CCC / SSS

Seminar săpt. 5: Op. cu sp. vec. Aplicații liniare

3. $(\mathbb{R}^3, +, \cdot)/_{\mathbb{R}}$

$V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0\}$

$V_2 = \langle\{(1, -1, 2), (3, 1, 0)\}\rangle$

a) Să se descrie V_2 printr-un sis. de ec. liniare

b) Precizați câte un reper în $V_1, V_2, V_1 + V_2, V_1 \cap V_2$

c) Este $V_1 + V_2$ sumă directă?

Sol.:

a)

$$\operatorname{rg}\begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 0 \end{pmatrix} = 2 = \max \Rightarrow \{u, v\} \text{ SLI} \Rightarrow \dim V_2 = 2$$

$$\begin{array}{cc} u & v \end{array}$$

$\forall x \in V_2, \exists a, b \in \mathbb{R}$ a.î. $x = au + bv$

$(x_1, x_2, x_3) = a(1, -1, 2) + b(3, 1, 0)$

$= (a + 3b, b - a, 2a)$

$$(*)\begin{cases} a + 3b = x_1 \\ -a + b = x_2 \\ 2a = x_3 \end{cases} \qquad \left(\begin{array}{cc|c} 1 & 3 & x_1 \\ -1 & 1 & x_2 \\ 2 & 0 & x_3 \end{array}\right)$$

$(*)$ sistem compatibil (SC) $\Leftrightarrow \operatorname{rang} A = \operatorname{rang} \bar{A} = 2$

$$\Delta_c = \begin{vmatrix} 1 & 3 & x_1 \\ -1 & 1 & x_2 \\ 2 & 0 & x_3 \end{vmatrix} = 0 = \begin{vmatrix} 1 & \cancel{3} & \cancel{x_1} \\ \cancel{0} & 4 & x_1 + x_2 \\ \cancel{0} & -6 & x_3 - 2x_1 \end{vmatrix} = 4(x_3 - 2x_1) + 6(x_1 + x_2) = 0$$

$-2x_1 + 6x_2 + 4x_3 = 0$

$V_2 = \{x \in \mathbb{R}^3 \mid -x_1 + 3x_2 + 2x_3 = 0\} = S(A')$

$A' = (-1 \quad 3 \quad 2)$

$\dim V_2 = 3 - \operatorname{rg} A' = 3 - 1 = 2$

b) $V_1 = \{(x_1, x_2, -2x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} = \langle \underbrace{\{(1,0,-2),(0,1,1)\}}_{\mathcal{R}_1} \rangle$

$\mathcal{R}_1$ SG

$\dim V_1 = 3 - rg(2 \ \ -1 \ \ 1) = 3 - 1 = 2 = card\, \mathcal{R}_1$ $\Rightarrow \mathcal{R}_1$ reper în V_1

$\mathcal{R}_2 = \{u, v\}$ reper în V_2

Met. 1:

$\mathcal{R}_1$ SLI

$rg\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{pmatrix} = 2$ max $\Rightarrow$

$\Rightarrow \mathcal{R}_1$ SLI și SG

Met. 2:

V n-dim

$\mathcal{R} = \{e_1, \dots, e_n\}$

UAE.: 1) $\mathcal{R}$ reper

2) $\mathcal{R}$ SLI

3) $\mathcal{R}$ SG

<u>T. Grassmann</u>:

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

$V_1 \cap V_2 = \{x \in \mathbb{R}^3 \mid \begin{cases} -x_1 + 3x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}\} = S(A'')$

$A'' = \begin{pmatrix} -1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

$\dim(V_1 \cap V_2) = 3 - rg A'' = 3 - 2 = 1$

$\left.\begin{array}{l} \dim(V_1 + V_2) = \dim \mathbb{R}^3 = 3 \\ V_1 + V_2 \subset \mathbb{R}^3 \text{ sp. vec.} \end{array}\right| \Rightarrow V_1 + V_2 = \mathbb{R}^3$

$\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reper în $\mathbb{R}^3$ $\Rightarrow$

$\Rightarrow \mathcal{R}_0$ reper în $V_1 + V_2$

$\left.\begin{array}{l} V_1 + V_2 \subset \mathbb{R}^3 \text{ sp. vec.} \\ \dim(V_1 + V_2) = \dim \mathbb{R}^3 = 3 \end{array}\right| \Rightarrow$

$\Rightarrow V_1 + V_2 = \mathbb{R}^3$

c) $\dim(V_1 \cap V_2) = 1 \Rightarrow V_1 \cap V_2 \neq \{0_V\} \Rightarrow$

$\Rightarrow V_1 + V_2$ <u>nu</u> este suma directă

<u>Prop.</u>: $V' \subseteq V$ sp.

