$\boxed{C_1}$ nesingulară / inversabilă $\Leftrightarrow \det \neq 0$ $A^{-1} = \frac{1}{\det A}\cdot A^*$ $A \to A^T \to A^*$ $A^*_{ij} = (-1)^{i+j}\delta_{ij}$

$\det(A^{-1}) = \frac{1}{\det A}$ $\det(A^*) = (\det A)^{n-1}$

$(GL(n,\mathbb{K}) = \{A \in \mathcal{M}_n(\mathbb{K}) / \det A \neq 0\}, \cdot)$ grup general liniar $(SL(n) = \{A \in GL(n,\mathbb{K}) / \det A = 1\}, \cdot)$ grup special liniar

$(O(n) = \{A \in \mathcal{M}_n(\mathbb{K}) / A \cdot A^T = I_n\}, \cdot)$ grup ortogonal $(SO(n) = \{A \in O(n) / \det A = 1\}, \cdot)$ grup special ortogonal

$SO(n) = O(n) \cap SL(n)$

formă eșalon $A = \begin{pmatrix} x & - & \text{pivoți} & \\ 0 & x & & \vdots \\ 0 & 0 & x & \\ 0 & 0 & 0 \ldots & 0 \end{pmatrix}$ formă eșalon redusă – toți pivoții sunt 1 – deasupra pivoților toate elem. sunt 0

Gauss-Jordan: $(A|I_n) \sim (C|B)$, C = forma eșalon redusă pt. A. Dacă $\exists A^{-1}$, atunci $C = I_n$, $B = A^{-1}$

$\boxed{C_2}$ $P_A(X) = \det(A - XI_n) = (-1)^n[X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} - \ldots + (-1)^n \sigma_n]$ polinom caracteristic

σ_k = suma minorilor diagonali de ordin K

$n = 2 \Rightarrow P_A(X) = X^2 - Tr(A)X + \det A$; $n = 3 \Rightarrow P_A(X) = (-1)^3[X^3 - \underbrace{\sigma_1}_{TrA} X^2 + \sigma_2 X - \underbrace{\sigma_3}_{\det A}]$ >

$\sigma_2 = Tr(A^*) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Hamilton-Cayley: $P_A(A) = (-1)^n [A^n - \sigma_1 A^{n-1} + \sigma_2 A^{n-2} - \ldots + (-1)^n \sigma_n I_n] = O_n$

$A^2 = Tr(A) \cdot A - \det(A) \cdot I_2$

$A^n = x_n A + y_n I_2, \forall n \geq 1$, $x_1 = 1, y_1 = 0, x_2 = TrA, y_2 = -\det A$, $A^{n+1} = A^n \cdot A \to \underline{x_{n+1}} A + \underline{y_{n+1}} I_2 =$

$= x_n A^2 + y_n A = x_n(\sigma_1 A - \sigma_2 I_2) + y_n A = \underline{(\sigma_1 x_n + y_n)} A \underline{- x_n \sigma_2} I_2 \Rightarrow x_{n+1} - \sigma_1 x_n + x_{n-1} \sigma_2 = 0$

ex: $x_{n+1} - 3x_n + 2x_{n-1} = 0$, $x_1 \to 1$, $x_2 \to 3$; ec. carac.: $t^2 - 3t + 2 = 0 \Rightarrow t \in \{1, 2\}$

$x_n = c_1 t_1^n + c_2 t_2^n = c_1 + c_2 \cdot 2^n$ $c_1 + 2c_2 = 1$ & $c_1 + 4c_2 = 3 \Rightarrow c_1 = -1, c_2 = 1 \Rightarrow x_n = -1 + 2^n$

$y_n = -\sigma_2 \cdot x_{n-1} = 2 - 2^n$

T. Laplace $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix} \leftarrow$ $p = 2, l_1, l_2$ fixate; $\det A = (-1)^{1+2+1+2} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \ldots + (-1)^{1+2+3+4} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}$

$\boxed{C_3}$ $\mathcal{M}_n^s(\mathbb{R}) = W = \{A \in \mathcal{M}_n(\mathbb{R}) / A = A^T\}$ ssp. vect. al matricelor simetrice

$\mathcal{M}_n^a(\mathbb{R}) = W' = \{A \in \mathcal{M}_n(\mathbb{R}) / A = -A^T\}$ ssp. vect. al matr. antisimetrice

