Numerical Analysis

Prof.dr. hab. Bostan Viorel

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On a decimal computer or calculator, we store x by instead storing σ, e and $\overline{x}$.

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For example on calculator HP-15, the number of digits in mantissa is 10 and

$$-99 \leq e \leq 99$$

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Notice, that the first digit in mantissa is always 1.

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We will concentrate on a particular form of computer floating point number, that is called the **IEEE floating point standard**.

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		e				
		-2	-1	0	1	2
$\overline{x}$	$(1.00)_2$	$(0.25)_{10}$	$(0.5)_{10}$	$(1)_{10}$	$(2)_{10}$	$(4)_{10}$
	$(1.01)_2$	$(0.3125)_{10}$	$(0.625)_{10}$	$(1.25)_{10}$	$(2.5)_{10}$	$(5)_{10}$
	$(1.10)_2$	$(0.375)_{10}$	$(0.75)_{10}$	$(1.5)_{10}$	$(3)_{10}$	$(6)_{10}$
	$(1.11)_2$	$(0.4375)_{10}$	$(0.875)_{10}$	$(1.75)_{10}$	$(3.5)_{10}$	$(7)_{10}$

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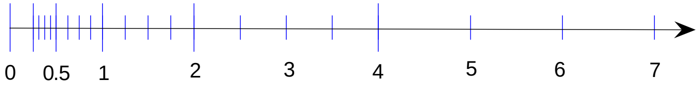

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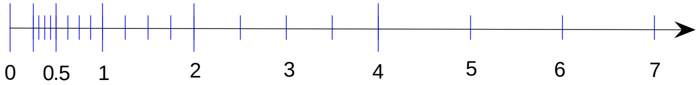

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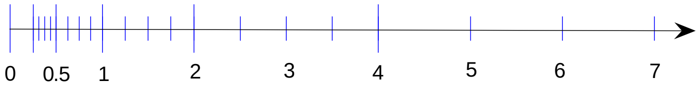

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These numbers are obtained if $e = e_{min}$ and the first digit of the significand is 0.

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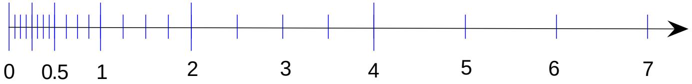

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b_1	b_2	$\ldots$	b_9	b_{10}	$\ldots$	b_{32}

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Number $x = 0$ is stored in the following way: $E = 0$, $\sigma = 0$ and $b_{10}b_{11}\ldots b_{32} = (00\ldots0)_2$.

IEEE single precision floating point representation

$E = (b_2 \dots b_9)_2$	e	x
$(00000000)_2 = (0)_{10}$	$-(127)_{10}$	$\pm(0.b_{10} \dots b_{32})_2 \cdot 2^{-126}$
$(00000001)_2 = (1)_{10}$	$-(126)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{-126}$
$(00000010)_2 = (2)_{10}$	$-(125)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{-125}$
$\vdots$	$\vdots$	$\vdots$
$(01111111)_2 = (127)_{10}$	$(0)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{0}$
$(10000000)_2 = (128)_{10}$	$(1)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{1}$
$\vdots$	$\vdots$	$\vdots$
$(11111101)_2 = (253)_{10}$	$(126)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{126}$
$(11111110)_2 = (254)_{10}$	$(127)_{10}$	$\pm(1.b_{10} \dots b_{32})_2 \cdot 2^{127}$
$(11111111)_2 = (255)_{10}$	$(128)_{10}$	$\pm\infty$, *dacă* $b_i = 0$ NaN, *otherwise*

