CCCC
SSSS

Seminar săpt. 1

Determinanți; rang

Lucru individual: 5, 6, 7, 8, 9, pag 3

1. Fie $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

a) Calc. $\det A = \Delta$ b) Dem. $\Delta = 0 \iff a+b+c=0$ sau $a=b=c$

Sol.:

a) $$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\cdot \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} =$$

$$= (a+b+c)\cdot \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} =$$

$$= (a+b+c)\cdot 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} =$$

$$= (a+b+c)\left[(c-b)(b-c) - (a-b)(a-c)\right] =$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-ac-bc) =$$

$$= -\frac{1}{2}(a+b+c)\left[(a^2-2ab+b^2)+(a^2-2ac+c^2)+(b^2-2bc+c^2)\right] =$$

$$\Delta = -\frac{1}{2}(a+b+c)\left[(a-b)^2+(a-c)^2+(b-c)^2\right]$$

b) $\Delta=0 \iff a+b+c=0$ sau $(a-b)^2+(a-c)^2+(b-c)^2=0$

$$\iff \begin{cases} a-b=0 \\ a-c=0 \\ b-c=0 \end{cases} \Rightarrow a=b=c$$

2. Fie $A=\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det(A)=V(a,b,c)$

<u>Sol.</u>:

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = 1\cdot(-1)^{1+1}\cdot\begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} =$$

$$=(b-a)(c-a)(c+\cancel{a}-b-\cancel{a})$$

$$V(a_1,\dots,a_n)=\prod_{1\le i<j\le n}(a_j-a_i)$$

2/pag3.
$$\begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & & \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{vmatrix}=0;\ a_1,\dots,a_n \text{ distincte 2 câte 2}$$

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x & a_1 & a_2 & \dots & a_n \\ x^2 & a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & & \\ x^n & a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} = (a_1-x)(a_2-x)\dots(a_n-x)\cdot\underbrace{\prod_{1\le i<j\le n}(a_j-a_i)}_{\neq 0}=0 \Rightarrow$$

$\Rightarrow$ polinom de grad n cu rădăcinile $a_1,\dots,a_n$.

3. Fie $A=\begin{pmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

Calculați $\det(A^*)$

<u>Sol.</u>: $\det(A^*)=\det(A)^{n-1}=\det(A)^2$

$$\Delta=\begin{vmatrix} 1+a^2 & 0+ba & 0+ca \\ 0+ab & 1+b^2 & 0+cb \\ 0+ca & 0+bc & 1+c^2 \end{vmatrix}$$

$1 \quad 1' \quad 2 \quad 2' \quad 3 \quad 3'$

$|1\ 2\ 3| \quad + \quad |1'\ 2\ 3|$

$|1\ 2\ 3'| \quad + \quad |1'\ 2\ 3'|=0$

$|1\ 2'\ 3| \quad + \quad |1'\ 2'\ 3|=0$

$|1\ 2'\ 3'|=0 + |1'\ 2'\ 3'|=0$

$$\Delta=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}+\begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix}+\begin{vmatrix} 1 & ba & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix}+\begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & cb \\ 0 & 0 & c^2 \end{vmatrix}=$$

$=1+a^2+b^2+c^2$

Deci $\det(A^*)=(1+a^2+b^2+c^2)^2$

4. Fie $A=\begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

a) Det. m a.î. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

b) Pt. $m=0$, calc. A^{-1}. Precizați mai multe metode

<u>Sol.</u>: a) $A, A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det A, \det A^{-1} \in \mathbb{Z}$

$\det A^{-1}=\dfrac{1}{\det A}$ $\Big| \Rightarrow \det A=\pm 1$

$$\begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} \overset{C_2-C_1}{=\!=} \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = -1\cdot(-1)^{3+1}\cdot\begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix}=$$

$$= -[(-3\cdot 1) - (3m+4)(m-1)] = 3m^2 + m - 1$$

Caz 1) $\det A = 1 \Rightarrow 3m^2 + m - 1 = 1$

$$3m^2 + m - 2 = 0 \Rightarrow m_1 = -1$$

$$m_2 = \frac{2}{3} \notin \mathbb{Z}$$

Caz 2) $\det A = -1 \Rightarrow 3m^2 + m - 1 = -1$

$$3m^2 + m = 0$$

$$m(3m+1) = 0 \Rightarrow m_1 = 0$$

Deci $m \in \{0, -1\}$ $\qquad m_2 = -\frac{1}{3} \notin \mathbb{Z}$

b)
$$A = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$A^*_{ij} = (-1)^{i+j} \delta_{ij}$$

$a^*_{11} = 1 \quad a^*_{12} = -4 \quad a^*_{13} = -1$

$a^*_{21} = -1 \quad a^*_{22} = 4 \quad a^*_{23} = 2$

$a^*_{31} = -1 \quad a^*_{32} = 3 \quad a^*_{33} = 1$

$$A^{-1} = \cancel{\frac{1}{\det A}} \cdot A^* = \det^{-1}(A) \cdot A^* = -1 \cdot A^* = \begin{pmatrix} -1 & 4 & 1 \\ 1 & -4 & -2 \\ 1 & 3 & -1 \end{pmatrix}$$