$<S> = \{x \in V / x = a_1 x_1 + \ldots + a_n x_n, x_1, \ldots, x_n \in S\}$. Dacă $V = <S> \Rightarrow S = SG$ (sist. de generatori)

$S \subseteq V$ subm. nevidă, $(V,+,\cdot)/_{\mathbb{K}}$ • S = SLi $\Leftrightarrow \forall x_1, \ldots, x_n \in S$, $\forall a_1, \ldots, a_n \in \mathbb{K}$ a.î. $a_1 x_1 + \ldots + a_n x_n = 0_V \Rightarrow a_i =$

S = SLD $\Leftrightarrow \exists x_1, \ldots, x_n \in S$, $\exists a_1, \ldots, a_n \in \mathbb{K}$, nu toți nuli a.î. $a_1 x_1 + \ldots + a_n x_n = 0_V$

S = bază $\Leftrightarrow$ SLi + SG

• ∀ subm. a unui SLi e SLi   • ∀ supram. a unui SLD este SLD   • ∀ supram. a unui SG este SG

$\forall B_1, B_2$ baze în V $\Rightarrow |B_1| = |B_2| = n = \dim_K V$

• $\dim_K V = n$ → nr. max. de vectori care form. SLi / nr. minim de vect. care form. SG

ca un sist. să fie SLi $\Leftrightarrow$ rang A = max. (det. liniar indep.

$\dim(S(A)) = \dim \mathbb{K} - rgA$, $S(A) = \{x \in \mathbb{K} / AX = 0\}$

• S = SG $\Leftrightarrow \forall x \in V, \exists a_1, \ldots, a_n \in \mathbb{K}$ a.î. $x = a_1 x_1 + \ldots + a_n x_n$

$\boxed{C_4}$ T. schimbului: $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect., f. gen. $\{x_1, \ldots, x_n\}$ SG, $\{y_1, \ldots, y_m\}$ SLi $\Rightarrow \{y_1, \ldots, y_m\}$ SG

card ∀SG (finit) ≥ card ∀SLi

$$\begin{tikzcd} \mathcal{R} = \{e_1, \ldots, e_n\} \arrow[r, "A"] \arrow[rr, bend right, "C"'] & \mathcal{R}' = \{e_1', \ldots, e_n'\} \arrow[r, "B"] & \mathcal{R}'' = \{e_1'', \ldots, e_n''\} \end{tikzcd}$$

$\Rightarrow C = AB$

$A \in GL(n, \mathbb{K})$

$X = AX'$ $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $X' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$

$\mathcal{R}, \mathcal{R}'$ repere la fel orientate $\Rightarrow \det A > 0$

$V_1 + V_2 = \{v_1 + v_2 / v_1 \in V_1, v_2 \in V_2\} = <V_1 \cup V_2>$

$V_1 \oplus V_2$: sumă directă dacă $V_1 \cap V_2 = \{0_V\}$

$\boxed{C_5}$ Th. Grassmann

$(V,+,\cdot)/_{\mathbb{K}}$ sp. vect. finit generat și $V_1, V_2 \subset V$ ssp. vect. $\Rightarrow \dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$, unde $V_1 + V_2 = <V_1 \cup V_2>$

$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$

$V = V_1 \oplus V_2$, $\mathcal{R}_K$ reper în V_K $\Rightarrow \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper în V

$S(A) = \{X \in \mathbb{K}^n / AX = 0\} \Rightarrow S(A) \subseteq \mathbb{K}^n$ și $\dim S(A) = n - rgA$

$End(V) = \{f : V \to V / f \text{ liniară}\}$ $Aut(V) = \{f \in End(V) / f \text{ bij}\}$

$f : V_1 \to V_2$ aplicație liniară $\Leftrightarrow$ 1) $f(x+y) = f(x) + f(y)$; 2) $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in K$, $\forall x, y \in V_1$ $(\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y))$

f inj. $\Leftrightarrow Ker f = \{x \in V_1 / f(x) = 0_{V_2}\} = \{0_{V_1}\} \Leftrightarrow \dim V_1 = \dim Im f$

f surj. $\Leftrightarrow \dim Im f = \dim V_2$ f bij. $\Leftrightarrow \dim V_1 = \dim V_2$

f apl. lin:
a) f inj. $\Leftrightarrow \dim V_1 = rgA$
b) f surj. $\Leftrightarrow \dim V_2 = rgA$
c) f bij. $\Leftrightarrow A \in GL(n, \mathbb{K})$

