}_0$.

$$\begin{aligned}
(-2)\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) &= 0 \\
\sum_{i=1}^{n} y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^{n} x_i &= 0 \\
\frac{\sum_{i=1}^{n} y_i}{n} - \frac{\hat{\beta}_1 \sum_{i=1}^{n} x_i}{n} &= \hat{\beta}_0 \\
\bar{y} - \hat{\beta}_1 \bar{x} &= \hat{\beta}_0
\end{aligned}$$

Digression: Getting the Coefficients ($\hat{\beta}_1$)

- With the above partial derivative, the first-order condition for the second equation is given as:

$$(-2)\left(\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(x_i)\right) = 0$$

- This allows us to derive an expression for $\hat{\beta}_1$.

$$(-2)\sum_{i=1}^{n}(y_i {-} \hat{\beta}_0 - \hat{\beta}_1 x_i)(x_i) = 0$$

$$\sum_{i=1}^{n}(y_i {-} \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i)(x_i) = 0$$

$$\sum_{i=1}^{n}(y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}))(x_i) = 0$$

$$\sum_{i=1}^{n}(x_i)(y_i - \bar{y}) = \hat{\beta}_1 \sum_{i=1}^{n}(x_i)(x_i - \bar{x})$$

Digression: Getting the Coefficients ($\hat{\beta}_1$)

- Almost there:

$$\sum_{i=1}^{n}(x_i)(y_i - \bar{y}) = \hat{\beta}_1 \sum_{i=1}^{n}(x_i)(x_i - \bar{x})$$

$$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \hat{\beta}_1 \sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})}$$

$$\hat{\beta}_1 = \frac{Cov(x,y)}{Var(x)}$$

- This is exactly what we wanted to show.

OLS Regression in Practice

Least Squares Regression: Interpretation

- The regression line is: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$.
- Interpreting the slope coefficient $\hat{\beta}_1$: *On average*, a one-unit increase in X produces a $\hat{\beta}_1$ unit increase in Y.
- More generally, the so-called marginal effect of an infinitesimal change in X on Y, i.e., $\frac{\partial \hat{Y}}{\partial X} = \hat{\beta}_1$, is constant (independent of X) in case of OLS.
- The predicted value for X is $\hat{Y}$.
- Interpreting the intercept coefficient: When X is zero, the predicted value for $\hat{Y}$ is $\hat{\beta}_0$. Note that this may not be a meaningful quantity.
  - We will see next week that this is particularly important for interaction effects.
- The regression line always passes through two points:
  - Point 1: $(x_i = 0, y_i = \hat{\beta}_0)$. Why?
  - Point 2: $(x_i = \bar{x}, y_i = \bar{y})$. Why?

Least Squares Regression: Example

Year	VoteShare	Growth
1948	52.37	3.579
1952	44.595	.691
1956	57.764	-1.451
1960	49.913	.377
1964	61.344	5.109
1968	49.596	5.043
1972	61.789	5.914
1976	48.948	3.751
1980	44.697	-3.597
1984	59.17	5.440
1988	53.902	2.178
1992	46.545	2.662
1996	54.736	3.121
2000	50.265	1.219
2004	51.233	2.690
	$\bar{y} = 52.4578$	$\bar{x} = 2.4484$

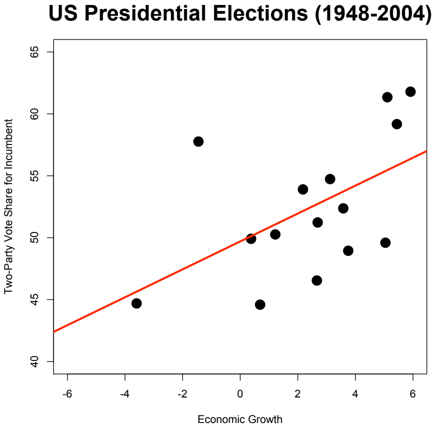
$(y_i - \bar{y})$	$(x_i - \bar{x})$
-0.088	1.131
-7.863	-1.757
5.306	-3.899
-2.545	-2.071
8.886	2.661
-2.862	2.595
9.331	3.466
-3.510	1.303
-7.761	-6.045
6.712	2.992
1.444	-0.270
-5.913	0.214
2.278	0.673
-2.193	-1.229
-1.225	0.242

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{111.559}{99.0181} = 1.127$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = 52.4578 - 1.127 \cdot 2.4484 = 49.699$$

