In conclusion, the simplest approach to estimable functions is to avoid them altogether by transforming the model into a full-rank model!

EXERCISES 3i

a LS est $= (X^T X)^- X^T X\beta$ + (linear Σ)

1. Prove that $\mathbf{a}'E[\hat{\boldsymbol\beta}]$ is an estimable function of $\boldsymbol\beta$.

$E[a^T\hat\beta] = a^T\alpha$

2. If $\mathbf{a}_1'\boldsymbol\beta, \mathbf{a}_2'\boldsymbol\beta, \ldots, \mathbf{a}_k'\boldsymbol\beta$ are estimable, prove that any linear combination of these is also estimable.

$\sum \theta_i h_i \in C(X^T)$

3. If $\mathbf{a}'\hat{\boldsymbol\beta}$ is invariant with respect to $\hat{\boldsymbol\beta}$, prove that $\mathbf{a}'\boldsymbol\beta$ is estimable.

4. Prove that $\mathbf{a}'\boldsymbol\beta$ is estimable if and only if

$a^T \in C(X^T)$ so $a^T = b^T X^T$

$$\mathbf{a}'(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{X} = \mathbf{a}'.$$

(Note that $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$.) $\Leftarrow$ obv. by $C(X)$ $\Rightarrow$ by $a'\beta = b'X\beta$

5. If $\mathbf{a}'\boldsymbol\beta$ is an estimable function, prove that

$a' \in C(X^T)$

$$\text{Var}[\mathbf{a}'\hat{\boldsymbol\beta}] = \sigma^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^-\mathbf{a}.$$

by $\hat\beta = (X^T X)^- X^T X\beta + (4)$

6. Prove that all linear functions $\mathbf{a}'\boldsymbol\beta$ are estimable if and only if the columns of $\mathbf{X}$ are linearly independent.

$C(X^T) = \mathbb{R}^p$

3.9.3 Introducing Further Explanatory Variables

If we wish to introduce further explanatory variables into a less-than-full-rank model, we can, once again, reduce the model to one of full rank. As in Section 3.7, we see what happens when we add $\mathbf{Z}\boldsymbol\gamma$ to our model $\mathbf{X}\boldsymbol\beta$. It makes sense to assume that $\mathbf{Z}$ has full column rank and that the columns of $\mathbf{Z}$ are linearly independent of the columns of $\mathbf{X}$. Using the full-rank model

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol\alpha + \mathbf{Z}\boldsymbol\gamma + \boldsymbol\varepsilon,$$

where $\mathbf{X}_1$ is $n \times r$ of rank r, we find that Theorem 3.6(ii), (iii), and (iv) of Section 3.7.1 still hold. To see this, one simply works through the same steps of the theorem, but replacing $\mathbf{X}$ by $\mathbf{X}_1$, $\boldsymbol\beta$ by $\boldsymbol\alpha$, and $\mathbf{R}$ by $\mathbf{I}_n - \mathbf{P}$, where $\mathbf{P} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1$ is the unique projection matrix projecting onto $\mathcal{C}(\mathbf{X})$.

3.9.4 Introducing Linear Restrictions

Referring to Section 3.8, suppose that we have a set of linear restrictions $\mathbf{a}_i'\boldsymbol\beta = 0$ $(i = 1, 2\ldots, q)$, or in matrix form, $\mathbf{A}\boldsymbol\beta = \mathbf{0}$. Then a realistic assumption is that these constraints are all estimable. This implies that $\mathbf{a}_i' = \mathbf{m}_i'\mathbf{X}$ for some $\mathbf{m}_i$, or $\mathbf{A} = \mathbf{MX}$, where $\mathbf{M}$ is $q \times n$ of rank q [as $q = \text{rank}(\mathbf{A}) \le \text{rank}(\mathbf{M})$