# Exercises 2

1 Let $f(y_1, y_2) = k^{-1} \exp\left(-\frac{2y_1^2 + y_2^2 + 2y_1y_2 - 22y_1 - 14y_2 + 65}{2}\right)$

(a) Show for appropriate $k$ this is the density of a normal.

Soln. $(y-\mu)^T \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{\Sigma} (y-\mu)$

$= 2y_1^2 + y_2^2 + 2y_1y_2 + \mu^T\Sigma\mu - 2y^T\Sigma\mu$

From $y_1, y_2$ terms:

$$4\mu_1 + 2\mu_2 = +22$$

$$2\mu_1 + 2\mu_2 = +14$$

$$\Rightarrow \mu_1 = 4, \mu_2 = 3$$

$\mu^T\Sigma\mu = 49$

$$k = \exp\left(-\frac{65-49}{2}\right)(2\pi)^{-1}\frac{\det^{1/2}\Sigma}{1}$$

(b) Find $\mu, \Sigma$.

Soln. See above.

2. Let $U$ have density $g$ and $A$ be nonsingular. Define $Y = A(U+c)$. Find the density of $y$.

Soln. By Radon-Nikodym, density $f$ of $Y$ is the unique fn (if it exists) satisfying for all Borel $S$ that:

$$\mathbb{P}\{Y \in S\} = \int_S f\,dy$$

Note the transform $t(U) = Y$ is invertible and bijective, so

$$\mathbb{P}\{Y \in S\} = \mathbb{P}\{U \in t^{-1}S\} = \int_{t^{-1}S} g\,du.$$

$$= \int_{t^{-1}S} g \circ t^{-1} \circ t\,du = \int_S g \circ t^{-1}\,dt_* u$$

The pushforward is given by

$$dt_* \mu = \frac{|\det(t^{-1})'| \, dy}{= \sqrt{|\det A|}}$$

Viewing these as differential n-forms.

3. Show the matrix

(a)

$$A = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

is PD for $\rho > -1/2$.

Soln. Prove by Schur complement.

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \text{ with } W = 1 \text{ PD},$$

$$\text{Then } X - YW^{-1}Z = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} - \rho^2$$

Consider the vector $(a, b)$ on this:

$$(a^2+b^2)(1-\rho^2) + \rho(1-\rho)2ab$$

$$(1-\rho)[(1+\rho)(a^2+b^2) + \rho 2ab]$$

Since $a^2+b^2 \geq 2ab$ by AM-GM,

for $\rho > -1/2$,

$(1-\rho) \geq 0$

$((1+\rho)(a^2+b^2) + 2\rho ab) \geq 0$

(b) Let $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Find $\Sigma^{1/2}$.

Soln. $\det(\Sigma - \lambda I) = 0$

$= (1-\lambda)^2 - \rho^2$

$= (1-\lambda-\rho)(1-\lambda+\rho)$

$\lambda = 1-\rho,\ 1+\rho,$

Solve for $v_i$, $(\Sigma - \lambda_i I)v_i = 0$,

$\begin{pmatrix} -\rho & \rho \\ \rho & -\rho \end{pmatrix} v = 0 \rightarrow (1,1)$

$\begin{pmatrix} \rho & \rho \\ \rho & \rho \end{pmatrix} v = 0 \rightarrow (1,-1)$

$$\Sigma = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

So $\Sigma^{1/2} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

assuming $|\rho| \leq 1$ (for $\mathbb{R}$-valued solutions)

## Exercises 2b

1. Let $X \sim N(\mu, \Sigma)$ with $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

and $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_x \sigma_y \rho \\ \sigma_x \sigma_y \rho & \sigma_y^2 \end{pmatrix}$.

What is its MGF?

