by A.2.1]. Since $\mathbf{A}\boldsymbol{\beta} = \mathbf{M}\mathbf{X}\boldsymbol{\beta} = \mathbf{M}\boldsymbol{\theta}$, we therefore find the restricted least squares estimate of $\boldsymbol{\theta}$ by minimizing $||\mathbf{Y} - \boldsymbol{\theta}||^2$ subject to $\boldsymbol{\theta} \in \mathcal{C}(\mathbf{X}) = \Omega$ and $\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$, that is, subject to

$$\boldsymbol{\theta} \in \mathcal{N}(\mathbf{M}) \cap \Omega \quad (= \omega, \text{ say}).$$

If $\mathbf{P}_\Omega$ and $\mathbf{P}_\omega$ are the projection matrices projecting onto Ω and ω, respectively, then we want to find $\hat{\boldsymbol{\theta}}_\omega = \mathbf{P}_\omega \mathbf{Y}$. Now, from B.3.2 and B.3.3,

$$\mathbf{P}_\Omega - \mathbf{P}_\omega = \mathbf{P}_{\omega^\perp \cap \Omega},$$

where $\omega^\perp \cap \Omega = \mathcal{C}(\mathbf{B})$ and $\mathbf{B} = \mathbf{P}_\Omega \mathbf{M}'$. Thus

$$\begin{aligned} \hat{\boldsymbol{\theta}}_\omega &= \mathbf{P}_\omega \mathbf{Y} \\ &= \mathbf{P}_\Omega \mathbf{Y} - \mathbf{P}_{\omega^\perp \cap \Omega} \mathbf{Y} \\ &= \hat{\boldsymbol{\theta}}_\Omega - \mathbf{B}(\mathbf{B}'\mathbf{B})^- \mathbf{B}'\mathbf{Y}. \end{aligned}$$

EXERCISES 3j

$C(Z) \perp C(X)$ so $PZ = 0$

1. If $\mathbf{P}$ projects onto $\mathcal{C}(\mathbf{X})$, show that $\mathbf{Z}'(\mathbf{I}_n - \mathbf{P})\mathbf{Z}$ is nonsingular. rank = size = dim Z.

2. Prove that if $\mathbf{X}_1$ is $n \times r$ of rank r and consists of a set of r linearly independent columns of $\mathbf{X}$, then $\mathbf{X} = \mathbf{X}_1 \mathbf{L}$, where $\mathbf{L}$ is $r \times p$ of rank r.

   defn of LI.

3. Prove that $\mathbf{B}$ has full column rank [i.e., $(\mathbf{B}'\mathbf{B})^- = (\mathbf{B}'\mathbf{B})^{-1}$].

   B3.4 + C

4. If $\mathbf{X}$ has full rank and $\hat{\boldsymbol{\theta}}_\omega = \mathbf{X}\hat{\boldsymbol{\beta}}_H$, show that

$$\hat{\boldsymbol{\beta}}_H = \hat{\boldsymbol{\beta}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'(\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}')^{-1}\mathbf{A}\hat{\boldsymbol{\beta}}.$$

   [This is a special case of (3.38).]

5. Show how to modify the theory above to take care of the case when the restrictions are $\mathbf{A}\boldsymbol{\beta} = \mathbf{c}$ ($\mathbf{c} \neq \mathbf{0}$).

   Use $\beta - \beta_0$ for any soln β_0 to $A\beta = c$ & reparam.

3.10 GENERALIZED LEAST SQUARES

Having developed a least squares theory for the full-rank model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $E[\boldsymbol{\varepsilon}] = \mathbf{0}$ and $\text{Var}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{I}_n$, we now consider what modifications are necessary if we allow the ε_i to be correlated. In particular, we assume that $\text{Var}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{V}$, where $\mathbf{V}$ is a *known* $n \times n$ positive-definite matrix.

Since $\mathbf{V}$ is positive-definite, there exists an $n \times n$ nonsingular matrix $\mathbf{K}$ such that $\mathbf{V} = \mathbf{K}\mathbf{K}'$ (A.4.2). Therefore, setting $\mathbf{Z} = \mathbf{K}^{-1}\mathbf{Y}$, $\mathbf{B} = \mathbf{K}^{-1}\mathbf{X}$, and