since $\mathbf{P}_\Omega \mathbf{X}\boldsymbol{\beta}_0 = \mathbf{X}\boldsymbol{\beta}_0$ and $\mathbf{A}\boldsymbol{\beta}_0 = \mathbf{c}$. Therefore, canceling $\mathbf{X}\boldsymbol{\beta}_0$ and multiplying both sides by $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ leads to $\hat{\boldsymbol{\beta}}_H$ of (3.38). Clearly, this gives a minimum as $\|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_H\|^2 = \|\tilde{\mathbf{Y}} - \mathbf{X}\hat{\boldsymbol{\gamma}}_H\|^2$.

$Y \sim I\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} + \varepsilon$ st $(1\ 1\ 1)\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \pi$

EXERCISES 3g

1. (a) Find the least squares estimates of α and β in Example 3.5 using the two approaches described there. What is the least squares estimate of γ?

   (b) Suppose that a further constraint is introduced: namely, $\alpha = \beta$. Find the least squares estimates for this new situation using both methods.

$\hat{\lambda} + A^T(AA^T)^{-1}(c - A\hat{\lambda})$

By Pythagoras, NTS $Y - \hat{Y} \perp \hat{Y} - \hat{Y}_H$

2. By considering the identity $\mathbf{Y} - \hat{\mathbf{Y}}_H = \mathbf{Y} - \hat{\mathbf{Y}} + \hat{\mathbf{Y}} - \hat{\mathbf{Y}}_H$, prove that

$$\|\mathbf{Y} - \hat{\mathbf{Y}}_H\|^2 = \|\mathbf{Y} - \hat{\mathbf{Y}}\|^2 + \|\hat{\mathbf{Y}} - \hat{\mathbf{Y}}_H\|^2.$$

→ Holds for any $\hat{Y}_H$ actually by LS. ⇒ solutions $\hat{\lambda} - Y$

3. Prove that

$$\text{Var}[\hat{\boldsymbol{\beta}}_H] = \sigma^2 \left\{ (\mathbf{X}'\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}' \left[\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'\right]^{-1} \mathbf{A}(\mathbf{X}'\mathbf{X})^{-1} \right\}.$$

Hence deduce that → linear Xform of $\hat{\beta}$.

$$\text{var}[\hat{\beta}_{Hj}] \leq \text{var}[\hat{\beta}_j],$$

where $\hat{\beta}_{Hj}$ and $\hat{\beta}_j$ are the jth elements of $\hat{\boldsymbol{\beta}}_H$ and $\hat{\boldsymbol{\beta}}$, respectively.

4. Show that

$$\|\mathbf{Y} - \hat{\mathbf{Y}}_H\|^2 - \|\mathbf{Y} - \hat{\mathbf{Y}}\|^2 = \sigma^2 \hat{\boldsymbol{\lambda}}_H' \left(\text{Var}[\hat{\boldsymbol{\lambda}}_H]\right)^{-1} \hat{\boldsymbol{\lambda}}_H.$$

$n \geq p \geq q$ ⇐ Apply (2), then 3.37 left-multiplied by X.

5. If $\mathbf{X}$ is $n \times p$ of rank p and $\mathbf{B}$ is $p \times q$ of rank q, show that rank$(\mathbf{XB}) = q$.

rank$(XB) \leq q$, NTS other direction: $N(X) = \emptyset$ so 0

3.9 DESIGN MATRIX OF LESS THAN FULL RANK

3.9.1 Least Squares Estimation

dim $C(XB) = q$

When the techniques of regression analysis are used for analyzing data from experimental designs, we find that the elements of $\mathbf{X}$ are 0 or 1 (Chapter 8), and the columns of $\mathbf{X}$ are usually linearly dependent. We now give such an example.

EXAMPLE 3.6 Consider the randomized block design with two treatments and two blocks: namely,

$$Y_{ij} = \mu + \alpha_i + \tau_j + \varepsilon_{ij} \qquad (i = 1, 2;\ j = 1, 2),$$