Examples

Contents

1 Programmatic Assertions for Examples (Fail Loudly)

For every worked example and for every executed event within it (trade, settlement, rebucketing, layoff), run the following assertions immediately *after* the state transition. Denote by Δ a one–event change from pre– to post–state, and use small numerical tolerances $\varepsilon_{\text{cash}}, \varepsilon_{\text{qty}} > 0$ for floating-point and integer checks, respectively.

Notation. Let $\mathcal{P}$ be the set of parties touched by the event; for each $j \in \mathcal{P}$, ΔC_j is the cash change for j and Δq_j is the change in the count of tickets of the relevant bucket/instrument on j's balance sheet. For dealers/VBTs, x is face inventory (tickets when S=1) and $E = C + Mx$ is equity at VBT mid.

1.1 C1. Double–entry and conservation (per event)

$$\left|\sum_{j\in\mathcal{P}} \Delta C_j\right| \le \varepsilon_{\text{cash}},$$
$$\left|\sum_{j\in\mathcal{P}} \Delta q_j\right| \le \varepsilon_{\text{qty}}.$$

For settlement, also assert that the total face paid equals aggregate recovery $R \cdot D$ and all matured tickets are deleted. (Event tables in Sec. 6.)[1]

1.2 C2. Quote bounds and pin detection

For the quoting dealer in the affected bucket (with outside quotes A, B):

$$b_c(x) \;\ge\; B, \qquad a_c(x) \;\le\; A.$$

Additionally flag pins:

$$a_c(x) = A \iff \text{ask clipped/pinned to outside}, \qquad b_c(x) = B \iff \text{bid clipped/pinned to outside}.$$

(Interior/clipping definitions in Sec. 7.5.)[2]

1.3 C3. Feasibility pre-checks for interior executions

Before applying an *interior* fill, assert:

- Customer SELL (dealer BUY): $x + S \le X^*$ and $C \ge b_c(x)$.
- Customer BUY (dealer SELL): $x \ge S$.

(Commit-to-quote feasibility in Sec. 7.6.)[3]

1.4 C4. Pass-through invariants at outside pins

If the event is executed via pass-through to the VBT at a pin,

$$\text{ask pinned at } A \text{ or bid pinned at } B \quad\Rightarrow\quad (x', C') = (x, C) \text{ for the dealer.}$$

(Events 9–10: dealer state unchanged under atomic pass-through.)[4]

[1] See per–event tables and the proportional-recovery settlement in Sec. 6: every cash leg has a mirrored counter-leg; matured tickets are extinguished.

[2] Sec. 7.5 defines interior quotes $a(x), b(x)$ and clipping to outside A, B; the inequalities must hold identically.

[3] Sec. 7.6 disallows shorting/borrowing; BUYing one ticket requires capacity and cash when the dealer buys; SELLing one requires inventory when the dealer sells.

[4] Events 9–10 in Sec. 6 specify that when pinned to A or B the dealer executes the customer and offsets with the VBT at the same price, leaving (x, C) unchanged.

1.5 C5. Equity basis by role

At every checkpoint (pre/post event, end-of-period snapshots):

Dealer/VBT: $E = C + Mx$ (at the bucket's current M); Trader: book assets and liabilities at face.

(Equity conventions in Sec. 5.2.)[5]

1.6 C6. Anchor-update timing discipline (VBT)

At the end of period t (after settlement) compute the bucket loss rate ℓ_t and update (M_{t+1}, O_{t+1}); during period $t+1$ order flow, do not mutate these anchors:

$$(M_{t+1}, O_{t+1}) = f(\ell_t;\ M_t, O_t), \qquad \partial(M_{t+1}, O_{t+1})/\partial(\text{order flow at } t+1) = 0.$$

(Linear loss-based rule and timing in Sec. 9.)[6]

Implementation notes (for examples)

1. **Attach checks per example.** After each scripted event in Examples 1–5 (and any additional examples), run C1–C6 and print a labeled failure that names: example, event index, party set $\mathcal{P}$, and the violated identity.
2. **Ticket-level reconciliation.** In C1, reconcile by *IDs* (issuer+maturity), not just counts; registry size must be unchanged by trades and reduced only by settlement deletions.
3. **Tolerances.** Use $\varepsilon_{\text{cash}} \approx 10^{-10}$ for floating-point arithmetic and $\varepsilon_{\text{qty}} = 0$ for integer ticket counts (unless the implementation stores counts in floating type, in which case use 10^{-10}).
4. **Pin classification.** For C2/C4, emit a boolean flag `is_pinned_ask`, `is_pinned_bid`; C4 must only trigger when either flag is true.
5. **Event types.** Map example events to spec events: trades $\rightarrow$ Events 1–2 (or 9–10 if pinned), settlement $\rightarrow$ Events 3–8, rebucketing $\rightarrow$ Events 11–12.

2 EXAMPLE 1 (revised): selling a migrating claim and dealer rebucketing with explicit double–entry

We work through two periods with one highlighted non–dealer agent and the Long/Mid dealers active. Outside anchors are fixed over these periods. ▷ See also the worked outline in the Examples document. *(ref.)*

Basic setup (consistent with the simulator)

- Ticket size: $S = 1$.
- Fixed outside anchors over these two dates: $M^L = M^M = 1.00$, $O^L = 0.30$, $O^M = 0.20$. Hence $A^L = 1.15,\ B^L = 0.85$ and $A^M = 1.10,\ B^M = 0.90$. ▷ Spec. Secs. 4.2, 7; Examples Ex. 1.
- Bucket ranges follow the baseline: Short $\tau \in \{1, 2, 3\}$; Mid $\tau \in \{4, \ldots, 8\}$; Long $\tau \geq 9$. ▷ Spec. Sec. 3.2.
- Two dates and a migrating ticket $T^\star$: at t_1, $\tau(T^\star) = 9$ (Long); at t_2, $\tau(T^\star) = 8$ (Mid).

Dealers use the L1 kernel per bucket: $V = Ma + C$, $K^* = \lfloor V/M \rfloor$, $X^* = SK^*$, $\lambda = S/(X^* + S)$, $I = \lambda O$, $p(x) = M - \frac{O}{X^*+2S}(x - \frac{X^*}{2})$, $a(x) = p(x) + \frac{I}{2}$, $b(x) = p(x) - \frac{I}{2}$. ▷ Spec. Sec. 7.

---

[5] Sec. 5.2: market makers (dealers/VBTs) are marked to outside mid M; traders book claims and obligations at face.

[6] Sec. 9: anchors for $t+1$ depend only on realized losses in t; there is no contemporaneous dependence on dealer inventory or order flow.

Balance sheets at t_1 (before trading)

Long dealer D^L (bucket L). Inventory $a_0^L = 1$ (ticket L_0), cash $C_0^L = 1$. Then $V_0^L = 2$, $K^{*L} = 2$, $X^{*L} = 2$, ladder $x^L \in \{0, 1, 2\}$, $N^L = 3 \Rightarrow \lambda^L = 1/3$, $I^L = 0.10$. Midline $p^L(x) = 1 - \frac{0.30}{2+2}(x-1) = 1 - 0.075(x-1)$. At $x^L = 1$: $a^L(1) = 1.05$, $b^L(1) = 0.95$ (no clipping). ▷ Examples Ex. 1; Spec. Secs. 7.2–7.5.

Mid dealer D^M (bucket M). Inventory $a_0^M = 1$, cash $C_0^M = 1$; $V_0^M = 2$, $K^{*M} = 2$, $X^{*M} = 2$. $N^M = 3 \Rightarrow \lambda^M = 1/3$, $I^M = \frac{1}{3} \cdot 0.20 \approx 0.0666$. $p^M(x) = 1 - \frac{0.20}{4}(x-1) = 1 - 0.05(x-1)$. At $x^M = 1$: $a^M(1) \approx 1.0333$, $b^M(1) \approx 0.9667$. ▷ Examples Ex. 1.

Agent A_1 and the migrating ticket $T^\star$. A_1 holds one Long ticket $T^\star$ (issuer A_2) and cash $C_1(t_1) = 1.05$. At t_1 it must pay 2 to its single creditor $j(A_1)$ (one–creditor rule); shortfall $= 2 - 1.05 = 0.95 > 0 \Rightarrow$ sell one Long ticket by the liquidity rule. ▷ Spec. Secs. 5.2, 10.1; Examples Ex. 1.

Period t_1: A_1 sells $T^\star$ to D^L

With $x^L = 1$, $b_c^L(1) = 0.95$. Interior dealer BUY feasible: $x^L + S = 2 \le X^{*L} = 2$ and $C^L = 1 \ge 0.95$. Execute at $p = 0.95$:

$$\text{Dealer } D^L: \ x^L \leftarrow 2, \ C^L \leftarrow 0.05; \qquad \text{Agent } A_1: \ C_1 \leftarrow 2.00.$$

Quotes for D^L at end of t_1 (now $x^L = 2$): $p^L(2) = 1 - 0.075(1) = 0.925$, hence $a^L(2) = 0.975$, $b^L(2) = 0.875$ (no clipping). ▷ Examples Ex. 1; Spec. Secs. 7.6–7.7.

Between t_1 and t_2: $T^\star$ migrates Long→Mid and is rebucketed

At t_2 the remaining maturity of $T^\star$ declines from 9 to 8, so it moves to the Mid bucket. Because a dealer holds $T^\star$, we process the boundary crossing as an *internal sale at the Mid bucket mid* $M^M = 1$, from D^L to D^M, i.e. a double-entry transfer at M_{Mid}. ▷ Spec. **Event 11**.

Cash and inventory updates at the rebucketing instant (internal sale at M^M).

$$\begin{aligned}
&\text{Long dealer } (D^L): \quad a^L: \ 2 \to 1, \quad C^L: \ 0.05 \to 1.05;\\
&\text{Mid dealer } (D^M): \quad a^M: \ 1 \to 2, \quad C^M: \ 1.00 \to 0.00.
\end{aligned}$$

Explicit double–entry / equity-at-mid check. Compute equity at mid (per bucket) $E = C + Ma$ immediately before vs. after the internal sale:

$$\begin{aligned}
E^L_{\text{before}} &= C^L + M^L a^L = 0.05 + 1 \cdot 2 = 2.05,\\
E^L_{\text{after}} &= 1.05 + 1 \cdot 1 = 2.05;\\
E^M_{\text{before}} &= C^M + M^M a^M = 1.00 + 1 \cdot 1 = 2.00,\\
E^M_{\text{after}} &= 0.00 + 1 \cdot 2 = 2.00.
\end{aligned}$$

Hence equity (at mid) for both dealers is unchanged by the rebucketing transfer — the cash leg moves with the ticket at M_{Mid}, conserving accounting identities. ▷ Matches Spec. Event 11; double-entry preserved.

