Item Response Model from scratch with Rcpp

vkyo23

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Contents

1 Preface

In the repository, I have published the code to implement the Bayesian two parameter logistic item response model (2PL IRT) from scratch using `Rcpp` and `RcppArmadillo`.

In this RMarkdown file, I will describe the model and show an example of analysis using those codes[1].

IMPORTANT NOTE: As this is a transcription by a student who is still studying IRT for his own notes, there will be errors in various parts. Please let me know if there are any mistakes.

[1] The description and implementation of model is relied on *Handbook of Item Response Theory Volume 2: Statistical Tools* by Wim J. van der Linden (2016). https://www.routledge.com/Handbook-of-Item-Response-Theory-Volume-2-Statistical-Tools/Linden/p/book/9780367221041

2 Likelihood and posteriors

Suppose there are $i = 1, \dots, N$ individuals and $j = 1, \dots, J$ items (or exams). The probability of an individual i's correct answer ($y_{ij} = 1$, otherwise 0) to an item j is given by

$$p_{ij} = \Pr(y_{ij} = 1 \mid \alpha_j, \beta_j, \theta_i) = \text{logit}^{-1}(\beta_j \theta_i - \alpha_j).$$

where α_j, β_j is the **difficulty** and **discrimination** of an item j, and θ_i is the ability of i. It is straightforward to show the likelihood f:

$$f(y \mid \alpha, \beta, \theta) = \prod_{i=1}^{N} \prod_{j=1}^{J} p_{ij}^{y_{ij}} (1 - p_{ij})^{1 - y_{ij}}. \tag{1}$$

For the above generative model, Bayesian estimation is used to estimate the latent variables, the parameters α_j, β_j and θ_i. From the Bayes rule, we can write the posterior distribution as

$$\pi(\Theta \mid y) \propto f(y \mid \Theta) \pi(\Theta), \tag{2}$$

where $\pi(\Theta)$ is **prior distribution**, $f(y \mid \Theta)$ is likelihood function and $\pi(\Theta \mid y)$ is **posterior distribution**.

Applying these to the above example, we will derive the posterior distribution of the item response theory model. First, we set prior distributions for α, β, θ

$$\alpha_j \sim N(a_0, A_0) \tag{3}$$
$$\beta_j \sim N(b_0, B_0) \tag{4}$$
$$\theta_i \sim N(0, 1). \tag{5}$$

Then, we can write the conditional posterior distribution for α, β, θ from equation (1) ~ (5), given the rest parameters:

$$\begin{aligned} \pi(\alpha_j \mid y, \beta, \theta) &\propto f(y \mid \alpha, \beta, \theta) \pi(\alpha_j) \\ &= f(y \mid \alpha, \beta, \theta) \times N(\alpha_j \mid a_0, A_0) \\ &= \prod_{i=1}^{N} \left[p_{ij}^{y_{ij}} (1 - p_{ij})^{1 - y_{ij}} \right] N(\alpha_j \mid a_0, A_0) \quad \forall j = 1, \dots J. \end{aligned} \tag{6}$$

$$\begin{aligned} \pi(\beta_j \mid y, \alpha, \theta) &\propto f(y \mid \alpha, \beta, \theta) \pi(\beta_j) \\ &= f(y \mid \alpha, \beta, \theta) \times N(\beta_j \mid b_0, B_0) \\ &= \prod_{i=1}^{N} \left[p_{ij}^{y_{ij}} (1 - p_{ij})^{1 - y_{ij}} \right] N(\beta_j \mid b_0, B_0) \quad \forall j = 1, \dots J. \end{aligned} \tag{7}$$

$$\begin{aligned} \pi(\theta_i \mid y, \alpha, \beta) &\propto f(y \mid \alpha, \beta, \theta) \pi(\theta_i) \\ &= f(y \mid \alpha, \beta, \theta) \times N(\theta_i \mid 0, 1) \\ &= \prod_{j=1}^{J} \left[p_{ij}^{y_{ij}} (1 - p_{ij})^{1 - y_{ij}} \right] N(\theta_i \mid 0, 1) \quad \forall i = 1, \dots N. \end{aligned} \tag{8}$$

Normally, we would use a Gibbs sampler to sample parameters from the posterior distribution, but since the conditional posteriors in (6) ~ (8) above are not in the form of the standard distributions, such as normal and gamma distributions, we cannot simply sample parameters using a Gibbs sampler. Therefore, we adopt the Metropolis-Hastings Algorithm, which allows us to perform MCMC even in such a case.