Least Squares Regression: Example

- OLS model estimation: $\widehat{VoteShare}_i = 49.699 + 1.127 \cdot Growth_i$
- The sum of squared residuals is minimized at $\sum_{i=1}^{n} e_i^2 = 311.1486$



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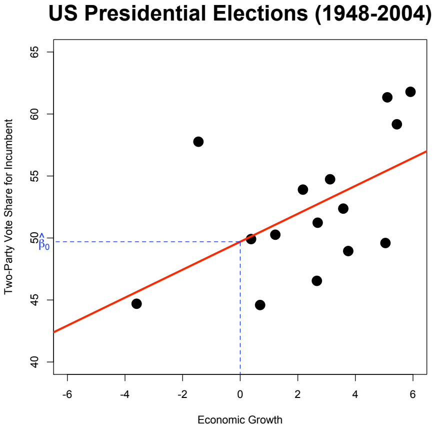

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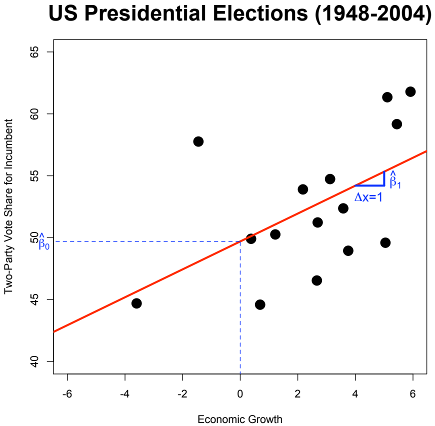


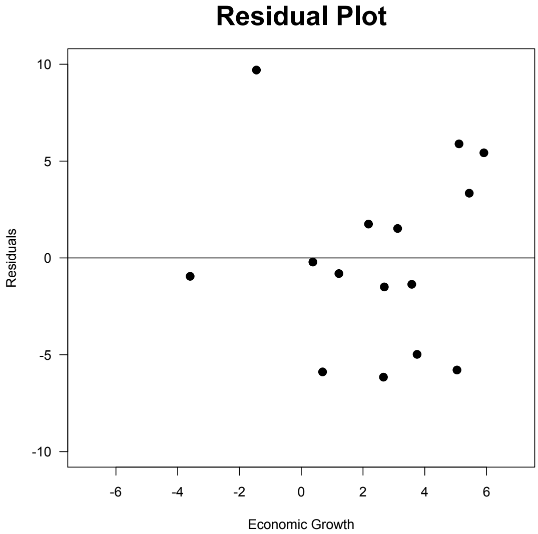
Regression Diagnostics

Regression Diagnostics: Residual Analysis

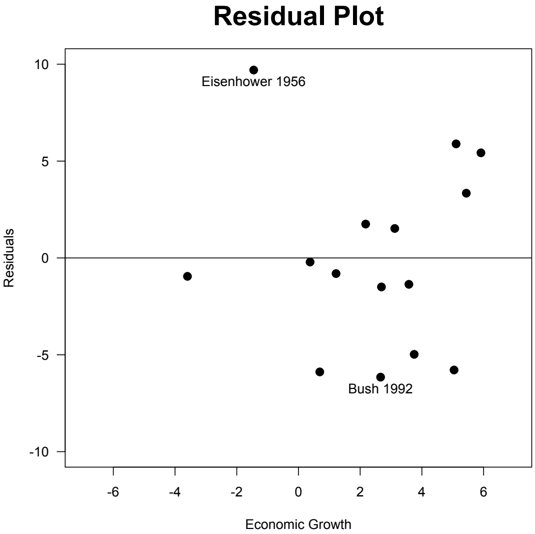
- A residual plot is a scatterplot of the regression residuals against the explanatory variable X or the predicted values $\hat{Y}$.
- The residual plot is a diagnostic plot as it helps us to detect patterns in the residuals.
- Patterns in residuals signal that systematic influences on Y still have not been captured by our model, or that our model misrepresents the data, or that errors do not have a constant variance.
- Punchline: Residual patterns diagnose model shortcomings.
- Ideally, residuals plots should look as if the pattern was generated by pure chance.
- By construction (first-order condition of $\hat{\beta}_0$), OLS residuals sum to zero:

$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}x_i) = \sum_{i=1}^{n}(y_i - \hat{y}_i) = \sum_{i=1}^{n} e_i = 0.$$

