(C2) - GA

Polinomul caracteristic. Teorema Hamilton-Cayley. Teorema Laplace. Sist. de ec. algebrice de ord 1 cu mai multe necunoscute.

Def $A \in \mathcal{M}_n(\mathbb{C})$

$$P_A(X) = \det(A - X I_n) = \begin{vmatrix} a_{11}-X & a_{12} \dots & a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn}-X \end{vmatrix} =$$

$$= (-1)^n \left[X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} + \dots + (-1)^n \sigma_n\right]$$

polinomul caracteristic asociat matricei A.

σ_k = suma minorilor diagonali de ordinul k, $k=\overline{1,n}$

$A = (a_{ij})_{i,j=\overline{1,n}}$

$\sigma_1 = a_{11} + \dots + a_{nn}$

$$\sigma_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \quad (\exists\, C_n^2 \text{ minori diag. de ord } 2)$$

$$\sigma_3 = \sum_{1 \le i < j < k \le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \quad (\exists\, C_n^3 \text{ minori diag. de ord } 3)$$

$\vdots$

$\sigma_n = \det(A)$

Cazuri particulare

a) $n = 2$

$$P_A(X) = \begin{vmatrix} a_{11}-X & a_{12} \\ a_{21} & a_{22}-X \end{vmatrix} = X^2 - \underbrace{(a_{11}+a_{22})}_{\sigma_1 = TrA} X + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\sigma_2 = \det A};$$

b) $n = 3$

$$P_A(X) = (-1)^3 \left[X^3 - \underbrace{\sigma_1}_{Tr(A)} X^2 + \underbrace{\sigma_2}_{Tr(A^*)} X - \underbrace{\sigma_3}_{\det A}\right]$$

Teorema Hamilton-Cayley

$\forall A \in \mathcal{M}_n(\mathbb{C})$

$$P_A(A) = O_n \iff (-1)^n \left[A^n - \sigma_1 A^{n-1} + \dots + (-1)^n \sigma_n I_n\right] = O_n$$

Cazuri part.

a) $n=2$ $\quad A^2 - \underset{(a_{11}+a_{22})}{\underset{\parallel}{\sigma_1}} A + \underset{a_{11}a_{22}-a_{12}a_{21}}{\underset{\parallel}{\sigma_2}} I_2 = O_2$

b) $n=3$ $\quad A^3 - \underset{Tr(A)}{\underset{\parallel}{\sigma_1}} A^2 + \underset{Tr(A^*)}{\underset{\parallel}{\sigma_2}} A - \underset{\det(A)}{\underset{\parallel}{\sigma_3}} I_3 = O_3$

Aplicatii ale T.H-C

(1) Calculul lui A^{-1} $(\det A \neq 0)$

Ex $\quad A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Sa se calc. A^{-1}.

SOL

TH-C: $A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$

$\sigma_1 = Tr A = 3$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

$\sigma_3 = \det A = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$

$A^3 - 3A^2 + A - I_3 = O_3 \quad | \cdot A^{-1} \Rightarrow A^2 - 3A + I_3 - \underline{A^{-1}} = O_3$

$A^{-1} = A^2 - 3A + I_3 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

(2) Calculul lui A^n.

$A \in \mathcal{M}_2(\mathbb{C})$ $\quad$ T.H-C $\quad A^2 = \sigma_1 A - \sigma_2 I_2$

$A^n = x_n A + y_n I_2$; $n=1$ $\; A^1 = 1 \cdot A + 0 I_2$ $\quad x_1 = 1, x_2 = \sigma_1$

$n=2$ $\; A^2 = \sigma_1 A - \sigma_2 I_2$ $\quad y_1 = 0, y_2 = -\sigma_2$

$A^{n+1} = A^n \cdot A \Rightarrow$

$x_{n+1} A + y_{n+1} I_2 = x_n(\sigma_1 \underline{A} - \sigma_2 I_2) + y_n \underline{A}$

$= (\sigma_1 x_n + y_n) A - \sigma_2 x_n I_2$

$$\left\{\begin{array}{l} x_{n+1} = \sigma_1 x_n + y_n \\ y_{n+1} = -\sigma_2 x_n \Rightarrow y_n = -\sigma_2 x_{n-1} \end{array}\right\} \Rightarrow x_{n+1} - \sigma_1 x_n + \sigma_2 x_{n-1} = 0$$

$\forall n \geq 2$.