3 Metropolis-Hastings Algorithm

Here, for simplicity of notation, I define an arbitrary parameter as δ_k. First, we sample the **candidate** δ_k^* from a random walk distribution as follows

$$\delta_k^* \sim N(\delta_k^{(t-1)}, \tau_\delta),$$

where $\delta_k^{(t-1)}$ is a sample from previous iteration and τ is a so-called tuning parameter. Next, we must calculate **the acceptance probability** which determines whether we accept the candidate δ_k^* or previous $\delta_k^{(t-1)}$ as a current sample. Following the discussion of Junker, Patz and VanHoudnos (2016, p.277)[2], the acceptance probability is given by:

$$ap = \min\left\{\frac{\pi(\delta_k^* \mid y, \eta) \cdot g(\delta_k^{(t-1)} \mid \delta_k^*)}{\pi(\delta_k^{(t-1)} \mid y, \eta) \cdot g(\delta_k^* \mid \delta_k^{(t-1)})}, 1\right\}$$

where $\pi(\cdot \mid y, \eta)$ is the posterior density, $g(\cdot \mid \cdot)$ is proposal density and η indicates rest parameters other than δ_k. If $u < ap$, we accept δ_k^* as a current sample, otherwise $\delta_k^{(t-1)}$ where $u \sim U(0, 1)$.

To calculate the acceptance probability, we first calculate the posterior density of the candidate sample, $\pi(\delta_k^* \mid y, \eta)$, and the previous sample, $\pi(\delta_k^{(t-1)} \mid y, \eta)$, respectively. For example, the case of α_j:

$$\begin{aligned}
\text{Candidate} \quad ... \quad \pi(\alpha_j^* \mid y, \beta, \theta) &\propto f(y \mid \alpha^*, \beta, \theta)\pi(\alpha_j^*) \\
&= f(y \mid \alpha^*, \beta, \theta) \times N(\alpha_j^* \mid a_0, A_0) \\
&= \prod_{i=1}^{N} \left[p_{ij}(\alpha_j^*)^{y_{ij}} \{1 - p_{ij}(\alpha_j^*)\}^{1-y_{ij}}\right] N(\alpha_j^* \mid a_0, A_0) \quad \forall j = 1, ... J.
\end{aligned}$$

$$\begin{aligned}
\text{Previous} \quad ... \quad \pi(\alpha_j^{(t-1)} \mid y, \beta, \theta) &\propto f(y \mid \alpha^{(t-1)}, \beta, \theta)\pi(\alpha_j^{(t-1)}) \\
&= f(y \mid \alpha^{(t-1)}, \beta, \theta) \times N(\alpha_j^{(t-1)} \mid a_0, A_0) \\
&= \prod_{i=1}^{N} \left[p_{ij}(\alpha_j^{(t-1)})^{y_{ij}} \{1 - p_{ij}(\alpha_j^{(t-1)})\}^{1-y_{ij}}\right] N(\alpha_j^{(t-1)} \mid a_0, A_0) \quad \forall j = 1, ... J.
\end{aligned}$$

Next, we calculate the proposal density $g(\alpha_j^* \mid \alpha_j^{(t-1)})$ and $g(\alpha_j^{(t-1)} \mid \alpha_j^*)$

$$\begin{aligned}
\text{Candidate} \quad \cdots \quad g(\alpha_j^* \mid \alpha_j^{(t-1)}) &= N(\alpha_j^* \mid \alpha_j^{(t-1)}, \tau_\alpha) \quad \forall \quad j = 1, \ldots, J \\
\text{Previous} \quad \cdots \quad g(\alpha_j^{(t-1)} \mid \alpha_j^*) &= N(\alpha_j^{(t-1)} \mid \alpha_j^*, \tau_\alpha) \quad \forall \quad j = 1, \ldots, J.
\end{aligned}$$

The same steps are applied to the case of other parameters β_j and θ_i.

[2]From the same book as in footnote 1.

Here, we convert the acceptance probability into natural logarithm for the convenience of computation, that is:

$$\log(ap) = \min\left\{\log[\pi(\delta_k^* \mid y, \eta)] + \log[g(\delta_k^{(t-1)} \mid \delta_k^*)] - \log[\pi(\delta_k^{(t-1)} \mid y, \eta)] - \log[g(\delta_k^* \mid \delta_k^{(t-1)}], 0\right\},$$

and if $\log(u) < \log(ap)$, we accept δ_k^* otherwise $\delta_k^{(t-1)}$.

4 Coding the sampler

In this section, I write down the code for parameter sampling.

4.1 Log-liklihood calculator `loglik.cpp`

First, I introduce helper function `loglik` which is enable us to calculate log-likelihood. The log-likelihood function is written as

$$l(y \mid \alpha, \beta, \theta) = \sum_{i=1}^{N}\sum_{j=1}^{J}[y_{ij}\log(p_{ij}) + (1 - y_{ij})\log(1 - p_{ij})].$$

```
// [[Rcpp::depends(RcppArmadillo)]]
#include <RcppArmadillo.h>
using namespace arma;
using namespace Rcpp;

// Log-likelihood function
arma::mat loglik(arma::mat Y, arma::vec alpha,
                 arma::vec beta, arma::vec theta) {
  //calculate beta_j * theta_i
  arma::mat temp = beta * theta.t();

  //beta_j * theta_i - alpha_j
  arma::mat temp2 = temp.each_col() - alpha;

  //exp(beta_j * theta_i - alpha_j)
  arma::mat exp_ = arma::exp(temp2.t());

  //inverse logit
  arma::mat p = exp_ / (1 + exp_);

  //calculate log-lik
  arma::mat log_lik = Y % arma::log(p) + (1 - Y) % arma::log(1 - p);

  return(log_lik);
}
```