Regression Diagnostics: Residual Analysis



Regression Diagnostics: Residual Analysis



Regression Diagnostics: Goodness-of-Fit

- How well does our model explain the variation in the dependent variable?
- Disaggregate the variance of Y into that part that we have explained by X and that part that we have not explained.
  - Explained sum of squares (ESS) $= \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$
  - Residual sum of squares (RSS) $= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} e_i^2$
  - Total sum of squares (TSS) $= \sum_{i=1}^{n} (y_i - \bar{y})^2$
- Remember: TSS=ESS+RSS

Regression Diagnostics: Goodness-of-Fit

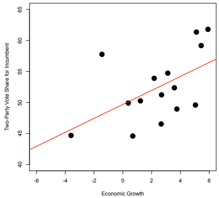
$$\frac{\text{Explained Variance}}{\text{Total Variance}} = 1 - \frac{\text{Unexplained Variance}}{\text{Total Variance}} = \text{Goodness-of-fit}$$

$$\frac{ESS}{TSS} = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS} = R^2$$

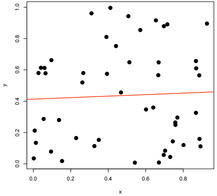
- Interpretation: Proportion of the total variance explained by the fitted model.
- The goodness-of-fit measure is bounded: $0 \leq R^2 \leq 1$
- For a bivariate linear regression model, R^2 is identical to the squared Pearson's r correlation coefficient of x and y.
- Note the following two caveats:
  - R^2 is not resistant to outliers. One outlier can distort the value.
  - R^2 increases with the addition of more explanatory variables.

Regression Diagnostics: Goodness-of-Fit

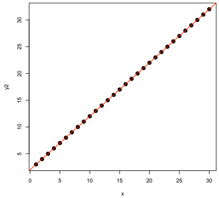
$R^2 = 0.288$



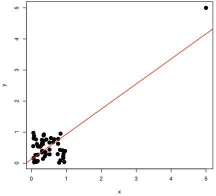
$R^2 = 0.002$



$R^2 = 1$



$R^2 = 0.664$



Regression Diagnostics: Goodness-of-Fit

- R^2 increases the more explanatory variables we add.
- This is due to the fact that the sum of squared residuals never goes up as more variables are added.
- The adjusted R^2, therefore, imposes a penalty for adding independent variables.
- If an independent variable is added to a regression, the RSS falls, but so do the degrees of freedom in the regression model.
- While the regular R^2 is bounded in $[0, 1]$, the adjusted R^2 even can become negative. This indicates a bad model fit.
- The adjusted R^2 for sample size, n, and k independent variables is defined as

$$\text{Adj. } R^2 = 1 - (1 - R^2)\left(\frac{n-1}{n-k-1}\right).$$

- Example: US Presidential Election Data.

$$R^2 = 0.808 \text{ and Adj. } R^2 = 1 - (1 - 0.808)\left(\frac{15-1}{15-1-1}\right) = 0.793$$

Transformation and Nonlinearity

What does linear in Linear Regression actually mean?

- The linearity assumption refers to linearity in parameters only.
- This allows for nonlinearities in variables.
- Suppose you want to model the following:

$$Y = \beta_0 X^{\beta_1} \cdot \epsilon$$

Then, $log(Y) = log(\beta_0) + \beta_1 log(X) + log(\epsilon)$

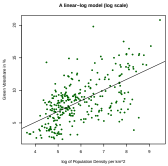
Which is $\tilde{Y} = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{X} + \tilde{\epsilon}$

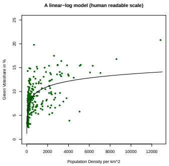
and can be estimated via OLS.

- Interpretation of the estimates $\hat{\tilde{\beta}}_1$ (and $\hat{\tilde{\beta}}_0$ respectively)?

Logarithmic Transformation of Variables

- In applied work, you will sometimes encounter a dependent variable or a covariate in logarithmically transformed:
  - Log-transformed covariates makes sense, if we theoretically expect an nonlinear decreasing impact of *X* on *Y*, e.g., the effect diminishes if *X* increases (e.g., GDP, district magnitude, pop. density).
  - Log-transformed dependent variable makes sense, if the residuals are not nearly normally distributed (e.g., rightly skewed) and inclusion of further variables did not help (to fix it afterwards).
- Our goal to make the relationship between two variables more linear through transforming (one or both of) them.
- Interpretation of coefficient changes with respect to untransformed variables.





Statistical Inference for Linear Models

Classical OLS Assumptions

Suppose we have the following bivariate linear model:

$$Y = \beta_0 + \beta_1 X + \epsilon$$

We need two assumptions to derive unbiased regression coefficients, $\hat{\beta}_0$ and $\hat{\beta}_1$.

