CCC Seminar săpt. 8
SSS

Forme pătratice. Formă canonică. Metoda Gauss. Metoda Jacobi

1. Fie $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$

a) $G = ?$ matricea asociată în raport cu $\mathcal{B}_0 = \{e_1, e_2, e_3\}$

b) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată

c) Să se aducă Q la o formă canonică, utilizând metoda Gauss, respectiv Jacobi. Este Q poz. definită?

Generalizare.

Sol.: a)

$$G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

b) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$g(x,y) = 2^{-1}\left(Q(x+y) - Q(x) - Q(y)\right)$$

$$g(x,y) = x_1y_1 + \frac{1}{2}(x_1y_2 + x_1y_3 + x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2) +$$

$$+ x_2y_2 + x_3y_3 \text{ — forma polară}$$

I) Metoda Jacobi

$\Delta_1 = \det(1) = 1 \neq 0$

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} \neq 0$$

$$\Delta_3 = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = 1 + \frac{1}{8} + \frac{1}{8} - \frac{3}{4} = \frac{1}{2} \neq 0$$

Deci $\exists\, \mathcal{R}'$ reper în $\mathbb{R}^3$ a.î.:

$$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2 = x_1'^2 + \frac{4}{3} x_2'^2 + \frac{3}{2} x_3'^2$$

sgn $(3,0) \Rightarrow Q$ este poz. definită

II) Metoda Gauss

Începem cu x_1

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 - \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 + \frac{1}{2}x_2x_3$$

$$= \text{———}''\text{———} + \frac{3}{4}\left(x_2^2 + x_3^2 + \frac{2}{3}x_2x_3\right)$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 - \frac{1}{12}x_3^2 + \frac{3}{4}x_3^2$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{2}{3}x_3^2$$

Fie schimbarea de reper:

$$\begin{cases} y_1 = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ y_2 = x_2 + \frac{1}{3}x_3 \\ y_3 = x_3 \end{cases}$$

$Q(x) = y_1^2 + \frac{3}{4}y_2^2 + \frac{2}{3}y_3^2$; sgn$(3,0)$

Pt. forma normală:

$$\begin{cases} z_1 = y_1 \\ z_2 = \frac{\sqrt{3}}{2} y_2 \\ z_3 = \sqrt{\frac{2}{3}}\, y_3 \end{cases}$$

$Q(x) = z_1^2 + z_2^2 + z_3^2$

Generalizare:

$$Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_i x_j \; ; \; \text{sgn}(n,0) \Rightarrow \text{poz. def.}$$

2. Fie $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

Să se aducă la o formă canonică (met. Gauss/Jacobi)

Precizați signatura.

<u>Sol.:</u>

$$G = \begin{pmatrix} 0 & \boxed{1} & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

$g_{12} = 1 \neq 0$

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \\ y_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}(y_1 + y_2) \\ x_2 = \frac{1}{2}(y_1 - y_2) \\ x_3 = y_3 \end{cases}$$

$$Q(x) = \frac{1}{2}(\underline{y_1^2} - y_2^2) \underline{- 6y_1}y_3$$

$$= \frac{1}{2}(y_1^2 - 12y_1y_3) - \frac{1}{2}y_2^2$$

$$= \frac{1}{2}(y_1 - 6y_3)^2 - 18y_3^2 - \frac{1}{2}y_2^2$$

Schimbare de reper:

$$\begin{cases} z_1 = y_1 - 6y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases}$$

$Q(z) = \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - 18z_3^2$

Signatura este $(1, 2)$; Q nu este poz. def.

3. Fie $Q: \mathbb{R}^4 \to \mathbb{R}$, $Q(x) = \underline{x_1^2} + x_2^2 + x_3^2 - 2x_4^2 \underline{- 2x_1x_2} + \underline{2x_1x_3} - \underline{2x_1x_4} + 2x_2x_3 - 4x_2x_4$

Să se aducă la f. can.

<u>Sol.:</u>

$$G = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & -2 \end{pmatrix}$$

$$Q(x) = (x_1 - x_2 + x_3 - x_4)^2 - \cancel{x_2^2} - \cancel{x_3^2} - x_4^2 + \cancel{2x_2x_3} - 2x_2x_4 + 2x_3x_4 +$$

$$+ \cancel{x_2^2} + \cancel{x_3^2} - 2x_4^2 - \cancel{2x_2x_3} - 4x_2x_4$$

$$= (x_1 - x_2 + x_3 - x_4)^2 \underline{-6x_2x_4} + 2x_3x_4 - 3x_4^2$$

Fie schimbarea de reper:

$$\begin{cases} y_1 = x_1 - x_2 + x_3 - x_4 \\ y_2 = x_2 + x_4 \\ y_4 = x_2 - x_4 \\ y_3 = x_4 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{1}{2}(y_2 + y_3) \\ x_3 = y_3 \\ x_4 = \frac{1}{2}(y_2 - y_4) \end{cases}$$

$$Q(x) = y_1^2 - \frac{3}{2}(y_2^2 - y_4^2) + \blacksquare (y_2y_3 - y_2y_4) - 3\left(\frac{1}{4}(y_2 - y_4)^2\right)$$