Recurența de ord 2.

Ex $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ a) Să se det $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ a.î. $A^n = x_n A + y_n I_2$.

Sol

T.H-C $A^2 = \sigma_1 A - \sigma_2 I_2$, $\sigma_1 = 3$, $\sigma_2 = 2$.

$A^2 - 3A + 2I_2 = O_2$

$x_{n+1} - 3x_n + 2x_{n-1} = 0$, $\forall n \geq 2$, $x_1 = 1$, $x_2 = \sigma_1 = 3$

Asociem ec. caracteristică:

$t^2 - 3t + 2 = 0 \Rightarrow (t-1)(t-2) = 0 \Rightarrow t_1 = 1$, $t_2 = 2$.

$x_n = c_1 t_1^n + c_2 t_2^n$, $\forall n \geq 1$.

$n=1 \Rightarrow$
$n=2$

$$\left\{\begin{array}{l} 1 = c_1 \cdot 1 + c_2 \cdot 2 \\ 3 = c_1 + c_2 \cdot 4 \end{array}\right. \ominus$$

$2 = / \quad 2c_2$

$\left\{\begin{array}{l} c_2 = 1 \\ c_1 = -1 \end{array}\right.$

$x_n = 2^n - 1$, $\forall n \geq 1$

$y_n = -\sigma_2 \cdot x_{n-1} = -2(2^{n-1} - 1) = 2 - 2^n$, $\forall n \geq 1$

$\Rightarrow A^n = (2^n - 1)A + (2 - 2^n)I_2$, $\forall n \geq 1$.

b) Fie $B = A^4 + A^3 + A^2 + A + I_2 = \underbrace{P_A(A)}_{O_2} \cdot C(A) + R(A) = 26A - 21I_2$

Să se det $a, b \in \mathbb{R}$ a.î. $B = aA + bI_2$.

$P = X^4 + X^3 + X^2 + X + 1$ $\quad P_A = X^2 - 3X + 2$; $P_A(A) = O_2$

$P = P_A \cdot C + \underset{aX+b}{R} = (X-1)(X-2)C + R$

$$\left\{\begin{array}{l} P(1) = a + b = 5 \\ P(2) = 2a + b = \frac{2^5 - 1}{2-1} = 31 \end{array}\right.$$

$a = 31 - 5 = 26$

$b = 5 - 26 = -21$

$R = 26X - 21$

③ Rez. de ec. binome în $\mathcal{M}_2(\mathbb{C})$

Ex Rez. în $\mathcal{M}_2(\mathbb{C})$ ec $X^4 = A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$ | det

SOL

$\det(X^4) = (\det X)^4 = \det A = 0 \Rightarrow \det X = 0$

T.H-C: $X^2 - Tr(X)X + \det(X) I_2 = O_2 \Rightarrow X^2 = Tr(X)X$

• Prop. $X^2 = \alpha X \Rightarrow X^n = \alpha^{n-1} X, \forall n \geq 2$

$X^4 = Tr(X)^3 \cdot X. \Rightarrow (*) A = Tr(X)^3 \cdot X$ | Tr

• Prop $Tr(\alpha \cdot X) = \alpha Tr(X), \forall \alpha \in \mathbb{C}$

$\Rightarrow Tr(A) = Tr^3(X) \cdot Tr(X) \Rightarrow \underset{1}{\underset{\parallel}{Tr(A)}} = Tr^4(X) \Rightarrow Tr X \in \{\pm 1; \pm i\}$

$(*) \Rightarrow X = \frac{1}{\pm 1} A = \pm A$

$X = \frac{1}{\pm i^3} A = \frac{1}{\mp i} A = \pm i A$

$X \in \{-A, A, -iA, iA\}$

OBS $X \in \mathcal{M}_2(\mathbb{R}) \Rightarrow X \in \{-A, A\}$.