Soln. $\mathbb{E} \exp(t \cdot X) =$

$$\exp\left(\frac{1}{2} t \cdot \mu + t^T \Sigma t / 2\right)$$

$$= \exp\left(\frac{1}{2}\left(t_1 \mu_1 + t_2 \mu_2 + t_1^2 \sigma_x^2 + 2 \sigma_x \sigma_y \rho t_1 t_2 + t_2^2 \sigma_y^2\right)\right)$$

2. If $Y \sim N(\mu, \Sigma)$ show $Y_i \sim N(\mu_i, \sigma_{ii})$.

Soln. By choice of $e_i^T Y$ as the Normal. Expand MGF.

3. Let $\mu = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

If $Y \sim N(\mu, \Sigma)$ then find the joint of $Z_1 = Y_1 + Y_2 + Y_3$ and $Z_2 = Y_1 - Y_2$.

Soln. Notice $Z = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{C} Y$

so $Z \sim N(C\mu, C\Sigma C^T)$.

4. Show $a^T Y \perp\!\!\!\perp b^T Y$ for $a \perp b$ and $Y \sim N(\mu, I)$.

Soln. Factorize the MGF:

$$M_{(a+b)^T Y}(t) = \exp\left(\tfrac{1}{2}(a+b)^T \mu + \tfrac{1}{2}(a+b)^T I (a+b)\right)$$

Since cross terms cancel,

$$M_{(a+b)^T Y}(t) = M_{a^T Y}(t)\, M_{b^T Y}(t)$$

5. Observe $(X_i, Y_i)$ bivariate normals, $n$ indep observations. Find the distrib of $(\bar{X}, \bar{Y})$.

Soln. If $(X_i, Y_i) \sim N(\mu, \Sigma)$ then its avg is $N(\mu, \Sigma/n)$

6. If $Y_1 + Y_2$, $Y_1 - Y_2$ are indep std normals, what is $Y = (Y_1, Y_2)$

Soln. Let $X_1 = Y_1 + Y_2$, $X_2 = Y_1 - Y_2$.

$$M_{X_1 - X_2}(t) = M_{X_1}(t)\, M_{-X_2}(t)$$

$$= \exp(t^2)$$

Same for $X_1 + X_2$. $\mathbb{E}[(X_1 - X_2)(X_1 + X_2)]$
$= 0$

But $t_1(X_1 - X_2) + t_2(X_1 + X_2)$

$$= \underbrace{(t_1 + t_2)X_1 - (t_2 - t_1)X_2}$$

Jointly gaussian.

So $Y \sim N(0, \Sigma)$ too.

7. Let $f(x,y) = \frac{1}{2\pi} \exp\left(\frac{-1}{2}(x^2+y^2)\right)$

$$\times \left[1 - \frac{xy}{(1+x^2)(1+y^2)}\right].$$

What are the marginals?

Soln. By symm. suff. to show for $X$.

$$\int f\,dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left(1 - \frac{xy}{(1+x^2)(1+y^2)}\right) dy$$

$$= \gamma(x)\left[\int \gamma(y)\,dy - \int y \frac{\gamma(y)\,x}{(1+x^2)(1+y^2)}\,dy\right]$$

$\gamma$ gaussian density.

But left term is 1
and right term is odd.

8. Let $Y_i \overset{iid}{\sim} N(0,1)$. Show
$\bar{Y} \perp\!\!\!\perp \sum_i (Y_i - \bar{Y})^2$.

Soln. Define $\bar{X} = (\bar{Y}, Y_1 - \bar{Y}, \dots, Y_n - \bar{Y})$.