Start of t_2: dealer pre–computation and quotes

Effect on D^L (bucket L). $V^L = M^L a^L + C^L = 1 \cdot 1 + 1.05 = 2.05$. Hence $K^{*L} = 2$, $X^{*L} = 2$, $N^L = 3$, $\lambda^L = 1/3$, $I^L = 0.10$. At $x^L = 1$: $p^L(1) = 1$, so $a^L(1) = 1.05$, $b^L(1) = 0.95$. Capacity and width return to their t_1 starting values because cash moved with the ticket. ▷ Examples Ex. 1; Spec. Secs. 7.2–7.5.

Effect on D^M (bucket M). $V^M = 1 \cdot 2 + 0 = 2 \Rightarrow K^{*M} = 2$, $X^{*M} = 2$, $N^M = 3$, $\lambda^M = 1/3$, $I^M \approx 0.0666$. At $x^M = 2$: $p^M(2) = 1 - \frac{0.20}{4} \cdot (1) = 0.95$, so $a^M(2) \approx 0.9833$, $b^M(2) \approx 0.9167$. Here Mid capacity does not rise; the arrival was paid at M^M, leaving V^M unchanged from t_1. ▷ Examples Ex. 1.

Summary

1. At t_1 the agent sells one Long ticket to D^L at the dealer bid; D^L moves to the top rung and inside quotes shift accordingly.
2. At the t_2 boundary the ticket migrates Long→Mid and is reallocated as an internal sale at M_{Mid}. Cash and inventory move one–for–one across dealers.
3. **Double–entry/equity check:** $E = C + Ma$ for both D^L and D^M is invariant across the rebucketing transfer. Quotes at t_2 reflect the new inventories with capacities unchanged.

3 EXAMPLE 2: Maturing debt and cross–bucket reallocation (with explicit kernel checks)

We consider three dealers, one per bucket $b \in \{S, M, L\}$ (Short, Mid, Long). All trade a single standardized ticket with size $S = 1$. Outside anchors are held fixed across the two periods of the example.

Outside quotes (fixed in this example)

For each bucket $b \in \{S, M, L\}$:

$$M^S = M^M = M^L = 1, \qquad O^S = 0.20,\ O^M = 0.30,\ O^L = 0.40,$$

$$A^b = M^b + \tfrac{1}{2}O^b, \qquad B^b = M^b - \tfrac{1}{2}O^b.$$

Thus

$$A^S = 1.10,\ B^S = 0.90; \qquad A^M = 1.15,\ B^M = 0.85; \qquad A^L = 1.20,\ B^L = 0.80.$$

Dealer kernel (per bucket, L1)

Given securities a^b (tickets) and cash C^b,

$$V^b = M^b a^b + C^b, \quad K^{*b} = \lfloor V^b / M^b \rfloor, \quad X^{*b} = S K^{*b},$$

ladder $x^b \in \{0, S, \ldots, X^{*b}\}$ with $N^b = X^{*b}/S + 1$, layoff probability $\lambda^b = 1/N^b = S/(X^{*b} + S)$, inside width $I^b = \lambda^b O^b$, and inventory–sensitive midline (with $S = 1$)

$$p^b(x) = M^b - \frac{O^b}{X^{*b} + 2}\left(x - \frac{X^{*b}}{2}\right), \qquad a^b(x) = p^b(x) + \tfrac{1}{2}I^b, \quad b^b(x) = p^b(x) - \tfrac{1}{2}I^b.$$

(Clipping to A^b, B^b is not needed below.)

3.1 Initial conditions at the start of t_1t1

Each bucket dealer starts "balanced": one ticket and one unit of cash.

Short D^S. $a^S = 1,\ C^S = 1 \Rightarrow V^S = 2,\ K^{*S} = 2,\ X^{*S} = 2$; ladder $x^S \in \{0, 1, 2\},\ N^S = 3$, $\lambda^S = \frac{1}{3}$, $I^S = \frac{1}{3} \cdot 0.20 \approx 0.0666$. At x^S=1: $p^S(1) = 1$, $a^S(1) \approx 1.0333$, $b^S(1) \approx 0.9667$.

Mid D^M. $a^M = 1,\ C^M = 1 \Rightarrow V^M = 2,\ K^{*M} = 2,\ X^{*M} = 2$; $N^M = 3$, $\lambda^M = \frac{1}{3}$, $I^M = 0.10$. At x^M=1: $p^M(1) = 1$, $a^M(1) = 1.05$, $b^M(1) = 0.95$.

Long D^L. $a^L = 1,\ C^L = 1 \Rightarrow V^L = 2,\ K^{*L} = 2,\ X^{*L} = 2$; $N^L = 3$, $\lambda^L = \frac{1}{3}$, $I^L \approx 0.1333$. At x^L=1: $p^L(1) = 1$, $a^L(1) \approx 1.0667$, $b^L(1) \approx 0.9333$.

3.2 End of t_1t1: settlement of the short ticket

The short ticket $S^\star$ held by D^S matures and repays in full. Update:

$$a^S : 1 \to 0, \qquad C^S : 1 \to 2, \qquad V^S = 2.$$

Start of t_2 pre–computation in S: $K^{*S} = 2,\ X^{*S} = 2,\ x^S = 0,\ N^S = 3,\ \lambda^S = \frac{1}{3},\ I^S \approx 0.0666$ and

$$p^S(0) = 1 - \frac{0.20}{2+2}(0-1) = 1.05, \quad a^S(0) \approx 1.0833, \quad b^S(0) \approx 1.0167.$$

3.3 t_2 order flow: cross–bucket reallocation via dealer–as–customer trades

Arrival 1 (in t_2t2): D^S BUYS one Mid ticket from D^M

Query D^M at $x^M = 1$: $a^M(1) = 1.05$. Execute one BUY at $p = 1.05$ (interior SELL feasible).

Post–trade states. *Selling bucket (Mid).*

$$a^M : 1 \to 0, \quad C^M : 1 \to 2.05, \quad V^M = 2.05, \quad K^{*M} = \lfloor 2.05 \rfloor = 2,\ X^{*M} = 2.$$

At the new inventory $x^M = 0$:

$$p^M(0) = 1 - \frac{0.30}{2+2}(0-1) = 1.075, \quad a^M(0) = 1.125, \quad b^M(0) = 1.025.$$

Buying institution (Short dealer acting as customer in Mid).

$$C^S : 2.00 \to 0.95.$$

Kernel independence assertion (home bucket S). The Short dealer's *home* bucket kernel uses only (a^S, C^S) and is unaffected by the acquired Mid ticket: a^S and x^S are unchanged by this cross–bucket purchase (they remain 0); only C^S changes, and the S–bucket capacity/quotes are recomputed purely from (a^S, C^S).[7]

Checks (assertions). (i) *Selling bucket ladder moves down:* $x^M : 1 \to 0$ and $p^M(0) > p^M(1)$ (here $1.075 > 1.00$). (ii) *Buying dealer's home bucket kernel is independent of the off–bucket asset:* a^S (and hence x^S) unchanged; only C^S updated.

[7] Architecturally, holdings outside a dealer's home bucket are ordinary assets and do not enter that bucket's kernel state.

Arrival 2 (later in t_2t2): D^M BUYS one Long ticket from D^L

Query D^L at $x^L = 1$: $a^L(1) \approx 1.0667$. Execute one BUY at $p \approx 1.0667$ (interior SELL feasible).

Post–trade states. *Selling bucket (Long).*

$$a^L : 1 \to 0, \quad C^L : 1 \to 2.0667, \quad V^L = 2.0667, \quad K^{*L} = \lfloor 2.0667 \rfloor = 2,\ X^{*L} = 2.$$

At the new inventory $x^L = 0$:

$$p^L(0) = 1 - \frac{0.40}{2+2}(0-1) = 1.10, \quad a^L(0) \approx 1.1667, \quad b^L(0) \approx 1.0333.$$

Buying institution (Mid dealer acting as customer in Long).

$$C^M : 2.05 \to 0.9833.$$

Kernel independence assertion (home bucket M). The Mid dealer's *home* bucket kernel (a^M, x^M) is unchanged by acquiring a Long ticket; only its cash C^M changes. The M–bucket ladder and quotes are recomputed solely from (a^M, C^M); the off–bucket Long holding does not enter the M kernel.

Checks (assertions). (i) *Selling bucket ladder moves down:* $x^L : 1 \to 0$ and $p^L(0) > p^L(1)$ (here $1.10 > 1.00$). (ii) *Buying dealer's home bucket kernel is independent of the off–bucket asset:* a^M (and hence x^M) unchanged; only C^M updated.

Architecture note (bucket independence)

Any agent, including another bucket's dealer, may be a customer of a bucket dealer. Holdings outside a dealer's home bucket are ordinary assets and do not affect its home–bucket kernel; all executions are intermediated by the relevant bucket dealer. The above assertions should be used as unit tests after each cross–bucket trade.

4 EXAMPLE 3: Outside-bid clipping toggle (A/B) on Example 3

This example is a minimal extension of Example 3 that runs the post-default state twice: once *without* the optional outside-bid non-negativity clip (B_{t+1} can be negative), and once *with* the clip $B_{t+1} \leftarrow \max\{0, B_{t+1}\}$. All primitives and intermediate computations up to the end of t_1 are identical to Example 3. *Purpose:* verify both code paths without changing the economic scenario.

Setup (identical to Example 3 up to the end of t_1)

At t_0: $M_0 = 1.0$, $O_0 = 0.30$, $A_0 = 1.15$, $B_0 = 0.85$. Dealer holds $a_0 = 3$ (all maturing at t_1) and $C_0 = 1$. At t_1, one ticket repays, two default (zero recovery), so loss rate $\ell_{t_1} = 2/3$. Anchors update via the linear rule:

$$M_{t_2} = M_{t_1} - \phi_M \ell_{t_1} = \tfrac{1}{3}, \qquad O_{t_2} = O_{t_1} + \phi_O \ell_{t_1} = 0.70.$$

Hence the raw outside quotes for t_2 are

$$A_{t_2} = M_{t_2} + \tfrac{1}{2} O_{t_2} = 0.6833, \qquad B_{t_2} = M_{t_2} - \tfrac{1}{2} O_{t_2} = -0.0167.$$

Dealer state carried into t_2: $a = 0$, $C = 2$. With the L1 kernel, $X^*_{t_2} = \lfloor 2/(1/3) \rfloor = 6$, $\lambda_{t_2} = 1/7$, $I_{t_2} = \lambda_{t_2} O_{t_2} = 0.10$,

$$p_{t_2}(x) = M_{t_2} - \frac{O_{t_2}}{X^*_{t_2} + 2}\Big(x - \tfrac{X^*_{t_2}}{2}\Big) = \tfrac{1}{3} - \frac{0.70}{8}(x - 3).$$

At the current inventory $x = 0$: $p_{t_2}(0) \approx 0.5958$, so interior quotes $a_{t_2} = p_{t_2}(0) + I_{t_2}/2 \approx 0.6458$, $b_{t_2} = p_{t_2}(0) - I_{t_2}/2 \approx 0.5458$. (These match Example 3.) *No other changes to Example 3 are made.* [8]

Variant A (no outside-bid clip)

Leave B_{t_2} as computed above (possibly negative). The dealer's interior quotes are unchanged from the setup:

$$A_{t_2} = 0.6833, \quad B_{t_2} = -0.0167, \qquad a_{t_2} \approx 0.6458, \quad b_{t_2} \approx 0.5458.$$

Because $b_{t_2} > B_{t_2}$ and $a_{t_2} < A_{t_2}$, clipping plays no role for interior quotes at $x = 0$.