$\dim V_1 = \dim Ker f + \dim Im f$ $V_1 \simeq V_2 \Leftrightarrow \dim V_1 = \dim V_2$

$\mathcal{R}_1 = \{e_1, \ldots, e_n\} \xrightarrow{A} \mathcal{R}_2 = \{e_1', \ldots, e_m'\}$, $A = [f]_{\mathcal{R}_1, \mathcal{R}_2}$, $\underline{Y = AX}$, $\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$f(e_i) = \sum_{j=1}^{m} a_{ji} e_j'$

$$\begin{tikzcd} \mathcal{R}_1 \arrow[r, "A"] \arrow[d, "C"'] & \mathcal{R}_2 \arrow[d, "\Delta"] \\ \mathcal{R}_1' \arrow[r, "A'"] & \mathcal{R}_2' \end{tikzcd}$$

$\Rightarrow \underline{A' = \Delta^{-1} A C}$ $rg A = rg A'$

$\boxed{C7}$ $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect. $(V^* = \{f : V \to \mathbb{K} \,/\, f$ ap. liniară$\}, +, \cdot)/\mathbb{K}$ spațiul vectorial dual lui V

$(f+g)(x) := f(x) + g(x)$ $\quad (\alpha f)(x) := \alpha f(x)$ $\quad V \simeq V^*$

$x \in V$ s.m. vector propriu $\iff \exists \lambda \in \mathbb{K}$ a.î. $f(x) = \lambda x$, λ = val. proprie

Proiecții & simetrii $\qquad$ p=proiecția $\Rightarrow p \circ p = p$

$p : V_1 \oplus V_2 \to V_1 \oplus V_2$ apl. lin., proiecție pe $V_1 \iff p(\underset{\overset{\cap}{V_1}}{x_1} + \underset{\overset{\cap}{V_2}}{x_2}) = x_1$

$s \in End(V)$ s s.m. simetrie $\iff s \circ s = id\,V$

p proiecție pe $V_1 \iff s = 2p - id_V$ este simetrie față de V_1

$\boxed{C8}$ $g : V \times V \to \mathbb{K}$ s.m. formă biliniară $\iff g$ e liniară în fiecare argument i.e. $g(\alpha x + \beta y, z) =$
$= \alpha g(x,z) + \beta g(y,z)$; $g(x, \alpha y + \beta z) = \alpha g(x,y) + \beta g(x,z)$

g s.m. f. simetrică $\iff g(x,y) = g(y,x)$ / antisimetrică $\iff g(x,y) = -g(y,x)$

g simetrică & liniară $\Rightarrow$ biliniară $\qquad L^s(V,V;\mathbb{K}) = \{g \in L(V,V;\mathbb{K}) \,/\, g$ sim$\}$

matricea asociată $G = (g_{ij})_{i,j=\overline{1,n}}$, $g_{ij} = g(e_i, e_j)$, $\mathcal{R} = \{e_1, \dots, e_n\}$

$\mathcal{R} \xrightarrow{C} \mathcal{R}' \Rightarrow G' = C^T G C$ $\quad$ rg G' = rg G invariant la sch. reperelor

Aplicația $Q : V \to \mathbb{K}$ form. pătratică $\iff \exists g \in L^s(V,V;\mathbb{K})$ a.î. $Q(x) = g(x,x)$, $\forall x \in V$

$Q(x) = \sum_{i=1}^{n} g_{ii} x_i^2 + 2 \sum_{i<j} g_{ij} x_i x_j$ / g = nedegenerată $\iff$ Ker g $= \{0_V\}$ / det $G \neq 0$

$x \in$ Ker g $\iff g(x, e_1) = \dots = g(x, e_n) = 0$

Q = pozitiv definită $\iff Q(x) > 0$, $\forall x \neq 0_V$ și $Q(x) = 0 \iff x = 0_V$

g forma polară asociată. g = poz. def. $\iff$ Q poz. def.

$g \in L^s(V,V;\mathbb{K})$ g = poz. def $\Rightarrow$ g nedegenerată

Q poz. def $\iff$ signatura $(n, 0)$

$\boxed{C9}$ $g : V \times V \to \mathbb{R}$ s.m. produs scalar $\iff$ g formă biliniară simetrică + pozitiv definită.