$$M_{\bar{X}}(t) = \mathbb{E}\exp\left(t_0 \bar{Y} + (Y - \bar{Y})^T t\right)$$

$$= \mathbb{E}\exp\left(t^T Y - \frac{1_n^T Y}{n}\left(1_n^T t - t_0\right)\right)$$

$$= \mathbb{E}\exp\Bigg(\underbrace{\left(t - \frac{(1_n^T t - t_0)}{n} 1_n\right)}_{\alpha}^T Y\Bigg)$$

$$= \exp\left(\frac{1}{2}\|\alpha\|_2^2\right)$$

$$= \exp\left(\frac{1}{2}\left(\|t\|_2^2 - \frac{2(1_n^T t)^2}{n} + t_0\frac{1_n^T t}{n} + \frac{(1_n^T t - t_0)^2}{n}\right.\right.$$

$$= \exp\left(\frac{1}{2}\left(\|t\|_2^2 - \frac{(1_n^T t)^2}{n} - t_0 \frac{1_n^T t}{n} + \frac{t_0^2}{n}\right)\right)$$

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Scratch above. Consider

$$C = \begin{pmatrix} & 1/n \, 1_n^T \\ & I_n - \frac{1_n 1_n^T}{n} \end{pmatrix}$$

Then $\bar{X} = CY$

and $MGF_X(t) = \exp\left(\frac{1}{2} t^T C C^T t\right)$

$$= \exp\left(\frac{1}{2} t^T \begin{pmatrix} 1/n & 0 \\ 0 & I_n - \frac{1_n 1_n^T}{n} \end{pmatrix} t\right)$$

$$= MGF_{\bar{y}}(t_0) \, MGF_{y-\bar{y}}(t_{1:n})$$

$\rightarrow \underline{\text{\checkmark}}$.

9. Let $X \sim N(0, I_3)$ and define

$$Y = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} X.$$

What is the distrib of $Y$?

Soln. Since $X$ ~~transform~~ is ortho,

$$Y \sim N(0, I_n).$$

## Exercises 2e

1. Are pairwise indep. normals w/ MVN joint mutually independent?

Soln. Yes, b/c cov $= 0$ off-diag.

2. Let $Y \sim N(\mu \mathbb{1}_n, \Sigma)$ where $\Sigma = (1-\rho) I_n + \rho J_n$ and $J_n = \mathbb{1}_n \mathbb{1}_n^T$. Let $\rho > \frac{-1}{n-1}$.

When $\rho = 0$, $\bar{Y}$ and $Y - \bar{Y}$ are indep. What about $\rho \neq 0$?

Soln.
$$\begin{pmatrix} \bar{Y} \\ Y - \bar{Y} \end{pmatrix} = \begin{pmatrix} 1_n^T / n \\ I - \frac{1_n 1_n^T}{n} \end{pmatrix} Y$$

So by thm 2.2,

indep if

$$0 = \frac{1_n^T}{n} \left[ (1-\rho) I + \rho J \right] (I - J/n)$$

$$= \underbrace{\frac{\rho 1_n^T}{n} J}_{\rho 1_n^T} (I - J/n)$$

$\rightarrow$ Not 0 $\rightarrow$ Not indep.

3. Suppose $Y \sim N(\mu, \Sigma)$ with

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.$$ For

what $\rho$ values are

$$Z_1 = Y_1 + Y_2 + Y_3$$

$$Z_2 = Y_1 - Y_2 - Y_3$$

- indep?

Soln. $Z = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}}_{A} Y$

$Z_1 \perp\!\!\!\perp Z_2$ when $\underbrace{A \Sigma A^T}_{=} = \cancel{\ } 0$ off diag.

$$\begin{pmatrix} 1+\rho & 2\rho+1 & 1+\rho \\ 1-\rho & 1 & -\rho-1 \end{pmatrix} A^T$$

$$= \begin{pmatrix} 3+4\rho & -1-2\rho \\ -1-2\rho & 1 \end{pmatrix} \rightarrow \rho = -1/2$$

# Exercises 2d

1. Show that the MGF of $\sum_{i=1}^{n} d_i Z_i^2$ for $Z_i \sim N(0,1)$ is $\prod (1-2td_i)^{-1/2}$

Soln. Follows by $Z_1^2$ MGF $(1-2t)^{-1/2}$ w/ independence.

