as $\sum_i r_{ii}d_{ii} = [(n-p)/n]\,\mathrm{tr}(\mathbf{D}) = 0$. Now $\gamma_2 > -2$ (A.13.1), so that $\mathrm{var}[\mathbf{Y}'\mathbf{A}\mathbf{Y}]$ is minimized when $d_{ij} = 0$ for all i, j. Thus in both cases we have minimum variance if and only if $\mathbf{A} = \mathbf{R}$. □

This theorem highlights the fact that a uniformly minimum variance quadratic unbiased estimate of σ^2 exists only under certain restrictive conditions like those stated in the enunciation of the theorem. If normality can be assumed ($\gamma_2 = 0$), then it transpires that (Rao [1973: p. 319]) S^2 is the minimum variance unbiased estimate of σ^2 in the entire class of unbiased estimates (not just the class of quadratic estimates).

Rao [1970, 1972] has also introduced another criterion for choosing the estimate of σ^2: *minimum norm quadratic unbiased estimation* (MINQUE). Irrespective of whether or not we assume normality, this criterion also leads to S^2 (cf. Rao [1970, 1974: p. 448]).

EXERCISES 3c

1. Suppose that $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$, where $\mathbf{X}$ is $n \times p$ of rank p.

   (a) Find $\mathrm{var}[S^2]$. Per (3.14), $2\sigma^4/(n-p)$

   (b) Evaluate $E[(\underline{\mathbf{Y}'\mathbf{A}_1\mathbf{Y}} - \sigma^2)^2]$ for

   $E X^2 - 2\sigma^2 E X + \sigma^4$
   $$\mathbf{A}_1 = \frac{1}{n-p+2}[\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'].$$
   → simplify → $\dfrac{2\sigma^4}{n-p+2}$

   $\dfrac{2\sigma^4(n-p)}{(n-p+2)^2} + \sigma^4/(n-p+2)^2$

   (c) Prove that $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$ is an estimate of σ^2 with a smaller mean-squared error than S^2. → denom is larger, +2

   (Theil and Schweitzer [1961])

2. Let $Y_1, Y_2, \ldots, Y_n$ be independently and identically distributed with mean θ and variance σ^2. Find the nonnegative quadratic unbiased estimate of σ^2 with the minimum variance.

   Thm 3.4 applies since diagonal P is equal as $X = 1_n$

3.4 DISTRIBUTION THEORY

Until now the only assumptions we have made about the ε_i are that $E[\boldsymbol{\varepsilon}] = \mathbf{0}$ and $\mathrm{Var}[\boldsymbol{\varepsilon}] = \sigma^2\mathbf{I}_n$. If we assume that the ε_i are also normally distributed, then $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$ and hence $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$. A number of distributional results then follow.

THEOREM 3.5 *If $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$, where $\mathbf{X}$ is $n \times p$ of rank p, then:*

(i) $\hat{\boldsymbol{\beta}} \sim N_p(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$.

(ii) $(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{X}'\mathbf{X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma^2 \sim \chi^2_p$.