IEEE double precision floating point representation

$$x = \sigma \cdot 1.a_1 a_2 \ldots a_{52} \cdot 2^e.$$

with $E = e + 1023$

σ	E			$\bar{x}$		
b_1	b_2	$\ldots$	b_{12}	b_{13}	$\ldots$	b_{64}

IEEE double precision floating point representation

$E = (b_2 \dots b_{12})_2$	e	x
$(00000000000)_2 = (0)_{10}$	$-(1023)_{10}$	$\pm(0.b_{13}\dots b_{64})2^{-1022}$
$(00000000001)_2 = (1)_{10}$	$-(1022)_{10}$	$\pm(1.b_{13}\dots b_{64})2^{-1022}$
$(00000000010)_2 = (2)_{10}$	$-(1021)_{10}$	$\pm(1.b_{13}\dots b_{64})2^{-1021}$
$\vdots$	$\vdots$	$\vdots$
$(01111111111)_2 = (1023)_{10}$	$(0)_{10}$	$\pm(1.b_{13}\dots b_{64})2^{0}$
$(10000000000)_2 = (1024)_{10}$	$(1)_{10}$	$\pm(1.b_{13}\dots b_{64})2^{1}$
$\vdots$	$\vdots$	$\vdots$
$(11111111101)_2 = (2045)_{10}$	$(1022)_{10}$	$\pm(1.b_{13}\dots b_{64})2^{1022}$
$(11111111110)_2 = (2046)_{10}$	$(1023)_{10}$	$\pm(1.b_{13}\dots b_{64})_2^{1023}$
$(11111111111)_2 = (2047)_{10}$	$(1024)_{10}$	$\pm\infty$, $b_i = 0$ NaN, *otherwise*

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(\underbrace{111\ldots11}_{\text{23 ones}})_2 &= (1.11\ldots1)_2 \cdot 2^{23} \\
&= 2^{23} + 2^{22} + \ldots + 2^{0} \\
&= 2^{24} - 1
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For double precision $M = 2^{53} \approx 9.0 \cdot 10^{15}$

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Thus α drops off the end of the floating point representation in the machine.

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$$1 + \delta > 1$$

in the arithmetic of the machine.

For any number $0 < \alpha < \delta$ the result of $1 + \alpha$ is exactly 1 in the machines arithmetic.

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The machine epsilon has been replacing it in recent years.

Chopping and rounding in decimal

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We write a computer floating point number z as

$$z = \sigma \cdot \overline{z} \cdot 10^e \equiv \sigma \cdot (a_1.a_2 \ldots a_n)_{10} \cdot 10^e$$

with $a_1 \neq 0$, so that there are n decimal digits in the significand.

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The floating point chopped version of x is given by

$$fl(x) = \sigma \cdot \overline{x} \cdot 10^e \equiv \sigma \cdot (a_1.a_2 \ldots a_n)_{10} \cdot 10^e,$$

where we assume that e fits within the bounds required by the computer or calculator.

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A simplified formula is

$$fl(x) = \begin{cases} \sigma \cdot (a_1.a_2 \ldots a_n)_{10} \cdot 10^e, & \textit{if} \quad a_{n+1} < 5 \\ \sigma \cdot (a_1.a_2 \ldots a_n)_{10} \cdot 10^e + (0.00 \ldots 01)_{10}, & \textit{if} \quad a_{n+1} \geq 5 \end{cases}$$

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The term $(0.00 \ldots 01)_{10}$ denotes 10^{-n+1}, giving the ordinary sense of rounding with which you are familiar.

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With $x \neq fl(x)$ and rounding, the error $x - fl(x)$ is negative for half the values of x, and it is positive for the other half of possible values of x.

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There is also an extended representation, having $n = 69$ digits in its significand.

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To obtain the binary representation, convert each hexadecimal digit to a four digit binary number according to the table on next slide:

MATLAB floating point representation

Format hex	Format binary	Format hex	Format binary
0	0000	8	1000
1	0001	9	1001
2	0010	*a*	1010
3	0011	*b*	1011
4	0100	*c*	1100
5	0101	*d*	1101
6	0110	*e*	1110
7	0111	*f*	1111

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b_1	b_2	$\ldots$	b_{12}	b_{13}	$\ldots$	b_{64}

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$$\begin{aligned} err(x_A) &= e - \frac{19}{7} \approx 0.003996 \\ rel(x_A) &\approx \frac{0.003996}{e} \approx 0.00147 \end{aligned}$$

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with $b \geq 0$. In this, $\overrightarrow{r}(t)$ is the vector position of the projectile; and the final term in the equation represents friction force in air. If there is an error in this a model of a physical situation, then the numerical solution of this equation is not going to improve the results.

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We need to be aware of these effects and to so arrange the computation as to minimize the effects.

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For example, the Taylor polynomial approximation

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The numerical integration

$$\int_0^1 f(x)dx \approx \frac{1}{N}\sum_{j=1}^{N} f\left(\frac{j}{N}\right)$$

contains an approximation error.

Sources of Error

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Even if theoretically an algorithm can run for indefinite time, after a finite (usually specified) number of iterations the algorithm will be stopped.