2. Let $Y \sim N(0, I_n)$ and $A$ be symm.

(a). What is the MGF of $Y$?

Soln. $MGF_y(t) = \int p(y) \exp(y^T A y t)\, dy$

$$= \frac{1}{(2\pi)^{n/2}} \int \exp\left(-\frac{1}{2} y^T (I - 2At) y\right) dy$$

For small $t$, $I \succeq 2At$ so this is a Normal tilt, and the integral is the partition, $\det(I - 2At)^{-1/2}$.

(b) If $A$ is idempotent & rank $r$, how does the answer improve?

Soln. $MGF_Y(t) = (1-2t)^{-r/2}$ by Corr. 2.8 & MGF of $\chi^2_r$.

(c) What happens if $Y \sim N(0, \Sigma)$?

Soln. By eigen $\Sigma = Q \Lambda Q^T$ w/ $Q$ full rank and unitary. Using

Let $\Sigma^{1/2} = Q \Lambda^{1/2} Q^T$.

If $\Lambda$ is not full rank, defn $\Lambda^{-1/2}$ w/ $0^{-1/2} := 0$. then let $Z = \Sigma^{-1/2} Y$

so $Z \sim N\left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\right)$

where $r = \operatorname{rank} \Sigma$. As

$$Y^T A Y = Z^T (\Sigma^{1/2})^T A \Sigma^{1/2} Z$$

We can truncate matrices $\Sigma^{1/2}$, A, and wlog take $r = n$.
So then $\Sigma^{1/2}$ is full rank. By 2a,

symm, $\Sigma^{T/2} = \Sigma^{1/2}$

$$MGF_{Y^TAY}(t) = MGF_{Z^T(\Sigma^{1/2})^T A (\Sigma^{1/2}) Z}(t)$$

$$= \det\left(I - 2(\Sigma^{1/2})^T A \Sigma^{1/2} t\right)$$

$$= \det\left(\Sigma^{+1/2}\Sigma^{-1/2} - 2\Sigma^{\frac{1}{2}} A \Sigma^{1/2} t\right)$$

$$= \det(\Sigma^{-1/2}) \underbrace{\det(\cdots)\det(}_{x=1}\Sigma^{1/2})$$

$$= \det(I - 2A\Sigma t)$$

3. Find a, b st

$$a(Y_1 - Y_2)^2 + b(Y_1 + Y_2)^2 \sim \chi^2_2.$$

Soln. $= (a+b)\underbrace{(Y_1^2 + Y_2^2)}_{\to \chi^2_2}$

$+ (a+b)(Y_1 Y_2)$

$\to a = b = 1/2$. Only soln. by inspecting MGF.

4. Let $Y_i$ be iid $N(0,1)$. $i \in [n]$

Defn $Y_{n+1} \equiv Y_1$.

When is $\sum_{i=1}^{n} (Y_{i+1} - Y_i)^2$

$\chi^2$-distributed?

Soln. Sum is $Y^T A^T A Y$ for

$A = \text{band}(0 \to 1, 1 \to -1)$

(i.e. the matrix w/ terms

$$a_{ij} = \begin{cases} 1 & \text{if } j-i \equiv 0 \bmod n \\ -1 & \text{if } j-i \equiv 1 \bmod n \end{cases})$$

Then $A^T A = \text{band}(-1 \to -1, 0 \to 2, 1 \to -1)$.

$\frac{1}{3}(A^T A)$ is rank-2 idempotent but by inspection of the $(n,1)$-st entry, $(A^T A)^2 \neq (A^T A)\alpha$ for any $\alpha$. for any $n > 3$.

5. Let $A, B$ be symmetric. Find joint MGF of $Y^T A Y$, $Y^T B Y$. Use this to characterize when the two forms are independent.