(V,g) s.v.e.r. $\quad$, $\mathcal{R} = \{e_1, \dots, e_n\}$ reper $\to$ ortogonal $\iff g(e_i, e_j) = 0$, $\forall i \neq j$.
$\qquad \searrow$ ortonormat $\iff g(e_i, e_j) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

$\mathcal{R} \xrightarrow{C} \mathcal{R}'$ repere ortonormate $\Rightarrow [C \in O(n) \iff C \cdot C^T = I_n]$

prod. vectorial $x \times y$ astfel: I) $S = \{x, y\}$ SLD $\Rightarrow x \times y = 0$ $\quad$ II) $S = \{x,y\}$ SLI $\Rightarrow$

1) $\|x \times y\|^2 = \begin{vmatrix} g_0(x,x) & g_0(x,y) \\ g_0(y,x) & g_0(y,y) \end{vmatrix}$ $\quad$ 2) $g_0(x \times y, y) = 0$, $g_0(x \times g, x) = 0$

3) $\mathcal{R} = \{x, y, x \times y\}$ e reper pozitiv orientat

$x \times y = -y \times x$

$z \wedge x \wedge y = g_0(z, x \times y) = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

$x \in V$, $x^\perp = <\{x\}>^\perp = \{y \in V \,/\, g(x,y) = 0\} \subset V$

$U^\perp = \{y \in V \,/\, g(x,y) = 0, \forall x \in U\}$ $\qquad U, U^\perp \subset V$ subsp. vect.

$\boxed{C10}$ $(E_i, <\cdot,\cdot>_i)$, $i = \overline{1,2}$ s.v.e.r. $f : E_1 \to E_2$ liniară, f aplicație ortogonală $\iff$
$<f(x), f(y)>_2 = <x, y>_1$

$f \in End(E)$; f transf. ortogonală $\iff <f(x), f(y)> = <x, y>$

$\|x\|_1 = \sqrt{<x,x>_1}$, f apl. ort. $\Rightarrow \|f(x)\|_2 = \|x\|_1$, $\forall x \in E_1$ și f inj.

$O(E) = \{f : E \to E$ liniară / f transf. ortogonală$\}$

$f \in O(E) \iff \|f(x)\| = \|x\|$, $\forall x \in E$

$n = 1 \to O(E) = \{id_E, -id_E\}$

$n = 2 \Rightarrow \mathcal{R} = \{e_1, e_2\}$ reper ortonormat în E $\qquad$ toate răd. polin. car. sunt reale

$Sim(E) = \{f \in End(E) \,/\, f$ simetric$\}$; $f \in Sim(E) \iff A = [f]_{\mathcal{R},\mathcal{R}}$ = simetrică, $\forall \mathcal{R}$ = rep. orto

$A = A^T \longrightarrow Q : E \to \mathbb{R}$, $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

$\searrow$ $f \in Sim(E)$ $\quad [f]_{\mathcal{R},\mathcal{R}} = A$

$<x, f(x)> = Q(x)$

C11 $f \in End(E)$ simetric $\Rightarrow \dim V_{\lambda_i} = m_i$, λ_i = val. proprie

$E = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_K}$ $\quad \mathcal{R}' = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_K$

$(\mathcal{A}, V/_{\mathbb{K}}, \varphi)$ spațiu afin: $\mathcal{A} \neq \emptyset$; $V/_{\mathbb{K}}$ sp. vect. director ; $\varphi : A \times A \to V$ struct. afină ($\varphi(A,B) + \varphi(B,C) = \varphi(A,C)$; $\exists\, O \in \mathcal{A}$ a.î. $\varphi_0 : \mathcal{A} \to V$, $\varphi_0(A) = \varphi(O,A)\ \forall A \in \mathcal{A}$)

$\varphi(A,B) = \overrightarrow{AB}$; $\dim \mathcal{A} = \dim V = m$

- Ec. unei drepte afine în $\mathbb{R}^n$. $A \in \mathcal{D}$. $V_{\mathcal{D}} = <\{u\}>$, $u \neq 0_{\mathbb{R}^n}$

$\forall M \in \mathcal{D} \Rightarrow \overrightarrow{AM} \in V_{\mathcal{D}}$ $\quad \overrightarrow{OA} = \sum_{i=1}^{m} a_i e_i$; $A(a_1, \ldots, a_m)$

