Obs! $\dim_{\mathbb{K}} V = n$

a) n = nr. max. de vectori din V care form. SLI

b) n = nr. min. de vectori din V care form. SG

$\frac{CCC}{SSS}$ Seminar săpt. 3

Sis. de ec. lin. Spații vect.

1. $\begin{cases} x + \alpha y + z = 1 \\ \alpha x - y + z = 1 \\ x + y - z = 2 \end{cases}$ Să se rezolve. Discuție după $\alpha \in \mathbb{R}$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & \alpha & 1 & 1 \\ \alpha & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

$$\begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \overset{C_3+C_2}{=\!=} \begin{vmatrix} 1 & \alpha & 1+\alpha \\ \alpha & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (1+\alpha)\cdot(-1)^{1+3}\cdot \begin{vmatrix} \alpha & -1 \\ 1 & 1 \end{vmatrix} =$$

$= (1+\alpha)^2$

Caz 1) $\Delta \neq 0 \iff (1+\alpha)^2 \neq 0 \iff \alpha \neq -1 \iff \alpha \in \mathbb{R} \setminus \{-1\}$

$rg A = rg \bar{A} = 3$ (maxim) $\Rightarrow$ SCD (sol. unică; cu Kramer)

$x = \frac{\Delta x}{\Delta}$

$$\Delta x = \begin{vmatrix} 1 & \alpha & \alpha+1 \\ 1 & -1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = (\alpha+1)\cdot(-1)^{1+3}\cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3(\alpha+1)$$

$x = \frac{3}{\alpha+1}$

$y = \frac{\Delta y}{\Delta}$

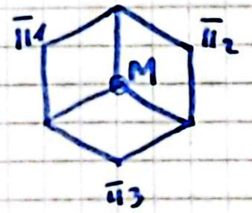
$$\Delta y = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \alpha-1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \alpha - 1 ; \quad y = \frac{\alpha-1}{(\alpha+1)^2}$$

$$z=\frac{\Delta_z}{\Delta},\ \Delta_z=\begin{vmatrix}1&\alpha&1\\ \alpha&-1&1\\ 1&1&2\end{vmatrix}=\begin{vmatrix}1&\alpha&1\\ \alpha-1&-\alpha-1&0\\ -1&1-2\alpha&0\end{vmatrix}=1\cdot(-1)^{1+3}\cdot\begin{vmatrix}\alpha-1&-\alpha-1\\ -1&1-2\alpha\end{vmatrix}=$$

$$=(\alpha-1)(1-2\alpha)-(\alpha+1)=-2\alpha^2+2\alpha-2=-2(\alpha^2-\alpha+1)$$

$$z=-\frac{2(\alpha^2-\alpha+1)}{(\alpha+1)^2}$$

Sol. unică: $(x,y,z)=\left\{\left(\frac{3}{\alpha+1},\frac{3(\alpha-1)}{(\alpha+1)^{\alpha}},\frac{-2(\alpha^2-\alpha+1)}{(\alpha+1)^2}\right)\right\}$



coord. punctului M

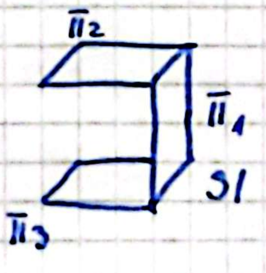
Caz 2) $\Delta=0 \Rightarrow \alpha=-1$

$$\bar{A}=\left(\begin{array}{ccc|c}\boxed{\begin{matrix}1&-1\\ -1&-1\end{matrix}}&\begin{matrix}1\\ 1\end{matrix}&\begin{matrix}1\\ 1\end{matrix}\\ 1\quad 1&-1&2\end{array}\right),\ rg\,A\neq 3$$

$$\Delta_p=\begin{vmatrix}1&-1\\ -1&-1\end{vmatrix}=-2\neq 0\Rightarrow rg\,A=2$$

$$\Delta_c=\begin{array}{r}\\ \\ B\rightarrow\end{array}\begin{vmatrix}1&-1&1\\ -1&-1&1\\ 1&1&2\end{vmatrix}=\begin{vmatrix}1&0&0\\ -1&-2&2\\ 1&2&1\end{vmatrix}=\begin{vmatrix}-2&2\\ 2&1\end{vmatrix}=-6$$

$rg\,\bar{A}=3\neq rg\,A \Rightarrow SI$



4. ΔABC, a, b, c - lungimile lat.