Variant B (with outside-bid non-negativity clip)

Apply the optional clip from the specification:

$$B_{t_2} \leftarrow \max\{0, B_{t_2}\} = 0, \qquad A_{t_2} \text{ unchanged at } 0.6833.$$

This toggles only the stored outside bid; the dealer's *interior* a_{t_2}, b_{t_2} remain as above because they depend on $p(\cdot)$ and I, not directly on (A, B) unless a pin is hit.

Sanity check that actually exercises the pin logic (optional)

To force a pass-through SELL (and thus exercise B on the trade path), run a one-off stress tick at the start of t_2 by temporarily setting the dealer cash to $C^{\text{stress}} = 0$ (inventory still $x = 0$). Then process a *customer SELL* (one ticket). Interior BUY is infeasible (fails the cash check $C \geq b_{t_2}$), hence the commit-to-quote rule routes to the VBT at the outside bid:

$$\text{execution price} = \begin{cases} B_{t_2} = -0.0167, & \text{Variant A (no clip)},\\ B_{t_2} = 0, & \text{Variant B (clip)}.\end{cases}$$

In both subruns the dealer's state is unchanged by the pass-through (x', C' equal to x, C^{stress}), per the atomic offset rule; only the paid price differs, which confirms that the clip toggle is effective in the code path that uses B.[9]

What to assert in tests

1. With identical state ($x = 0, C = 2$), interior quotes at t_2 are identical across Variants A/B (no dependence on the B clip until a pin is hit).
2. The stored outside bid equals -0.0167 in Variant A, and 0 in Variant B.
3. For a forced pass-through SELL at t_2, the executed price equals the stored outside bid in each variant, with the dealer's (x, C) unchanged by the trade.

[8] Numerical values and the loss-based anchor rule are exactly those in Example 3. See "EXAMPLE 3: Default at maturity, VBT anchor update, and dealer quotes."

[9] This micro-step is solely to hit the boundary logic deterministically; it is not part of the economic narrative of Example 3. Restore $C = 2$ afterwards.

Provenance. Numbers and flow replicate Example 3; the optional clip follows the specification's VBT update section (loss-based anchors with optional non-negativity for B) and the dealer commit-to-quote rules.

5 EXAMPLE 4: Dealer reaches inventory limit and VBT layoff occurs

We consider one bucket traded by a dealer (inside quotes, inventory) and a value-based trader (VBT, outside quotes with deep capacity). All tickets have face value $S = 1$.

Primitives

- Outside anchors (fixed in this example): $M = 1.0$, $O = 0.30$.
- Outside (VBT) quotes:

$$A \;=\; M + \frac{O}{2} \;=\; 1.15, \qquad B \;=\; M - \frac{O}{2} \;=\; 0.85.$$

Dealer kernel (L1)

For dealer securities a (tickets), cash C, mid M and $S = 1$:

$$V \;=\; Ma + C, \qquad K^* \;=\; \left\lfloor \frac{V}{M} \right\rfloor, \qquad X^* \;=\; K^*,$$

$$x \in \{0, 1, \ldots, X^*\}, \quad N = X^* + 1, \quad \lambda = \frac{1}{N}, \quad I = \lambda O,$$

$$p(x) = M - \frac{O}{X^* + 2}\left(x - \frac{X^*}{2}\right), \qquad a(x) = p(x) + \frac{I}{2}, \;\; b(x) = p(x) - \frac{I}{2}.$$

Commit-to-quote at the left boundary (ask-side pin). For a customer BUY (dealer must sell one ticket): if $x \geq 1$ an interior sale is feasible. If $x < 1$ (no inventory), the dealer pins the operational ask to A and executes by atomic pass-through to the VBT at price A; the dealer's state does not change:

$$\Delta x \;=\; 0, \qquad \Delta C \;=\; 0.$$

(This is Event 10 in the event table.)

Initial dealer state at t_0

Balanced start:

$$a_0 = 2, \qquad C_0 = 2.$$

With $M = 1$:

$$V_0 = 4, \quad X_0^* = \left\lfloor \frac{4}{1} \right\rfloor = 4, \quad x \in \{0, 1, 2, 3, 4\}, \; x_0 = 2, \quad N_0 = 5, \; \lambda_0 = \tfrac{1}{5}, \; I_0 = \lambda_0 O = 0.06.$$

Midline

$$p_0(x) = 1 - \frac{0.30}{4 + 2}\,(x - 2) = 1 - 0.05(x - 2).$$

At $x_0 = 2$:

$$a_0 = 1 + 0.03 = 1.03, \qquad b_0 = 1 - 0.03 = 0.97.$$

Step 1 at t_0: first customer BUY (interior)

Interior SELL is feasible since $x_0 \geq 1$. Execute at $p_1 = a_0 = 1.03$.

$$x_1 = x_0 - 1 = 1, \qquad C_1 = C_0 + p_1 = 3.03.$$

Capacity unchanged ($X_1^* = 4$). Midline at $x_1 = 1$: $p_1(1) = 1.05$; quotes

$$a_1 = 1.08, \qquad b_1 = 1.02.$$

Step 2 at t_0: second customer BUY (interior to boundary)

Interior SELL is feasible since $x_1 \geq 1$. Execute at $p_2 = a_1 = 1.08$.

$$x_2 = x_1 - 1 = 0, \qquad C_2 = C_1 + p_2 = 4.11, \qquad X_2^* = 4.$$

Counterfactual (informational) interior quotes at $x_2 = 0$ (not executable without inventory):

$$p_2^{\text{int}}(0) = 1 - 0.05(0 - 2) = 1.10, \quad a_2^{\text{int}} = 1.13, \quad b_2^{\text{int}} = 1.07.$$

Step 3 at t_0: third customer BUY (VBT layoff at the boundary)

At $x_2 = 0 < 1$ an interior sale is infeasible. By the boundary rule, pin the operational ask to A and execute via VBT at

$$p_3 = A = 1.15.$$

Pass-through invariants (dealer):

$$x_3 = x_2 = 0, \qquad C_3 = C_2 = 4.11, \qquad \Delta x = 0, \;\; \Delta C = 0.$$

(The VBT supplies one ticket at A; the dealer's state is unchanged.)

Interpretation

1. Starting from a balanced shelf ($x_0 = 2$ on $\{0, \ldots, 4\}$), two interior BUYs move the dealer to $x = 0$ while cash rises $2 \to 4.11$; capacity and inside width remain $X^* = 4$, $I = 0.06$.
2. At the boundary $x = 0$, interior quotes are *informational only*; operational execution is pinned to $A = 1.15$ and routed to the VBT (Event 10), leaving (x, C) unchanged.
3. The pinned operational ask persists until the dealer rebuilds inventory (e.g. by buying tickets back from customers).

6 EXAMPLE 5 (revised): Dealer earns over time and inventory grows (with invariant checks)

We consider one bucket with a dealer trading against customers and a value-based trader (VBT). Tickets are homogeneous.

Primitives

- Ticket size: $S = 1$.
- Outside anchors (fixed in this example): $M = 1.0$, $O = 0.30$.
- VBT outside quotes: $A = M + \frac{O}{2} = 1.15$, $\quad B = M - \frac{O}{2} = 0.85$.

Dealer kernel (L1)

For dealer holdings (a, C) (tickets, cash) and mid M with $S = 1$:

$$V = Ma + C, \qquad K^* = \left\lfloor \tfrac{V}{M} \right\rfloor, \qquad X^* = SK^*,$$

$$N = \tfrac{X^*}{S} + 1, \qquad \lambda = \tfrac{S}{X^*+S}, \qquad I = \lambda O,$$

$$p(x) = M - \frac{O}{X^* + 2S}\left(x - \frac{X^*}{2}\right), \qquad a(x) = p(x) + \tfrac{I}{2}, \quad b(x) = p(x) - \tfrac{I}{2}.$$

All trades below are interior (no clipping; no layoff).

Initial dealer state at t_0

Balanced position:

$$a_0 = 2, \quad C_0 = 2.$$

With $M = 1$:

$$V_0 = 1 \cdot 2 + 2 = 4, \quad K_0^* = \lfloor 4 \rfloor = 4, \quad X_0^* = 4.$$

Ladder $x \in \{0, 1, 2, 3, 4\}$ with $x_0 = 2$; $N_0 = 5 \Rightarrow \lambda_0 = 1/5 = 0.2$.

$$I_0 = \lambda_0 O = 0.06, \qquad p_0(x) = 1 - \frac{0.30}{4+2}(x - 2) = 1 - 0.05(x - 2).$$

At $x_0 = 2$:

$$p_0(2) = 1, \qquad a_0 = 1.03, \qquad b_0 = 0.97.$$

Invariant check (initialization).

$$E_0 = C_0 + Ma_0 = 2 + 2 = 4 \quad (= V_0), \quad a_0 \geq b_0, \quad 0.85 \leq 0.97 \leq 1.03 \leq 1.15.$$

Trade path at t_0: three customer trades

1. Customer **SELL** (dealer BUYS one) at the bid b_0.
2. Customer **BUY** (dealer SELLS one) at the ask a_1.
3. Customer **SELL** (dealer BUYS one) at the bid b_2.