$A(1,2,3)$, $v = (3,1,1) \Rightarrow \mathcal{D} : \frac{x_1 - 1}{3} = \frac{x_2 - 2}{1} = \frac{x_3 - 3}{1} = t \Leftrightarrow \begin{cases} x_1 = 3t + 1 \\ x_2 = t + 2 \\ x_3 = t + 3 \end{cases}$ ec. param.

ec. generală a planului: $a x_1 + b x_2 + c x_3 + d = 0$

C12 $V_{\mathcal{D}_1} = <\{u\}>$, $u = (u_1, u_2, u_3)$; $V_{\mathcal{D}_2} = <\{v\}>$, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \ldots)$

$\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\Leftrightarrow \begin{vmatrix} u_1 & v_1 & b_1 - a_1 \\ u_2 & v_2 & b_2 - a_2 \\ u_3 & v_3 & b_3 - a_3 \end{vmatrix} \neq 0$ $\quad \mathcal{D} \perp \mathcal{D}_K, K = \overline{1,2}$ perp. comună

$\overrightarrow{P_1P_2} = (b_1 - a_1 + v_1 s - u_1 t\ , b_2 - a_2 + v_2 s - u_2 t, b_3 - a_3 + v_3 s - u_3 t)$

$\begin{cases} <\overrightarrow{P_1P_2}, u> = 0 \\ <\overrightarrow{P_1P_2}, v> = 0 \end{cases}$ ← prod. scalar $\Rightarrow t, s.$ $\quad P_1 \in \mathcal{D}_1$, $P_2 \in \mathcal{D}_2$ (scriem fiecare termen cu ec. param.

Aria unui Δ în $n = 3$

$\{u, v\}$ SLI $\quad \|u \times v\| = \|u\| \cdot \|v\| \cdot \sin\alpha$ $\quad \mathcal{A}_{\Delta ABC} = \frac{1}{2} \cdot \|\overrightarrow{AB} \times \overrightarrow{AC}\|$

Elipsa : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ $\quad a^2 = b^2 + c^2$, $a > b$ (LG al pct. P) $PF + PF' = 2a$

Hiperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad d^2 = a^2 + b^2$, $c > a$, $|PF - PF'| = 2a$, $a > 0$

Parabola : $x_2^2 = 2p x_1$ $\quad \frac{d(P,F)}{d(P,d)} = 1$

Γ conică

P_0 = centru $\Leftrightarrow [\forall P \in \Gamma\ , s_{P_0}(P) \in \Gamma]$

C13 $\Gamma : f(x) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 + 2b_1 x_1 + 2b_2 x_2 + c = 0$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T$ ($rg A \geq 1$) , $\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$ $\quad \delta = \det A$, $\Delta = \det \tilde{A}$

I. $\delta \neq 0$ $\exists!$ P_0 centrul conicei

II. $\delta = 0$ (Γ nu are centru unic)

Δ (natura)	δ (genul)	Tipul conicei
$\Delta \neq 0$	$\delta > 0$	$\emptyset$, Elipsă
	$\delta < 0$	Hiperbolă
	$\delta = 0$	Parabolă
$\Delta = 0$	$\delta > 0$	Pct. dublu
	$\delta < 0$	Drepte concurente
	$\delta = 0$	$\emptyset$, drepte $\parallel$

Sfera : $(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = R^2$

Elipsoidul : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$

Hiperboloid cu o pânză : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$

Hiperboloid cu 2 pânze : $-\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$

Paraboloidul eliptic : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3 \geq 0$

Paraboloid hiperbolic : $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3$

Cilindrul – eliptic $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, $x_3 \in \mathbb{R}$

– hiperbolic $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$, $x_3 \in \mathbb{R}$

