EXERCISES 3k

1. Let $Y_i = \beta x_i + \varepsilon_i$ $(i = 1, 2)$, where $\varepsilon_1 \sim N(0, \sigma^2)$, $\varepsilon_2 \sim N(0, 2\sigma^2)$, and ε_1 and ε_2 are statistically independent. If $x_1 = +1$ and $x_2 = -1$, obtain the weighted least squares estimate of β and find the variance of your estimate. Apply Ex 3.9.

2. Let Y_i $(i = 1, 2, \ldots, n)$ be independent random variables with a common mean θ and variances σ^2/w_i $(i = 1, 2, \ldots, n)$. Find the linear unbiased estimate of θ with minimum variance, and find this minimum variance.

3. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, and let Y_i have a $N(i\theta, i^2\sigma^2)$ distribution for $i = 1, 2, \ldots, n$. Find the weighted least squares estimate of θ and prove that its variance is σ^2/n.

By BLUE, this is the WLS of $Y = 1\theta + \varepsilon_i$. Apply 3.9 again.

4. Let $Y_1, Y_2, \ldots, Y_n$ be random variables with common mean θ and with dispersion matrix $\sigma^2\mathbf{V}$, where $v_{ii} = 1$ $(i = 1, 2, \ldots, n)$ and $v_{ij} = \rho$ $(0 < \rho < 1;\ i, j = 1, 2, \ldots, n;\ i \neq j)$. Find the generalized least squares estimate of θ and show that it is the same as the ordinary least squares estimate. *Hint*: $\mathbf{V}^{-1}$ takes the same form as $\mathbf{V}$.

Same as OLS b/c $X^T V^{-1} X = X^T X$ for $X = 1$

(McElroy [1967])

$V = (1-\rho)I + \rho J$

if $V^{-1} = aI + bJ$,

$V^{-1}V = (1-\rho)aI + \rho a J + (1-\rho)bJ + b\rho n J$

$a = \frac{1}{1-\rho}$

$\frac{\rho}{1-\rho} + (1-\rho)b + b\rho n = 0$ solve for b.

5. Let $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, where $\mathbf{X}$ is $n \times p$ of rank p and $\mathbf{V}$ is a known positive-definite $n \times n$ matrix. If $\boldsymbol{\beta}^*$ is the generalized least squares estimate of $\boldsymbol{\beta}$, prove that

   Follows by expanding $V^{-1} = LL^T$ & $n \times m$ OLS optimality

   (a) $Q = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}^*)'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}^*)/\sigma^2 \sim \chi^2_{n-p}$.

   (b) Q is the quadratic nonnegative unbiased estimate of $(n-p)\sigma^2$ with minimum variance.

   (c) If $\mathbf{Y}^* = \mathbf{X}\boldsymbol{\beta}^* = \mathbf{P}^*\mathbf{Y}$, then $\mathbf{P}^*$ is idempotent but not, in general, symmetric. $X(X^T V^{-1} X)^{-1} X^T V^{-1}$ idem inspectably; asymm → choose $X \neq X^T$

6. Suppose that $E[\mathbf{Y}] = \boldsymbol{\theta}$, $\mathbf{A}\boldsymbol{\theta} = \mathbf{0}$, and $\text{Var}[\mathbf{Y}] = \sigma^2\mathbf{V}$, where $\mathbf{A}$ is a $q \times n$ matrix of rank q and $\mathbf{V}$ is a known $n \times n$ positive-definite matrix. Let $\boldsymbol{\theta}^*$ be the generalized least squares estimate of $\boldsymbol{\theta}$; that is, $\boldsymbol{\theta}^*$ minimizes $(\mathbf{Y} - \boldsymbol{\theta})'\mathbf{V}^{-1}(\mathbf{Y} - \boldsymbol{\theta})$ subject to $\mathbf{A}\boldsymbol{\theta} = \mathbf{0}$. Show that

$$\mathbf{Y} - \boldsymbol{\theta}^* = \mathbf{V}\mathbf{A}'\boldsymbol{\gamma}^*,$$

   where $\boldsymbol{\gamma}^*$ is the generalized least squares estimate of $\boldsymbol{\gamma}$ for the model $E[\mathbf{Y}] = \mathbf{V}\mathbf{A}'\boldsymbol{\gamma}$, $\text{Var}[\mathbf{Y}] = \sigma^2\mathbf{V}$.

(Wedderburn [1974])

~~Ex Q~~ Solve constrained opt w/ lagrange.

3.11 CENTERING AND SCALING THE EXPLANATORY VARIABLES

It is instructive to consider the effect of centering and scaling the x-variables on the regression model. We shall use this theory later in the book.