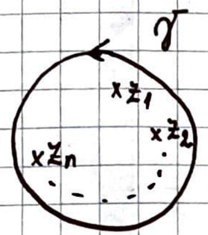
6. TEOREMA REZIDUURILOR ȘI A SEMIREZIDUURILOR

NOȚIUNI TEORETICE

T. Reziduurilor



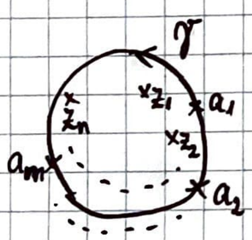
γ - închisă, netedă
(netedă pe porțiuni)

- f - olomorfă pe $\operatorname{Int} \gamma \setminus \{z_1, \dots, z_n\}$ → poli, puncte singulare esențiale *
- f - continuă pe γ

OBS! aici * nu zicem nimic de puncte aparente, deoarece nu ar avea sens. Integrala pt. aceste puncte este 0

$$\Longrightarrow I = \int_\gamma f(z)\,dz = 2\pi j \sum_{k=1}^{n} \operatorname{Rez}(f; z_k)$$

T. Semireziduurilor



γ - închisă, netedă
(netedă pe porțiuni)

- f - olomorfă pe $\operatorname{Int} \gamma \setminus \{z_1, z_2, \dots, z_n\}$
- f - continuă pe $\gamma \setminus \{a_1, a_2, \dots, a_m\}$ → poli simpli

$$\Longrightarrow I = \int_\gamma f(z)\,dz = 2\pi j \sum_{k=1}^{n} \operatorname{Rez}(f; z_k) + \pi j \sum_{k=2}^{n} \operatorname{Rez}(f; a_k)$$

PROBLEME

① Calculați $\displaystyle\int_{|3z+j|=1} \underbrace{\frac{e^{\pi j z}}{z(3z+j)^3}}_{f(z)}\, dz$

$\underline{z_0 = 0}$

- punct aparent $\quad \lim\limits_{z\to z_0} f(z)$ - finită
- poli $\quad \frac{\ }{0}$
- puncte esențiale $\quad e^{\frac{}{0}}$, sh $\frac{}{0}$, ch $\frac{}{0}$, sin $\frac{}{0}$, cos $\frac{}{0}$

Pe ciornă înlocuim z_0 cu 0:

$\underline{z_0 = 0}$ pol simplu $\quad \left(\frac{1}{0\cdot j^3}\right) \quad 3z+j = 0 \Rightarrow \underline{z_1 = \frac{-j}{3}}$ pol triplu

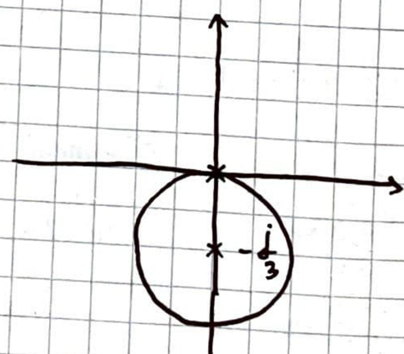
$$\left(\frac{e^{\frac{\pi}{3}}}{\left(-\frac{j}{3}\right)\cdot 0^3}\right)$$

$|3z+j| = 1 \quad |:3$

$\left|z - \left(-\frac{j}{3}\right)\right| = \frac{1}{3}$

$|z - z_0| = r$

$(\gamma) = \Gamma\left(-\frac{j}{3}; \frac{1}{3}\right)$



Avem puncte pe frontieră $\Rightarrow$ suntem obligați să folosim T. semirez.

T. semirez $\Longrightarrow \quad I = 2\pi j\, \mathrm{Rez}\left(f; \frac{-j}{3}\right) + \pi j \cdot \mathrm{Rez}(f; 0)$

z_0 - pol de ordin n

$$\mathrm{Rez}(f; z_0) = \frac{1}{(n-1)!}\cdot \lim_{z\to z_0} \left((z-z_0)^n \cdot f(z)\right)^{(n-1)}$$

$$\underset{\uparrow\, n=1}{\mathrm{Rez}(f; z_0) =} \frac{1}{0!}\cdot \lim_{z\to 0} z^1 \cdot \frac{e^{\pi j z}}{z(3z+j)^3} = \frac{1}{j^3} = j$$

sau DOAR pt. polii simpli

$$Rez(f;0) = \frac{e^{\pi j z}}{z'(3z+j)^3}\Bigg|_{z=0} =$$

Se derivează acel numitor care s-ar anula pt. z_0 și se calculează în z_0 dat.