Sources of Error

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Some simple rules to decrease the risk of having a bug in the code:

- Break programs into small testable subprograms;
- Run test cases for which you know the outcome;
- When running the full code, maintain a skeptical eye on the output, checking whether the output is reasonable or not.

Sources of Error

6. **Rounding/chopping Error**.

Sources of Error

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We later look at the effects of such errors.

Sources of Error

7. Finitness of precision errors

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All the numbers stored in computer memory are subject to the finiteness of allocated space for storage.

Pendulum Example

Original problem in engineering or in science to be solved:

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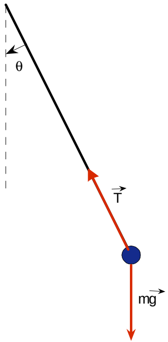

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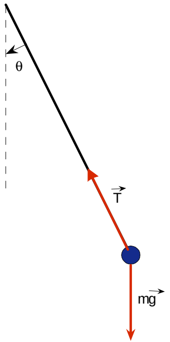

Model this physical problem mathematically.

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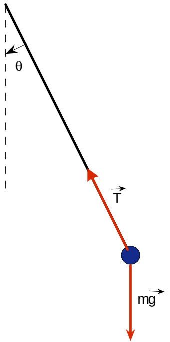

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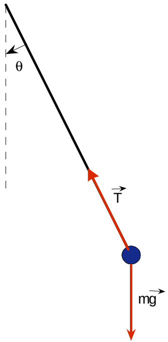

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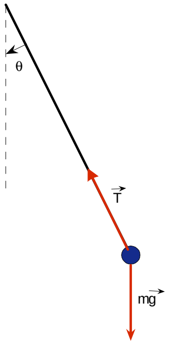

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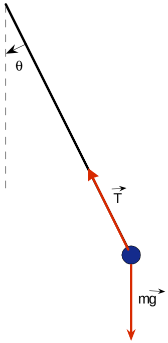
$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{l}\sin\theta \end{cases}$$

Pendulum Example

Problem of continuous mathematics:



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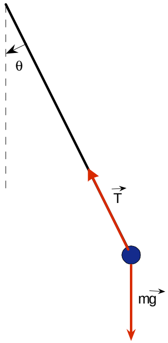

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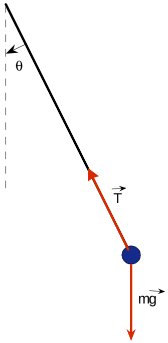

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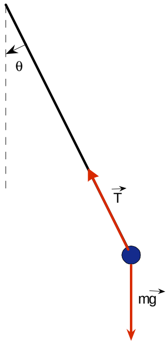
- Modeling Errors
- Physical Errors

Pendulum Example

Mathematical Algorithms:



Pendulum Example

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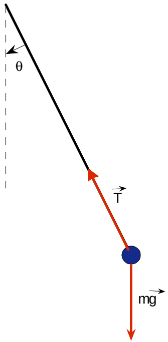

$$\begin{cases} \theta_{n+1} = \theta_n + h\omega_{n+1} \\ \omega_{n+1} = \omega_n - h\frac{g}{l}\sin(\theta_n) \end{cases}$$

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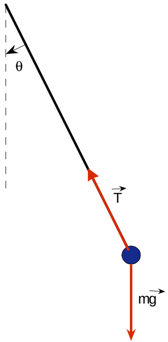

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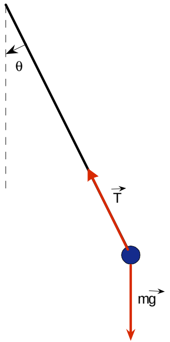
- Discretisation Errors
- Finiteness of Algorithm Errors

Pendulum Example

Computer Implementation:



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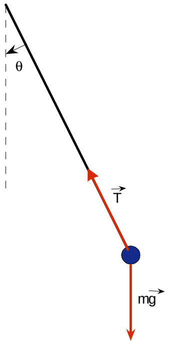


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        Omega = Omega - H*g/L*sin(Theta);
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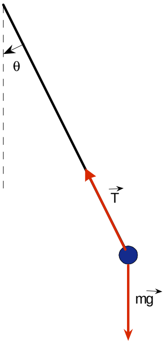


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- Rounding / Chopping Errors
- Bugs in the Code
- Finite Precision Errors