4.2 Sampling α_j `alpha_sample.cpp`

```
// [[Rcpp::depends(RcppArmadillo)]]
#include <RcppArmadillo.h>
#include "loglik.h"
using namespace arma;
using namespace Rcpp;

// SAMPLING ALPHA
NumericVector alpha_sample(arma::mat Y, arma::vec alpha_old, arma::vec beta_old,
                           arma::vec theta_old, double a0, double A0, double MH_alpha) {
  // NOTE:
  // _star -> candidate
  // _old -> previous

  int J = Y.n_cols; //# of alpha
  arma::vec alpha_star(J); //candidate sample for alpha
  arma::vec log_prop_star(J); //log proposal density for alpha_star
  arma::vec log_prop_old(J); //log proposal density for alpha_old

  //Sample theta_star and log proposal density.
  for (int j = 0; j < J; j++) {
    alpha_star[j] = R::rnorm(alpha_old[j], MH_alpha);
    log_prop_star[j] = R::dnorm(alpha_star[j], alpha_old[j], MH_alpha, true);
    log_prop_old[j] = R::dnorm(alpha_old[j], alpha_star[j], MH_alpha, true);
  }

  //log-likelihood
  arma::rowvec loglik_star = colSums(as<NumericMatrix>(wrap(loglik(Y, alpha_star, beta_old, theta_old))
                                     true);
  arma::rowvec loglik_old = colSums(as<NumericMatrix>(wrap(loglik(Y, alpha_old, beta_old, theta_old))),
                                    true);

  //log prior density
  arma::rowvec log_dnorm_star =  dnorm(as<NumericVector>(wrap(alpha_star)), a0, A0, true);
  arma::rowvec log_dnorm_old = dnorm(as<NumericVector>(wrap(alpha_old)), a0, A0, true);

  //log posterior density
  arma::rowvec log_pd_star = loglik_star + log_dnorm_star;
  arma::rowvec log_pd_old = loglik_old + log_dnorm_old;

  //log acceptance probability
  arma::vec log_densfrac = log_pd_star.t() + log_prop_old - log_pd_old.t() - log_prop_star;
  NumericVector log_ap = pmin(as<NumericVector>(wrap(log_densfrac)), 0);
  NumericVector log_u = log(runif(J, 0, 1));

  //save samples
  NumericVector sample = ifelse(log_u < log_ap, as<NumericVector>(wrap(alpha_star)),
                                as<NumericVector>(wrap(alpha_old)));

  return(sample);
}
```

4.3 Sampling β_j beta_sample.cpp

```
// [[Rcpp::depends(RcppArmadillo)]]
#include <RcppArmadillo.h>
#include "loglik.h"
using namespace arma;
using namespace Rcpp;

// SAMPLING BETA
NumericVector beta_sample(arma::mat Y, arma::vec alpha_old, arma::vec beta_old,
                          arma::vec theta_old, double b0, double B0, double MH_beta) {

  // NOTE:
  // _star -> candidate
  // _old -> previous

  int J = Y.n_cols; //# of beta
  arma::vec beta_star(J); //candidate sample for beta
  arma::vec log_prop_star(J); //log proposal density for beta_star
  arma::vec log_prop_old(J); //log proposal density for beta_old

  //Sample theta_star and log proposal density.
  for (int j = 0; j < J; j++) {
    beta_star[j] = R::rnorm(beta_old[j], MH_beta);
    log_prop_star[j] = R::dnorm(beta_star[j], beta_old[j], MH_beta, true);
    log_prop_old[j] = R::dnorm(beta_old[j], beta_star[j], MH_beta, true);
  }

  //log-likelihood
  arma::rowvec loglik_star = colSums(as<NumericMatrix>(wrap(loglik(Y, alpha_old, beta_star, theta_old)))
                                     true);
  arma::rowvec loglik_old = colSums(as<NumericMatrix>(wrap(loglik(Y, alpha_old, beta_old, theta_old))),
                                    true);

  //log prior density
  arma::rowvec log_dnorm_star =  dnorm(as<NumericVector>(wrap(beta_star)), b0, B0, true);
  arma::rowvec log_dnorm_old = dnorm(as<NumericVector>(wrap(beta_old)), b0, B0, true);

  //log posterior density
  arma::rowvec log_pd_star = loglik_star + log_dnorm_star;
  arma::rowvec log_pd_old = loglik_old + log_dnorm_old;

  //log acceptance probability
  arma::vec log_densfrac = log_pd_star.t() + log_prop_old - log_pd_old.t() - log_prop_star;
  NumericVector log_ap = pmin(as<NumericVector>(wrap(log_densfrac)), 0);
  NumericVector log_u = log(runif(J, 0, 1));

  //save samples
  NumericVector sample = ifelse(log_u < log_ap, as<NumericVector>(wrap(beta_star)),
                                as<NumericVector>(wrap(beta_old)));

  return(sample);
}
```