$$= \frac{e^{\pi j z}}{(3z+j)^3}\Bigg|_{z=0} = \frac{1}{j^3} = j$$

Se poate aplica doar dacă această funcție $\neq 0$ pt. $z = z_0$

Aplicând formula pt. z_0 - pol de ordin n, obținem:

$$Rez\left(f; \frac{-j}{3}\right) = \frac{1}{2!}\lim_{z\to -j/3}\left(\left(z+\frac{j}{3}\right)^3 \cdot \frac{e^{\pi j z}}{(3z+j)^3}\right)^{(2)} =$$

$$= \frac{1}{2}\lim_{z\to -j/3}\left(\frac{(3z+j)^3}{27}\cdot\frac{e^{\pi j z}}{z(3z+j)^3}\right)^{(2)} = \frac{1}{54}\cdot\lim_{z\to -j/3}\left(\frac{e^{\pi j z}}{z}\right)^{(2)}$$

$$\left(\frac{e^{\pi j z}}{z}\right)' = \frac{\pi j\cdot e^{\pi j z}\cdot z - 1\cdot e^{\pi j z}}{z^2} = \frac{e^{\pi j z}(\pi j z - 1)}{z^2}$$

$$\left(\frac{e^{\pi j z}}{z}\right)'' = \frac{e^{\pi j z}(-\pi^2 z^2 - 2\pi j z + 2)}{z^3}$$

$$Rez\left(f; \frac{-j}{3}\right) = \frac{1}{54}\cdot\frac{e^{\pi j z}(-\pi^2 z^2 - 2\pi j z + 2)}{z^3}\Bigg|_{z=-j/3}$$

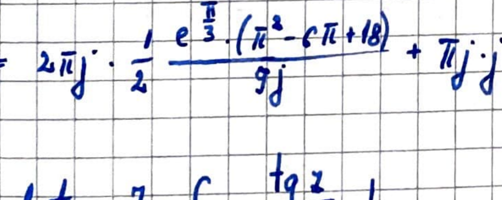
$$= \frac{1}{54}\cdot\frac{e^{\frac{\pi}{3}}\cdot\left(\pi^2\cdot\frac{1}{9} - \frac{2\pi}{3} + 2\right)}{-j^3/27} = \frac{1}{2}\cdot e^{\frac{\pi}{3}}\cdot\frac{\pi^2 - 6\pi + 18}{9j}$$

$$I = 2\pi j\cdot\frac{1}{2}\cdot\frac{e^{\frac{\pi}{3}}\cdot(\pi^2 - 6\pi + 18)}{9j} + \pi j\cdot j = \frac{\pi e^{\frac{\pi}{3}}\cdot(\pi^2 - 6\pi + 18)}{9} - \pi$$

2 Calculați $I = \int_{|z|=\pi}\frac{tg\, z}{z^2}\,dz$

$$f(z) = \frac{\sin z}{z^2\cdot\cos z}$$

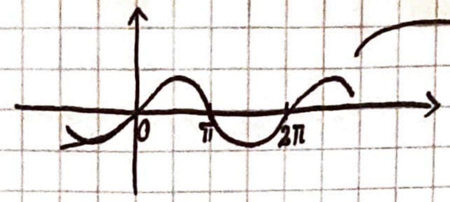
OBS!



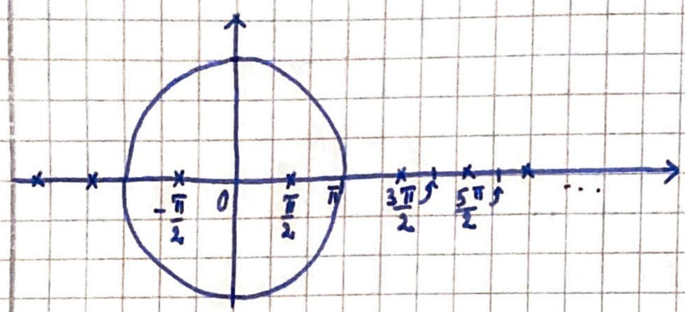
$\cos z = 0 \iff z_k = \frac{(2k+1)\pi}{2}, \ k\in\mathbb{Z}$

$z' = 0$ $\left(\frac{0}{0\,0\cdot 1}\right)$

pol simplu

OBS! 

$\sin z = 0 \iff z_k = k\pi\ ,\ k \in \mathbb{Z}$



Nu avem puncte pe frontieră $\Rightarrow$ T. reziduurilor $\Rightarrow$

$$\Rightarrow I = 2\pi j\left(\text{Rez}\left(f; -\frac{\pi}{2}\right) + \text{Rez}(f; 0) + \text{Rez}\left(f; \frac{\pi}{2}\right)\right)$$

$$\text{Rez}(f;0) \underset{n=1}{=} \lim_{z\to 0} \cancel{z^1} \cdot \frac{\sin z}{\cos z \cdot \cancel{z^2}} = \frac{1}{1} = 1$$

($\frac{\sin z}{z} \to 1$)

OBS! Dacă m-ar duce bidonul să fac:

$$\cancel{\text{Rez}(f;0) = \frac{\sin z}{(z^2)'\cos z}\Big|_{z=0} = \frac{\sin z}{2z\cdot\cos z}\Big|_{z=0} = \frac{0}{0}}$$

