consistent estimate of $\boldsymbol{\beta}$ if and only if the smallest eigenvalue of $\mathbf{X}'\mathbf{X}$ tends to infinity. This condition on the smallest eigenvalue is a mild one, so that the result has wide applicability. Eicker also proves a theorem giving necessary and sufficient conditions for the asymptotic normality of each $\hat{\beta}_j$ (see Anderson [1971: pp. 23–27]). BLUE terms are uncorr when design cols are ortho! then $X^TX = \Lambda$ so $\hat{\beta}$ is a rotation of ε's $\Lambda^{-1}X^T$ is a rotate-and scale.

EXERCISES 3b

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & -2 \end{pmatrix}$

1. Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ $(i = 1, 2, \ldots, n)$, where $E[\boldsymbol{\varepsilon}] = \mathbf{0}$ and $\text{Var}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{I}_n$. Find the least squares estimates of β_0 and β_1. Prove that they are uncorrelated if and only if $\overline{x} = 0$.

2. In order to estimate two parameters θ and ϕ it is possible to make observations of three types: (a) the first type have expectation θ, (b) the second type have expectation $\theta + \phi$, and (c) the third type have expectation $\theta - 2\phi$. All observations are subject to uncorrelated errors of mean zero and constant variance. If m observations of type (a), m observations of (b), and n observations of type (c) are made, find the least squares estimates $\hat{\theta}$ and $\hat{\phi}$. Prove that these estimates are uncorrelated if $m = 2n$. → As above design

3. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\theta, \sigma^2)$. Find the linear unbiased estimate of θ with minimum variance.

4. Let $Y \sim 1_n \Theta + \varepsilon$, BLUE is $\bar{Y}$ because ortho

$$Y_i = \beta_0 + \beta_1(x_{i1} - \overline{x}_1) + \beta_2(x_{i2} - \overline{x}_2) + \varepsilon_i \qquad (i = 1, 2, \ldots, n),$$

where $\overline{x}_j = \sum_{i=1}^n x_{ij}/n$, $E[\boldsymbol{\varepsilon}] = \mathbf{0}$, and $\text{Var}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{I}_n$. If $\hat{\beta}_1$ is the least squares estimate of β_1, show that

By 3.11, $\text{var}\,\hat{\beta} = \sigma^2(X^TX)^{-1}$,

$$\text{var}[\hat{\beta}_1] = \frac{\sigma^2}{\sum_i (x_{i1} - \overline{x}_1)^2 (1 - r^2)},$$

where r is the correlation coefficient of the n pairs (x_{i1}, x_{i2}).

Conclusion follows by using Block matrix inv to inspect the 11-th entry.

3.3 UNBIASED ESTIMATION OF σ^2

We now focus our attention on σ^2 $(= \text{var}[\varepsilon_i])$. An unbiased estimate is described in the following theorem.

THEOREM 3.3 *If* $E[\mathbf{Y}] = \mathbf{X}\boldsymbol{\beta}$, *where* $\mathbf{X}$ *is an* $n \times p$ *matrix of rank* r $(r \leq p)$, *and* $\text{Var}[\mathbf{Y}] = \sigma^2 \mathbf{I}_n$, *then*

$$S^2 = \frac{(\mathbf{Y} - \hat{\boldsymbol{\theta}})'(\mathbf{Y} - \hat{\boldsymbol{\theta}})}{n - r} = \frac{RSS}{n - r}$$