4.4 Sampling θ_i theta_sample.cpp

```
// [[Rcpp::depends(RcppArmadillo)]]
#include <RcppArmadillo.h>
#include "loglik.h"
using namespace arma;
using namespace Rcpp;

// SAMPLING THETA
NumericVector theta_sample(arma::mat Y, arma::vec alpha_old, arma::vec beta_old,
                           arma::vec theta_old, double MH_theta) {

  // NOTE:
  // _star -> candidate
  // _old -> previous

  int I = Y.n_rows; //# of theta
  arma::vec theta_star(I); //candidate sample for theta
  arma::vec log_prop_star(I); //log proposal density for theta_star
  arma::vec log_prop_old(I); //log proposal density for theta_old

  //Sample theta_star and log proposal density.
  for (int i = 0; i < I; i++) {
    theta_star[i] = R::rnorm(theta_old[i], MH_theta);
    log_prop_star[i] = R::dnorm(theta_star[i], theta_old[i], MH_theta, true);
    log_prop_old[i] = R::dnorm(theta_old[i], theta_star[i], MH_theta, true);
  }

  //log-likelihood
  arma::vec loglik_star = rowSums(as<NumericMatrix>(wrap(loglik(Y, alpha_old, beta_old, theta_star))),
                                  true);
  arma::vec loglik_old = rowSums(as<NumericMatrix>(wrap(loglik(Y, alpha_old, beta_old, theta_old))),
                                 true);

  //log prior density
  arma::vec log_dnorm_star =  dnorm(as<NumericVector>(wrap(theta_star)), 0, 1, true);
  arma::vec log_dnorm_old = dnorm(as<NumericVector>(wrap(theta_old)), 0, 1, true);

  //log posterior density
  arma::vec log_pd_star = loglik_star + log_dnorm_star;
  arma::vec log_pd_old = loglik_old + log_dnorm_old;

  //log acceptance probability
  arma::vec log_densfrac = log_pd_star + log_prop_old - log_pd_old - log_prop_star;
  NumericVector log_ap = pmin(as<NumericVector>(wrap(log_densfrac)), 0);
  NumericVector log_u = log(runif(I, 0, 1));

  //save samples
  NumericVector sample = ifelse(log_u < log_ap, as<NumericVector>(wrap(theta_star)),
                                as<NumericVector>(wrap(theta_old)));

  return(sample);
}
```

4.5 Sampler `sampler_irt.cpp`

In MCMCpack, reparameterization of estimated parameters is done. Specifically, θ_i is standardized with mean 0 and sd 1, and α_j, β_j are alsop adjusted accordingly. Specifically, this is.

$$
\begin{aligned}
\theta_i^{adj} &= \frac{\theta_i - \overline{\theta}}{s_\theta} \\
\alpha_j^{adj} &= \beta_j \overline{\theta} - \alpha_j \\
\beta_j^{adj} &= \beta_j s_\theta,
\end{aligned}
$$

where s_θ is standard deviation of θ and $\overline{\theta}$ is the mean. In my sampler, I have also incorporated these processes properly.

```
// [[Rcpp::depends(RcppArmadillo)]]
#include <RcppArmadillo.h>
#include "alpha_sample.h"
#include "beta_sample.h"
#include "theta_sample.h"
using namespace Rcpp;
using namespace arma;

// SAMPLER
// [[Rcpp::export]]
List sampler_irt(arma::mat datamatrix, arma::vec alpha, arma::vec beta,
                 arma::vec theta, double a0, double A0,
                 double b0, double B0,
                 double MH_alpha, double MH_beta, double MH_theta,
                 int iter, int warmup, int thin, int refresh) {

  int total_iter = iter + warmup; // total iteration
  int sample_iter = iter / thin; // # of samples to save

  arma::mat Y = datamatrix; // rename datamatrix to Y
  int I = Y.n_rows; // # of individuals
  int J = Y.n_cols; // # of items

  // rename
  arma::vec alpha_old = alpha;
  arma::vec beta_old = beta;
  arma::vec theta_old = theta;

  // create storages for parameters
  NumericMatrix theta_store(I, sample_iter);
  NumericMatrix alpha_store(J, sample_iter);
  NumericMatrix beta_store(J, sample_iter);


  // WARMUP
  Rcout << "Warmup:   " << 1 << " / " << total_iter << " [ " << 0 << "% ]\n";
  for (int g = 0; g < warmup; g++) {
    if ((g + 1) % refresh == 0) {
```

```
      double gg = g + 1;
      double ti2 = total_iter;
      double per = std::round((gg / ti2) * 100);
      Rcout << "Warmup:   " << (g + 1) << " / " << total_iter << " [ " << per << "% ]\n";
    }
    theta = theta_sample(Y, alpha_old, beta_old, theta_old, MH_theta);
    theta_old = theta;
    alpha = alpha_sample(Y, alpha_old, beta_old, theta_old, a0, A0, MH_alpha);
    alpha_old = alpha;
    beta = beta_sample(Y, alpha_old, beta_old, theta_old, b0, B0, MH_beta);
    beta_old = beta;
  }

  // SAMPLING
  double gg = warmup + 1;
  double ti2 = total_iter;
  double per = std::round((gg / ti2) * 100);
  Rcout << "Sampling: " << gg << " / " << total_iter << " [ " << per << "% ]\n";
  for (int g = warmup; g < total_iter; g++) {
    if ((g + 1) % refresh == 0) {
      double gg = g + 1;
      double ti2 = total_iter;
      double per = std::round((gg / ti2) * 100);
      Rcout << "Sampling: " << (g + 1) << " / " << total_iter << " [ " << per << " % ]\n";
    }
    theta = theta_sample(Y, alpha_old, beta_old, theta_old, MH_theta);
    theta_old = theta;
    alpha = alpha_sample(Y, alpha_old, beta_old, theta_old, a0, A0, MH_alpha);
    alpha_old = alpha;
    beta = beta_sample(Y, alpha_old, beta_old, theta_old, b0, B0, MH_beta);
    beta_old = beta;

    if (g % thin == 0) {
      double th = thin;
      double wu = warmup;
      double gg = g;
      double ggg = (g - warmup) / thin;

      // Reparameterization (fix theta with mean 0 and sd 1)

      NumericVector theta_nv = as<NumericVector>(wrap(theta_old));
      NumericVector alpha_nv = as<NumericVector>(wrap(alpha_old));
      NumericVector beta_nv = as<NumericVector>(wrap(beta_old));

      NumericVector theta_std = (theta_nv - mean(theta_nv)) / sd(theta_nv);
      NumericVector alpha_std = beta_nv * mean(theta_nv) - alpha_nv;
      NumericVector beta_std = beta_nv * sd(theta_nv);

      theta_store(_, ggg) = theta_std;
      alpha_store(_, ggg) = alpha_std;
      beta_store(_, ggg) = beta_std;
    }
  }
```

```
  List L = List::create(Named("alpha") = alpha_store,
                        Named("beta") = beta_store,
                        Named("theta") = theta_store);
  return(L);
}
```