– parabolic $x_2^2 = 2p x_1$, $x_3 \in \mathbb{R}$

Conul pătratic

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0$$

$(\mathbb{R}^3,+,\cdot)/_{\mathbb{R}}$, $\mathcal{R}=\{e_1'=(0,1,1), e_2'=(0,\alpha,1), e_3'=(\alpha,1,0)\}$. $\alpha=?$ a.î. $\mathcal{R}$ = reper în $\mathbb{R}^3$ $\Rightarrow \mathcal{R}=SLI \Rightarrow A=\begin{pmatrix}0&0&\alpha\\1&\alpha&1\\1&1&0\end{pmatrix}$ și $rg\,A=\max \Rightarrow \det A\neq 0 \Rightarrow \alpha\in\mathbb{R}\setminus\{0,1\}$
card $\mathcal{R}=3=\dim_{\mathbb{R}}\mathbb{R}^3 \Rightarrow \mathcal{R}$ = reper ; Pt. $\alpha=-1$, coord. lui $x=(1,1,1)$ în rap. cu $\mathcal{R}$? $x=\underline{c_1}e_1'+\underline{c_2}e_2'+\underline{c_3}e_3'$. Dacă pt. un alt α, $\det A=0 \Rightarrow \mathcal{R}=SLD$. alegem 2 vect. din $\mathcal{R}$ a.î. să form. SLI $rg\begin{pmatrix}0&-1\\1&1\\1&0\end{pmatrix}=2=\max \Rightarrow$ e SLI
$\Rightarrow \mathcal{R}=\{(0,1,1),(-1,1,0)\}$ completăm cu încă un v. ca să obținem reper. $\begin{pmatrix}0&-1&1\\1&1&0\\1&0&0\end{pmatrix}$
• Dacă v are coord. $X=(x_1,x_2,x_3)$ în rap. cu reperul $\mathcal{R}$ și X' cu rep. $\mathcal{R}' \Rightarrow X=AX' \longrightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} \Rightarrow x_1=\dots$ $x_2=\dots$ $x_3=\dots$

$f(x)=(2x_1+x_3, x_1+x_2+2x_3, 6x_1+3x_3)$. $Ker f=\{x\in\mathbb{R}^3 \mid AX=\begin{pmatrix}0\\0\\0\end{pmatrix}$ sau $f(x)=0\}$ → sistem de ec. $\Rightarrow Ker f=\{(x_1,3x_1,-2x_1) / x_1\in\mathbb{R}\} \Rightarrow$
$\Rightarrow \underline{(1,3,-2)}$ reper în Ker f/baza în Ker f. $Y=AX$. Extindem $\mathcal{R}$ la un reper în $\mathbb{R}^3$. $rg\begin{pmatrix}1&1&0\\3&0&0\\-2&0&1\end{pmatrix}=\max \Rightarrow SLI$
$\mathcal{R}\cup\{e_1,e_3\}$ reper în $\mathbb{R}^3 \Rightarrow \{f(e_1),f(e_3)\}$ reper baza în Im f (T.dim.) $\overset{=3}{\dim \mathbb{R}^3}=\dim Im f+\overset{=1}{\dim Ker f} \Rightarrow \dim Im f=2$ | $A=\begin{pmatrix}2&0&1\\1&1&2\\6&0&3\end{pmatrix}$
• val. proprii. $P_A(\lambda)=\det(A-\lambda I_3)=\lambda(1-\lambda)(\lambda-5)$. $\underline{\lambda_1=0}$, $m_1=1$. $V_{\lambda_1}=\{x \mid f(x)=0\}=Ker f=<\{(1,3,-2)\}>$; $\dim V_{\lambda_1}=1=m_1$
$\underline{\lambda_2=1}$, $m_2=1$. $V_{\lambda_2}=\{x \mid f(x)=x\} \Leftrightarrow (A-I_3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \Rightarrow <\{(0,1,0)\}>$, $\dim V_{\lambda_2}=m_2$; $\underline{\lambda_3=5}$, $m_3=1$, $V_{\lambda_3}=\{x / f(x)=5x\}\dots$
3 val. proprii dist. $\Rightarrow$ se poate diagonaliza. $\mathcal{R}'=\{(1,3,-2),(0,1,0),(1,\frac{7}{4},3)\}$. $A'=[f]_{\mathcal{R}'\mathcal{R}'}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&5\end{pmatrix}$

