Strong Alternatives

Recall by weak duality we can easily construct system of inequalities

$$\{x \mid f_i(x) \leq 0\} \qquad (P)$$

$$\{\lambda \geq 0 \mid g(\lambda) = \inf_x \lambda^T f(x) > 0\} \quad (D)$$

such that at most one of (P), (D) is nonempty / feasible.

Strict Inequalities:

For $i \in [m]$ define:

$$\{x \in D \mid f_i(x) < 0, \; Ax = b\} \quad (P)$$

$$\{(\lambda, \nu) \mid \lambda \geq 0, \; \lambda \neq 0, \; g(\lambda, \nu) \geq 0\} \quad (D)$$

where

$$g(\lambda, \nu) = \inf_x \lambda^T f(x) + \nu^T (Ax - b)$$

Then (P) $\leftrightarrow$ (D) are strong alternatives, in that exactly one of them is feas, if a Slater-like condition is met,

$$\exists \tilde{x} \in \text{relint } D \text{ with } A\tilde{x} = b$$

Convert strict to standard form via new slack variable:

$$p^* = \min_{x, s} \quad s$$
$$\text{st} \quad f_i(x) - s \leq 0$$
$$Ax = b$$

where $p^* < 0 \iff$ (P) feas.

Note this satisfies slater w/ $(\tilde{x}, \tilde{s})$ st $\tilde{s} > \max_i f_i(\tilde{x})$.

Then:

$$d^* = p^* = \max_{\lambda, \nu} \inf_s s - \lambda^T s + g(\lambda, \nu)$$

$$= \max_{\lambda, \nu} \begin{cases} g(\lambda, \nu) & \lambda^T 1 = 1 \\ -\infty & o/w. \end{cases}$$

where by slater the optimum is attained by (λ^*, ν^*) st:

$g(\lambda^*, \nu^*) = p^*$, $\lambda^* \geq 0$, $1^T \lambda^* = 1$.

No to show strong alt:

If (P) is infeas, then $p^* \geq 0$.

In turn, (D) satisfied w/ (λ^*, ν^*)

if (D) is feas. w/ (λ, ν),

then $(\tilde{\lambda}, \tilde{\nu}) = (\lambda, \nu) / \|\lambda\|_1$

is dual feas, well-def'd (since $\|\lambda\|_1 \neq 0$)

and has $g(\tilde{\lambda}, \tilde{\nu}) = \frac{1}{\|\lambda\|_1} g(\lambda, \nu) \geq 0$

so $p^* \geq 0$ implying (P) <u>infeas.</u>

<u>Non-strict Inequalities</u>

(P) $\{x \mid f_i(x) \leq 0, \; Ax = b\}$

(D) $\{(\lambda, \nu) \mid \lambda \geq 0, \; g(\lambda, \nu) > 0\}$

Requires both $x \in$ relint Dom $Ax = b$

<u>and</u> that p^* from before is attained, e.g. if $\max_i f_i(x) \to \infty$ as $x \to \infty$

This yields SD w/ an attained primal & dual. By the same proof we have strong alternatives, where attainment in the primal was used to force $p^* > 0$ when (P) is infeasible (a non-compact feasible set might still allow $p^* \geq 0$).

<u>Example:</u>

Linear:

$$Ax \leq b \quad (P)$$

$$\left.\begin{aligned} \lambda &\geq 0 \\ A^T\lambda &= 0 \\ b^T\lambda &< 0 \end{aligned}\right\} (D)$$

strict-linear

$$Ax < b \quad (P)$$

$$\left.\begin{array}{l} \lambda \geq 0 \\ \lambda \neq 0 \\ A^T\lambda = 0 \\ b^T\lambda \leq 0 \end{array}\right\} (D)$$

Farkas

$$\left.\begin{array}{l} Ax = b \\ x \geq 0 \end{array}\right\} (P)$$

$$\left.\begin{array}{l} A^T y \geq 0 \\ b^T y < 0 \end{array}\right\} (D)$$

Ellipsoids.

Define m ellipses open $\mathcal{E}_i = \{x \mid f_i(x) < 0\}$

$$f_i(x) = x^T A_i x + b_i^T x + c_i, \quad A_i \text{ PD}$$

When is the intersection of $\{\mathcal{E}_i\}$ with nonempty interior?

$\iff$ (P) $\{x \mid \forall i \;\; f_i(x) < 0\}$ feas.

then (D) $\{\lambda \geq 0 \mid \lambda \neq 0, \; g(\lambda) \geq 0\}$

is a strong alternative, where

$$g(\lambda) = \begin{cases} -b(\lambda)^T A(\lambda)^{+} b(\lambda) + c(\lambda) & A(\lambda) \succeq 0 \\ & b(\lambda) \in \\ & \mathrm{Span}(A(\lambda)) \\ -\infty & \text{o/w} \end{cases}$$

where $b(\lambda) = \sum_i \lambda_i b_i$

$A(\lambda) = \sum_i \lambda_i A_i$

$c(\lambda) = \sum_i \lambda_i c_i$

For $\lambda \succeq 0$, $\lambda \neq 0$, as

$A_i \succ 0$, $A(\lambda) \succ 0$ so

$A(\lambda)^{+} = A(\lambda)^{-1}$ thus

$$g(\lambda) = -b(\lambda)^T A(\lambda)^{-1} b(\lambda) + c(\lambda)$$

Defining $\mathcal{E}_\lambda = \{ x \mid \underbrace{x^T A(\lambda) x + x^T b(\lambda) + c(\lambda)}_{f_\lambda(x) =} \leq 0 \}$

Note $\bigcap_i \mathcal{E}_i \subseteq \mathcal{E}_\lambda$ as

$f_i(x) \leq 0 \Rightarrow f_\lambda(x) \leq 0$

for $\lambda \geq 0$.

Thus, "geometric weak duality" here

Conversely, note

$$\inf_x f_\lambda(x) = -b(\lambda)^T A(\lambda)^{-1} b(\lambda) + c(\lambda)$$

by completing the square.

if this is ≥ 0 then

$\mathcal{E}_\lambda$ must have empty interior

(but $\mathcal{E}_\lambda$ is not necessarily $\emptyset$).

The implication is exact for any intersection of ellipsoids $\bigcap_i \mathcal{E}_i$ w/ empty int, another ellipsoid $\mathcal{E}_\lambda$ can contain it w/ empty int itself.

This isn't necessary b/c in <u>higher dimension</u> the statement isn't true: