Exercises 1a

1. Let X be a random vector and a a const vector with the same dimension. Show

$$\mathbb{E}\left[(X-a)(X-a)^T\right] = \operatorname{var} X + (\mathbb{E}X - a)(\mathbb{E}X-a)^T$$

Then, if $[\operatorname{var} X]_{ij} = \sigma_{ij}$, deduce

$$\mathbb{E}\|X-a\|^2 = \sum_i \sigma_i + \|\mathbb{E}X - a\|^2$$

Soln. $\mathbb{E}\left[(X-a)(X-a)^T\right]$

$$= \mathbb{E}\left[XX^T - 2X^Ta + aa^T\right]$$

$$= \operatorname{var} X + \mathbb{E}[X]\mathbb{E}[X^T] + \mathbb{E}\left[-2X^Ta + aa^T\right]$$

$$= \operatorname{var} X + (\mathbb{E}X - a)(\mathbb{E}X - a)^T$$

where we apply the computational formula

$$\operatorname{var} X = \mathbb{E}\left[XX^T\right] - \mathbb{E}X\,\mathbb{E}X^T$$

Let $\operatorname{var} X = \Sigma$,

$$\mathbb{E}\|X-a\|^2 = \mathbb{E}\operatorname{tr}\left((X-a)(X-a)^T\right)$$

$$= \operatorname{tr}\mathbb{E}\left[(X-a)(X-a)^T\right]$$

$$= \operatorname{tr}\left(\Sigma + (\mathbb{E}X - a)(\mathbb{E}X - a)^T\right)$$

$$= \operatorname{tr}\Sigma + \|\mathbb{E}X - a\|^2$$

2. If X, Y are random vectors and a, b are constants then show

$$\operatorname{cov}(X-a, Y-b) = \operatorname{cov}(X, Y).$$

Soln. $\operatorname{cov}(X-a, Y-b)$

$$= \operatorname{cov}(X-a, Y) - \cancel{\operatorname{cov}(X-a, b)}^{\,0}$$

$$= \operatorname{cov}(X, Y) - \cancel{\operatorname{cov}(a, Y)}^{\,0}$$

$$= \operatorname{cov}(X, Y)$$

3. Let $X = (X_1, \ldots X_n)^T$ be a random vector. Define

$Y_i = X_i - X_{i-1}$ with $X_0 = 0$.

If Y_i are mutually indep w/ unit variance, find $\operatorname{var} X$.

Soln. Equivalently, write $X_i = \sum_{j=1}^{i} Y_j$

$$[\operatorname{var} X]_{ij} = \mathbb{E}\Big[\Big(\sum_{k=1}^{i} Y_k - \mathbb{E}\sum_{k=1}^{i} Y_k\Big)\Big(\sum_{k=1}^{j} Y_k - \mathbb{E}\sum_{k=1}^{j} Y_k\Big)\Big]$$

$$= \sum_{k=1}^{i\wedge j} \operatorname{var} Y_k + \cancel{\sum_{k\neq m} \mathbb{E}\big[(Y_k - \mathbb{E}Y_k)(Y_m - \mathbb{E}Y_m)\big]}$$

0

$$= i \wedge j$$

4. Let $X_{i+1} = \rho X_i$ for a const ρ and $i \in [n-1]$. If $\operatorname{var} X_1 = \sigma^2$, find $\operatorname{var} X$.

Soln. $X_i = \rho^{i-1} X_1$

$$[\operatorname{var} X]_{ij} = \mathbb{E}\big[(X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)\big]$$

$$= \rho^{i+j-2} \operatorname{var} X_1 = \rho^{i+j-2}\sigma^2$$

Exercises 1b

1 Suppose $X = (X_1 \ X_2 \ X_3)$ have common mean μ and variance

$$\operatorname{var} X = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 1/4 & 1 \end{pmatrix}$$

Then, find $\mathbb{E}[X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2]$.

Soln. Use $\mathbb{E}[X^T A X] = \operatorname{tr}(\Sigma A) + \mu^T A \mu$ where $\Sigma = \operatorname{var} X$

with $\mu = 1_3 \mu$ and

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix}.$$

$\mathbb{E}[X^T A X] = \sigma^2$ by lots of cancellations.

2. Let X_i be independent vars for $i \in [n]$ with common mean μ and variances σ_i^2. Show that $\sum_i (X_i - \bar{X})^2 / (n(n-1))$

is an unbiased estimate of var $\bar{x}$.

Soln. Note $\sum_i (X_i - \bar{X})^2 = \|AX\|_2^2$

$$\text{for } A_{ij} = \begin{cases} 1 - 1/n & i=j \\ -1/n & o/w \end{cases}$$

and by construction $\mathbb{E} AX = 0$.

$$\text{So } \operatorname{var} AX = \mathbb{E}[X^T A^T A X]$$

$$= \operatorname{tr}(A\Sigma) + \mu^2 \underbrace{(1_n A^T A 1_n)}_{0}$$

where $\Sigma = \operatorname{var} X$
$= \operatorname{diag}(\sigma_i)_i$

$$= \sum_i \sigma_i^2 (1 - 1/n),$$

done b/c $\operatorname{var} \bar{X} = \frac{1}{n^2} \sum_i \sigma_i^2$.

3. Consider the setting of the previous exercise. Define the weighted $\bar{X}_w = \sum_i w_i X_i$ where $w \in \Delta_n$.

(a) Show that $\bar{X}_w$ is a MVUE when $w_i \propto \sigma_i^{-2}$ and find the minimum variance.

Soln. By indep, $\operatorname{var} \bar{X}_w = \sum_i \sigma_i^2 w_i^2$

Using KKT for $\max_w \operatorname{var} \bar{X}_w$ s.t. $w \in \Delta_n$

yields the condition $w_i = a/\sigma_i^2$ where $a = \left(\sum_i \sigma_i^{-2}\right)^{-1}$.

Then $\operatorname{var} \bar{X}_w = \sum_i w_i^2 \operatorname{var} X_i = a \sum_i w_i = a$

(b) Let $S_w^2 = \sum_i w_i (X_i - \bar{X}_w)^2/(n-1)$.

Show S_w^2 is an unbiased est for a.

Soln. Let $W = 1_n w^T$. Let

$$A = (I - W)^T (\operatorname{diag} w)(I - W).$$

Note that since $w \in \Delta_n$, $(I - W) 1_n = 0$

By constr. $X^T A X = S_w^2 (n-1)$

So given our identity for inner products, assuming $\operatorname{var} X = \Sigma = \operatorname{diag} \sigma^2$

$$(n-1)^* \, \mathbb{E} S_w^2 = \langle \Sigma, A \rangle + \cancelto{0}{\mu^2 1_n^T A 1_n}$$

$$= \langle \Sigma, \operatorname{diag} w \rangle - 2 \langle \Sigma, (\operatorname{diag} w) W \rangle + \langle \Sigma, W^T (\operatorname{diag} w) W \rangle$$

$$= \langle \sigma^2, w \rangle - 2 \langle \sigma^2, w^2 \rangle + \langle \Sigma, w 1_n^T (\operatorname{diag} w) 1_n w^T \rangle$$

$$= na - 2a + \underbrace{(w^T \Sigma w)}_{a} \underbrace{(1_n^T (\operatorname{diag} w) 1_n)}_{1}$$

$$= (n-1) a$$

4. n rvs X_i, $i \in [n]$ share mean $\mu \neq 0$ and var σ^2. The corr b/w any pair is ρ.

(a) Show $\frac{-1}{n-1} \le \rho \le 1$.

Soln. Show LB first.

$$\operatorname{var} \bar{X} = \frac{1}{n^2}\left(\sum_i \operatorname{var} X_i + 2\sum_{i<j} \operatorname{cov}(X_i, X_j)\right)$$

$$= \frac{\sigma^2}{n} + \binom{n}{2}\frac{2}{n^2}\rho\sigma^2$$

Then $\operatorname{var}\bar{X} \geq 0 \iff (n-1)\rho \geq -1$.

Elaborating on UB.

$$\sigma^2\rho = \operatorname{cov}(X_i, X_j)$$

$$\leq (\operatorname{var} X_i \operatorname{var} X_j)^{1/2}$$

$$\leq 1 \text{ by Cauchy-Schwarz.}$$

(b) Suppose $Q = a\sum_i X_i^2 + b\left(\sum_i X_i\right)^2$ is σ^2 in expectation. Then show

$$Q = \frac{\sum_i (X_i - \bar{X})^2}{(1-\rho)(n-1)}$$

Soln. Notice $Q = a X^T I X + b (1_n^T X)^2$

$= X^T (a I + b 1_n 1_n^T) X$.

Since $\mu_x = 1_n \mu$

$$\sigma^2 = \mathbb{E} Q = \underbrace{(a+b) n \sigma^2 + 2\binom{n}{2} b \rho \sigma^2}_{\text{must be } \sigma^2}$$

$$+ \underbrace{\mu_x^T (a I + b 1_n 1_n^T) \mu_x}_{\text{must be } 0}$$

holds for nonzero μ,

solving $(a+b) n + n(n-1) b \rho = 1$

and $-a n \mu^2 = b n^2 \mu^2$,

$b n (n-1)(\rho - 1) = 1$ and $a = -bn$.

This gives the desired form of Q.

5. Let $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ for $i \in [n]$.

Define

$S^2 = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2$ and

$$Q = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 .$$

(a) Show $\operatorname{var} S^2 = 2\sigma^4/(n-1)$

Soln. Let $A = I - \frac{1}{n} 1_n 1_n^T$. Then

$S^2 = X^T A X$ so by Thm 1.6.

$$\operatorname{var} S^2 = (\mu_4 - 3\mu_2^2) \|\operatorname{diag} A\|_2^2$$

$$+ 2\mu_2^2 \operatorname{tr} A^2$$

$$+ 4\mu_2 \cancel{\mu_1 1_n^T A 1_n \mu_1}^{\,0} + \mu_3 \cancel{(\dots)}^{\,0}$$

$$= (3\sigma^4 - 3\sigma^4) \|\cdots\|$$

$$+ \underbrace{2\sigma^4 \operatorname{tr} A^2}_{2\sigma^4 \times \frac{1}{(n-1)^2}\left(n(1-1/n)^2 + \frac{n(n-1)}{n^2}\right)}$$

$$= 2\sigma^4/_{n-1}$$

(b) Show Q is unbiased est. for σ^2.

Soln. Let $D_k^{(n-1)}$ be the matrix of ones on the k-th band of size $(n-1)\times(n-1)$. Then

$$A = \underbrace{\begin{pmatrix} D_{+1}^{(n-1)} & \begin{matrix} 0_{n-2} \\ 1 \end{matrix} \\ 0 & 1 \end{pmatrix}}_{D} - I$$

w/ e elementary basis vectors and $t = 2\mathbf{1}_n - e_1 - e_n$, note two complicated facts:

$$A^T A = \operatorname{diag} t - D_{+1}^{(n)} - D_{-1}^{(n)}$$

$$A \mathbf{1}_n = 0$$

$$Q = \frac{1}{2(n-1)} X^T A^T A X.$$

$$EQ = \left\langle A^T A, \sigma^2 I \right\rangle + \cancelto{0}{\mu \mathbf{1}_n^T A^T A \mathbf{1}_n \mu} \Big/ 2(n-1)$$

$$= \sigma^2 \langle A, A \rangle / 2(n-1)$$

$$= \sigma^2$$

(c) Find the variance of Q.

Soln. Apply Thm 1.6.

$$\mu_1 = \mu, \mu_2 = \sigma^2, \mu_3 = 0, \mu_4 = 3\sigma^2,$$

so only μ_2 term remains:

$$\operatorname{Var} Q = \frac{1}{4(n-1)^2}\left[2\sigma^4 \operatorname{tr}\left((A^T A)^2\right)\right]$$

by $A^T A$ calculation above,

$$\operatorname{tr}(A^T A)^2 = 2(3n-4),$$

$$\text{so } \operatorname{var} Q = \frac{3\sigma^4}{n} + o_p(1)$$

Exercises 1c

1. If X, Y have the same var, then show $\text{cov}(X+Y, X-Y) = 0$. Use this to provide an example of dependent but uncorrelated variables.

Soln. By linearity, cov is 0.

$$X = \begin{cases} +1 & \text{wp } 2/3 \\ -1 & \text{wp } 1/3 \end{cases} \qquad Y = \begin{cases} +1 & \text{wp } 1/3 \\ -1 & \text{wp } 2/3 \end{cases}$$

Then $X+Y$, $X-Y$ are dependent but uncorrelated. Dependence example: Assume indep for contradiction.

$$\mathbb{E}\left[(X+Y)^2(X-Y)\right]$$
$$= \mathbb{E}\left[(X+Y)\underbrace{(X^2-Y^2)}_{0}\right]$$
$$= 0$$

vs

$$\mathbb{E}\left[(X+Y)^2(X-Y)\right]$$
$$= \mathbb{E}\left[(X+Y)^2\right]\mathbb{E}\left[(X-Y)\right] > 0$$

2. Let X, Y be binary rvs where $\mathbb{P}\{X=i, Y=j\}=p_{ij}$. Then $\mathrm{cov}(X,Y)=0$ $\iff X \perp\!\!\!\perp Y$.

Soln. ($\Leftarrow$) is immediate.