Trade 1: customer SELL; dealer BUY at b_0 **Feasibility.** Interior BUY feasible if $(x_0 + S) \leq X_0^*$ and $C_0 \geq b_0$:

$$x_0 + 1 = 3 \leq 4, \qquad C_0 = 2 \geq 0.97 \quad \Rightarrow \quad \text{feasible.}$$

Execution. $p = b_0 = 0.97$. **Update.**

$$x_1 = 3, \qquad C_1 = 2 - 0.97 = 1.03.$$

Kernel recomputation.

$$V_1 = 1 \cdot 3 + 1.03 = 4.03, \quad K_1^* = \lfloor 4.03 \rfloor = 4, \quad X_1^* = 4.$$

$$p_1(3) = 1 - 0.05(1) = 0.95, \quad a_1 = 0.95 + 0.03 = 0.98, \quad b_1 = 0.95 - 0.03 = 0.92.$$

Invariant check (after Trade 1).

$$E_1 = C_1 + Ma_1 = 1.03 + 3 = 4.03 \quad (= V_1), \quad a_1 \geq b_1, \quad 0.85 \leq 0.92 \leq 0.98 \leq 1.15.$$

Trade 2: customer BUY; dealer SELL at a_1 **Feasibility.** Interior SELL feasible if $x_1 \geq S$: $x_1 = 3 \geq 1$. **Execution.** $p = a_1 = 0.98$. **Update.**

$$x_2 = 2, \qquad C_2 = 1.03 + 0.98 = 2.01.$$

Kernel recomputation.

$$V_2 = 1 \cdot 2 + 2.01 = 4.01, \quad K_2^* = \lfloor 4.01 \rfloor = 4, \quad X_2^* = 4.$$

At $x_2 = 2$:

$$p_2(2) = 1, \qquad a_2 = 1.03, \qquad b_2 = 0.97.$$

Invariant check (after Trade 2).

$$E_2 = C_2 + Ma_2 = 2.01 + 2 = 4.01 \quad (= V_2), \quad a_2 \geq b_2, \quad 0.85 \leq 0.97 \leq 1.03 \leq 1.15.$$

Trade 3: customer SELL; dealer BUY at b_2 **Feasibility.** Interior BUY feasible if $(x_2 + S) \leq X_2^*$ and $C_2 \geq b_2$:

$$x_2 + 1 = 3 \leq 4, \qquad C_2 = 2.01 \geq 0.97 \quad \Rightarrow \quad \text{feasible.}$$

Execution. $p = b_2 = 0.97$. **Update.**

$$x_3 = 3, \qquad C_3 = 2.01 - 0.97 = 1.04.$$

Kernel recomputation.

$$V_3 = 1 \cdot 3 + 1.04 = 4.04, \quad K_3^* = \lfloor 4.04 \rfloor = 4, \quad X_3^* = 4.$$

At $x_3 = 3$ (same slope as in Trade 1 step): $p_3(3) = 0.95 \Rightarrow a_3 = 0.98,\ b_3 = 0.92$. *Invariant check (after Trade 3).*

$$E_3 = C_3 + Ma_3 = 1.04 + 3 = 4.04 \quad (= V_3), \quad a_3 \geq b_3, \quad 0.85 \leq 0.92 \leq 0.98 \leq 1.15.$$

Outcome

Initial vs. final:

$$(a_0, C_0, E_0) = (2, 2, 4) \;\longrightarrow\; (a_3, C_3, E_3) = (3, 1.04, 4.04).$$

Inventory grows $2 \to 3$, and equity rises $4 \to 4.04$. All steps satisfy:

$$E_t = C_t + Ma_t, \quad a_t(x_t) \geq b_t(x_t), \quad B \leq b_t(x_t) \leq a_t(x_t) \leq A.$$

Notes for implementation tests

At each arrival, assert:

1. **Feasibility**:
   BUY (dealer sells): $x \geq S$.
   SELL (dealer buys): $x + S \leq X^*$ and $C \geq b(x)$.
2. **Accounting invariants**: $E = C + Ma$ equals the computed V; inventory in face units equals the ticket count a when $S = 1$.
3. **Quote sanity**: $a(x) \geq b(x)$ and $B \leq b(x) \leq a(x) \leq A$ (no outside breach when interior).
4. **Capacity consistency**: $K^* = \lfloor (Ma + C)/M \rfloor$ and $X^* = SK^*$ are recomputed after each trade.

These assertions should pass for the three trades above; any failure indicates a kernel, bookkeeping, or rounding bug.

7 EXAMPLE 6: Bid–side pass–through (dealer lays off at the outside bid)

We consider one bucket with a dealer (inside quotes, inventory) and a value–based trader (VBT) that posts outside quotes with deep capacity. A customer *SELL* arrives when the dealer cannot buy one more ticket. Per the commit–to–quote rules, the dealer pins to the outside bid B and executes by atomic pass–through to the VBT; the dealer's own state (x, C) is unchanged. This is Event 9 in the spec.[10] [1]

Primitives (common to both subcases)

- Ticket size S (specified per subcase).
- Outside anchors held fixed: $M = 1.0$, $O = 0.30$; outside quotes $A = M + \frac{O}{2} = 1.15$, $B = M - \frac{O}{2} = 0.85$.
- Dealer kernel (per bucket):

$$V = Ma + C, \quad K^* = \left\lfloor \frac{V}{M} \right\rfloor, \quad X^* = SK^*, \quad \lambda = \frac{S}{X^* + S}, \; I = \lambda O,$$

$$p(x) = M - \frac{O}{X^* + 2S}\left(x - \frac{X^*}{2}\right), \quad a(x) = p(x) + \frac{I}{2}, \;\; b(x) = p(x) - \frac{I}{2},$$

$$a_c(x) = \min\{A, a(x)\}, \qquad b_c(x) = \max\{B, b(x)\}.$$

- BUY–side feasibility (dealer buys one): interior feasible iff $x + S \leq X^*$ *and* $C \geq S \cdot b_c(x)$. Otherwise pin to B and pass through to VBT (Event 9). [1]

Subcase A: capacity binding at the right edge ($x = X^*$)

Setup. Let $S = 1$. Choose dealer inventory and cash

$$a_0 = 2, \quad C_0 = 0.10 \quad \Rightarrow \quad V_0 = Ma_0 + C_0 = 2.10, \; K_0^* = \lfloor 2.10 \rfloor = 2, \; X_0^* = 2.$$

Ladder $x \in \{0, 1, 2\}$ with $x_0 = a_0 = 2 = X_0^*$. Then $\lambda_0 = \frac{1}{X_0^*+1} = \frac{1}{3}$, $I_0 = \lambda_0 O = 0.10$,

$$p(2) = 1 - \frac{0.30}{2+2}(2-1) = 0.925, \qquad b_c(2) = \max\{0.85, \; 0.925 - 0.05\} = 0.875.$$

Arrival and feasibility. A customer *SELLS* one ticket to the dealer. Interior BUY fails because $x_0 + S = 3 > X_0^* = 2$ (capacity bound).[11] [1]

Execution (Event 9). Pin to $B = 0.85$ and execute by atomic pass–through:

$$\text{price } p = B, \qquad \Delta x_D = 0, \; \Delta C_D = 0, \quad \Delta x_{VBT} = +1, \; \Delta C_{VBT} = -0.85,$$

$$\Delta \text{Cash}_{\text{Trader}} = +0.85, \quad \Delta \text{Tickets}_{\text{Trader}} = -1.$$

Dealer state remains $(x, C) = (2, \; 0.10)$. [1]

[10] See Sec. 7 (kernel), Sec. 7.6 (feasibility and pins), and Event 9 in Sec. 6 of the specification.

[11] Here cash is also insufficient ($C_0 = 0.10 < b_c(2)$), but the binding constraint is $x + S > X^*$.

Subcase B: cash binding with $S > 1$ (general S)

Setup. Let $S = 5$. Choose dealer state

$$x_0 = 0 \quad (\Rightarrow a_0 = 0), \qquad C_0 = 3.40.$$

Then

$$V_0 = 0 + 3.40, \quad K_0^* = \lfloor 3.40 \rfloor = 3, \quad X_0^* = SK_0^* = 15,$$
$$\lambda_0 = \frac{S}{X_0^* + S} = \frac{5}{20} = 0.25, \qquad I_0 = \lambda_0 O = 0.075.$$

Midline at $x_0 = 0$:

$$p(0) = 1 - \frac{0.30}{15 + 2 \cdot 5}\left(0 - \frac{15}{2}\right) = 1 + \frac{0.30 \cdot 7.5}{25} = 1.09, \qquad b_c(0) = \max\{0.85,\ 1.09 - 0.0375\} = 1.0525.$$

Arrival and feasibility. A customer *SELLS* one ticket. Capacity is ample: $x_0 + S = 5 \leq X_0^* = 15$. Cash is the binding constraint:

$$C_0 \ < \ S \cdot b_c(0) \ = \ 5 \times 1.0525 \ = \ 5.2625,$$

so interior BUY is infeasible. [1]

Execution (Event 9). Pin to $B = 0.85$ and pass through:

$$\text{price } p = B, \qquad \Delta x_D = 0,\ \Delta C_D = 0, \quad \Delta x_{VBT} = +1 \text{ ticket (face } S),\ \Delta C_{VBT} = -S{\cdot}B = -4.25,$$
$$\Delta \text{Cash}_{\text{Trader}} = +S \cdot B = +4.25, \quad \Delta \text{Tickets}_{\text{Trader}} = -1 \text{ (face } S).$$

Dealer remains $(x, C) = (0,\ 3.40)$. [1]

Test assertions (both subcases)

- **Dealer invariants (Event 9):** $\Delta x_D = 0$, $\Delta C_D = 0$.
- **Clipping bound:** $b_c(x) \geq B$ (holds numerically above).
- **Double–entry:** trader's cash gain equals VBT cash outflow (scaled by S when $S > 1$); ticket flows mirror cash flows. [1]

References

[1] *Specification of the Dealer Module for the Kalecki Ring (Bilancio Simulation)*, secs. 6–7 (events, feasibility), 9 (anchors), 11 (event loop).

8 EXAMPLE: Edge rung without interior clipping (approach to outside pins)

This example demonstrates that under the L1 kernel the interior quotes remain strictly inside the outside quotes for all feasible inventories $x \in [0, X^*]$. As x approaches the boundaries, the quotes $\{a(x), b(x)\}$ approach $\{A, B\}$ but do not reach them; clipping is inactive. Outside pins occur only at the boundary via pass–through (Sec. 7.6).

Primitives and kernel

Fix ticket size $S > 0$, outside anchors (M, O) with outside quotes

$$A = M + \frac{O}{2}, \qquad B = M - \frac{O}{2}.$$

For dealer securities a (tickets), cash C, and S (we identify face inventory $x = aS$ with a when $S = 1$):

$$V = Ma + C, \qquad K^* = \left\lfloor \frac{V}{M} \right\rfloor, \qquad X^* = SK^*, \qquad \lambda = \frac{S}{X^* + S}, \qquad I = \lambda O.$$

Inventory–sensitive midline and interior quotes (Sec. 7.4–7.5):

$$p(x) = M - \frac{O}{X^* + 2S}\left(x - \frac{X^*}{2}\right), \qquad a(x) = p(x) + \frac{I}{2}, \;\; b(x) = p(x) - \frac{I}{2}.$$

Clipped quotes (inactive in the interior in this kernel):

$$a_c(x) = \min\{A, a(x)\}, \qquad b_c(x) = \max\{B, b(x)\}.$$

Impossibility of interior clipping (one–line proof)

At the left edge $x = 0$,

$$a(0)-A = p(0)+\frac{I}{2}-\left(M+\frac{O}{2}\right) = \frac{O}{2}\left(\frac{X^*}{X^* + 2S} + \frac{S}{X^* + S} - 1\right) = -\frac{O}{2}\,\frac{X^*S}{(X^* + 2S)(X^* + S)} \;<\; 0,$$

so $a(0) < A$. At one rung before the right edge, $x = X^* - S$,

$$b(X^*-S) - B = \frac{O}{2}\left(\frac{4S}{X^* + 2S} - \frac{S}{X^* + S}\right) = \frac{O}{2}\,\frac{S(3X^* + 2S)}{(X^* + 2S)(X^* + S)} \;>\; 0,$$

so $b(X^*-S) > B$. Hence interior clipping $a(x) = A$ or $b(x) = B$ cannot occur at any feasible interior rung. (Pins to A or B arise only at the boundary via pass–through.)

