get

$$\begin{pmatrix} \boldsymbol{\theta} \\ \hline \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \hline \mathbf{H} \end{pmatrix} \boldsymbol{\beta} = \left(\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \tau_1 \\ \tau_2 \end{pmatrix},$$

$X\hat{\beta}_1 = X\hat{\beta}_2$

where the augmented matrix now has five linearly independent columns. Thus given $\boldsymbol{\theta}$, $\boldsymbol{\beta}$ is now unique. □

$X(\beta - \hat{\beta}_1) = X(\beta - \hat{\beta}_2)$

EXERCISES 3h

1. Suppose that $\mathbf{X}$ does not have full rank, and let $\hat{\boldsymbol{\beta}}_i$ $(i = 1, 2)$ be any two solutions of the normal equations. Show directly that

$$\|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1\|^2 = \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_2\|^2.$$

$\Sigma^T \Sigma$ both sides.

2. If the columns of $\mathbf{X}$ are linearly dependent, prove that there is no matrix $\mathbf{C}$ such that $\mathbf{CY}$ is an unbiased estimate of $\boldsymbol{\beta}$.

$\forall$ unbiased est of β, ask for $\beta + N(X)$. $\underbrace{X(\hat{\beta}_1 - \hat{\beta}_2) = 0}$

3.9.2 Estimable Functions

Must be the case, else Normal eq can be violated.

Since $\hat{\boldsymbol{\beta}}$ is not unique, $\boldsymbol{\beta}$ is not estimable. The question then arises: What can we estimate? Since each element θ_i of $\boldsymbol{\theta}$ $(= \mathbf{X}\boldsymbol{\beta})$ is estimated by the ith element of $\hat{\boldsymbol{\theta}} = \mathbf{PY}$, then every linear combination of the θ_i, say $\mathbf{b}'\boldsymbol{\theta}$, is also estimable. This means that the θ_i form a linear subspace of estimable functions, where $\theta_i = \mathbf{x}_i'\boldsymbol{\beta}$, $\mathbf{x}_i'$ being the ith row of $\mathbf{X}$. Usually, we define estimable functions formally as follows.

***Definition* 3.1** *The parametric function* $\mathbf{a}'\boldsymbol{\beta}$ *is said to be estimable if it has a linear unbiased estimate,* $\mathbf{b}'\mathbf{Y}$*, say.*

We note that if $\mathbf{a}'\boldsymbol{\beta}$ is estimable, then $\mathbf{a}'\boldsymbol{\beta} = E[\mathbf{b}'\mathbf{Y}] = \mathbf{b}'\boldsymbol{\theta} = \mathbf{b}'\mathbf{X}\boldsymbol{\beta}$ identically in $\boldsymbol{\beta}$, so that $\mathbf{a}' = \mathbf{b}'\mathbf{X}$ or $\mathbf{a} = \mathbf{X}'\mathbf{b}$ (A.11.1). Hence $\mathbf{a}'\boldsymbol{\beta}$ is estimable if and only if $\mathbf{a} \in \mathcal{C}(\mathbf{X}')$.

EXAMPLE 3.8 If $\mathbf{a}'\boldsymbol{\beta}$ is estimable, and $\hat{\boldsymbol{\beta}}$ is any solution of the normal equations, then $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is unique. To show this we first note that $\mathbf{a} = \mathbf{X}'\mathbf{b}$ for some $\mathbf{b}$, so that $\mathbf{a}'\boldsymbol{\beta} = \mathbf{b}'\mathbf{X}\boldsymbol{\beta} = \mathbf{b}'\boldsymbol{\theta}$. Similarly, $\mathbf{a}'\hat{\boldsymbol{\beta}} = \mathbf{b}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{b}'\hat{\boldsymbol{\theta}}$, which is unique. Furthermore, by Theorem 3.2, $\mathbf{b}'\hat{\boldsymbol{\theta}}$ is the BLUE of $\mathbf{b}'\boldsymbol{\theta}$, so that $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is the BLUE of $\mathbf{a}'\boldsymbol{\beta}$. □