$\sin 0 = 0$, deci nu putem aplica

$$\text{Rez}\left(f; -\frac{\pi}{2}\right) = \frac{\sin z}{z^2 \cdot \cos z'}\Big|_{z=-\frac{\pi}{2}} = \frac{\sin z}{z^2\cdot(-\sin z)}\Big|_{z=-\frac{\pi}{2}} = \frac{-1}{z^2} = \frac{-1}{\frac{\pi^2}{4}} = \frac{-4}{\pi^2}$$

$$\text{Rez}\left(f; \frac{\pi}{2}\right) = \frac{\sin z}{z^2(\cos z)'}\Big|_{z=\frac{\pi}{2}} = \frac{\sin z}{z^2(-\sin z)}\Big|_{z=\frac{\pi}{2}} = -\frac{1}{z^2} = \frac{-4}{\pi^2}$$

(3) Fie $f(z) = \dfrac{z^{12}\cdot \sin\frac{3\pi j}{z}}{(z^2+2)^5\cdot(z^2+jz+6)}$

(a) Natura punctelor singulare

(b) $\displaystyle\int_{|z|=2} f(z)\,dz$

R: a) $z_0 = 0$ punct singular esențial

$z^2 + 2 = 0 \Rightarrow z^2 = -2 = (\sqrt{2}j)^2$

$\underline{z_{1,2} = \pm\sqrt{2}j}$ - poli de ordinul 5

$z^2 + jz + 6 = 0$

$\Delta = -1 - 24 = -25 = (5j)^2$

$$z_{3,4} = \frac{-j \pm 5j}{2} \begin{cases} z_3 = 2j & \text{pol simplu} \\ z_4 = -3j & \text{punct singular aparent} \end{cases}$$

capcană!

$$(z^2 + jz + 6) = (z - 2j)(z + 3j)$$

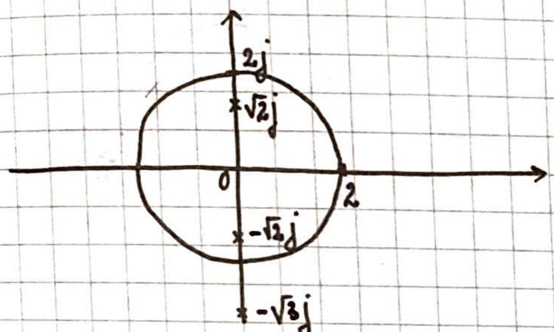
$$\left(\frac{(3j)^{12} \cdot \sin\frac{3\pi j}{3j}}{((2j)^2 + 2)^5 \cdot 0 \cdot 5j}\right)$$

$$\left(\frac{0}{0}\right)$$

$$\lim_{z \to -3j} \frac{z^{12} \sin\frac{3\pi j}{z}}{(z^2+2)^5 \cdot (z - 2j)(z + 3j)}$$

$$\lim_{z \to -3j} \frac{\sin\frac{3\pi j}{z}}{z + 3j} \overset{l'H}{=} \lim_{z \to -3j} \frac{\frac{-3\pi j}{z^2}}{1} = \frac{-3\pi j}{-9} \text{ - finită}$$

$$\operatorname{Rez}(f; -3j) = 0$$



$|z| = 2 \iff |z - 0| = 2$

$\gamma = \Gamma(0; 2)$

$$\xRightarrow{\text{T. semirez}} J = 2\pi j\left(\operatorname{Rez}(f; \sqrt{2}j) + \operatorname{Rez}(f; 0) + \operatorname{Rez}(f; -\sqrt{2}j)\right) + \pi j \operatorname{Rez}(f; 2j)$$

Folosim teorema de ora trecută, care era independentă de γ!

T

$$\underbrace{\operatorname{Rez}(f; \sqrt{2}j) + \operatorname{Rez}(f; 0) + \operatorname{Rez}(f; -\sqrt{2}j)} + \operatorname{Rez}(f; 2j) + \underbrace{\operatorname{Rez}(f; -3j)}_{=0} + \operatorname{Rez}(f; \infty) = 0$$

$$J = -\pi j \operatorname{Rez}(f; 2j) - 2\pi j \operatorname{Rez}(f; \infty)$$

$$\operatorname{Rez}(f; 2j) = \lim_{z \to 2j} (z - 2j) \cdot \frac{z^{12} \sin\frac{3\pi j}{z}}{(z^2+2)^5 (z - 2j)(z + 3j)} = \frac{(2j)^{12} \sin\frac{3\pi j}{2j}}{(-4+2)^5 \cdot 5j} = \frac{-2^{12}}{(-2)^5 \cdot j \cdot 5} = \frac{2^7}{5j}$$

> La infinit avem formulă pentru reziduu:
> $$\operatorname{Rez}(f; \infty) = -\operatorname{Rez}\left(\frac{1}{z^2} \cdot f\left(\frac{1}{z}\right); 0\right)$$

$$f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^{12}} \cdot \sin 3\pi j z}{\left(\frac{1}{z^2} + 2\right)^5 \left(\frac{1}{z^2} + \frac{1}{z} + 6\right)} = \frac{\frac{z^{12}}{z^{12}} \cdot \sin 3\pi j z}{(1 + 2z^2)^5 (1 + jz + 6z^2)}$$