Seminar săpt. 11

Transformări ortogonale. Endomorfisme simetrice

8. $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0,R_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) f transf. ort. de speță 2 $f = s \circ R_\varphi$

b) $\varphi = ?$ axa de rotație?

c) Să se det. R ortonormat, $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

Sol:

a) $f \in O(\mathbb{R}^3) \Leftrightarrow A \in O(3)$

$A \cdot A^t = I_3$ ✓

$\det A = -1 \Rightarrow f$ este de speță 2

b) $Tr A = 1 = -1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 2 \Rightarrow \cos\varphi = 1 \Rightarrow \varphi = 0^\circ$

Axa: $f(x) = -x$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_2, x_1, x_3)$

$f(x) = -x \Rightarrow \left. \begin{array}{l} x_2 = -x_1 \\ x_1 = -x_2 \end{array} \right| \Rightarrow x_1 = -x_2$

$x_3 = -x_3 \Rightarrow x_3 = 0$

$(x_1, x_2, x_3) = (-x_2, x_2, 0) = x_2(-1, 1, 0)$

$e_1 = \frac{1}{\sqrt{2}}(-1, 1, 0)$ versorul axei de rotație

c) $e_1^\perp = \{x \in \mathbb{R}^3 \mid g_0(x, e_1) = 0\} = \{(x_1, x_1, x_3) \mid x_1, x_3 \in \mathbb{R}\} =$

$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$ $\qquad = \langle\{\underbrace{(1,1,0)}_{f_2}, \underbrace{(0,0,1)}_{f_3}\}\rangle$

prod. scalar canonic

$\{f_2, f_3\}$ reper în $e_1^\perp$

$\langle f_2, f_3 \rangle = 0 \Rightarrow$ reper ortonormat, altfel aplicăm Gram-Schmidt

$e_2=\frac{1}{\sqrt{2}}(1,1,0)$

$e_3=(0,0,1)$

dreaptă plan

$\mathcal{R}=\{e_1,e_2,e_3\}$ reper ortonormat, $\mathbb{R}^3=\langle\{e_1\}\rangle\oplus\langle\{e_1\}\rangle^{\perp}$

$$[f]_{\mathcal{R},\mathcal{R}}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$[f]_{\mathcal{R},\mathcal{R}}=$

2/2. $(\mathbb{R}^3,g_0)$ s.v.e.r, $u=(1,-1,2)$

a) $\langle\{u\}\rangle^{\perp}=?$ Precizați un reper ortonormat

b) Să se det. transf. ortogonală, de speța 1, care este rotație de $\sphericalangle$ orientat $\varphi=\frac{\pi}{2}$ și axa $\langle\{u\}\rangle$

Sol.:

a) $u^{\perp}=\{x\in\mathbb{R}^3 \mid g_0(x,u)=0\}=\{x\in\mathbb{R}^3 \mid x_1-x_2+2x_3=0\}=$

(x_1,x_1+2x_3,x_3)

$x_2=x_1+2x_3$

$=\langle\{\underset{f_2}{(1,1,0)},\underset{f_3}{(0,2,1)}\}\rangle$

$\{f_2,f_3\}$ reper arbitrar, $\{e_2',e_3'\}$ ortogonal

$e_2'=f_2=(1,1,0)$

$e_3'=f_3-\frac{\langle f_3,e_2'\rangle}{\langle e_2',e_2'\rangle}\cdot e_2'=(0,2,1)-\frac{2}{2}(1,1,0)=(-1,1,1)$

$\{e_2=\frac{1}{\sqrt{2}}(1,1,0),\ e_3=\frac{1}{\sqrt{3}}(-1,1,1)\}$ reper ortonormat

$e_1=\frac{1}{\sqrt{6}}(1,-1,2)$ versorul axei

$\mathbb{R}^3=u\oplus u^{\perp}$

$\mathcal{R}=\{e_1,e_2,e_3\}$ reper ortonormat în $\mathbb{R}^3$ a.î.

$$A'=[f]_{\mathcal{R},\mathcal{R}}=\begin{pmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R = \{\frac{1}{\sqrt{6}}(1,-1,2), \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(-1,1,1)\}$

$A' = C^{-1}AC \Rightarrow A = CA'C^t$

$$C = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \end{pmatrix}$$

$$A = \frac{1}{6}\begin{pmatrix} 1 & \sqrt{3} & -\sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ \sqrt{3} & \sqrt{3} & 0 \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{6}\begin{pmatrix} 1 & -1-2\sqrt{6} & 2-\sqrt{6} \\ -1+2\sqrt{6} & 1 & -2-\sqrt{6} \\ 2+\sqrt{6} & -2-\sqrt{6} & 4 \end{pmatrix}$$

$$f: \mathbb{R}^3 \to \mathbb{R}^3, f(x) = \frac{1}{6}(x_1 + x_2(-1-2\sqrt{6}) + x_3(2-\sqrt{6}),$$
$$x_1(-1+2\sqrt{6}) + x_2 + x_3(-2-\sqrt{6}),$$
$$x_1(2+\sqrt{6}) + x_2(-2-\sqrt{6}) + 4x_3)$$