4.6 R wrapper function

I also write a wrapper function `irt_cpp` for the sampler.

```
# This code is from "irt_cpp.R"
irt_cpp <- function(datamatrix, iter = 2000, warmup = 1000, thin = 1, refresh = 100,
                    seed, init, tuning_par, prior) {

  # datamatrix -> individual * item matrix (matrix)
  # iter -> # of iterations (int)
  # warmup -> # of burn-in (int)
  # thin -> Save sample every [thin] iteration (int)
  # refresh -> Output the status of sampling every [refresh] iteration (int)
  # seed -> seed value (double)
  # init -> initial values (list, please name correctly as below!)
  #         -> alpha: init for alpha
  #         -> beta: init for beta
  #         -> theta: init for theta
  # tuning_par -> tuning parameter (list, please name correctly as below!)
  #               -> alpha: tau for alpha
  #               -> beta: tau for beta
  #               -> theta: tau for theta
  #
  # prior -> priors (list, please name correctly as below!)
  #         -> a0: prior means for alpha
  #         -> A0: prior sd for alpha
  #         -> b0: prior means for beta
  #         -> B0: prior sd for beta

  cat("\n=================================================================\n")
  cat("Run Metropolis-Hasting Sampler for 2PL item response model...\n\n")
  cat("  Observations:", nrow(datamatrix) * ncol(datamatrix),"\n")
  cat("    Number of individuals:", nrow(datamatrix), "\n")
  cat("    Number of items:", ncol(datamatrix), "\n")
  cat("    Total correct response:", sum(as.numeric(datamatrix), na.rm = TRUE), "/",
    nrow(datamatrix) * ncol(datamatrix),
    "[", round(sum(as.numeric(datamatrix), na.rm = TRUE) / (nrow(datamatrix) *
                                                         ncol(datamatrix)), 2) * 100, "%]","\n\n
  cat("  Priors: \n")
  cat("    alpha ~", paste0("N(", prior$a0, ", ", prior$A0, "),"),
    "beta ~", paste0("N(", prior$b0, ", ", prior$B0, "),"),
    "theta ~ N(0, 1).\n")
  cat("================================================================\n\n")

  # Preparation
  ## Measure starting time
```

```r
stime <- proc.time()[3]
## Set seed
set.seed(seed)

# Run sampler
mcmc <- sampler_irt(datamatrix = Y,
                    alpha = init$alpha,
                    beta = init$beta,
                    theta = init$theta,
                    a0 = prior$a0,
                    A0 = prior$A0,
                    b0 = prior$b0,
                    B0 = prior$B0,
                    MH_alpha = tuning_par$alpha,
                    MH_beta = tuning_par$beta,
                    MH_theta = tuning_par$theta,
                    iter = iter,
                    warmup = warmup,
                    thin = thin,
                    refresh = refresh)

# Generate variable labels
label_iter <- paste0("iter_", 1:(iter/thin))
alpha_lab <- paste0("alpha_", colnames(datamatrix))
beta_lab <- paste0("beta_", colnames(datamatrix))
theta_lab <- paste0("theta_", rownames(datamatrix))
colnames(mcmc$alpha) <- colnames(mcmc$beta) <- colnames(mcmc$theta) <- label_iter
rownames(mcmc$alpha) <- alpha_lab
rownames(mcmc$beta) <- beta_lab
rownames(mcmc$theta) <- theta_lab

# Redefine quantile function
lwr <- function(x) quantile(x, probs = 0.025, na.rm = TRUE)
upr <- function(x) quantile(x, probs = 0.975, na.rm = TRUE)
mean_ <- function(x) mean(x, na.rm = TRUE)
median_ <- function(x) median(x, na.rm = TRUE)

# Calculate statistics
alpha_post <- data.frame(parameter = alpha_lab,
                         mean = apply(mcmc$alpha, 1, mean_),
                         median = apply(mcmc$alpha, 1, median_),
                         lwr = apply(mcmc$alpha, 1, lwr),
                         upr = apply(mcmc$alpha, 1, upr))
beta_post <- data.frame(parameter = beta_lab,
                        mean = apply(mcmc$beta, 1, mean_),
                        median = apply(mcmc$beta, 1, median_),
                        lwr = apply(mcmc$beta, 1, lwr),
                        upr = apply(mcmc$beta, 1, upr))
theta_post <- data.frame(parameter = theta_lab,
                         mean = apply(mcmc$theta, 1, mean_),
                         median = apply(mcmc$theta, 1, median_),
                         lwr = apply(mcmc$theta, 1, lwr),
                         upr = apply(mcmc$theta, 1, upr))
```

```
  # Aggregate
  result <- list(summary = list(alpha = alpha_post,
                               beta = beta_post,
                               theta = theta_post),
                 sample = list(alpha = mcmc$alpha,
                              beta = mcmc$beta,
                              theta = mcmc$theta))
  etime <- proc.time()[3]
  cat(crayon::yellow("Done: Total time", round(etime - stime, 1), "sec\n"))
  return(result)
}
```

