EEEE Seminar săpt. 2: T. Hamilton-Cayley, T. Laplace, f. eşalon, alg. Gauss-Jordan
SSSS

1. Calculați A^{-1} folosind T. Hamilton-Cayley, respectiv alg. Gauss-Jordan

b) $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

Metoda 1: T. Hamilton-Cayley

$p_A(x) = \det(A - x \cdot I_3) =$

$= \begin{vmatrix} 2-x & -1 & 3 \\ 0 & 4-x & 1 \\ 3 & 1 & 5-x \end{vmatrix} = (-1)^3 (x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3)$

$\sigma_1 = Tr(A) = 2+4+5 = 11$

$\sigma_2 = \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix} =$

$= 20 - 1 + 10 - 9 + 8 = 19 + 9 = 28$

$\sigma_3 = \det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 5 \end{vmatrix} \xlongequal{C_2 - 4C_3} \begin{vmatrix} 2 & -13 & 3 \\ 0 & 0 & 1 \\ 3 & -19 & 5 \end{vmatrix} =$

$= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 2 & -13 \\ 3 & -19 \end{vmatrix} = -(-38+39) = -1$

$p_A(x) = -(x^3 - 11x^2 + 28x + 1)$

$A^3 - 11A^2 + 28A + I_3 = O_3 \quad | \cdot A^{-1} \Rightarrow$

$\Rightarrow A^2 - 11A + 28I_3 + A^{-1} = O_3$

$A^{-1} = -A^2 + 11A - 28I_3$

Metoda 2: Alg. Gauss-Jordan:

$$\boxed{\begin{array}{l} \det(A) \neq 0 \\ (A \mid I_3) \sim (I_3 \mid A^{-1}) \end{array}}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \overset{\frac{1}{2}L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} & 0 & \frac{2}{5} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{20} & -\frac{3}{5} & -\frac{1}{4} & \frac{2}{5} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{5} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 12 & 5 & 8 \end{array}\right) \overset{\substack{L_2 - \frac{1}{4}L_3 \\ L_1 - \frac{3}{2}L_3}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{35}{2} & -\frac{15}{2} & 12 \\ 0 & 1 & 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 12 & 5 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & -8 & 11 \\ 0 & 1 & 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 12 & 5 & 8 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -19 & -8 & 11 \\ -3 & -1 & 2 \\ 12 & 5 & 8 \end{pmatrix}$$

2. Fie $A = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 0 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ Det. o formă eș. pe linii și f. eș. redusă. Det. rg A.

__Sol.:__ $\begin{pmatrix} 3 & 1 & -1 & 0 \\ 0 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 3 & 1 & -1 & 0 \end{pmatrix} \overset{L_3 - 3L_1}{\sim}$

$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 1 & -4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 2 & -3 & -1 \end{pmatrix} \overset{L_3 - 2L_2}{\sim}$

3 pivoți $\Rightarrow$ $\Rightarrow$ rg $A = 3$

$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 5 & 5 \end{pmatrix} \overset{\frac{L_3}{5}}{\sim} \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & -4 & -3 \\ 0 & 0 & \boxed{1} & 1 \end{pmatrix} \underset{L_2 + 4L_3}{\overset{L_1 - L_3}{\sim}} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$ a) $p_A = ?$
b) $A^{100} = ?$ (TH-C)

<u>Sol.</u>: $p_A(X) = \det(A - X I_4) = X^4 - \sigma_1 X^3 + \sigma_2 X^2 - \sigma_3 X + \sigma_4$