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad \bar{A} = \left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)$$

$\forall \Delta ABC, \exists! (x_0, y_0, z_0)$ a. î. $x_0, y_0, z_0 \in (-1, 1)$

Sol.: $\det(A) = 0 - (acb + bac) = -2abc \neq 0$

$$\Delta_x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = a^3 - (ac^2 + ab^2) = -a(b^2 + c^2 - a^2)$$

$$x = \frac{\Delta x}{\Delta} = \frac{-\cancel{a}(b^2 + c^2 - a^2)}{-2\cancel{a}bc} = \frac{b^2 + c^2 - a^2}{2bc}$$

Th. cos.: $a^2 = b^2 + c^2 - 2bc \cos A$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc} = \frac{b^2 + c^2 - a^2}{2bc} = \cos \measuredangle A$$

$m(\measuredangle A) \in (0, \pi) \Rightarrow \cos A \in (-1, 1)$

Analog, $y = \cos B$, $z = \cos C$ deci sol. unică

este $(x_0, y_0, z_0) = (\cos A, \cos B, \cos C)$

3. $$\begin{cases} x + y + z = 0 \\ ax + by + cz = 0 \\ (b+c)x + (a+c)y + (a+b)z = 0 \end{cases}$$

Pt. $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ sis. omogen mereu comp.

Să se rez. pt. $a \neq b$ $(a, b, c \in \mathbb{R})$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ b+c & a-b & a-c \end{vmatrix} = \begin{vmatrix} b-a & c-a \\ a-b & a-c \end{vmatrix} = (b-a)(c-a) \cdot \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$$

$$\Delta_p = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b-a \neq 0 \Rightarrow rg\, A = 2$$

$$\Delta_c = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ b+c & a+c & 0 \end{vmatrix} = 0 \Rightarrow rg\, \bar{A} = 2$$

Deci avem SCN

x și y nec. principale

$z = \alpha \in \mathbb{R}$ nec. sec.

$$\begin{cases} x + y = -\alpha \quad | \cdot (-a) \\ ax + by = -c\alpha \end{cases} \Rightarrow (b-a)y = \alpha(a-c)$$

$$\Rightarrow y = \frac{\alpha(a-c)}{b-a}$$

$$\Rightarrow x = -\alpha - \frac{\alpha(a-c)}{b-a} = -\alpha \cdot \frac{b-c}{b-a}$$

$$(x, y, z) \in \left\{ \left(\frac{\alpha(c-b)}{b-a}, \frac{\alpha(a-c)}{b-a}, \alpha \right) \middle| \alpha \in \mathbb{R} \right\}$$

Spații vectoriale (L3.GA.2025)

6. Fie sp. vec $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

a) Fie sist. de vec. $S = \{(1, m, 1), (m, 1, 1), (1, 0, m)\} \subset \mathbb{R}^3$, $m \in \mathbb{R}$

1) $m = ?$ a.î. $S = SLI$

2) $m = ?$ a.î. $S = SLD$

3) Dacă $m = 2$, atunci S este bază

b) Fie sis. de vec. $S' = \{(1, a_1, a_1^2), (1, a_2, a_2^2), (1, a_3, a_3^2)\} \subset \mathbb{R}^3$ și $a_1, a_2, a_3 \in \mathbb{R}$. Ce relație verif. a_1, a_2, a_3 a.î. S' să fie bază?

Sol.:

a) 1) Fie $a, b, c \in \mathbb{R}$

$$a(1,m,1)+b(m,1,1)+c(1,0,m)=0_{\mathbb{R}^3}$$

$$(a,am,a)+(bm,b,b)+(c,0,cm)=(0,0,0)$$

$$(*)\begin{cases} a+bm+c=0 \\ am+b+0=0 \\ a+b+cm=0 \end{cases} \qquad \bar{A}=\left(\begin{array}{ccc|c} 1 & m & 1 & 0 \\ m & 1 & 0 & 0 \\ 1 & 1 & m & 0 \end{array}\right)$$

$$\Delta=\begin{vmatrix} 1 & m & 1 \\ m & 1 & 0 \\ 1 & 1 & m \end{vmatrix}=\begin{vmatrix} 1-m & m & 1 \\ m-1 & 1 & 0 \\ 0 & 1 & m \end{vmatrix}=(m-1)\begin{vmatrix} -1 & m & 1 \\ 1 & 1 & 0 \\ 0 & 1 & m \end{vmatrix}=$$

$$=(m-1)\begin{vmatrix} -1 & m & 1 \\ 0 & m+1 & 1 \\ 0 & 1 & m \end{vmatrix}=(m-1)\cdot(-1)\cdot\begin{vmatrix} m+1 & 1 \\ 1 & m \end{vmatrix}=$$

$$=-(m-1)(m^2+m-1)$$

$\Delta \neq 0 \iff m \in \mathbb{R}\setminus\{1, \frac{-1\pm\sqrt{5}}{2}\} \iff (*)$ SCD $\Rightarrow$