Soln. Use the same procedure as in 2(A), yielding

$$MGF_{(Y^TAY,\, Y^TBY)}(s,t) = \det(I - 2sA - 2tB)^{-1/2}.$$

If $AB = 0$,

Notice $(I - 2sA)(I - 2tB)$

$$= I - 2sA - 2tB + \underbrace{4stAB}_{0}$$

$$= I - 2sA - 2tB.$$

But then MGF of pair = product of MGFs (det distributes over product).

$\rightarrow$ indep.

# Miscellaneous Exercises 2.

1. Let $\varepsilon \sim N(0, I_3)$ and $Y_{i+1} = \rho Y_i + \varepsilon_i$ with $Y_0 \sim N(0, \sigma_0^2)$.

(a) Find $\operatorname{var} Y_n$

Soln. Note $Y_0, \varepsilon_i$ are mutually indep.

$$\operatorname{var} Y_i = \rho^2 \operatorname{var} Y_{i-1} + \sigma^2$$

$$\operatorname{cov}(Y_i, Y_{i+1}) = \rho \operatorname{var} Y_i$$

$$Y_{i+2} = \rho^2 Y_i + \rho \varepsilon_{i+1} + \varepsilon_{i+2}$$

$$\operatorname{cov}(Y_i, Y_{i+2}) = \rho^2 \operatorname{var} Y_i$$

mirror

$$\operatorname{var} Y = \begin{pmatrix} \rho^2\sigma_0^2 + \sigma^2 & & \\ \rho \cdots & \rho^4\sigma_0^2 + (\rho^2+1)\sigma^2 & \\ \rho^2 \cdots & \rho \cdots & \rho^6\sigma_0^2 + (\rho^4+\rho^2+1)\sigma^2 \end{pmatrix}$$

(b) What is the distrib of $Y$?

Soln. $Y$ is MVN b/c

$Y_i$ are jointly Normal.

$a^T Y = b^T \begin{pmatrix} \varepsilon \\ Y_0 \end{pmatrix}$ for some $b^T$

and any $a^T$.

2. Let $Y \sim N(0, I_n)$. Define:

$X = AY$, $U = BY$, $V = CY$

for matrices $A, B, C$. Suppose

$\mathrm{cov}(X, U) = \mathrm{cov}(X, V) = 0$.

Then show $X \perp\!\!\!\perp U+V$.

Soln. Since everything is normal,

$X \perp\!\!\!\perp U$, $X \perp\!\!\!\perp V$. Thus

$X \perp\!\!\!\perp U+V$.

3. $i \in [n]$, $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$.

Show $\bar{Y} \perp\!\!\!\perp \underbrace{\sum_i (Y_i - Y_{i+1})^2}_{Q}$.

Soln. $\bar{Y} = 1/n \, \mathbb{1}^T Y$

$$Q = Y^T A^T A Y,$$

$$A = \text{circulant}(0 \to 1, 1 \to -1)$$

Neat: $A^T = \text{circulant}(-1 \to -1, 0 \to 1)$

Let $\mathcal{F}$ be the DFT. Then

If $c_A$ = first column of $A$:

$$A = \mathcal{F}^{-1} \operatorname{diag}(\mathcal{F} c_A) \mathcal{F}$$

$$\therefore A^T A = \mathcal{F}^{-1} \left( \operatorname{diag}(\mathcal{F} c_A) \times \operatorname{diag}(\mathcal{F} c_{A^T}) \right) \mathcal{F}$$

$= \text{circulant}(c_A * c_{A^T}) \to$ convolution.