Def $A \in \mathcal{M}_n(\mathbb{C})$

• minor de ordin p ($p \leq n$)

$\det(A_{IJ}) = \begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_p} \\ \vdots & & \\ a_{i_p j_1} & \dots & a_{i_p j_p} \end{vmatrix}$ $I = \{i_1, .., i_p\}$

$J = \{j_1, .., j_p\}$

$1 \leq i_1 < \dots < i_p \leq n$

$1 \leq j_1 < \dots < j_p \leq n$.

• minor complementar lui $\det(A_{I,J})$

$\det(A_{\bar{I}\bar{J}})$, unde $\bar{I} = \{1, .., n\} \setminus I$, $\bar{J} = \{1, .., n\} \setminus J$

(am suprimat liniile $l_{i_1}, \dots, l_{i_p}$
coloanele $c_{j_1}, \dots, c_{j_p}$)

- complementul algebric al minorului $\det(A_{IJ})$

$$(-1)^{i_1+\dots+i_p+j_1+\dots+j_p} \det(A_{\bar{I},\bar{J}})$$

<u>Teorema Laplace</u>

$\det(A)$ = suma produselor minorilor de ordinul p cu complemenții algebrici corespunzători, pentru p linii fixate (resp. p coloane fixate)

$$\det(A) = \sum_{1 \leq j_1 < \dots < j_p \leq n} (-1)^{i_1+\dots+i_p+j_1+\dots+j_p} \det(A_{IJ}) \cdot \det(A_{\bar{I}\bar{J}})$$

$$= \sum_{1 \leq i_1 < \dots < i_p \leq n} (-1)^{i_1+\dots+i_p+j_1+\dots+j_p} \det(A_{IJ}) \det(A_{\bar{I}\bar{J}})$$

<u>OBS</u> $p=1 \Rightarrow$ dezv. de pe o linie / resp de pe o coloană.

<u>Exemplu</u>

$$A = \begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix} \qquad \begin{array}{l} p=2 \\ l_1, l_2 \text{ fixate.} \end{array}$$

Calculați $\det A$, utilizând Th. Laplace.

<u>SOL</u>

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} +$$

$$+ (-1)^{1+2+1+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2+2+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} +$$

$$+ (-1)^{1+2+2+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2+3+4} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}$$

$$= 0 + (-1) 1 \cdot 18 + 1 \cdot 12 + 1 \cdot 7 - 1 \cdot 5 + (-1) \cdot 1$$

$$= -18 + 12 + 7 - 5 - 1 = \boxed{-5}$$

Sisteme de ecuații de ordinul 1 cu mai multe necunosc.

Fie sist $\circledast$ $AX \overset{!}{=} B$ $\quad$ $A \in \mathcal{M}_{m,n}(\mathbb{K})$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ $\quad$ $(\mathbb{K},+,\cdot)$ corp com.

$X \in \mathcal{M}_{n,1}(\mathbb{K})$ $\quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$B \in \mathcal{M}_{m,1}(\mathbb{K})$ $\quad B = \begin{pmatrix} b_1 \\ b_m \end{pmatrix}$

$S(A) = \{ x = (x_1, .., x_n) \in \mathbb{K}^n \mid AX = B \}$

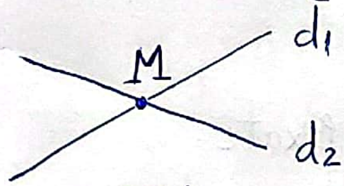
mulțimea sol. sist. $\circledast$

$\boxed{\text{OBS}}$ a) $S(A) \neq \phi$ $\begin{cases} \text{SCD} & \exists! \text{ soluție.} \\ \text{SCN} & \text{mai multe sol. / o inf.} \end{cases}$

b) $S(A) = \phi$ $\quad$ SI

Cazuri particulare

a) $n = 2$ $\quad \begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{cases}$ $\quad \cap$ 2 drepte în plan.

