Endomorfisme simetrice

Teoremă (C_{10} - AG)

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in Sim(E)$

$\Rightarrow$ toate rădăcinile polinomului caracteristic sunt reale

Dem. $\mathcal{R} = \{e_1, \dots, e_n\}$ reper ortonormat în E.

$A = [f]_{\mathcal{R},\mathcal{R}}$ și $P(\lambda) = \det(A - \lambda I_n) = 0$. Fie λ rădăcină.

Fie $AX = \lambda X$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$(A - \lambda I_n)X = O_{n,1}$ este SLO

$$\begin{pmatrix} a_{11}-\lambda & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Înmulțim la stânga cu matricea:

$$\begin{pmatrix} \overline{x_1} & \dots & 0 \\ \vdots & & \\ 0 & \dots & \overline{x_n} \end{pmatrix}$$

Prin calcul obținem:

$$\begin{pmatrix} \overline{x_1} & \dots & 0 \\ 0 & \dots & \overline{x_n} \end{pmatrix} \begin{pmatrix} (a_{11}-\lambda)x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \dots + (a_{nn}-\lambda)x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Rezultă

$$\begin{cases} (a_{11}-\lambda)x_1\overline{x_1} + a_{12}x_2\overline{x_1} + \dots + a_{1n}x_n\overline{x_1} = 0 \\ \vdots \\ a_{n1}x_1\overline{x_n} + a_{n2}x_2\overline{x_n} + \dots + (a_{nn}-\lambda)x_n\overline{x_n} = 0 \end{cases} \quad \oplus$$

$$\sum_{k,j=1}^{n} a_{kj}x_k\overline{x_j} = \lambda \underbrace{\sum_{k=1}^{n} x_k\overline{x_k}}_{\in \mathbb{R}} \qquad (\text{Prop: } z\cdot\overline{z} = |z|^2 \in \mathbb{R})$$

$$\|$$

$$\sum_{k<j} a_{kj}x_k\overline{x_j} + \sum_{k>j} a_{kj}x_k\overline{x_j} + \sum_{k=1}^{n} a_{kk}x_k\overline{x_k} \qquad (A = A^T \in \mathcal{M}_n(\mathbb{R}))$$

$$\sum_{k<j} a_{kj}\underbrace{(x_k\overline{x_j} + x_j\overline{x_k})}_{\in \mathbb{R}} + \sum_{k=1}^{n} a_{kk}\underbrace{x_k\overline{x_k}}_{\in \mathbb{R}} = \lambda \sum_{k=1}^{n} x_k\overline{x_k} \Rightarrow \lambda \in \mathbb{R}$$

Teorema de descompunere polară

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r

$\forall f \in Aut(E) \Rightarrow \exists h \in Sim(E)$, $\exists t \in O(E)$ a.î. $f = h \circ t$

OBS

$\forall A \in GL(n, \mathbb{R})$, $\exists B \in \mathcal{M}_n(\mathbb{R})$, $B = B^T$, $\exists C \in O(n)$ a.î. $A = B \cdot C$

Lemă

$f \in Sim(E)$, poz. def ($[f]_{R,R}$ poz definită sau Q forma pătratică asociată poz. definită) $\Rightarrow$

$\exists h \in Sim(E)$ poz. def a.î. $f = h^2$

Dem (Lemă) $R = \{e_1, .., e_n\}$ reper orton. a.î. $A_f = [f]_{R,R}$

$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ (f este diagonalizabil)

$Q_f : E \to \mathbb{R}$ f. pătratică asociată

$Q_f(x) = \lambda_1 x_1^2 + ... + \lambda_n x_n^2$, $x = \sum_{i=1}^{n} x_i e_i$ (sign este $(n, 0)$)

Q_f este p. def $\Rightarrow \lambda_1 > 0, .., \lambda_n > 0$

Fie $h \in End(E)$, $[h]_{R,R} = A_h = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$

$h \in Sim(E)$

$A_{h^2} = A_h \cdot A_h = A_f$, $Q_h(x) = \sqrt{\lambda_1} x_1^2 + ... + \sqrt{\lambda_n} x_n^2$

este poz. def $\Rightarrow h$ este poz. def. și

$f = h^2$

Dem (teoremă)

Fie $R=\{e_1,\dots,e_n\}$ reper orton, $A_f=[f]_{R,R}\in GL(n,R)$

Fie $\tilde f\in \mathrm{End}(E)$ a.î. $A_{\tilde f}=A_f\cdot A_f^T \overset{not}{=} B$ (*)

$B=B^T \Rightarrow \tilde f\in \mathrm{Sim}(E)$ (**)

Dem că $\tilde f$ este poz. definită.

Fie $Q_{\tilde f}: E\longrightarrow \mathbb{R}$ forma pătratică asociată

$$\underset{i=\overline{1,n}}{Q_{\tilde f}(e_i)} = \langle e_i, \tilde f(e_i)\rangle = \langle e_i, \sum_{j=1}^{n} b_{ji}\, e_j\rangle =$$

$$= \sum_{j=1}^{n} b_{ji}\,\langle e_i, e_j\rangle = b_{ii} \overset{(*)}{=} \sum_{k=1}^{n} a_{ik}a_{ik} = \sum_{k=1}^{n} a_{ik}^2 > 0$$

($\langle e_i, e_j\rangle = \delta_{ij}$)

(linia i a lui A_f nu poate fi nulă, $A_f\in GL(n,R)$)

Deci $Q_{\tilde f}(x)>0,\ \forall x\neq 0_E \Rightarrow \tilde f$ p.def $\xrightarrow[(**)]{\text{Lema}}$

$\exists\, h\in \mathrm{Sim}(E)$, poz def a.î. $\tilde f = h^2$

$$B = A_f\cdot A_f^T = A_h\cdot A_h$$

Fie $t=h^{-1}\circ f$. Dem că $t\in O(n)$

$$A_t\, A_t^T = A_{h^{-1}\circ f}\left(A_{h^{-1}\circ f}\right)^T = A_{h^{-1}}\cdot A_f \left(A_{h^{-1}}\cdot A_f\right)^T$$

$$= A_{h^{-1}}\cdot \underbrace{A_f\cdot A_f^T}_{A_{\tilde f}\, =\, A_{h^2}}\cdot A_{h^{-1}} = A_{h^{-1}}\cdot A_h\cdot A_h\cdot A_{h^{-1}} = I_n.$$

(h sim)

$$\Rightarrow t\in O(n)$$

Deci $\underset{\mathrm{Aut}(E)}{\underset{\cap}{f}} = \underset{\mathrm{Sim}(E)}{\underset{\cap}{h}}\circ \underset{O(E)}{\underset{\cap}{t}}$.