Numerical illustration (approach to pins, no clipping)

Let $S = 1$, $M = 1.0$, $O = 0.30$. Suppose the dealer has $a_0 = 4$ tickets and $C_0 = 0.90$. Then

$$V_0 = 4.90, \quad K^* = \lfloor 4.90 \rfloor = 4, \quad X^* = 4, \quad \lambda = \frac{1}{5} = 0.2, \quad I = \lambda O = 0.06, \quad A = 1.15,\; B = 0.85.$$

Midline (Sec. 7.4): $p(x) = 1 - \frac{0.30}{4+2}\,(x - 2) = 1 - 0.05(x - 2)$.

Left side: one rung before running out of inventory ($x = 1$).

$$p(1) = 1.05, \quad a(1) = p(1) + \tfrac{I}{2} = 1.05 + 0.03 = 1.08 \;<\; A = 1.15,$$

interior SELL is feasible ($x \geq S$), and no pass–through applies. The *gap to the outside ask* is $A - a(1) = 0.07$.

Right side: one rung before capacity ($x = X^* - S = 3$).

$$p(3) = 0.95, \quad b(3) = p(3) - \tfrac{I}{2} = 0.95 - 0.03 = 0.92 \;>\; B = 0.85,$$

interior BUY is feasible provided cash covers $b(3)$ (Sec. 7.6); no pass–through. The *gap to the outside bid* is $b(3) - B = 0.07$.

Harness checks

1. **Inside bounds (no interior clipping).** Assert $a(x) < A$ and $b(x) > B$ for all $x \in \{0, S, 2S, \ldots, X^* - S\}$.
2. **Approach to pins.** Verify the gaps $A - a(S)$ and $b(X^* - S) - B$ are positive and shrink as X^* decreases, but remain strictly >0.
3. **Feasibility (no pass–through).** At $x = S$ check interior SELL is feasible; at $x = X^* - S$ check $x + S \le X^*$ and $C \ge b(x)$ so interior BUY is feasible.
4. **Boundary pins (for contrast).** At $x = 0$ (no inventory) a BUY order pins to A and is routed to VBT; at $x = X^*$ a SELL order pins to B and routes to VBT (Sec. 7.6). Those are the only cases where execution occurs at A or B.

Implementation note. The clipping operators $a_c(x) = \min\{A, a(x)\}$, $b_c(x) = \max\{B, b(x)\}$ are harmless guards; with the L1 width rule $I = \lambda O$ and the midline in Sec. 7.4 they never bind on interior rungs. Pins to A or B arise through the feasibility rules (pass–through) at the boundaries.

9 EXAMPLE: Guard at very low mid $M \le M_{\min}$

This test verifies that when the outside mid M falls below a small threshold $M_{\min}$, the dealer kernel applies the *Guard*: set $X^* := 0$, the inventory ladder collapses to $\{0\}$, quotes are *pinned* to the outside levels (A, B), and only pass–through executions (to the VBT) are possible on both sides.[12] [1]

Primitives and Guard

- Ticket size $S = 1$.
- Choose a small guard threshold $M_{\min} = 0.02$ and anchors with $M = 0.01 \le M_{\min}$, spread $O = 0.01$, so outside quotes are

$$A = M + \frac{O}{2} = 0.015, \qquad B = M - \frac{O}{2} = 0.005.$$

- Dealer pre–Guard state (any a, C will do; Guard overrides capacity): take $a_0 = 3$ tickets, $C_0 = 2.00$.

What the Guard enforces (kernel level). Even though the naive capacity formula would give $K^* = \lfloor V/M \rfloor$ with $V = Ma_0 + C_0 = 0.01 \cdot 3 + 2 = 2.03$ (so $\lfloor 2.03/0.01 \rfloor = 203$), the *Guard* overrides:

$$M \le M_{\min} \quad \Longrightarrow \quad X^* := 0, \quad \text{ladder } x \in \{0\}, \quad a_c(x) \equiv A, \quad b_c(x) \equiv B.$$

Hence the interior schedule $\{p(x), a(x), b(x)\}$ is operationally disabled; the dealer quotes the outside and can only route orders to the VBT. [1]

Pass–through on both sides (Events 9–10)

We now fire one SELL and one BUY arrival. In both, the dealer's state is unchanged by design.

[12] Guard and kernel recap: Sec. 7.2 sets $X^* := 0$ when $M \le M_{\min}$; Secs. 7.5–7.6 define clipped quotes and boundary pass–through; Events 9–10 give the pass–through double–entry.

Arrival 1: customer *SELLS* 1 ticket (Event 9).

- **Pricing/feasibility under Guard:** bid is pinned $b_c \equiv B = 0.005$. Capacity is $X^* = 0$, so interior dealer BUY is *not* feasible anyway ($x + S \le X^*$ fails for any $x \ge 0$).
- **Execution (atomic pass–through at B):**

$$p = B = 0.005, \quad \Delta x_D = 0,\ \Delta C_D = 0; \quad \Delta x_{VBT} = +1,\ \Delta C_{VBT} = -0.005; \quad \Delta \mathrm{Cash}_{\mathrm{Trader}} = +0.005.$$

Arrival 2: customer *BUYS* 1 ticket (Event 10).

- **Pricing/feasibility under Guard:** ask is pinned $a_c \equiv A = 0.015$. Regardless of the dealer's inventory a_0, the Guard regime routes to VBT (operational freeze of interior execution).
- **Execution (atomic pass–through at A):**

$$p = A = 0.015, \quad \Delta x_D = 0,\ \Delta C_D = 0; \quad \Delta x_{VBT} = -1,\ \Delta C_{VBT} = +0.015; \quad \Delta \mathrm{Cash}_{\mathrm{Trader}} = -0.015.$$

Dealer/VBT/trader double–entry (after both arrivals)

	Dealer D	VBT
Inventory change	0	$+1 - 1 = 0$
Cash change	0	$-0.005 + 0.015 = +0.010$

The customer side receives $+0.005$ on the SELL and pays 0.015 on the BUY, consistent with the two executions at (B, A). Dealer (x, C) is unchanged in both trades, as required by pass–through. [1]

Harness checks

1. **Guard activation:** with $M \le M_{\min}$ assert $X^* = 0$ and ladder $\{0\}$.
2. **Pinned quotes:** assert $a_c(x) \equiv A$, $b_c(x) \equiv B$ (no interior schedule used).
3. **Pass–through only:** for any arrival side, execution price equals outside (A for BUY, B for SELL), and dealer deltas satisfy $\Delta x_D = \Delta C_D = 0$.
4. **Double–entry:** customer cash changes equal and opposite VBT cash changes; ticket transfers mirror cash legs (Events 9–10).

References

[1] *Specification of the Dealer Module for the Kalecki Ring (Bilancio Simulation)*, Sec. 7.2 (Guard and capacity), Secs. 7.5–7.6 (clipping and pins), Events 9–10 (pass–through).

10 EXAMPLE: Partial-recovery default with multiple claimant types

We consider one issuer I whose tickets (face $S = 1$) mature at period t. The maturing cohort is held by three distinct claimant types: the bucket dealer D, the value-based trader (VBT), and a non-dealer trader K. Settlement follows the proportional recovery rule:

$$R_I(t) \;=\; \min\!\Big(1,\ \frac{C_I(t)}{D_I(t)}\Big), \qquad D_I(t) \;=\; S \cdot N_I(t),$$

and for each holder h with q_h maturing tickets on I, the payment is $q_h\, S\, R_I(t)$; all those tickets are then deleted, and the issuer's post–settlement cash is $C_I(t^+) = C_I(t) - R_I(t)\, D_I(t)$.

Primitives

- Ticket size (face): $S = 1$.
- Total number of maturing tickets on issuer I: $N_I(t) = 8 \Rightarrow D_I(t) = 8$.
- Issuer cash before settlement: $C_I(t) = 3 \;\Rightarrow\; R_I(t) = \dfrac{3}{8} = 0.375 \in (0,1)$.
- Holder composition of the maturing cohort:

$$q_D = 3 \quad \text{(dealer)}, \qquad q_V = 2 \quad \text{(VBT)}, \qquad q_K = 3 \quad \text{(trader } K\text{)}, \qquad q_D + q_V + q_K = 8.$$

Starting balance sheets (slice at the settlement instant)

We show only the cash and the specific maturing tickets; all other items are omitted.

Entity	Assets (before)	Liabilities (before)
Issuer I	Cash $C_I(t) = 3$	$N_I(t) = 8$ maturing tickets (face $S = 1$)
Dealer D	Cash C_D; $q_D = 3$ maturing $(I,\cdot)$	—
VBT	Cash C_V; $q_V = 2$ maturing $(I,\cdot)$	—
Trader K	Cash C_K; $q_K = 3$ maturing $(I,\cdot)$	—

Settlement: proportional recovery with $R = \frac{3}{8}$

Each holder $h \in \{D, \text{VBT}, K\}$ receives $q_h\, S\, R$ in cash and the corresponding q_h maturing tickets are deleted.

$$\begin{aligned}
\text{Dealer } D : \Delta\text{Cash} &= q_D \cdot S \cdot R = 3 \times 1 \times \tfrac{3}{8} = \tfrac{9}{8} = 1.125, \quad q_D : 3 \to 0,\\
\text{VBT} : \Delta\text{Cash} &= q_V \cdot S \cdot R = 2 \times 1 \times \tfrac{3}{8} = \tfrac{6}{8} = 0.75, \quad q_V : 2 \to 0,\\
\text{Trader } K : \Delta\text{Cash} &= q_K \cdot S \cdot R = 3 \times 1 \times \tfrac{3}{8} = \tfrac{9}{8} = 1.125, \quad q_K : 3 \to 0.
\end{aligned}$$

Issuer cash after payouts:

$$C_I(t^+) = C_I(t) - R \cdot D_I(t) = 3 - \tfrac{3}{8} \cdot 8 = 0.$$

All issuer liabilities in this maturing cohort are extinguished; the default is recorded with recovery $R = \frac{3}{8}$.