5 Example: 106th US Senate roll-call vote analysis

As an example, I will apply item response theory using data of the 106th US Senate roll-call vote. The data is from the {MCMCpack} package[3]. In the field of political science, IRT is frequently used to measure the policy positions (a.k.a. ideal points) of political actors using roll call voting and judgment data. For a discussion of the relevance of spatial voting models to IRT, see Clinton, Jackman & Rivers (2004, APSR)[4].

First, load {Rcpp} package, compile `sampler_irt.cpp` and data.

```
library(Rcpp)
sourceCpp("../cpp/sampler_irt.cpp")
data(Senate, package = "MCMCpack")
```

To run the sampler, we must convert the data into **roll-call matrix (individual * item matrix)**.

```
# Check data (data frame)
Senate[1:10, 1:10]
```

```
              id statecode   state party     member rc1 rc2 rc3 rc4 rc5
SESSIONS   49700        41 ALABAMA     1   SESSIONS   1   0   0   0   1
SHELBY     94659        41 ALABAMA     1     SHELBY   1   0   0   0   1
MURKOWSKI  14907        81  ALASKA     1  MURKOWSKI   1   0   0   0   1
STEVENS    12109        81  ALASKA     1    STEVENS   1   0   0   0   1
KYL        15429        61 ARIZONA     1        KYL   1   0   0   0   1
MCCAIN     15039        61 ARIZONA     1     MCCAIN   1   0   0   0   1
HUTCHINSON 29306        42 ARKANSA     1 HUTCHINSON   1   0   0   0   1
 [ reached 'max' / getOption("max.print") -- omitted 3 rows ]
```

```
# → unnecessary variables are recorded, so drop

# Drop some variables and convert into matrix
Y <- as.matrix(Senate[, 6:ncol(Senate)])
```

Also, we set **initial values** for sampling, **tuning parameters** and **priors**. In this analysis, I set the priors as follows:

[3]See more detail: https://cran.r-project.org/web/packages/MCMCpack/MCMCpack.pdf

[4]Clinton, J., Jackman, S., & Rivers, D. (2004). The statistical analysis of roll call data. American Political Science Review, 98(2), 355-370.

$$\alpha_j \sim N(0, 10), \quad \beta_j \sim N(1, 0.2).$$

Then, $a_0 = 0, A_0 = 10, b_0 = 1, B_0 = 0.2$. And I supply very flat initial values – all inits are set 0.1.

```
# Note: We must name the list of element correctly as below to insert the list to wrapper function
# Initial values
init <- list(alpha = rep(0.1, ncol(Y)),
             beta = rep(0.1, ncol(Y)),
             theta = rep(0.1, nrow(Y)))

# Tuning parameters
tuning_par <- list(alpha = 0.4,
                   beta = 0.4,
                   theta = 0.4)

# Priors
prior <- list(a0 = 0, # prior mean for alpha
              A0 = 10, # prior sd for alpha
              b0 = 1, # prior mean for beta
              B0 = 0.2) # prior sd for beta
```

Yes! We are ready to run MCMC!