$$= y_1^2 - \frac{9}{4}y_2^2 + \frac{3}{4}y_4^2 + 6y_2y_3 - y_3y_4 + \frac{6}{4}y_2y_4$$

$$= y_1^2 - \frac{9}{4}\left(y_2^2 - \frac{8}{3}y_2y_3 - \frac{2}{3}y_2y_4\right) - y_3y_4 + \frac{3}{4}y_4^2$$

$$= y_1^2 - \frac{9}{4}\left[\left(y_2 - \frac{4}{3}y_3 - \frac{1}{3}y_4\right)^2 - \frac{16}{9}y_3^2 - \frac{1}{9}y_4^2 - \frac{8}{9}y_3y_4\right] - y_3y_4 + \frac{3}{4}y_4^2$$

$$= y_1^2 - \frac{9}{4}\left(y_2 - \frac{4}{3}y_3 - \frac{1}{3}y_4\right)^2 + \underline{4y_3^2} + y_4^2 + \underline{y_3y_4}$$

$$= y_1^2 - \frac{9}{4}\left(y_2 - \frac{4}{3}y_3 - \frac{1}{3}y_4\right)^2 + 4\left(y_3 + \frac{1}{8}y_4\right)^2 + \frac{15}{16}y_4^2$$

$$\begin{cases} z_1 = y_1 \\ z_2 = y_2 - \frac{4}{3}y_3 - \frac{1}{3}y_4 \\ z_3 = y_3 + \frac{1}{8}y_4 \\ z_4 = y_4 \end{cases}$$

$$Q(x) = z_1^2 - \frac{9}{4}z_2^2 + 4z_3^2 + \frac{15}{16}z_4^2$$

Signatura este $(3, 1)$

4. Fie $Q:\mathbb{R}^3\to\mathbb{R}$ f. pătr. și $G=\begin{pmatrix}1&2&1\\2&3&2\\1&2&1\end{pmatrix}$ matr. asoc. în

rap. cu $\mathcal{R}_0=\{e_1,e_2,e_3\}$. Să se det. o f. canonică și reperul în ~~[illegible]~~ care se realizează.

<u>Sol.:</u> $Q(x)=\underline{x_1}^2+3x_2^2+x_3^2+\underline{4x_1x_2}+\underline{2x_1x_3}+4x_2x_3$

$$=(x_1+2x_2+x_3)^2-4x_2^2-\cancel{x_3^2}-\cancel{4x_2x_3}+3x_2^2+\cancel{x_3^2}+\cancel{4x_2x_3}$$

$$=(x_1+2x_2+x_3)^2-x_2^2$$

$$\begin{cases}x_1'=x_1+2x_2+x_3\\x_2'=x_2\\x_3'=x_3\end{cases}\qquad \mathcal{R}_0\xrightarrow{C}\mathcal{R}$$

$$x=\sum_{i=1}^{3}x_ie_i=\sum_{j=1}^{3}x_j'\underbrace{e_j'}_{\sum_{i=1}^{3}c_{ij}e_i}=\sum_{i=1}^{3}\left(\sum_{j=1}^{3}c_{ij}x_j'\right)e_i$$

$$X=CX'$$

$$\begin{cases}x_1=x_1'-2x_2'-x_3'\\x_2=x_2'\\x_3=x_3'\end{cases}\qquad C=\begin{pmatrix}1&-2&-1\\0&1&0\\0&0&1\end{pmatrix}$$

$\mathcal{R}=\{e_1'=(1,0,0),\ e_2'=(-2,1,0),\ e_3'=(-1,0,1)\}$

$Q(x)=x_1'^2-x_2^2$

Signatura este $(1,1)$

8/3. Fie $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1 y_1 - x_2 y_2 - x_1 y_3 - x_3 y_1 + 2x_2 y_3 + 2x_3 y_2$

a) $g \in L^2(\mathbb{R}^3, \mathbb{R}^3, \mathbb{R})$ formă bilin. simetrică

b) $G = ?$ în rap. cu $R_0 = \{e_1, e_2, e_3\}$

c) $\ker g = ?$ Este g nedegenerată

d) Să se afle $G' = ?$ în rap. cu $R' = \{e_1' = (1,1,1), e_2' = (1,2,1), e_3' = (0,0,1)\}$

Sol.:

a/b) $G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} = G^T \Rightarrow g$ simetrică

c) $\ker g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0, \forall y \in \mathbb{R}^3\}$

$x \in \ker g \Rightarrow (*) \begin{cases} g(x, e_1) = 0 \Rightarrow x_1 - x_3 = 0 \\ g(x, e_2) = 0 \Rightarrow -x_2 + 2x_3 = 0 \\ g(x, e_3) = 0 \Rightarrow -x_1 + 2x_2 = 0 \end{cases}$

(*) SLO

$\Delta = \det G = \begin{vmatrix} \cancel{0} & \cancel{0} & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 0 \end{vmatrix} = (-1)^4 \cdot (-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = -3$

$\Rightarrow (x_1, x_2, x_3) = 0_{\mathbb{R}^3} \Rightarrow \ker g = \{0_{\mathbb{R}^3}\} \Rightarrow g$ nedegenerată

d) $R_0 \xrightarrow{C} R'$

$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$G' = C^T G C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$