ccc Seminar nr. 7 : Vectori/valori proprii . Diagonalizare

sss

5/1 $f : \mathbb{R}^4 \to \mathbb{R}^4$, $f(x) = (x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4)$

a) Să se afle valorile proprii

b) Precizați care sunt subspațiile proprii

c) $\exists$ un reper $\mathcal{R}$ în $\mathbb{R}^4$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ este diagonală?

Sol.: $\mathcal{R}_0 = \{e_1^0, \dots, e_4^0\}$ reper canonic în $\mathbb{R}^4$

$$[f]_{\mathcal{R}_0,\mathcal{R}_0} = A$$

$$AX = Y$$

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 + x_4 \\ -\prime\prime- \\ x_4 \\ x_4 \end{pmatrix}$$

$$P_A(\lambda) = \det(A - \lambda I_4) =$$

$$= \begin{vmatrix} -\lambda & 1 & -1 & 1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = \lambda^2 (1-\lambda)^2 = 0$$

$\lambda_1 = 0$, $m_1 = 2$

$\lambda_2 = 1$, $m_2 = 2$

b) $V_{\lambda_1} = \{x \in \mathbb{R}^4 \mid f(x) = \lambda_1 x\} = \ker f$

$$AX = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_1} = 4 - rg A = 4 - 2 = 2 = m_1$$

$$\begin{cases} x_2 - x_3 + x_4 = 0 \\ x_4 = 0 \end{cases} \Rightarrow x_2 = x_3$$

$$V_{\lambda_1} = \{(x_1, x_2, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$$
$$= \{x_1(1,0,0,0) + x_2(0,1,1,0) \mid x_1, x_2 \in \mathbb{R}\}$$
$$= \langle\{\underbrace{(1,0,0,0)}_{e_1}, \underbrace{(0,1,1,0)}_{e_2}\}\rangle$$

$\mathcal{R} = \{e_1, e_2\}$ reper în V_{λ_1}

$$V_{\lambda_2} = \{x \in \mathbb{R}^4 \mid f(x) = \lambda_2 x = x\}$$
$$AX = X$$
$$(A - I_4)X = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_2} = 4 - rg(A - I_4)$$

$$(A - I_4)X = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + x_2 - x_3 + x_4 = 0 \\ -x_3 + x_4 = 0 \Rightarrow x_4 = x_3 \end{cases} \Rightarrow x_2 = x_1$$

$$V_{\lambda_2} = \{(x_1, x_1, x_3, x_3) \mid -''-\} = \{x_1(1,1,0,0) + x_3(0,0,1,1) \mid -''-\} =$$
$$= \langle\{\underbrace{(1,1,0,0)}_{e_3}, \underbrace{(0,0,1,1)}_{e_4}\}\rangle$$

c) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 = \{e_1, e_2, e_3, e_4\}$ reper în $\mathbb{R}^4$ de vec. proprii

$[f]_{\mathcal{R},\mathcal{R}} = ?$

$f(e_1) = 0$

$f(e_2) = 0$

$f(e_3) = \lambda_2 \cdot e_3 = e_3$

$f(e_4) = e_4$

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} \boxed{0} & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} = A'$$

λ_1

λ_2

$$A' = C^{-1} A C$$

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}, \quad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = C A' C^{-1}$$

$$A^n = C (A')^n C^{-1} = C A' C^{-1}$$

5/4. $f \in End(\mathbb{R}^3)$

$\lambda_1 = 3, \quad v_1 = (-3, 2, 1)$

$\lambda_2 = -2, \quad v_2 = (-2, 1, 0)$

$\lambda_3 = 1, \quad v_3 = (-6, 3, 1)$

$\llcorner$ val. proprii $\quad \llcorner$ vec. proprii corespunzători

$$A = [f]_{\mathcal{R}_0, \mathcal{R}_0} = ?$$

Sol.: $f(v_1) = \lambda_1 v_1 = 3 v_1$

$f(v_2) = \lambda_2 v_2 = -2 v_2$

$f(v_3) = \lambda_3 v_3 = v_3$

v_1, v_2, v_3 vec. proprii corесp. la val. proprii distincte $\xRightarrow{\text{Prop.}}$

$\Rightarrow \{v_1, v_2, v_3\}$ SLI

$\dim \mathbb{R}^3 = 3$ $\Big\} \Rightarrow \mathcal{R} = \{v_1, v_2, v_3\}$ reper

$$A' = [f]_{\mathcal{R}, \mathcal{R}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}, \quad C = ?$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$A' = C^{-1} A C$

$A = C A' C^{-1}$