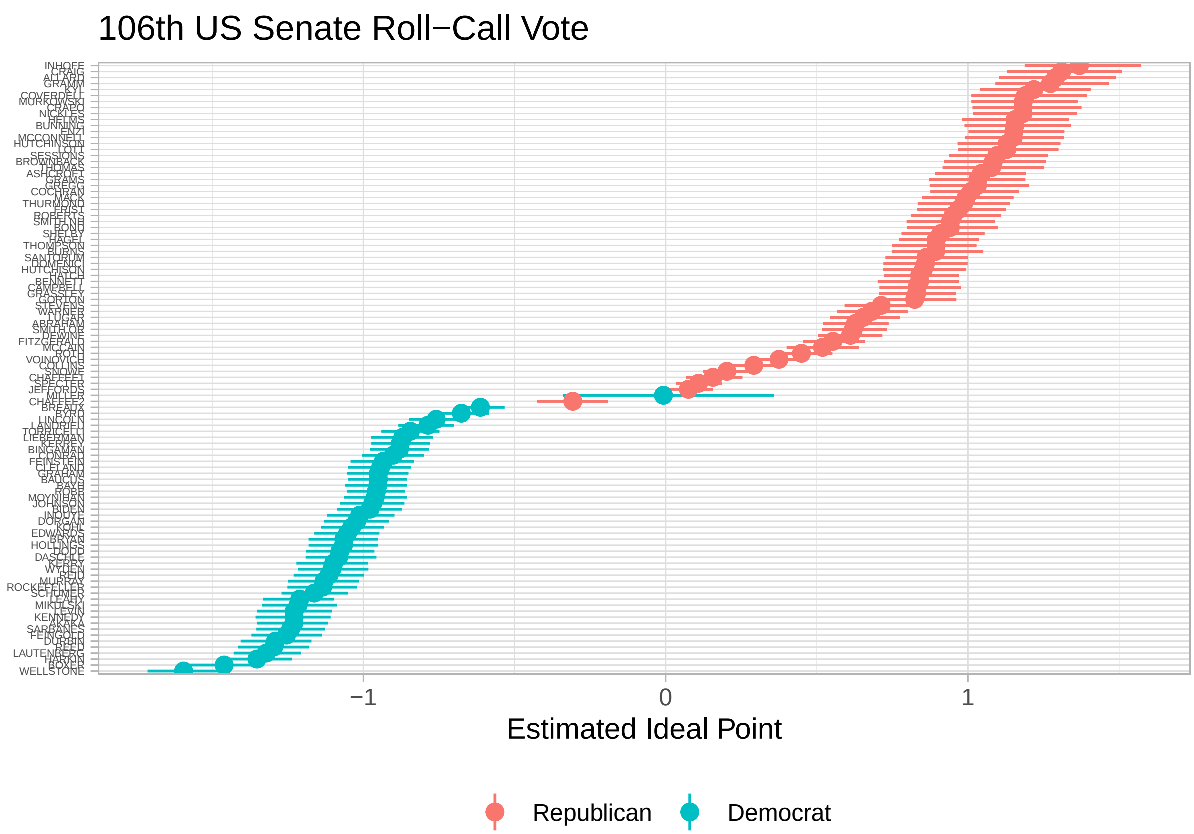
```
# Run
fit <- irt_cpp(datamatrix = Y, # roll-call matrix
               iter = 5000, # number of sampling
               warmup = 3000, # burn-in
               thin = 5, # save samples every [thin] steps
               refresh = 1000, # output the status of sampling every [refresh] steps
               seed = 1, # seed value
               init = init, # initial values
               tuning_par = tuning_par, # tuning parameters
               prior = prior) # priors
```

```
================================================================
Run Metropolis-Hasting Sampler for 2PL item response model...

  Observations: 68544
    Number of individuals: 102
    Number of items: 672
    Total correct response: 42458 / 68544 [ 62 %]

  Priors:
    alpha ~ N(0, 10), beta ~ N(1, 0.2), theta ~ N(0, 1).
================================================================

Warmup:   1 / 8000 [ 0% ]
Warmup:   1000 / 8000 [ 13% ]
Warmup:   2000 / 8000 [ 25% ]
Warmup:   3000 / 8000 [ 38% ]
Sampling: 3001 / 8000 [ 38% ]
Sampling: 4000 / 8000 [ 50 % ]
```

```
Sampling: 5000 / 8000 [ 63 % ]
Sampling: 6000 / 8000 [ 75 % ]
Sampling: 7000 / 8000 [ 88 % ]
Sampling: 8000 / 8000 [ 100 % ]
Done: Total time 99.1 sec
```

Foo! Sampler finished. Next, we extract the result and plot the senators' ideal point.

```
# For data handling
library(tidyverse)

# Extract the result
theta <- fit$summary$theta %>%
  mutate(name = rownames(Y),
         party = Senate$party)

# Plot
theta %>%
  ggplot(aes(y = reorder(name, mean), x = mean, color = factor(party))) +
  geom_pointrange(aes(xmin = lwr, xmax = upr)) +
  theme_light() +
  xlab("Estimated Ideal Point") +
  ylab("") +
  ggtitle("106th US Senate Roll-Call Vote") +
  scale_color_discrete(limits = c("1", "0"),
                       label = c("Republican", "Democrat")) +
  theme(legend.position = "bottom",
        legend.direction = "horizontal",
        legend.title = element_blank(),
        axis.text.y = element_text(size = 4))
```