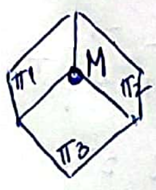


SCD

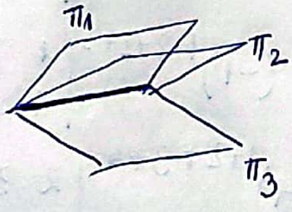
SCN

SI

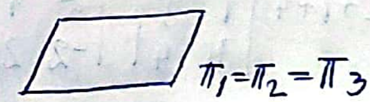
b) $n = 3$ $\quad \begin{cases} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \end{cases}$ $\quad \cap$ a 3 plane în spațiu.



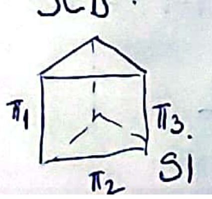
SCD.



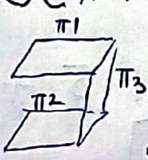
SCdN



SCdN.

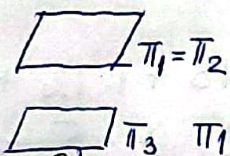


SI



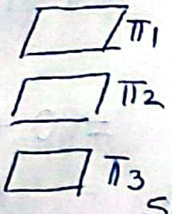
$\pi_1 \parallel \pi_2$

SI



$\pi_1 \parallel \pi_3$

SI



SI

Cazul general

(*) $AX = B$ $\qquad \bar{A} = (A|B)$ matricea extinsă

- Dacă $m = n$ și $\det A \neq 0$.

$$A^{-1} \cdot A X = A^{-1} B \Rightarrow X = A^{-1} B$$

$$(x_1, \dots, x_n) = \left(\frac{\Delta x_1}{\Delta}, \dots \frac{\Delta x_n}{\Delta} \right)$$

$\Delta = \det A$, Δ_k se obține din A înlocuind c_k cu coloana t. liberi.

T. Kronecker - Capelli

Un sist (*) este compatibil $\Leftrightarrow rg A = rg \bar{A}$

T. Rouché

Un sist. (*) este compatibil $\Leftrightarrow$ toți minorii caracteristici (dc ∃) sunt nuli

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Fie $\Delta_p = \det(A_{I,J}) \neq 0$, $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_r\}$, $rg A = r$

minor principal $\qquad r \leq \min\{m, n\}$.

Δ_c se obține prin bordare cu col. t. liberi și (minor caracteristic) adăugarea unei linii $l_i \in \{1, \dots, n\} \setminus I$

1) Dc ∃ un $\Delta_c \neq 0$ SI

2) Dc toți $\Delta_c = 0$ $\quad rg A = rg \bar{A} = r$. SC

Ip. (fără a restrânge generalitatea, ev. renumerotăm)

$$\Delta_p = \begin{vmatrix} a_{11} \dots & a_{1r} \\ \vdots & \\ a_{r1} \dots & a_{rr} \end{vmatrix}$$

(**) sist. format din primele r ec.
(celelalte ec sunt combinații liniare ale primelor r ec)

2a) Dacă $m > n$ (nr ec $>$ nr nec).

a_1) $\text{rg}A = \text{rg}\bar{A} = r = n$ SCD ($\nexists$ nec. secundare)

a_2) $\text{rg}A = \text{rg}\bar{A} = r < n$ SCN.

$x_1, \dots, x_r$ nec. principale, $x_{r+1}, \dots, x_n$ nec. secundare

nec. principale se exprimă în funcție de cele secundare

2b) Dacă $m < n$ (nr ec $<$ nr nec).

$\text{rg}A = \text{rg}\bar{A} = r \le m$ SCN.

<u>Obs</u> SLO

(*) $AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m,1}(\mathbb{K})$

(T) Un SLO este totdeauna compatibil.

a) $m = n$ $\to$ $\det A \neq 0$ $\exists!$ sol nulă

$\searrow$ $\det A = 0$ $\exists$ și sol nenule.

b) $m \neq n$. b_1) $m > n$ $\text{rg}A = \text{rg}\bar{A} = n$ SCD.

b_2) $m < n$ $\text{rg}A = \text{rg}\bar{A} \le m$ SCN.

<u>Def</u> 2 sist s.n. echivalente $\Leftrightarrow$ au aceeași mult de sol

<u>Prop</u> (*) și (**) sunt echivalente.