($\Rightarrow$)

$$\mathrm{cov}(X,Y) = \mathbb{E}[XY - X\mathbb{E}Y - Y\mathbb{E}X] + \mathbb{E}X\mathbb{E}Y$$

$$\mathbb{E}X = p_{10}+p_{11} \quad \mathbb{E}Y = p_{01}+p_{11}$$

$$\mathbb{E}[XY] = p_{11}$$

Then $\mathrm{cov}(X,Y)=0$

$$\iff p_{11} = \underbrace{(p_{10}+p_{11})}_{\mathbb{P}\{X=1\}}\underbrace{(p_{11}+p_{01})}_{\mathbb{P}\{Y=1\}},$$

def of indep.

3. Let X be a rv w/ a symm density function and zero mean, show $\mathrm{cov}(X, X^2)=0$.

Soln. $\mathrm{cov}(X,X^2) = \mathbb{E}[X(X^2-\mathbb{E}X^2)]$

$$= \mathbb{E}X^3 - \mathbb{E}X\,\mathbb{E}X^2$$

$\mathbb{E}X = 0$, so right term is 0.

$\mathbb{E}X^3 = 0$ by symm.

↳ Assuming it exists

→ All 0.

4. Let X, Y, Z on $[-1, 1]$ be rvs with joint density

$$f(x, y, z) = \frac{1}{8}(1 + xyz).$$

Show X, Y, Z are pairwise independent but not mutually so.

Soln. By symm suff to show for X, Y.

$$\mathbb{E}\left[\exp(tX + sY)\right] = \int f(x,y,z)\, e^{tX} e^{sY}\, dx\,dy\,dz$$

$$= \underbrace{\int \frac{1}{2} e^{tX} e^{sY}\, dx\,dy}_{\text{factorizes} \Rightarrow \text{indep.}} \quad (xyz \text{ is odd})$$

But $\mathbb{E}[\exp(tX + sY + uZ)]$

$\neq \mathbb{E}[e^{tX}]\mathbb{E}[e^{sY}]\mathbb{E}[e^{uZ}]$

Miscellaneous Exercises 1

1 Prove the law of total variance.

Soln. $\mathbb{E}\,\mathrm{var}(X|Y) + \mathrm{var}\,\mathbb{E}[X|Y]$

$= \mathbb{E}\left[\mathbb{E}[X^2|Y] - \mathbb{E}^2[X|Y]\right] + \mathbb{E}\,\mathbb{E}[X|Y]^2 - \mathbb{E}^2\mathbb{E}[X|Y]$

$= \mathbb{E}X^2 - \mathbb{E}\mathbb{E}^2[X|Y] + \mathbb{E}\mathbb{E}^2[X|Y] - \mathbb{E}^2X$

$= \mathbb{E}X^2 - \mathbb{E}^2X = \mathrm{var}\,X$

2. Let $X = (X_1\ X_2\ X_3)^T$ with

$$\mathrm{var}\,X = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

(a) Find $\mathrm{var}(X_1 - 2X_2 + X_3)$.

Soln. $t = (1\ -2\ 1)^T$, $\mathrm{var}(t^T X)$

$= t^T(\mathrm{var}\,X)t = 22$

(b) Let $Y_1 = X_1 + X_2$ $Y_2 = X_1 + X_2 + X_3$.

Find $\operatorname{var} Y$.

Soln. $Y = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{B} X$.

Again,

$$\operatorname{var} Y = B^T (\operatorname{var} X) B$$

$$= \begin{pmatrix} 12 & 15 \\ 15 & 21 \end{pmatrix}$$

3. Let $X_1, \ldots X_i \ldots X_n$ be rvs with a common mean μ and suppose $\operatorname{cov}(X_i, X_j) = 0$ for $j > i+1$.