$\dim V' = \dim V = n \Rightarrow$ ~~dim~~ $V' = V$

8. $(\mathbb{R}^4, +, \cdot)|_{\mathbb{R}}$, $V = \{x \in \mathbb{R}^4 \mid \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_4 = 0 \end{cases}\}$

a) Det. $V' \subset \mathbb{R}^4$ a.î. $V \oplus V' = \mathbb{R}^4$

b) Precizați un reper $\mathcal{R}'' = \mathcal{R}' \cup \mathcal{R}$ în $\mathbb{R}^4$ a.î. $\mathcal{R}$ reper în V și $\mathcal{R}'$ reper în V'

c) Aflați coord. lui $x = (1, 2, -1, 3)$ în rap. cu $\mathcal{R}'$ și descompuneți x în rap cu $\mathbb{R}^4 = V \oplus V'$

d) Generalizare pt. un spațiu n-dim

Sol.:

a) și b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$V = S(A)$; $\dim V = 4 - rg A = 2 = 2$

$\begin{cases} x_1 + x_2 = -s - t \\ x_1 = -t \end{cases}$, unde $x_3 = s$, $x_4 = t$

$x_2 = -s - t + t = -s$

$V = \{(-t, -s, s, t) \mid s, t \in \mathbb{R}\} = \langle \underbrace{\{(\overset{u}{-1, 0, 0, 1}), (\overset{v}{0, -1, 1, 0})\}}_{\mathcal{R}} \rangle$

$(-t, -s, s, t) = t(-1, 0, 0, 1) + s(0, -1, 1, 0)$

$\left.\begin{array}{l} \mathcal{R} \text{ SG} \\ \dim V = 2 = |\mathcal{R}| \end{array}\right| \Rightarrow \mathcal{R}$ reper în V

$\mathcal{R} = \{u, v\}$ reper în V

Extindem $\mathcal{R}$ la un reper $\mathcal{R}''$ în $\mathbb{R}^4$

$rg \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4 = max$

$\mathcal{R}'' = \mathcal{R} \cup \underbrace{\{e_1, e_3\}}_{= \mathcal{R}'}$

$V' = \langle \mathcal{R}' \rangle$ $\qquad \mathbb{R}^4 = V \oplus V'$

c) $x=(1,2,-1,3)$ coord în rap. cu $\mathcal{R}''=\{u,v,e_1,e_3\}$

$\exists\, a,b,c,d\in\mathbb{R}$ a.î. $x=au+bv+ce_1+de_3$

$(1,2,-1,3)=(-a,0,0,a)+(0,-b,b,0)+(c,0,0,0)+(0,0,d,0)$

$=(-a+c,-b,b+d,a)$

$$\begin{cases}-a+c=1\\ -b=2\\ b+d=-1 \Rightarrow d=1\\ a=3\end{cases} \Rightarrow \begin{cases}a=3\\ b=-2\\ c=4\\ d=1\end{cases}$$

$(a,b,c,d)=(3,-2,4,1)$ coord lui x în rap. cu $\mathcal{R}''$

$$x=\underbrace{(au+bv)}_{\substack{\in V\\ w}}+\underbrace{(ce_1+de_3)}_{\substack{\in V'\\ w'}}$$

$w=3(-1,0,0,1)-2(0,-1,1,0)=(-3,2,-2,3)$

$w'=4(1,0,0,0)+1(0,0,1,0)=(4,0,1,0)$

d) $(\mathbb{R}^n,+,\cdot)/_{\mathbb{R}}$

$$V=\left\{x\in\mathbb{R}^n \,\middle|\, \begin{cases}x_1+\ldots+x_n=0\\ x_1+x_n=0\end{cases}\right\}$$

$$A=\begin{pmatrix}\boxed{\begin{matrix}1 & 1\\ 1 & 0\end{matrix}} & \begin{matrix}\ldots & 1 & 1\\ \ldots & 0 & 1\end{matrix}\end{pmatrix},\ rg\,A=2$$

$\dim V=n-rg\,A=n-2$

$$\begin{cases}x_1+x_2=-x_3-x_4-\ldots-x_n\\ x_1=-x_n\end{cases} \Rightarrow x_2=-x_3-x_4-\ldots-x_{n-1}$$

$V=\{(-x_n,-x_3-x_4-\ldots-x_{n-1},x_3,x_4,\ldots,x_n)\}$

$=\langle\{(0,-1,1,0,\ldots,0),(0,-1,0,1,\ldots,0),\ldots,(-1,0,\ldots,0,1)\}\rangle$

$$\operatorname{rg}\begin{pmatrix} 0 & 0 & & 0 & -1 & 0 & 0 \\ -1 & -1 & & -1 & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 & 1 & 0 \end{pmatrix}$$