4/3. $(\mathbb{R}^3, g_0)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = g_0(x, u)u$, unde $u = (1,-1,2)$

a) Să se arate că $f \in \mathrm{Sim}(\mathbb{R}^3)$, $f = ?$

b) Să se afle $Q: \mathbb{R}^3 \to \mathbb{R}$ formă pătratică asociată. Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h (adică o schimbare de repere ortonormate)

Sol.:

a) $g_0(x, u) = x_1 - x_2 + 2x_3$

$f(x) = (x_1 - x_2 + 2x_3, -x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + 4x_3)$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^t \Rightarrow f \in \mathrm{Sim}(\mathbb{R}^3)$$

$\langle f(x), x \rangle = Q(x)$

b) $Q(x)=x_1^2+x_2^2+4x_3^2-2x_1x_2+4x_1x_3-4x_2x_3$

c) Metoda valorilor proprii

$$P_A(\lambda)=\det(A-\lambda I_3)=\begin{vmatrix}1-\lambda & -1 & 2\\ -1 & 1-\lambda & -2\\ 2 & -2 & 4-\lambda\end{vmatrix}\overset{L_1+L_2}{=}\begin{vmatrix}-\lambda & -\lambda & 0\\ -1 & 1-\lambda & -2\\ 2 & -2 & 4-\lambda\end{vmatrix}=$$

$$=-\lambda\begin{vmatrix}1 & 1 & 0\\ -1 & 1-\lambda & -2\\ 2 & -2 & 4-\lambda\end{vmatrix}=-\lambda\left(\begin{vmatrix}1-\lambda & -2\\ -2 & 4-\lambda\end{vmatrix}-\begin{vmatrix}-1 & -2\\ 2 & 4-\lambda\end{vmatrix}\right)=$$

$=-\lambda(-4+4-5\lambda+\lambda^2-(\lambda-4+4))$

$=-\lambda(\lambda^2-6\lambda)$

$=-\lambda^2(\lambda-6)=0$

$\lambda_1=0,\ m_1=2 \Rightarrow \dim V_{\lambda_1}=2$

$\lambda_2=6,\ m_2=1 \Rightarrow \dim V_{\lambda_2}=1$

$\mathbb{R}^3=V_{\lambda_1}\oplus V_{\lambda_2}$

$\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$

$V_{\lambda_1}=\{x\in\mathbb{R}^3\mid AX=0X\}=\ker f=\{(x_1,x_1+2x_3,x_3)\}=\langle\{\underbrace{(1,1,0)}_{f_1},\underbrace{(0,2,1)}_{f_2}\}\rangle$

$x_1-x_2+2x_3=0 \Rightarrow x_2=x_1+2x_3$

$e_1=f_1=(1,1,0)$

$e_2=f_2-\frac{\langle f_2,e_1\rangle}{\langle e_1,e_1\rangle}e_1=(0,2,1)-\frac{2}{2}(1,1,0)=(-1,1,1)$

$e_1'=\frac{1}{\sqrt2}(1,1,0),\ e_2'=\frac{1}{\sqrt3}(-1,1,1)$

$\mathcal{R}_1=\{e_1',e_2'\}$ - reper ortonormat în V_{λ_1}

$V_{\lambda_2}=\{x\in\mathbb{R}^3\mid AX=6X\}$

$$B\to\begin{pmatrix}-5 & -1 & 2\\ -1 & -5 & -2\\ 2 & -2 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\det B = \begin{vmatrix} -5 & -6 & -3 \\ -1 & -6 & -3 \\ 2 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{cases} -5x_1 - x_2 = -2x_3 \\ -x_1 - 5x_2 = 2x_3 \end{cases} \quad (+)$$

$$-6(x_1 + x_2) = 0 \Rightarrow x_2 = -x_1 \; ; \ldots$$

$$V_{\lambda_2} = \langle \{ (\tfrac{1}{2}, -\tfrac{1}{2}, 1) \} \rangle = \langle \{ (1, -1, 2) \} \rangle$$

$$\mathcal{R}_2 = \{ e_3' = \tfrac{1}{\sqrt{6}} (1, -1, 2) \}$$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormat în $\mathbb{R}^3$ a.î.

$$[f]_{\mathcal{R}, \mathcal{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad Q(x) = 6x_3'^2 \text{ , sgn } (1, 0)$$

$$\mathcal{R}_0 = \{ e_1^0, e_2^0, e_3^0 \} \xrightarrow{C} \mathcal{R} = \{ e_1' = \tfrac{1}{\sqrt{2}} (1, 1, 0), e_2' = \tfrac{1}{\sqrt{3}} (-1, 1, 1), e_3' = \tfrac{1}{\sqrt{6}} (1, -1, 2) \}$$

$$h \in O(\mathbb{R}^3), \; h(e_i^0) = e_i', \; \forall i = \overline{1,3}$$

$$C = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & 1 \\ \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix}$$