$\Rightarrow \exists!\ (a,b,c)=(0,0,0)$

2) $S=$ SLD $\iff m \in \{1, \frac{-1\pm\sqrt{5}}{2}\}$ ((*) are și sol. nenule)

Prop.: $(V,+,\cdot)|_{\mathbb{K}}$, $\dim_{\mathbb{K}} V=n$

$B=\{v_1,\dots,v_n\}$

Atunci UAE:

1) B bază

2) B este SLI

3) B este SG

3) $B_0=\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$ bază canonică

$S=\{(1,2,1),(2,1,1),(1,0,2)\}$ este SLI (cf. 1)

$|B_0|=3=\dim_{\mathbb{R}}\mathbb{R}^3=3$ $\Rightarrow$

$\overset{prop.}{\Rightarrow}$ S e bază

Obs! Dem. că $S = SG$

$\mathbb{R}^3 \underset{\subseteq}{\supseteq} \langle S \rangle$

Fie $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \Rightarrow \exists\, a, b, c \in \mathbb{R}$ a.î.

a.î. $(x_1, x_2, x_3) = a(1,2,1) + b(2,1,1) + c(1,0,2)$

$$\begin{cases} a + 2b + c = x_1 \\ 2a + b = x_2 \\ a + b + 2c = x_3 \end{cases} \qquad \bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 2 & 1 & 0 & x_2 \\ 1 & 0 & 2 & x_3 \end{array}\right)$$

$\Delta = -1(4+2-1) = -5 \neq 0 \Rightarrow rg\, A = rg\, \bar{A} = 3 \Rightarrow SCD$

deci există sol. și deci $S = SG$

b) $$A = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

S' bază $\Leftrightarrow S'$ SLI (deoarece $|S'| = \dim_{\mathbb{R}} \mathbb{R}^3 = 3$)

S' SLI $\Leftrightarrow \Delta \neq 0 \Leftrightarrow a_1, a_2, a_3$ distincte

9. Fie $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$

a) $B = \{(1,2), (3,4)\}$ bază

b) $S = \{(1,2), (3,4), (4,2)\}$ este SLD, SG

c) $S' = \{(1,4)\}$ este SLI, nu e SG

Să se extindă la o bază

d) $S'' = \{(1,-1), (2,3), (3,2), (1,4)\}$ este SG

Să se extragă o bază din S''

Sol.:

a) $B_0 = \{e_1 = (1,0),\ e_2 = (0,1)\}$, $\dim_{\mathbb{R}} \mathbb{R}^2 = 2$

$|B| = \dim_{\mathbb{R}} \mathbb{R}^2 = 2$

B - bază $\overset{\text{rem.}}{\Leftrightarrow}$ B este SLI

$rg \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = 2$ (maxim) $\Rightarrow$ SLI deci B este bază

b) $S = B \cup \{(4,2)\}$

$|S| = 3 \Rightarrow SLD$

(2 este nr. max. de vectori care formează SLI

B - bază $\Rightarrow$ B este SG și orice supramulțime a lui este SG

deci S este SG

c) $\left.\begin{array}{l} S' = \{(1,4)\} \\ (1,4) \neq 0_{\mathbb{R}^2} \end{array}\right| \Rightarrow SLI$ (curs)

2 este nr. min. de vec. care să formeze un SG

$|S'| = 1 \Rightarrow S'$ nu este SG

$rg\begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} = 2 \Rightarrow S' \cup \{(1,0)\}$ este $SLI \xRightarrow{prop.} S' \cup \{(1,0)\}$ bază

d) $S'' = \{(1,-1), (2,3), (3,2), (1,4)\}$

$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 1 \\ -1 & 3 & 2 & 4 \end{array}\right)$ $B' = \{(1,-1), (2,3)\}$ SLI maximal al lui S'' (nu este unic)

$B' \xRightarrow{prop} B'$ bază $\xRightarrow{prop.} B'$ SG

$\left.\begin{array}{l} B' \subset S'' \\ B' \text{ SG} \end{array}\right| \Rightarrow S''$ este SG (supramulțime)