Set
$$Q_1 = \sum_i (X_i - \bar{X})^2$$

$$Q_2 = \sum_i (X_{i+1} - X_i)^2$$
with $X_{n+1} = X_1$

show $\mathbb{E}\left[\dfrac{3Q_1 - Q_2}{n(n-3)}\right] = \operatorname{var} \bar{X}$

Soln. $Q_1 = X^T A_1^T A_1 X \qquad A_1 = I - \dfrac{1_n 1_n^T}{n}$

$Q_2 = X^T A_2^T A_2 X$

$A_2 = I - D$ where $D = D_{+1} - D_{-n+1}$

Note $D^T D = I$

$\mathbb{E}[3Q_1 - Q_2)$

$= \mathbb{E}[X^T \underbrace{(3A_1^T A_1 - A_2 A_2)}_{A} X]$

$= \langle 3A_1^T A_1 - A_2^T A_2, \operatorname{var} X \rangle$

$+ \underbrace{\cancel{\mu 1_n^T A^T A 1_n \mu}}_{0}$

$= \langle I - \frac{1}{n} 1_n 1_n^T + D^T + D, \operatorname{var} X \rangle$

after much simplification.

Since off-$(-1, 0, +1)$ bands are 0 for var X,

$$= \frac{n-1}{n} \langle I + D_{+1} + D_{-1}, \operatorname{var} X \rangle$$

OTOH

$$\operatorname{var} \bar{X} = \frac{1}{n^2} \operatorname{var} \mathbb{1}_n^T X$$

$$= \frac{1}{n^2} \mathbb{1}_n^T (\operatorname{var} X) \mathbb{1}_n^T$$

$$= \frac{1}{n^2} \langle I + D_{+1} + D_{-1}, \operatorname{var} X \rangle$$

by same off-diagonal argument.

This yields

$$\mathbb{E}\left[\frac{3 Q_1 - Q_2}{\underbrace{n(n-1)}_{\text{NOT } (n-3)\ ???}}\right] = \operatorname{var} \bar{X}$$

4. Given indep. X_1, X_2, X_3 on $[-1,1]$ w/ uniform density, what's the variance of

$$(X_1-X_2)^2+(X_2-X_3)^2+(X_3-X_1)^2$$

Soln. This is $[X^T A^T A X] = V$

for $A = I - \underbrace{D_{+1} - D_{-2}}_{D = D_{+1} + D_{-2}}$

Apply Thm 1.6 formula

$\mu_1 = 0 \quad \mu_2 = \int_{-1}^{1} x^2/2 \, dx = 1/3 \quad \mu_3 = 0 \quad \mu_4 = \int_{-1}^{1} x^4/2 \, dx = 1/5$

$$\Rightarrow \operatorname{var} V = \left(1/5 - 1/3\right) 3^2 + 2/q \, \underbrace{\operatorname{tr} A^2}_{2(3)+2(3)}$$

$= \ldots$

5. For $X_i \overset{iid}{\sim} N(0, \sigma^2)$ and $i \in [n]$,

find $\mathrm{cov}(X^T A X, X^T B X)$, w/ A, B symm.

Soln.

Note $\mathbb{E} X^T A X = (\mathrm{tr} A)\sigma^2$.

Similarly for B.

Expanding the definition,

$\mathrm{cov}(X^T A X, X^T B X) =$

$$\underbrace{\mathbb{E}[X^T A X X^T B X]}_{\text{expand explicitly.}} - \sigma^4 \, \mathrm{tr} A \, \mathrm{tr} B$$

$$\mathbb{E} \sum_{ij} a_{ij} x_i x_j \sum_{ij} b_{ij} x_i x_j$$

$$= \mathbb{E} \sum_{ijkl} a_{ij} b_{kl} x_i x_j x_k x_l$$

$$= \mathbb{E} \sum_{ik} a_{ii} b_{kk} x_i^2 x_k^2 + \sum_{i \neq j} \sum_{k \neq l} a_{ij} b_{kl} x_i x_j x_k x_l$$

Latter term will be 0 if any term

$x_i x_j x_k x_l$ is multiplicity 1,

so restrict to cases where

$j=k$ & $i=l$ or $j=l$ & $i=k$

(all four can't be equal, as $i \neq j$, $k \neq l$)

$$= \mathbb{E} \sum_{ik} a_{ii} b_{kk} x_i^2 x_k^2 + 2 \sum_{i \neq j} a_{ij} b_{ji} x_i^2 x_j^2$$

Push $\mathbb{E}$ in, $\mathbb{E}\, x_i^2 x_k^2$

$= \mathbb{E} x_i^2\, \mathbb{E} x_k^2$

$= \sigma^4$

$$= (\operatorname{tr} A \operatorname{tr} B)\sigma^4 + 2\sigma^4 \operatorname{tr}(AB)$$

Which cancels w/ our earlier term.

We didn't even need symmetry...