$(\mathbb{R}^3,g_0)$, $V=<\{(1,-1,1),(2,-1,3),(1,3,5)\}>$. Det. $V^\perp$ și preciz. $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$ reper ortonormat în $\mathbb{R}^3$ ($\mathcal{R}_1$ reper ort. în V) ($\mathcal{R}_2$ —"— $V^\perp$)
$rg\begin{pmatrix}1&2&1\\-1&-1&3\\1&3&5\end{pmatrix}=2 \Rightarrow SLD \Rightarrow V=<\{(1,-1,1),(2,-1,3)\}>$ SLI
$V^\perp=\{y\in V \mid g_0(x,y)=0, \forall x\in V\}=\left\{y\in V \middle| \begin{array}{l}y_1-y_2+y_3=0\\2y_1-y_2+3y_3=0\end{array}\right\}=<\{(-2,-1,1)\}>$ $\mathcal{R}_2=\{e_3'=\frac{1}{\sqrt6}(-2,-1,1)\}$
Găsim rep. ortonormat în V (G.S.). $e_1=(1,-1,1) \Rightarrow e_1'=\frac{1}{\sqrt3}(1,-1,1)$; $e_2=(2,-1,3)-\frac{6}{3}(1,-1,1)=(0,1,1)$ $\quad \frac{6}{3}=\frac{(2,-1,3)\cdot(1,-1,1)}{(1,-1,1)\cdot(1,-1,1)}$
$\Rightarrow e_2'=\frac{1}{\sqrt2}(0,1,1) \Rightarrow \mathcal{R}_1=\{e_1',e_2'\} \Rightarrow \mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$. Baza ortonorm. pt. $V=\{e_1',e_2'\}=\mathcal{R}_1$
• $p: \mathbb{R}^3=V\oplus V^\perp \to V$ proiecția ortog. pe V. Aflați $p(0,1,0)$
$p(0,1,0)=\underbrace{a(1,-1,1)+b(0,1,1)}_{pr.\ pe\ V}+c(-2,-1,1)=0 \Rightarrow a=-\frac{1}{3}, b=\frac{1}{2}, c=-\frac{1}{6}$
$\Rightarrow p(0,1,0)=-\frac{1}{3}(1,-1,1)+\frac{1}{2}(0,1,1)$
sau $p(0,1,0)=((0,1,0)\cdot e_1')e_1'+((0,1,0)\cdot e_2')e_2'$
• $s: \mathbb{R}^3=V\oplus V^\perp \to V$ simetria ortogonală față de $V^\perp$. Aflați $s(-1,0,1)$. Mai întâi se calc. proiecția
$s(v)=p(v)\cdot 2-v$

$g: \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ formă biliniară; $G=\begin{pmatrix}2&2&0\\2&3&-\frac{1}{2}\\0&-\frac{1}{2}&\frac{5}{4}\end{pmatrix}$ matricea asociată în rap. cu reperul canonic $\mathcal{R}_0$. Este $(\mathbb{R}^3,g)$ sp. vect. euclidian? $G={}^TG \Rightarrow G$ simetrică $\Rightarrow g\in Sim(\mathbb{R}^3\times\mathbb{R}^3)$
ca să fie s.v.e.r. $\Rightarrow g$ = bilin. simetrică, poz. def.
$Q=2x_1^2+3x_2^2+\frac{5}{4}x_3^2+4x_1x_2-1x_2x_3=2(\underbrace{x_1+x_2}_{y_1})^2+(\underbrace{x_2-\frac{1}{2}x_3}_{y_2})^2+\underbrace{x_3^2}_{y_3} \Rightarrow Q(x)=2y_1^2+y_2^2+y_3^2$
sgn(3,0) poz. def. $\Rightarrow$ s.v.e.r. (câți y cu + ; câți y cu −)

$Q: \mathbb{R}^3\to\mathbb{R}$, $Q(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+3x_2x_3$; g = forma polară asociată lui Q. Este g un produs scalar?
$\Rightarrow G=\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&1&\frac{3}{2}\\\frac{1}{2}&\frac{3}{2}&1\end{pmatrix}$ $\left.\begin{array}{l}G={}^TG \Rightarrow G \text{ simetrică}\\Q(x)>0\end{array}\right\} \Rightarrow g$ = prod. scalar $\Rightarrow (\mathbb{R}^3,g)$ sp. v. e.
sgn(2,1) $\Rightarrow$ nu e prod. scalar

$(\mathbb{R}^3,(\mathbb{R}^3,g_0),\varphi)$ sp. punctual euclidian, $P(1,1,1)$ și $\mathcal{D}: \frac{x_1-1}{1}=\frac{x_2}{0}=\frac{x_3-1}{1}$. $dist(P,\mathcal{D})=?$
$A(1,0,1), B(2,0,2)\in\mathcal{D}$, $d(P,\mathcal{D})=\frac{\|\overrightarrow{PA}\times\overrightarrow{PB}\|}{\|\overrightarrow{AB}\|}$; $\overrightarrow{PA}\times\overrightarrow{PB}=\begin{vmatrix}e_1&e_2&e_3\\x_A-x_P&y_A-y_P&z_A-z_P\\x_B-x_P&y_B-y_P&z_B-z_P\end{vmatrix}=-e_1+e_3=(-1,0,1)$
$\|\overrightarrow{AB}\|=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}=\sqrt2 \Rightarrow d(P,\mathcal{D})=\frac{\sqrt2}{\sqrt2}=1$ $\quad \|\overrightarrow{PA}\times\overrightarrow{PB}\|=\sqrt2$