After settlement (slice)

Entity	Assets (after)	Liabilities (after)
Issuer I	Cash $C_I(t^+) = 0$	All 8 matured tickets deleted (default recorded)
Dealer D	Cash $C_D + 1.125$; 0 of those tickets	—
VBT	Cash $C_V + 0.75$; 0 of those tickets	—
Trader K	Cash $C_K + 1.125$; 0 of those tickets	—

Consistency checks (to embed in tests)

1. **Cash conservation for the cohort:** $(1.125 + 0.75 + 1.125) = 3 = R \cdot D_I(t)$ equals the issuer's cash outflow.
2. **Ticket deletion:** the 8 maturing tickets on I are removed from all holders.
3. **Issuer cash to zero:** $C_I(t^+) = 0$ because $R < 1$ (partial recovery).
4. **Type symmetry:** dealer, VBT, and trader all receive the same per-ticket payout $S\,R$, scaled by their ticket counts q_h.

Optional link to anchors (next period). If this cohort defines the bucket's loss in period t, the bucket loss rate is $\ell_t = 1 - R = \frac{5}{8}$. If you use the loss-based rule, $M_{t+1} = M_t - \phi_M\, \ell_t$, $O_{t+1} = O_t + \phi_O\, \ell_t$, and derive A_{t+1}, B_{t+1} accordingly before dealer pre-computation at $t{+}1$.

11 EXAMPLE: Trader-held rebucketing without dealer–dealer transfer

We construct a two-date example to verify that when a *trader* holds a ticket that moves across a bucket boundary, only the ticket's `bucket_id` changes (no internal sale between dealers). This is executed in the event-loop step "Update maturities and buckets" and contrasts with Event 11, which applies only when a *dealer* holds the migrating ticket.

Primitives and bucket ranges

- Standard ticket size $S = 1$.
- Maturity buckets by remaining τ (baseline):

$$\text{Short: } \tau \in \{1, 2, 3\}, \quad \text{Mid: } \tau \in \{4, \ldots, 8\}, \quad \text{Long: } \tau \geq 9.$$

- Outside anchors M^b, O^b (per bucket b) and the per-bucket dealers $\{D^{\text{Short}}, D^{\text{Mid}}, D^{\text{Long}}\}$ exist as required by the baseline, but remain passive in this example.

Entities

- Trader T: holds cash C_T and *one* ticket $T^\star$ on issuer J.
- Dealers $D^{\text{Long}}, D^{\text{Mid}}$: arbitrary initial states (irrelevant here).
- VBTs for Long/Mid buckets: present but unused.

Timeline and states

Date t_1 (before the boundary). Remaining maturity of $T^\star$ is $\tau(t_1) = 9 \Rightarrow$ **Long** bucket. We show only the slice relevant to the migrating ticket:

Entity	Assets (slice)	Liabilities (slice)
Trader T	Cash C_T; 1 ticket (J, Long)	—
D^{Long}	(arbitrary)	—
D^{Mid}	(arbitrary)	—

Event-loop step between t_1 and t_2: update maturities and buckets. The engine decrements τ and recomputes `bucket_id` for every ticket. For $T^\star$: $\tau : 9 \to 8 \Rightarrow$ bucket changes **Long→Mid**. Since the *owner is a trader*, there is *no* internal dealer–dealer transaction: no cash changes hands, no dealer inventory moves.

Date t_2 (after rebucketing).

Entity	Assets (slice)	Liabilities (slice)
Trader T	Cash C_T; 1 ticket (J, Mid)	—
D^{Long}	(unchanged from t_1)	—
D^{Mid}	(unchanged from t_1)	—

What *does not* happen here (contrast to Event 11)

If a dealer had been the holder at the boundary, the rebucketing would be represented as an *internal sale* of the ticket from the old-bucket dealer to the new-bucket dealer at the new bucket's mid price, with mirrored cash and inventory legs (Event 11). None of that applies when a trader holds the ticket: the owner remains T, and only the `bucket_id` tag is updated.

Optional continuation at t_2 (for testing)

If the trader T now decides to *sell* $T^\star$ at t_2, the trade is executed with the *Mid* dealer at the current quoted bid $b_c^{\text{Mid}}(x^{\text{Mid}})$ using Event 1 (dealer buys one ticket at price p), with standard feasibility and state updates.

Harness checks

1. **Ownership invariance across rebucketing:** ticket owner remains T.
2. **No cash movement at the boundary:** total cash of all entities unchanged from t_1 to t_2.
3. **Dealer states unchanged:** (x, C) of D^{Long} and D^{Mid} identical at t_1 and t_2.
4. **Event discrimination:** assert that Event 11 triggers *only* when the current holder is a dealer; otherwise only the `bucket_id` rewrite executes in the event-loop's "Update maturities and buckets."

12 EXAMPLE: Partial recovery default with multiple claimant types

We consider one bucket in period t where a single issuer I has several tickets maturing. These maturing tickets are held by three distinct claimants: the bucket dealer D, the bucket value–based trader VBT, and a non–dealer trader K. Settlement follows the proportional–recovery rule.

Primitives and settlement rule

- Ticket size (face): $S = 1$.
- At maturity t, issuer I has $N_I(t)$ maturing tickets, total due $D_I(t) = S \cdot N_I(t)$.
- Issuer cash at settlement: $C_I(t) \in (0, D_I(t))$, so recovery $R_I(t) \in (0, 1)$ where

$$R_I(t) \;=\; \min\!\left(1, \frac{C_I(t)}{D_I(t)}\right) = \frac{C_I(t)}{D_I(t)}.$$

- Each holder $h \in \{D, \text{VBT}, K\}$ with q_h maturing tickets receives $q_h\, S\, R_I(t)$. All those tickets are deleted. Issuer cash updates to $C_I(t^+) = C_I(t) - R_I(t)D_I(t) = 0$.

Numerical configuration

Choose a simple split of the maturing cohort:

$$(q_D, q_{\text{VBT}}, q_K) = (2, 2, 1), \qquad N_I(t) = q_D + q_{\text{VBT}} + q_K = 5, \qquad S = 1.$$

Let the issuer bring $C_I(t) = 3$ units of cash to settlement, so

$$R_I(t) = \frac{C_I(t)}{D_I(t)} = \frac{3}{5} = 0.6.$$

Starting balance sheets (slice at settlement time t)

Entity	Assets	Liabilities
Issuer I	Cash $C_I(t) = 3$	5 maturing tickets (face 1 each)
Dealer D	Cash C_D; q_D=2 tickets (I, bucket)	—
VBT	Cash C_V; q_{VBT}=2 tickets (I, bucket)	—
Trader K	Cash C_K; q_K=1 ticket (I, bucket)	—

Settlement with proportional recovery $R = \frac{3}{5}$

Payouts to claimants:

Dealer D : $q_D SR = 2{\cdot}1{\cdot}0.6 = 1.2$, VBT : $q_{\mathrm{VBT}} SR = 2{\cdot}0.6 = 1.2$, Trader K : $q_K SR = 1{\cdot}0.6 = 0.6$.

All 5 maturing tickets are extinguished.

After settlement (slice)

Entity	Assets	Liabilities
Issuer I	Cash $C_I(t^+) = 3 - 0.6 \cdot 5 = 0$	All 5 liabilities deleted; default recorded
Dealer D	Cash $C_D + 1.2$; 0 of (I, bucket)	—
VBT	Cash $C_V + 1.2$; 0 of (I, bucket)	—
Trader K	Cash $C_K + 0.6$; 0 of (I, bucket)	—

Checks.

- *Cash conservation on the default cohort:* Total paid to claimants = $(1.2 + 1.2 + 0.6) = 3 = R\,D_I(t)$.
- *Issuer cash to zero:* $C_I(t^+) = 0$ because $R < 1$.
- *Ticket deletion:* all 5 maturing tickets on I are removed from holders' assets and from I's liabilities.

Bucket loss rate and (optional) VBT anchor update

If this bucket contains no other issuers maturing at t, then each matured ticket loses $1 - R = 0.4$ of face value, so the bucket loss rate is

$$\ell_t \;=\; \frac{\sum_{k \in \mathcal{T}_t^{\mathrm{mat}}} (1-R)}{\sum_{k \in \mathcal{T}_t^{\mathrm{mat}}} 1} \;=\; 1 - R \;=\; 0.4.$$

If the loss–based rule is active, next–period anchors update via $M_{t+1} = M_t - \phi_M\,\ell_t$, $O_{t+1} = O_t + \phi_O\,\ell_t$, and outside quotes A_{t+1}, B_{t+1} are recomputed accordingly. (Otherwise keep anchors fixed for this test.)

13 EXAMPLE: One-ticket trade pushes capacity across an integer

We set $S = 1$, $M = 1.0$, $O = 0.30$, so outside quotes are $A = 1.15$, $B = 0.85$. The dealer kernel uses

$$V = Ma + C, \qquad K^* = \lfloor V/M \rfloor, \qquad X^* = SK^*,$$

$$\lambda = \frac{S}{X^* + S}, \qquad I = \lambda O,$$

$$p(x) = M - \frac{O}{X^* + 2S}\left(x - \frac{X^*}{2}\right), \qquad a(x) = p(x) + \frac{I}{2}, \quad b(x) = p(x) - \frac{I}{2}.$$

Interior BUY (dealer buys) is feasible iff $x + S \le X^*$ and $C \ge b(x)$; interior SELL (dealer sells) is feasible iff $x \ge S$. (No clipping binds in the scenarios below.)

A. Up–jump: $K^* : 3 \to 4$ after one interior BUY

Initial state (just below integer). Choose

$$a_0 = 2, \quad C_0 = 1.97 \quad \Rightarrow \quad V_0 = a_0 + C_0 = 3.97, \quad K_0^* = \lfloor 3.97 \rfloor = 3,\ X_0^* = 3.$$

Inventory $x_0 = a_0 = 2$. Then

$$\lambda_0 = \frac{1}{X_0^* + 1} = \frac{1}{4} = 0.25, \qquad I_0 = \lambda_0 O = 0.075,$$

$$p_0(x) = 1 - \frac{0.30}{3+2}\Big(x - \tfrac{3}{2}\Big) = 1 - 0.06\,(x - 1.5).$$

At $x_0 = 2$: $p_0(2) = 0.97$, $b_0 = b(x_0) = 0.97 - \frac{I_0}{2} = 0.9325$. Interior BUY is feasible: $x_0 + 1 = 3 \le X_0^* = 3$ and $C_0 = 1.97 \ge b_0$.

Execution (customer SELL; dealer BUYS 1 at b_0). Update:

$$x_1 = x_0 + 1 = 3, \qquad C_1 = C_0 - b_0 = 1.97 - 0.9325 = 1.0375,$$

$$V_1 = Ma_1 + C_1 = 1 \cdot 3 + 1.0375 = 4.0375, \qquad K_1^* = \lfloor 4.0375 \rfloor = 4, \quad X_1^* = 4.$$

Discrete jump in λ and I.

$$\lambda_0 = \frac{1}{4} = 0.25 \ \longrightarrow\ \lambda_1 = \frac{1}{5} = 0.20, \qquad I_0 = 0.075 \ \longrightarrow\ I_1 = \lambda_1 O = 0.06.$$

(Width tightens after the jump because capacity increased.)