$\therefore A^T A = \text{circulant}(-1$

$$C_A = \begin{array}{ccccc} -1 & 0 & 1 & \cdots & n \\ \hline 0 & 1 & -1 & 0 & \end{array}$$

$$\text{flip} = \;—\; 0 \;\; -1 \;\; 1 \;\; 0 \;—$$

$$C_A^T = \;—\; 0 \;\; -1 \;\; 1 \;\; 0 \;—$$

idx $-1$: | idx $0$: | idx $1$
--- | --- | ---
$\begin{array}{ccc} -2 & -1 & 0 \\ \hline -1 & 1 & \text{flip} \\ & -1 \;\; \cancel{0} & C_A^T \\ \hline & -1 & \end{array}$ | $\vdots$ <br> $2$ | $-1$

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$$A^T A = \text{circulant}\left(\begin{array}{l} -1 \to -1 \\ 0 \to 2 \\ 1 \to -1 \end{array}\right)$$

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Above is neat but unrelated to the problem lol

By Thm 2.5, $\underbrace{AY \perp\!\!\!\perp \bar{Y}}_{\Rightarrow \|AY\|_2^2 \perp\!\!\!\perp \bar{Y}}$

If cov is 0,

$$\mathrm{cov}\left(\frac{1_n^T}{n} Y, AY\right)$$

$$= \frac{1_n^T}{n} \underbrace{\mathrm{cov}(Y, Y)}_{\sigma^2 I} A^T$$

$$= 0 \text{ b/c } A 1_n = 0$$

4. If $X, Y$ are indep. MVN

then show $aX + bY$ is

Soln. $\begin{pmatrix} X \\ Y \end{pmatrix}$ is MVN, so linear $\begin{pmatrix} aI & 0 \\ 0 & bI \end{pmatrix}$ xform is.

5. If $Y \sim N(0, I_n)$, $a \neq 0$ show $Y^T Y \mid a^T Y = 0 \sim \chi^2_{n-1}$.

Soln. Let $B^T \in \mathbb{R}^{n-1 \times n}$ be the orthonormal basis completion after $a$, wlog $\|a\|_2 = 1$. Then

$Q = [B \ \ a]$ is unitary w/

$$Q^T Y = \begin{bmatrix} B^T \\ a^T \end{bmatrix} Y.$$

$$Y^T Y = Y^T Q Q^T Y$$

$$= \|Q^T Y\|_2^2 = \left\| \begin{matrix} B^T Y \\ a^T Y \end{matrix} \right\|_2^2.$$

Conditioned on $a^T Y = 0$, this is $\|B^T Y\|_2^2$. By constr. $B^T Y \sim N(0, I_{n-1})$

So $\chi^2_{n-1}$ holds under nonstandard.

6. Let $Y \sim N(\mu 1_n, \Sigma)$ where $\Sigma = (1-\rho) I_n + \rho J_n$ w/ $\rho > -1/_{n-1}$.

Show $s^2 = \sum_i (Y_i - \bar{Y})^2 / 1-\rho \sim \chi^2_{n-1}$

Soln.

$$s^2 (1-\rho) = \| \underbrace{(I_n - \tfrac{1}{n} J_n) Y}_{} \|_2^2$$

$$\sim N(0, \Sigma') \text{ where}$$

$$\Sigma' = (1-\rho) I_n + \left(\rho + \frac{\rho - 1}{n} - \rho\right) J_n$$

$$= (1-\rho)\left[I_n - \tfrac{1}{n} J_n\right]$$

Above is just $(I_n - \frac{1}{n} J_n)(\Sigma)$, missing right mult by $(I_n - \frac{1}{n} J_n)$, but idempotent

So $S^2 = \| N(0, I_n - \frac{1}{n} J_n) \|_2^2$

by cancellation of $1-\rho$ terms.

Cov is rank $n-1$ & idempotent & symm, so $S^2 \sim \chi^2_{n-1}$.

7. Let $Y_i \overset{iid}{\sim} N(\mu, \Sigma)$.

Show $\frac{1}{n-1} \sum_k (Y^{(k)} - \bar{Y})(Y^{(k)} - \bar{Y})^T$

is an unbiased est. of $\Sigma$.