- A1: An almost trivial assumption is that coefficients (i.e., parameters) are linear.
- A2: We make a zero conditional mean assumption:

$$E(\epsilon_i \mid X) = 0$$

- For the multiple regression model, we also need to assume that there is no perfect collinearity of independent variables, i.e., that X is not a function of other independent variables in the model.
    - This is why with k categories we only included $k - 1$ dummy variables.
- These assumptions are sufficient to estimate unbiased coefficients, $\hat{\beta}_0$ and $\hat{\beta}_1$, with OLS.

Classical OLS Assumptions

To also estimate the variance of the coefficients, we need to make additional assumptions.

- A3: We assume constant variance, which is known as homoskedasticity, regardless of the values of X:

$$Var(\epsilon_i \mid X) = \sigma^2$$

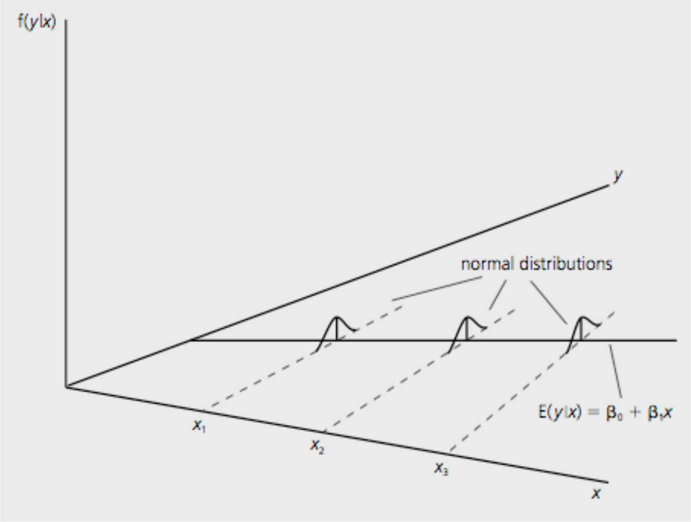
- A4: We assume no correlation among any pair of error terms:

$$Cov(\epsilon_i, \epsilon_j \mid X_i, X_j) = 0 \quad \forall\, i \neq j$$

- A5: We assume normality of the error term:

$$\epsilon_i \mid X \sim \mathcal{N}(0, \sigma^2)$$

Classical OLS Assumptions



Standard Errors for Regression Coefficients

- If the zero conditional mean assumption, $E(\epsilon_i \mid X)$, holds, we get

$$E(\hat{\beta}_0) = \beta_0$$
$$E(\hat{\beta}_1) = \beta_1.$$

- Assuming normally distributed errors, $\epsilon \mid X \sim \mathcal{N}(0, \sigma^2)$, the OLS coefficients themselves are normally distributed.

$$\hat{\beta}_0 \sim \mathcal{N}\left(\beta_0, Var(\hat{\beta}_0)\right)$$
$$\hat{\beta}_1 \sim \mathcal{N}\left(\beta_1, Var(\hat{\beta}_1)\right)$$

- This allows us to calculate standard errors based on normal approximation.

Standard Errors for Regression Coefficients

- The standard error for our estimated slope coefficient, $\hat{\beta}_1$, is:

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$SE(\hat{\beta}_1) = \sqrt{\frac{\sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

- The standard error for our estimated intercept coefficient, $\hat{\beta}_0$, is:

$$Var(\hat{\beta}_0) = \frac{\sigma^2 \frac{\sum_{i=1}^{n} x_i^2}{n}}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$SE(\hat{\beta}_0) = \sqrt{\frac{\sigma^2 \frac{\sum_{i=1}^{n} x_i^2}{n}}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

- Before we can estimate standard errors, we need to estimate $\hat{\sigma}^2$ because we do not observe σ^2.

Standard Errors for Regression Coefficients

- However, the regression error, σ^2, is inherently unobservable, but can be estimated from the model residuals e_i.
- In the bivariate model an unbiased estimator for the error variance (aka *residual variance*) is given as

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} e_i^2}{n-2}.$$

- This can be generalized to get an unbiased estimator in a multiple regression model with k independent variables and one intercept (i.e., $k+1$ parameters):

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} e_i^2}{n-(k+1)} = \frac{\sum_{i=1}^{n} e_i^2}{n-k-1}.$$

- Thus, we get the root mean squared error (RMSE) aka standard error of the estimate (*How far is the model off on average?*) of Y as:

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n-k-1}}.$$