$V' = \langle \{e_n, e_2\} \rangle$

2/ rg 3

$f: \mathbb{R}^3 \to \mathbb{R}^3,\ f(x) = (x_1 + 2x_2 + x_3,\ 2x_1 + 5x_2 + 3x_3,\ -3x_1 - 7x_2 - 4x_3)$

a) f liniară b) $\ker f = ?$, $\operatorname{Im} f = ?$ Prec. câte un reper în $\ker f$ și în $\operatorname{Im} f$

Sol.: Metoda 1:

a) $f(ax + by) = a f(x) + b f(y)$

Metoda 2:

$f(x) = y \iff Y = AX$

$$A = \begin{pmatrix} x_1 + 2x_2 + x_3 \\ 2x_1 + 5x_2 + 3x_3 \\ -3x_1 - 7x_2 - 4x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

b) $\ker f = \{ x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3} \}$

$= \{ x \in \mathbb{R}^3 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} = S(A)$

$\dim \ker f = 3 - \operatorname{rg} A = 3 - 2 = 1$

$$\begin{cases} \boxed{x_1 + 2x_2} + x_3 = 0 \\ \boxed{2x_1 + 5x_2} + 3x_3 = 0 \\ -3x_1 - 7x_2 - 4x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 = -\alpha \quad | \cdot 2 \\ 2x_1 + 5x_2 = -3\alpha \end{cases} \quad (+)$$

$x_2 = -\alpha \Rightarrow x_1 = \alpha$

$\ker f = \{ (\alpha, -\alpha, \alpha) \mid \alpha \in \mathbb{R} \} = \langle \overbrace{\{(1, -1, 1)\}}^{R_1} \rangle$

__Teorema dim.__ $f: V \to W$ lin.

$$\dim V = \dim \ker f + \dim \operatorname{Im} f$$
$$\underset{3}{\shortparallel} \qquad \underset{1}{\shortparallel} \qquad \underset{2}{\Downarrow}$$

__Metoda 1:__

$\mathcal{R}_1 = \{(1,-1,1)\}$ reper în ker f. Extindem la unul în $\mathbb{R}^3$

$$rg \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3 \quad \mathcal{R}_1 \cup \{e_2, e_3\}$$

$\mathcal{R}_2 = \{f(e_2), f(e_3)\}$ reper în Im f

$f((0,1,0)) = (2,5,-7)$ $\quad$ Im $f = \langle \{\mathcal{R}_2\} \rangle$

$f((0,0,1)) = (1,3,-4)$

__Metoda 2:__

$$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \text{ a.î. } f(x) = y\}$$

$$(*) \begin{cases} x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 - 7x_2 - 4x_3 = y_3 \end{cases} \quad A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & y_1 \\ 2 & 5 & 3 & y_2 \\ -3 & -7 & -4 & y_3 \end{array}\right)$$

Sis. (*) este compatibil, $\underset{2}{rg A} = rg \bar{A} \iff \Delta_c = 0$

$$\Delta_c = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = 0 = (y_1 + y_2 + y_3) \cdot (-1)^4 \cdot \begin{vmatrix} 2 & 5 \\ -3 & -2 \end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow y_1 + y_2 + y_3 = 0$

$$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 0\} = \{(y_1, y_2, -y_1 - y_2)\} =$$
$$= \langle \underbrace{\{(1,0,-1), (0,1,-1)\}}_{\mathcal{R}_2} \rangle$$

$\dim \operatorname{Im} f = 2 = |\mathcal{R}_2| \Rightarrow \mathcal{R}_2$ reper în Im f

3. $f:\mathbb{R}^2 \to \mathbb{R}^3,\ f(x) = (3x_1 - 2x_2,\ 2x_1 - x_2,\ -x_1 + x_2)$

a) f liniară b) f inj. c) $\operatorname{Im} f = ?$

<u>Sol.:</u>

a) $f(x) = y \iff Y = AX$

$$\begin{pmatrix} 3x_1 - 2x_2 \\ 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ ✓ f liniară

b) $\ker f = \{x \in \mathbb{R}^2 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\} = S(A)$

$(3,2)\ (2,1)$

$\operatorname{rg} A = 2$ (max) $\Rightarrow$ SCD $\dim \ker f = 2 - \operatorname{rg} A = 2 - 2 = 0$

$\exists!\ (0,0) \in \ker f \Rightarrow f$ inj. $\Rightarrow \ker f = \{0_{\mathbb{R}^2}\}$

$$(**) \begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \qquad \text{SC} \qquad \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0$$

$\forall y \in \operatorname{Im} f,\ \exists x \in \mathbb{R}^2$ a.î. $f(x) = y$

$(**)$ SG

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid 1 - y_2 + y_3 = 0\} = \{(y_1, y_2, y_2 - y_1)\} =$

$= \langle\{(1,0,-1), (0,1,1)\}\rangle$