Soln. Inspect $ij$-th term:

$$\Sigma_{ij} = \frac{1}{n-1} \sum_K (Y_i^{(k)} - \bar{Y}_i)(Y_j^{(k)} - \bar{Y}_j)$$

Each term is identically distributed:

$$\Sigma_{ij} = \frac{n}{n-1} \mathbb{E}(Y_i^{(1)} - \bar{Y})(Y_j^{(1)} - \bar{Y})$$

$$= \frac{n}{n-1} \mathbb{E}\left[ \left( \frac{n-1}{n} Y_i^{(1)} - \frac{1}{n} \sum_{k \neq i} Y_i^{(k)} \right) \times \left( \cdots j \cdots \right) \right]$$

$$= \frac{1}{n(n-1)} \mathbb{E}\left[ \underbrace{\left( (n-1) Y_i^{(1)} - \sum_{k \neq 1} Y_i^{(k)} \right) \left( (n-1) Y_j^{(1)} - \sum_{k \neq 1} Y_j^{(k)} \right)} \right]$$

$$(n-1)^2 \mathbb{E}\left[ Y_i^{(1)} Y_j^{(1)} \right]$$

$$- (n-1) \mathbb{E} Y_i^{(1)} \mathbb{E} \sum Y_j^{(k)}$$

$$- (n-1) \mathbb{E} Y_j^{(1)} \mathbb{E} \sum^{k \neq 1} Y_i^{(k)}$$

$$+ \sum_{k, m > 1} \mathbb{E} Y_i^{(k)} Y_j^{(m)} \quad k \neq 1$$

$$= \frac{1}{n(n-1)} \Big[ \mathbb{E}\, Y_i^{(1)} Y_j^{(1)} (n-1)^2$$

$$- \cancel{3}\, 2 (n-1)^2 \mu_i \mu_j$$

$$+ (n-1)\, \mathbb{E}\, Y_i^{(1)} Y_j^{(1)}$$

$$+ \underbrace{n(n-1) \mu_i \mu_j}_{= (n-1)^2 \mu_i \mu_j + (n-1) \mu_i \mu_j} \Big]$$

$$= \mathbb{E}\, Y_i^{(1)} Y_j^{(1)} - \mu_i \mu_j$$

$$= \mathrm{cov}(Y_i^{(1)}, Y_j^{(1)})$$

$$= \Sigma_{ij}$$

8. Let $Y \sim N(0, I_n)$ and $AB = BA = 0$ for symm. Idempotent matrices $A, B$. Show quadratic forms over $A, B, I - A - B$ are indep $\chi^2$.

Soln. For $A, B$, this follows by Thm 2.7 + Ex 2.12.

$A + B$ is still symm, idempotent. So by Ex 2.11 $I - A - B$ is symm, idemp, and makes indep $\chi^2$ as well from $A + B$.

We check indep w/ constituents $A, B$:

$$A(I-A-B) = \underbrace{\underbrace{A - A^2}_{A}}_{0} - \underbrace{AB}_{0}.$$

Same reasoning for $B$.

9. Let $(X_i, Y_i) \overset{iid}{\sim} N\left(\begin{pmatrix}\mu_x \\ \mu_y\end{pmatrix}, \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}\right)$.

Define $W = \begin{pmatrix} X \\ Y \end{pmatrix}$

(a) Find distrib. of $W$.

Soln. All bivariate are jointly Gaussian by $\perp\!\!\!\perp$ so

$$W \sim N\left(\begin{pmatrix}\mu_x 1_n \\ \mu_y 1_n\end{pmatrix}, \begin{pmatrix}\sigma_x^2 I_n & \rho\sigma_x\sigma_y I_n \\ \rho\sigma_x\sigma_y I_n & \sigma_y I_n\end{pmatrix}\right)$$

(b) Find $Y|X$.