Post-trade quotes at new ladder ($X_1^* = 4$). With $x_1 = 3$,

$$p_1(3) = 1 - \frac{0.30}{4+2}(3-2) = 0.95, \quad a_1(3) = 0.95 + \tfrac{I_1}{2} = 0.98, \quad b_1(3) = 0.95 - \tfrac{I_1}{2} = 0.92.$$

B. Down–jump: $K^* : 4 \to 3$ after one interior SELL

Initial state (just above integer). Choose

$$a_0 = 4, \quad C_0 = 0.02 \quad \Rightarrow \quad V_0 = 4.02, \quad K_0^* = \lfloor 4.02 \rfloor = 4,\ X_0^* = 4,$$

$x_0 = a_0 = 4$. Then

$$\lambda_0 = \frac{1}{X_0^* + 1} = \frac{1}{5} = 0.20, \qquad I_0 = \lambda_0 O = 0.06,$$

$$p_0(x) = 1 - \frac{0.30}{4+2}\Big(x - 2\Big) = 1 - 0.05\,(x - 2).$$

At $x_0 = 4$: $p_0(4) = 0.90$, so the interior ask is $a_0 = a(x_0) = 0.90 + \frac{I_0}{2} = 0.93$. Interior SELL is feasible since $x_0 \ge S$.

Execution (customer BUY; dealer SELLS 1 at a_0). Update:

$$x_1 = x_0 - 1 = 3, \qquad C_1 = C_0 + a_0 = 0.02 + 0.93 = 0.95,$$

$$V_1 = Ma_1 + C_1 = 1 \cdot 3 + 0.95 = 3.95, \qquad K_1^* = \lfloor 3.95 \rfloor = 3, \quad X_1^* = 3.$$

Discrete jump in λ and I.

$$\lambda_0 = \frac{1}{5} = 0.20 \;\longrightarrow\; \lambda_1 = \frac{1}{4} = 0.25, \qquad I_0 = 0.06 \;\longrightarrow\; I_1 = \lambda_1 O = 0.075.$$

(Width widens after the jump because capacity decreased.)

Post-trade quotes at new ladder ($X_1^* = 3$). With $x_1 = 3 = X_1^*$,

$$p_1(3) = 1 - \frac{0.30}{3+2}\Big(3 - \tfrac{3}{2}\Big) = 1 - 0.06\cdot 1.5 = 0.91, \quad a_1(3) = 0.91 + \tfrac{I_1}{2} = 0.9475, \quad b_1(3) = 0.91 - \tfrac{I_1}{2} = 0.8725.$$

Test assertions

1. (*Capacity jump*) Verify K^* changes by exactly ± 1 and $X^* = SK^*$ updates accordingly.
2. (*Width jump*) Verify $\lambda = S/(X^* + S)$ and $I = \lambda O$ jump discretely: $0.25 \to 0.20$ (up-jump) and $0.20 \to 0.25$ (down-jump) with $O = 0.30$.
3. (*Feasibility*) In A: $x + S \le X^*$ and $C \ge b(x)$; in B: $x \ge S$.
4. (*No clipping / no pass-through*) Check $a(x) < A$, $b(x) > B$ at the quoted x, and that the trade is interior.
5. (*State recomputation*) After execution, recompute $K^*, X^*, \lambda, I, p(\cdot)$ using the *updated* V and show the new quotes.

14 EXAMPLE: Minimal event–loop harness for arrivals

Purpose

Test the randomized order–flow subroutine in isolation: with fixed $\pi_{\text{sell}} \in (0,1)$ and $N_{\max} \in \mathbb{N}$, generate at most $N_{\max}$ one–ticket arrivals in a period, apply empty–set fallback (SELL→BUY or BUY→SELL), enforce "one trade per side per agent per period," and verify: (i) the empirical SELL share approaches π_{sell} *subject to* eligibility and fallback, (ii) post–trade removal from $\mathcal{S}_t/\mathcal{B}_t$ occurs, (iii) "no feasible execution" leaves the chosen agent in the eligibility set. This mirrors Sec. 11, Steps 3–4 of the event loop. :contentReference[oaicite:1]index=1

Harness setup (one bucket, fixed anchors)

- Ticket size $S = 1$. Outside anchors fixed in this test: $M = 1.0$, $O = 0.30$, so $A = 1.15$, $B = 0.85$. (VBT is present but only for feasibility/pass–through pins.) Kernel formulas and feasibility follow Sec. 7.2–7.7. :contentReference[oaicite:2]index=2
- Dealer initial state: $a_0 = 2$, $C_0 = 2$. Hence $V_0 = 4$, $K_0^* = \lfloor 4 \rfloor = 4$, $X_0^* = 4$, $N_0 = 5$, $\lambda_0 = 1/5$, $I_0 = \lambda_0 O = 0.06$. Midline $p(x) = 1 - \frac{0.30}{4+2}(x-2) = 1 - 0.05(x-2)$. At $x_0 = 2$: $a_0 = 1.03$, $b_0 = 0.97$. (No clipping.) :contentReference[oaicite:3]index=3
- Order–flow controls: choose $\pi_{\text{sell}} = 0.5$, $N_{\max} = 3$.
- Eligibility rule snapshots (Sec. 11, Step 3):
  - SELL set $\mathcal{S}_t$: traders with $\text{shortfall}_i(t) > 0$ *and* at least one ticket owned; one SELL removes i from $\mathcal{S}_t$ for period t.
  - BUY set $\mathcal{B}_t$: traders with $T_i^{\min}(t) - t \ge H$ and $\text{cash}_i(t) > \text{buffer}_B$; one BUY removes i from $\mathcal{B}_t$ for period t. :contentReference[oaicite:4]index=4

Agent pools for the micro–run

Construct tiny pools to force both *fallback* and *no–feasible* cases in a single period t:
- SELL side: one liquidity seller S_1 who owns one ticket and has shortfall > 0; thus $\mathcal{S}_t = \{S_1\}$ initially.

- BUY side: two investors B_1, B_2 with horizons satisfying the BUY condition. Set $\text{cash}_{B_1} = 2.00$ (rich) and $\text{cash}_{B_2} = 0.90$ (poor). Initially $\mathcal{B}_t = \{B_1, B_2\}$.

Within–period arrival loop (Sec. 11, Step 4)

Initialize $n = 1$. While $n \le N_{\max}$ and $\mathcal{S}_t \cup \mathcal{B}_t \neq \emptyset$:

1. Draw $Z_n \sim \text{Bernoulli}(\pi_{\text{sell}})$.
2. If $Z_n = 1$ (SELL preferred):
   - If $\mathcal{S}_t \neq \emptyset$, pick $i \in \mathcal{S}_t$ uniformly, query $b_c(x)$, check feasibility (interior BUY or pin to B); execute one SELL if feasible and remove i from $\mathcal{S}_t$.
   - Else ($\mathcal{S}_t = \emptyset$, but $\mathcal{B}_t \neq \emptyset$): **fallback** to BUY branch.
3. If $Z_n = 0$ (BUY preferred):
   - If $\mathcal{B}_t \neq \emptyset$, pick $i \in \mathcal{B}_t$ uniformly, query $a_c(x)$, test feasibility (dealer inventory or pin to A, and $\text{cash}_i \ge p$). If feasible, execute and remove i from $\mathcal{B}_t$.
   - Else ($\mathcal{B}_t = \emptyset$, but $\mathcal{S}_t \neq \emptyset$): **fallback** to SELL branch.
4. If *no feasible execution*, keep i in the eligibility set (Sec. 11, Step 4(f)). :contentReference[oaicite:5]index=5
5. Recompute dealer state/quotes if a trade executed (Sec. 11, Step 4(d)); increment n.

Worked micro–run (three arrivals in one period)

Take a particular draw sequence $Z_1 = 1$, $Z_2 = 1$, $Z_3 = 0$.

Arrival 1 ($Z_1 = 1$, SELL). $\mathcal{S}_t = \{S_1\}$. Execute interior dealer BUY at $p = b_c(x_0 = 2) = 0.97$ (feasible: $x_0 + 1 \le X_0^*$ and $C_0 \ge p$). Update dealer $x : 2 \to 3$, $C : 2 \to 1.03$; recompute quotes with X^* unchanged ($= 4$): $p(3) = 0.95 \Rightarrow a = 0.98,\ b = 0.92$. Remove S_1 from $\mathcal{S}_t \Rightarrow \mathcal{S}_t = \emptyset$. (Sec. 7.6–7.7.) :contentReference[oaicite:6]index=6

Arrival 2 ($Z_2 = 1$, SELL preferred but $\mathcal{S}_t = \emptyset$). Fallback to BUY. Suppose the random pick is B_2 (poor). Current ask $a_c(x = 3) = 0.98$. Feasibility fails because $\text{cash}_{B_2} = 0.90 < 0.98$. *No execution*; by rule, B_2 *stays* in $\mathcal{B}_t$ (Sec. 11, Step 4(f)). Dealer state and quotes unchanged. :contentReference[oaicite:7]index=7

Arrival 3 ($Z_3 = 0$, BUY). Pick B_1 (rich). $\text{cash}_{B_1} = 2.00 \ge a_c(x = 3) = 0.98$, interior dealer SELL feasible ($x \ge 1$). Execute at $p = 0.98$, update dealer $x : 3 \to 2$, $C : 1.03 \to 2.01$; recompute quotes: $p(2) = 1 \Rightarrow a = 1.03,\ b = 0.97$. Remove B_1 from $\mathcal{B}_t$.

Outcomes in this period.

- Executed trades: 1 SELL (Arrival 1), 1 BUY (Arrival 3).
- *Empty–set fallback* used at Arrival 2 (SELL→BUY).
- *No feasible execution* occurred for B_2; B_2 remained in $\mathcal{B}_t$ as required.
- *Post–trade removal* enforced: S_1 removed from $\mathcal{S}_t$; B_1 removed from $\mathcal{B}_t$.

Monte Carlo harness (many periods)

For diagnostics consistent with Sec. 12.3: fix $(\pi_{\text{sell}}, N_{\max})$, run T periods, and record

$$\widehat{\pi}_{\text{sell}}^{\text{exec}} = \frac{\text{executed SELL count}}{\text{executed SELL} + \text{executed BUY}}.$$

Under i.i.d. draws Z_n and stationary eligibility, the law of large numbers implies $\widehat{\pi}_{\text{sell}}^{\text{exec}} \to \pi_{\text{sell}}$ *modulated* by (i) empty–set fallback probabilities and (ii) the BUY feasibility rate (cash constraint). Instrument:

- F^{BUY} = fraction of BUY attempts that were feasible ($\text{cash}_i \ge p$);
- $q_{SB} = \Pr(\mathcal{S}_t = \emptyset,\ \mathcal{B}_t \neq \emptyset)$, $q_{BS} = \Pr(\mathcal{B}_t = \emptyset,\ \mathcal{S}_t \neq \emptyset)$.