$\sigma_1 = \operatorname{Tr}(A) = 0$

$$\sigma_2 = \underset{\text{"1 cu 2"}}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}} + \underset{\text{"1 cu 3"}}{\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix}} + \underset{\text{"1 cu 4"}}{\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}} + \underset{\text{"2 cu 3"}}{\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}} + \underset{\text{"2 cu 4"}}{\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}} + \underset{\text{"3 cu 4"}}{\begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix}} = 0$$

$$\text{"2 cu 4"}: \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} : \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix}$$

$$\sigma_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{vmatrix} \overset{C_2 = -2C_3}{=} 0$$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix}: C_1 = -C_3 \Rightarrow = 0$$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} \overset{C_1 = -C_3}{=} 0$$

$$\begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -2 \end{vmatrix} \overset{C_2 = -2C_3}{=} 0$$

$\sigma_3 = 0$

$\sigma_4 \overset{C_1 = C_4}{=\!=\!=} 0$

$p_A(X) = X^4$

$p_A(A) = A^4 = O_4 \Rightarrow A^{100} = O_4$

4. Fie $A=\begin{pmatrix}0 & 3\\ 1 & 2\end{pmatrix}$ și $B=A^5-3A^4+A-8I_2$

Det. a și $b \in \mathbb{R}$ a.î. $B=aA+bI_2$

<u>Sol.</u>: Metoda 1:

$$p=x^5-3x^4+x-8$$

$$p_A=X^2-2X-3=(x+1)(x-3)$$

$$p=C\cdot p_A+R=(x+1)(x-3)\cdot C+ax+b$$

$p(-1)=-a+b \qquad p(3)=3a+b$

$p(-1)=-13 \qquad p(3)=-5$

$$\begin{cases}-a+b=-13\\ 3a+b=-5\end{cases} \Rightarrow \begin{matrix}a=2\\ b=-11\end{matrix}$$

$$R=2X-11$$

$$p(A)=B=C(A)\cdot\overbrace{p_A(A)}^{=O_2}+R(A)$$

$$B=2A-11I_2$$

<u>Metoda 2:</u>

$A^2-2A-3I_2=O_2 \Rightarrow A^2=2A+3I_2 \mid\cdot A \Rightarrow$

$\Rightarrow A^3=2(2A+3I_2)+3A \Rightarrow A^3=7A+6I_2 \mid\cdot A \Rightarrow$

$\Rightarrow A^4=7(2A+3I_2)+6A \Rightarrow A^4=20A+21I_2 \mid\cdot A \Rightarrow$

$\Rightarrow A^5=20(2A+3I_2)+21A \Rightarrow A^5=61A+60I_2$

5. Fie $A=\begin{pmatrix}1 & -1 & 2 & -3\\ 1 & 0 & 2 & 5\\ 4 & 2 & 0 & 1\\ 3 & 1 & 0 & 2\end{pmatrix}$ Calc. $\det A$ folosind T. Laplace pentru $p=2$ și l_2, l_3 fixate

<u>Sol.</u>: $\det A=(-1)^{2+3+1+2}\cdot\begin{vmatrix}1 & 0\\ 4 & 2\end{vmatrix}\cdot\begin{vmatrix}2 & -3\\ 0 & 2\end{vmatrix}+$

$+(-1)^{2+3+1+3}\cdot\begin{vmatrix}1 & 2\\ 4 & 0\end{vmatrix}\cdot\begin{vmatrix}-1 & -3\\ 1 & 2\end{vmatrix}+$

$$+(-1)^{2+3+1+4}\cdot\begin{vmatrix}1&5\\4&1\end{vmatrix}\cdot\begin{vmatrix}-1&2\\1&0\end{vmatrix}+$$

$$+(-1)^{2+3+2+3}\cdot\begin{vmatrix}0&2\\2&0\end{vmatrix}\cdot\begin{vmatrix}1&-3\\3&2\end{vmatrix}+$$

$$+(-1)^{2+3+2+4}\cdot\begin{vmatrix}0&5\\2&1\end{vmatrix}\cdot\begin{vmatrix}1&2\\3&0\end{vmatrix}+$$

$$+(-1)^{2+3+3+4}\cdot\begin{vmatrix}2&5\\0&1\end{vmatrix}\cdot\begin{vmatrix}1&-1\\3&1\end{vmatrix}=8+8+38-44-60+8=$$

$$=-2$$

7. $A=\begin{pmatrix}1&3\\0&5\end{pmatrix}$ Calculați A^{2025} folosind TH-C

<u>Sol.</u>: $A^2-6A-5=O_2$

$A^n=x_nA+y_nI_2,\ \forall n\geq 1$

$A^{n+1}=A^n\cdot A$

$x_{n+1}-\underset{\overset{\parallel}{6}}{\sigma_1}x_n+\underset{\overset{\parallel}{5}}{\sigma_2}x_{n-1}=0$