Soln. By generic MVN conditioning,

$Y|X \sim$

$$N\Big(\mu_y 1_n + \rho\sigma_x\sigma_y I_n (\sigma_x^2 I_n)^{-1} (X - \mu_x 1_n),$$

$$\sigma_y^2 I_n - \rho^2\sigma_x^2\sigma_y^2 (\sigma_x^2)^{-1} I_n$$

$$= N\Big(\mu_y 1_n + \frac{\rho\sigma_y}{\sigma_x}(X - \mu_x 1_n),$$

$$\sigma_y^2(1-\rho^2) I_n\Big)$$

10. Suppose $Y \sim N_2(0, \Sigma)$.

Show $Y^T\Sigma^{-1}Y - Y_1^2/\Sigma_{11} \sim \chi_1^2$

Soln. Define $z = \Sigma^{-1/2} Y$.

So $z \sim N(0, I_2)$ & orthogonal

& stat is $z^T z - z_1^2 = z_2^2 \sim \chi^2_1$

11. Let $Y_i$ be MA(1) w/ parameter $\phi$ & noise $N(0, \sigma_\varepsilon^2)$.

How is $Y$ distributed?

Soln. $Y_i = \varepsilon_i + \phi Y_{i-1}$

$$\therefore Y_i = \sum_{j=0}^{i} \phi^{i-j} \varepsilon_j$$

Since each $Y_i$ is finite sum of $n$ Gaussians $\{\varepsilon_j\}_j$, all $Y_i$ are JG.

$\rightarrow$ MVN.

$$Y = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}}_{A} \varepsilon$$

$$\sim N(0, \sigma^2 A A^T)$$

13. Let $Y_i \overset{iid}{\sim} N(0,1)$ for $i \in [3]$ and find the MGF of

$$2(Y_1 Y_2 - Y_2 Y_3 - Y_3 Y_1)$$

and show an equiv. $\chi^2$ representation.

Soln: The above is

$$(Y_1 + Y_2 - Y_3)^2 - Y_1^2 - Y_2^2 - Y_3^2$$

$$y^T \Big( \underbrace{-\tfrac{I}{3} + \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}}_{A} \Big) y$$

So by (2.10) it's the eigenvalues of the above matrix, $\underline{2, -1, -1}$

so $y^T A y \sim \sum \lambda_i \chi^2_1 \quad \lambda_i$

13. Use Normal characterization through joint Gaussianity to show facts about its distribution

(a). $z_i \overset{iid}{\sim} N(0,1)$ then $z \sim N(0, I)$.

Soln. $a^T z = \sum a_i z_i \sim N(0, \|a\|_2^2)$ as desired.

(b) Show its MGF.

Sln. As $t^T X \sim N(t^T\mu, t^T\Sigma t)$

by characterization, and normal

MGF is the usual

$$\exp\left(s\mu + \frac{1}{2} s^2 \sigma^2\right),$$

apply to $t^T X$, $t$ fixed, choose $s=1$.

(c). Find its density.

Sln. $Y \sim N(\mu, \Sigma)$.

$$\Sigma^{-1/2}(Y - \mu) \text{ is standard,}$$

So just a product of Normals

→ Use change of variables.

14. [last term is odd in $Y_i$]

15. Let $Y \sim N_4(0, I)$

and $Q = Y_1 Y_2 - Y_3 Y_4$

(a) Why is $Q$ not $\chi^2$?

Soln. $Q$ is symm about $0$.

(b) Find the MGF of $Q$.

Soln. Note $Q$ is $Y^T A Y$ where

$$A = \left( \frac{-I_4 + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}{2} \right)$$

$\underline{\text{eigenvalues are } 0, 0, 1, -1}$

then use

$$MGF = \det(I - 2tA)^{-1/2}$$

Problem 2 of #4

16. < this was done in a previous exercise, pt 1.>

17. $Y \sim N(\mu, \Sigma)$,

show

$$\operatorname{var}(Y^T A Y) = \operatorname{tr}((A\Sigma)^2) 2 + 4\mu^T A \Sigma A \mu$$

(wlog $A$ symm).

Soln. Follows directly from Thm. 1.6.