Then the executed SELL share concentrates around

$$\pi_{\text{sell}} \cdot \Pr(\mathcal{S}_t \neq \emptyset) + (1 - \pi_{\text{sell}}) \cdot q_{BS} \quad \Big/ \quad \Big[\Pr(\text{an execution occurs})\Big],$$

where Pr(an execution occurs) increases with F^{BUY} and the nonemptiness of both sets. (Exact decomposition is implementation–specific; this harness is for empirical verification.) See Sec. 11 for the randomized loop and removal rules; Sec. 12.3 recommends this convergence check. :contentReference[oaicite:8]index=8

Assertions to include in the test

1. **Set discipline (per period):** once i executes a SELL (BUY), $i \notin \mathcal{S}_t$ ($i \notin \mathcal{B}_t$) for the remainder of period t. (Sec. 11, Step 3.) :contentReference[oaicite:9]index=9
2. **Fallback correctness:** if the preferred side's set is empty while the other is nonempty, the arrival is processed on the nonempty side with no resampling of Z_n. (Sec. 11, Step 4.) :contentReference[oaicite:10]index=10
3. **No–feasible retention:** if the chosen order cannot execute (e.g. investor cash $< a_c(x)$), the agent remains in the set. (Sec. 11, Step 4(f).) :contentReference[oaicite:11]index=11
4. **Kernel consistency:** after each executed trade, recompute $(X^*, \lambda, I, p, a_c, b_c)$ from the updated (a, C). (Sec. 7.2–7.7.) :contentReference[oaicite:12]index=12

Note. This harness isolates the order–flow logic; settlement (Sec. 11, Step 5) and VBT anchor updates (Sec. 9) are held fixed/omitted here by design. :contentReference[oaicite:13]index=13

15 EXAMPLE: Ticket–level transfer (no generic materialization)

Goal

Verify that on a customer BUY, the execution transfers a *specific* ticket ID (issuer + maturity) from the seller of record to the buyer: (i) interior BUY ⇒ transfer one ticket from the dealer's inventory; (ii) pinned BUY (pass–through) ⇒ transfer one ticket from the VBT's inventory. At no point may the system "create" a generic ticket. (Dealers/VBTs must track inventory at ticket level; BUYs transfer an existing ticket.) :contentReference[oaicite:1]index=1

Primitives and kernel (one bucket)

Ticket size $S = 1$. Outside anchors fixed: $M = 1.0$, $O = 0.30$, so $A = 1.15$, $B = 0.85$. Dealer kernel:

$$V = Ma + C, \quad K^* = \lfloor V/M \rfloor, \quad X^* = SK^*, \quad \lambda = \tfrac{S}{X^*+S}, \quad I = \lambda O,$$

$$p(x) = M - \frac{O}{X^* + 2S}\left(x - \frac{X^*}{2}\right), \qquad a(x) = p(x) + \tfrac{I}{2}, \quad b(x) = p(x) - \tfrac{I}{2}.$$

Feasibility: interior SELL (dealer sells to buyer) requires $x \geq S$; otherwise pin to A and route to VBT (dealer state unchanged). Interior BUY (dealer buys from customer) as in the spec. :contentReference[oaicite:2]index=2

Ticket universe at time t (IDs and owners)

We explicitly enumerate ticket IDs as triples $k = (\iota, \tau, \#)$ with issuer ι, maturity time τ, and a serial #. Remaining time `remaining_tau` $= \tau - t$ determines the bucket (here fixed). Initial inventory is:

Dealer D (owner): $x = 2$, $a = 2$, $C = 2$; ticket list

$$\mathcal{K}_D = \{k_1 = (I_1, t{+}6, \#101),\ k_2 = (I_2, t{+}5, \#202)\}.$$

VBT (owner): a deep list; for this test expose two concrete IDs in the bucket,

$$\mathcal{K}_{\text{VBT}} = \{v_1 = (I_3, t{+}7, \#303),\ v_2 = (I_4, t{+}6, \#404)\} \cup \{\text{many more}\}.$$

Buyer B: cash and issuer status will differ across the two subtests; initially holds no tickets.

Dealer pre–computation: $V = 4 \Rightarrow K^* = 4$, $X^* = 4$, $\lambda = \frac{1}{5}$, $I = 0.06$, $p(x) = 1 - \frac{0.30}{4+2}(x - 2) = 1 - 0.05(x - 2)$. At $x = 2$: $a(2) = 1.03$, $b(2) = 0.97$. (No clipping.) :contentReference[oaicite:3]index=3

Case A (interior BUY): Dealer transfers a specific ticket ID

Buyer state and issuer constraint. Let B have no preassigned asset–side issuer ($k(B)$ undefined), horizon satisfies BUY rule, and cash $C_B = 2.00 > \texttt{buffer}_B$ (so $B \in \mathcal{B}_t$). Per the spec, if a buyer has no issuer set, the BUY fixes $k(B)$ by the issuer of the *ticket actually transferred* from the seller's inventory. *Any* deterministic tie–breaker is acceptable (e.g. smallest τ). :contentReference[oaicite:4]index=4

Execution (customer BUY; interior). Dealer has $x = 2 \geq 1$, so an interior SELL is feasible. Quote $a(2) = 1.03$. Execute B's BUY of one ticket at $p = 1.03$ and *select* the concrete ticket to transfer. Use the tie–breaker "lowest maturity" in $\mathcal{K}_D$, so pick

$$k^\star = k_2 = (I_2, t{+}5, \#202).$$

Required state updates (double entry).

$$\begin{aligned}
&\text{Dealer:}\quad x \leftarrow 1,\ \ C \leftarrow 2 + 1.03 = 3.03,\ \ \mathcal{K}_D \leftarrow \{k_1\} \quad (\text{length falls } 2 \to 1);\\
&\text{Buyer:}\quad C_B \leftarrow 2.00 - 1.03 = 0.97,\ \ \mathcal{K}_B \leftarrow \{k^\star\},\ \ k(B) \text{ is set to issuer } I_2.
\end{aligned}$$

Assertions. (i) The transferred ID $k^\star$ belonged to $\mathcal{K}_D$ immediately before execution. (ii) After execution $k^\star \notin \mathcal{K}_D$ and $k^\star \in \mathcal{K}_B$. (iii) $|\mathcal{K}_D|$ decreases by exactly one; no new ticket is created anywhere. (iv) If B attempts any future BUY in this bucket, the chosen ticket *must* have issuer I_2 (single-issuer asset rule). :contentReference[oaicite:5]index=5

Case B (pinned BUY / VBT pass–through): VBT transfers a specific ID

Boundary set–up. Now place the dealer at $x = 0$ (e.g. after prior sells) with the same anchors $M = 1$, $O = 0.30$. By feasibility, interior SELL is impossible; the operational ask is pinned to $A = 1.15$ and the dealer must route to VBT. The dealer's state (x, C) must remain unchanged. :contentReference[oaicite:6]index=6

Buyer state. Use a buyer $\tilde{B}$ with no preassigned issuer and cash $C_{\tilde{B}} = 2.00$ (BUY-eligible).

Execution (customer BUY; pinned). Execute $\tilde{B}$'s BUY of one ticket at $p = A = 1.15$ by *transferring* a concrete VBT ID. Again use "lowest maturity" tie–breaker in $\mathcal{K}_{\text{VBT}}$:

$$v^\star = v_2 = (I_4, t{+}6, \#404).$$

Required state updates (double entry with pass–through).

$$\begin{aligned}
&\text{Dealer:}\quad x \leftarrow 0 \text{ (unchanged)},\;\; C \leftarrow C \text{ (unchanged)},\;\; \mathcal{K}_D \text{ unchanged};\\
&\text{VBT:}\quad \mathcal{K}_{\text{VBT}} \leftarrow \{v_1\} \cup \{\text{rest}\} \text{ (length falls by one)},\;\; C_{\text{VBT}} \leftarrow C_{\text{VBT}} + 1.15;\\
&\tilde{B}:\quad C_{\tilde{B}} \leftarrow 2.00 - 1.15 = 0.85,\;\; \mathcal{K}_{\tilde{B}} \leftarrow \{v^\star\},\;\; k(\tilde{B}) \text{ is set to issuer } I_4.
\end{aligned}$$

Assertions. (i) Dealer inventory and cash are exactly unchanged (pass–through property). (ii) The transferred ID $v^\star$ belonged to $\mathcal{K}_{\text{VBT}}$ immediately before execution. (iii) After execution $v^\star \notin \mathcal{K}_{\text{VBT}}$ and $v^\star \in \mathcal{K}_{\tilde{B}}$. (iv) No generic ticket is created; ownership changes are purely transfers. :contentReference[oaicite:7]index=7

Global invariants to test (unit tests)

1. **Ownership conservation.** Across $\{D, \text{VBT}\} \cup$ all traders, the multiset of ticket IDs is preserved under trading and settlement; only owners change (except deletion at maturity). :contentReference[oaicite:8]index=8
2. **Inventory length monotonicity (on SELL).** An executed dealer SELL reduces $|\mathcal{K}_D|$ by exactly 1 in interior trades; in pinned trades it reduces $|\mathcal{K}_{\text{VBT}}|$ by exactly 1, leaving $|\mathcal{K}_D|$ unchanged. Dealer/VBT cash updates match the price p (inside or outside). :contentReference[oaicite:9]index=9
3. **Single-issuer asset constraint.** If the buyer already had $k(i) = \bar{I}$, then the transferred ID must satisfy issuer $= \bar{I}$. If the buyer had no issuer, set $k(i)$ to the issuer of the transferred ID; all subsequent asset–side purchases by i must match this issuer. :contentReference[oaicite:10]index=10
4. **Feasibility conformity.** Interior vs. pinned paths must satisfy the kernel feasibility rules: interior SELL needs $x \geq S$; pinned SELL is routed at A with dealer state unchanged. :contentReference[oaicite:11]index=11
5. **Double-entry balance.** After each execution, (cash change at buyer) $= -p$ and (cash change at seller of record) $= +p$; the ticket ID moves exactly once in the same step. :contentReference[oaicite:12]index=12

Implementation note (determinism)

To avoid nondeterminism when multiple eligible IDs exist on the seller's side, implement a fixed tie–breaker (e.g. lowest maturity time τ, then smallest serial #). The choice does not alter kernel pricing or feasibility but makes unit tests reproducible. (BUY issuer selection and the "transfer an *existing* ticket" requirement are binding.) :contentReference[oaicite:13]index=13