We have clear estimate of the policy positions of each senator, well divided by party. Just to be sure, let's use `MCMCpack::MCMCirt1d` to see if our estimates is similar to it.

```
stime <- proc.time()[3]
fit_mcmc <- MCMCpack::MCMCirt1d(Y,
                                mcmc = 5000,
                                burnin = 3000,
                                thin = 5,
                                verbose = 1000)
```

```
MCMCirt1d iteration 1 of 8000


MCMCirt1d iteration 1001 of 8000


MCMCirt1d iteration 2001 of 8000


MCMCirt1d iteration 3001 of 8000


MCMCirt1d iteration 4001 of 8000
```

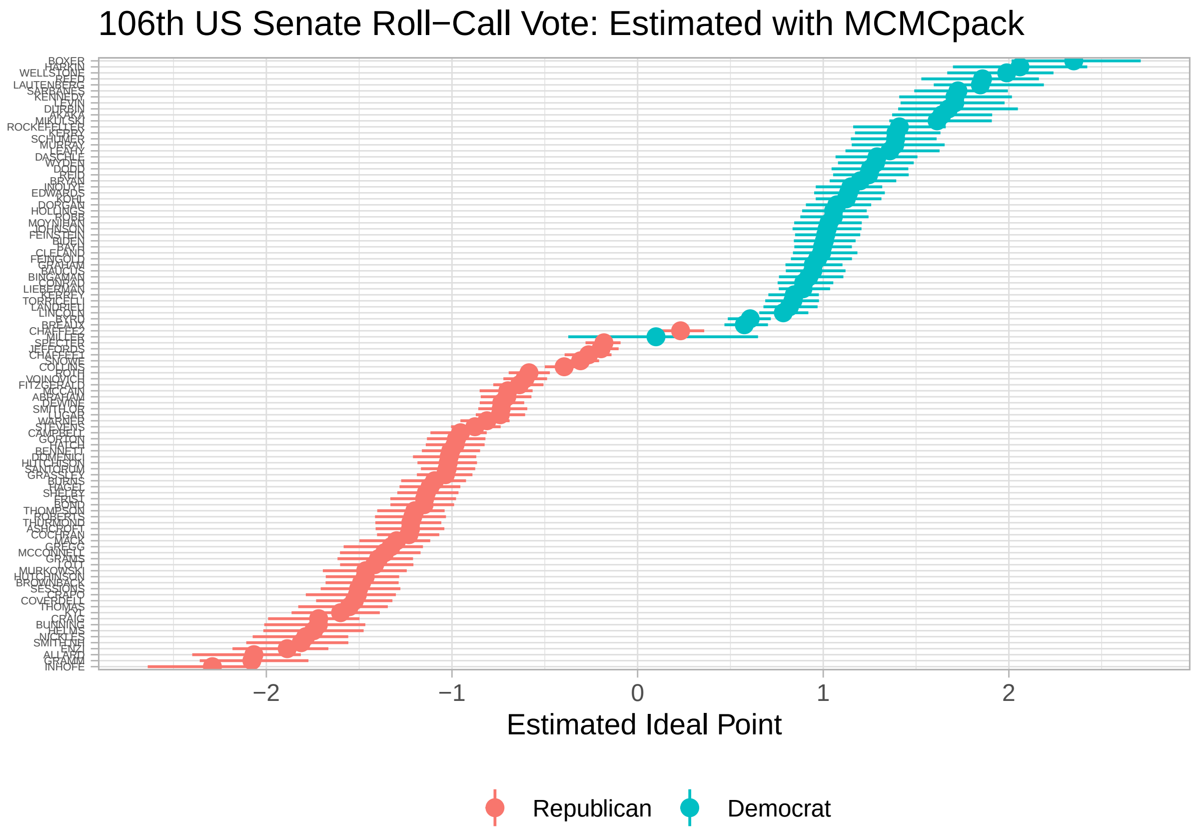
```
MCMCirt1d iteration 5001 of 8000


MCMCirt1d iteration 6001 of 8000


MCMCirt1d iteration 7001 of 8000
```

```
proc.time()[3] - stime
```

```
elapsed
 23.587
```

Very unfortunately, it is much faster to do IRT with MCMCpack. Also, it seems that MCMCpack uses probit execution instead of logit. I don't know if that affects the execution time, but it does seem very fast. Let's check the result.

```
theta_mcmc <- summary(fit_mcmc)$quantiles
theta_mcmc %>%
  as_tibble() %>%
  mutate(name = rownames(Y),
         party = Senate$party,
         mean = `50%`,
         lwr = `2.5%`,
         upr = `97.5%`) %>%
  ggplot(aes(y = reorder(name, mean), x = mean, color = factor(party))) +
  geom_pointrange(aes(xmin = lwr, xmax = upr)) +
  theme_light() +
  xlab("Estimated Ideal Point") +
  ylab("") +
  ggtitle("106th US Senate Roll-Call Vote: Estimated with MCMCpack") +
  scale_color_discrete(limits = c("1", "0"),
                       label = c("Republican", "Democrat")) +
  theme(legend.position = "bottom",
        legend.direction = "horizontal",
        legend.title = element_blank(),
        axis.text.y = element_text(size = 4))
```



The sign of θ_i is reversed, but both estimates are similar!