$\mathcal{D}_1: \frac{x_1-7}{1}=\frac{x_2-3}{2}=\frac{x_3-9}{-1}$, $\mathcal{D}_2: \frac{x_1-3}{-7}=\frac{x_2-1}{2}=\frac{x_3-1}{3}$; necoplanare + ec. perp. comune.
$A(7,3,9)\in\mathcal{D}_1$, $B(3,1,1)\in\mathcal{D}_2$; necoplanare $\Rightarrow \begin{vmatrix}1&-7&x_B-x_A\\2&2&y_B-y_A\\-1&3&z_B-z_A\end{vmatrix}\neq 0$
vectorul normal $n=d_1\times d_2=\begin{vmatrix}i&j&k\\1&2&-1\\-7&2&3\end{vmatrix}=8i+4j+16k \Rightarrow n=(8,4,16)$
Un pct. de pe perpendiculara comună este mij. [AB]: $M=(5,2,5)$. Ec. ⊥ comune trece prin $M(5,2,5)$ și are vec. de direcție $n=(8,4,16)$
$\frac{x-5}{8}=\frac{y-2}{4}=\frac{z-5}{16}$ ← ec. ⊥ comune

$(\mathbb{R}^2,(\mathbb{R}^2,g_0),\varphi)$ plan punctual euclidian și conica $\Gamma: f(x)=x_1^2+2x_1x_2+x_2^2\underline{-4}x_1\underline{-6}x_2+⑥=0$.
invarianții metrici și tipul conicei. Are centru unic? $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ $B=(\underline{-2}\ \underline{-3})$ $\tilde{A}=\begin{pmatrix}1&1&-2\\1&1&-3\\-2&-3&⑥\end{pmatrix}$
invarianți metrici: $\underline{\det A=\delta=0}$; $\det\tilde{A}=\Delta=-1$; $rg\,A=1$; $rg\,\tilde{A}=3$; $Tr\,A=2$
$\det A=0 \Rightarrow$ conică degenerată, neinversabilă $\Rightarrow$ parabolă $\to$ nu are centru unic $\Rightarrow$ nu are centru unic

$f\in End(\mathbb{R}^3)$ simetric, $P(\lambda)=(-1)^3(\lambda-1)^2(\lambda+2)$ polinom caracteristic. dim subsp. proprii, Tr(A), det A, $\forall\mathcal{R}$ reper în $\mathbb{R}^3$
$\underline{\lambda_1=1}$, $m_1=2$; $\underline{\lambda_2=-2}$, $m_2=1$; $A=[f]_{\mathcal{R},\mathcal{R}}$, $\dim V_{\lambda_1}$, $\dim V_{\lambda_2}=?$
$f\in Sim(\mathbb{R}^3) \Rightarrow A=[f]_{\mathcal{R},\mathcal{R}}$ se poate diagonaliza $\Rightarrow Tr\,A=Tr\,A'$, $\det A=\det A'$, unde $A'=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$
$Tr\,A=0$, $\det A=2$, $\dim V_{\lambda_1}=2$, $\dim V_{\lambda_2}=1$

$(\mathbb{R}^3,g_0)$, $f\in End(\mathbb{R}^3)$ rotație de unghi orientat $\varphi=\frac{\pi}{6}$ și axă $V^\perp$, unde $V=\{x\in\mathbb{R}^3 / x_1+x_2+x_3=0\}$. versorul axei, det A, Tr A, unde $A=[f]_{\mathcal{R},\mathcal{R}}$, $\forall\mathcal{R}$ reper ortonormat în $\mathbb{R}^3$.
$V^\perp=<\{(1,1,1)\}> \Rightarrow$ versorul este $u=\frac{1}{\sqrt3}(1,1,1)$
f rotație $\Rightarrow \det A=1$, $Tr\,A=1+2\cos\varphi=1+\sqrt3$