10. Fie $A = \begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$, $rg A = ?$

Sol.:
$$\begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} \underset{L_3 - L_2}{\overset{L_1 - L_2}{=\!=}} \begin{vmatrix} a-1 & 0 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & -a \end{vmatrix} = \begin{vmatrix} a-1 & 1 \\ -2 & -a \end{vmatrix} =$$

$$= -(a^2 + a) + 2 = -a^2 - a + 2a + 2a = (a+1)(-a+2)$$

Caz 1) $\Delta \neq 0 \Rightarrow a \in \mathbb{R} \setminus \{-1, 2\}$

$$rg A = 3$$

Caz 2) $\Delta=0$:

a) $a=-1$ avem $A=\begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} -1 & 1 \end{matrix} & 2 \end{pmatrix}$, $rg A=2$

b) $a=2$ avem $A=\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} -1 & 1 \end{matrix} & -1 \end{pmatrix}$, $rg A=2$

11. Fie $A=\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

Aflați a, b a.î. $rg A=2$

Sol.: $\begin{vmatrix} \boxed{1} & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{vmatrix}$

$$\Delta_3=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} \overset{L_2-2L_1}{=\!=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & a-6 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -4 & a-6 \\ 1 & 3 \end{vmatrix} = -6-a=0 \Rightarrow$$
$$\Rightarrow a=-6$$

$$\Delta_4=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} \overset{L_2-2L_1}{=\!=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -1 \\ 0 & 1 & b \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 1 & b \end{vmatrix} = -4b+1=0 \Rightarrow$$
$$\Rightarrow b=\frac{1}{4}$$

12. Fie $A \in \mathcal{M}_3(\mathbb{R})$, $A^{2023}-2023A-I_3=O_3$

a) $rg A=?$ b) $rg(2023A+I_3)=?$

Sol.: a) $A^{2023}=2023A+I_3 \Rightarrow (\det A)^{2023}=\det(2023A+I_3)$

$A(A^{2022}-2023)=I_3 \mid \det \Rightarrow$

$\Rightarrow \det A \cdot \det(A^{2022}-2023)=1 \Rightarrow$

$\Rightarrow \det A \neq 0 \Rightarrow rg A=3$

b) $\left.\begin{matrix} (\det A)^{2023}=\det(2023A+I_3) \\ \det A \neq 0 \end{matrix}\right| \Rightarrow \det(2023A+I_3)\neq 0 \Rightarrow$

$\Rightarrow rg(2023A+I_3)=3$

13. $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} \overset{\substack{L_1-L_4 \\ L_2-L_4 \\ L_3-L_4}}{=\!=} \begin{vmatrix} a^3-1 & 3a^2-3 & 3a-3 & 0 \\ a^2-1 & a^2+2a-3 & 2a-2 & 0 \\ a-1 & 2a-2 & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix} =$

$= (a-1)^3 \cdot \begin{vmatrix} a^2+a+1 & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = (a-1)^3 \cdot \begin{vmatrix} a^2+a-2 & 3(a-1) & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{vmatrix} =$

$= (a-1)^5 \cdot \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix} = (a-1)^6$

5/pag3. <u>Prop.</u>: $A \in \mathcal{M}_n(\mathbb{Z}_m)$, $(\mathbb{Z}_m, +, \cdot)$ inel

A inversabilă $\Leftrightarrow \det A \in U(\mathbb{Z}_m)$

$U(\mathbb{Z}_m) = \{\hat{a} \in \mathbb{Z}_m \mid (a, m) = 1\}$

a) $A = \begin{pmatrix} \hat{1} & \hat{2} \\ \hat{3} & \hat{4} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_5)$

$\begin{vmatrix} \hat{1} & \hat{2} \\ \hat{3} & \hat{4} \end{vmatrix} = \hat{1} \cdot \hat{4} - \hat{2} \cdot \hat{3} = \hat{4} - \hat{1} = \hat{3} \in U(\mathbb{Z}_5)$

$A^t = \begin{pmatrix} \hat{1} & \hat{3} \\ \hat{2} & \hat{4} \end{pmatrix} \quad A^* = \begin{pmatrix} \hat{4} & -\hat{2} \\ -\hat{3} & \hat{1} \end{pmatrix} = \begin{pmatrix} \hat{4} & \hat{3} \\ \hat{2} & \hat{1} \end{pmatrix}$

$A^{-1} = \det^{-1} A \cdot A^* = (\hat{3})^{-1} \cdot \begin{pmatrix} \hat{4} & \hat{3} \\ \hat{2} & \hat{1} \end{pmatrix} = \hat{2} \cdot \begin{pmatrix} \hat{4} & \hat{3} \\ \hat{2} & \hat{1} \end{pmatrix} = \begin{pmatrix} \hat{3} & \hat{1} \\ \hat{4} & \hat{2} \end{pmatrix}$

c) $A = \begin{pmatrix} \hat{2} & \hat{2} \\ \hat{1} & \hat{3} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_6)$

$\begin{vmatrix} \hat{2} & \hat{2} \\ \hat{1} & \hat{3} \end{vmatrix} = \hat{2} \cdot \hat{3} - \hat{2} \cdot \hat{1} = \hat{0} - \hat{2} = \hat{4} \notin U(\mathbb{Z}_6) \Rightarrow A$ neinv.

$A = A^t$ simetrică

$A = -A^t$ antisimetrică $\Rightarrow$ diag. este 0

Indicații pt. ultimul exercițiu:

$$AA^t = \alpha \cdot I_4$$

$$\alpha = u^2 + v^2 + s^2 + t^2$$

$$\det(A)^2 = \alpha^4$$