$\begin{cases}x_1=1\\x_2=6\end{cases}$

$t^2-6t+5=0\Rightarrow(t-1)(t-5)=0$

$x_n=c_1\cdot t_1^n+c_2\cdot t_2^n$

$=c_1+c_2\cdot 5^n$

$n=1\Rightarrow c_1+5c_2=1$

$n=2\Rightarrow c_1+25c_2=6 \quad (-)$

$-20c_2=-5\Rightarrow c_2=\frac{1}{4},\ c_1=-\frac{1}{4}$

$x_n=-\frac{1}{4}+\frac{1}{4}\cdot 5^n=\frac{1}{4}(5^n-1)$

$y_n=-\frac{5}{4}(5^{n-1}-1)$

$A^n=\frac{1}{4}(5^n-1)\cdot A-\frac{5}{4}(5^{n-1}-1)I_2$

8. Fie ec. $X^{2025} = A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Precizați nr. de sol. dacă:

a) $X \in \mathcal{M}_2(\mathbb{R})$

b) $X \in \mathcal{M}_2(\mathbb{C})$

<u>Sol.</u>: $\det^{2025}(X) = \det A = 0 \Rightarrow \det(X) = 0$

TH-C: $X^2 - Tr(X)X + \det(X)I_2 = 0$

$$X^2 = Tr(X)X \Rightarrow X^{2025} = Tr(X)^{2024} \cdot X$$

$Tr(\alpha A) = \alpha Tr(A)$

$$A = Tr(X)^{2024} \cdot X \mid Tr \Rightarrow$$

$$\Rightarrow Tr(A) = Tr(X)^{2024} \cdot Tr(X)$$

$$Tr(A) = Tr(X)^{2025} = 4 \quad (*)$$

a) $X \in \mathcal{M}_2(\mathbb{R})$

(*) are sol. unică, $Tr(X) = \sqrt[2025]{4} \Rightarrow X = \frac{1}{\sqrt[2025]{4^{2024}}} \cdot A$

b) $X \in \mathcal{M}_2(\mathbb{C})$

(*) are 2025 de soluții

└ teoremă pt. polinoame de grad n în $\mathbb{C}$ $\Rightarrow$

$\Rightarrow$ 2025 rădăcini, dacă coef. sunt reali

9. Fie ec. $x^3 + px + q = 0$, $x_1, x_2, x_3 \in \mathbb{C}$ sol. ec. ($p, q \in \mathbb{C}$)

Calculați Δ^2, unde

$$A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

<u>Sol.</u>:

$$AA^t = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = \begin{pmatrix} \underset{3}{S_0} & \underset{0}{S_1} & \underset{s_1^2 - 2s_2 = -2p}{S_2} \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$$

$S_K = x_1^K + x_2^K + x_3^K$, $K \in \mathbb{N}$

$$\begin{cases} s_1 = x_1 + x_2 + x_3 = 0 \\ s_2 = x_1x_2 + x_1x_3 + x_2x_3 = p \\ s_3 = x_1x_2x_3 = -q \end{cases}$$

$$AA^t = \begin{pmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{pmatrix}$$

$s_3 = -3q$, $s_4 = 2p^2$

$s_3 + p s_1 + 3q = 0$; $s_4 + p s_2 + q s_1 = 0$

$$\Delta^2 = \begin{vmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{vmatrix} = -4p^3 - 27q^2$$