C_6

Aplicații liniare

$(V_i, +, \cdot)/_{\mathbb{K}}$, $i=\overline{1,2}$ sp vect

$f: V_1 \to V_2$ s.n. aplicație liniară $\Leftrightarrow$ 1) $f(x+y) = f(x)+f(y)$

2) $f(\alpha x) = \alpha f(x)$, $\forall x, y \in V_1$, $\alpha \in \mathbb{K}$.

$\Leftrightarrow f(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f(x_i)$

$\operatorname{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\}$ nucleul lui f.

$\operatorname{Im} f = \{y \in V_2 \mid \exists x \in V_1$ ai $f(x)=y\}$ imag. lui f.

$\operatorname{Ker} f \subseteq V_1$ sspv ; $\operatorname{Im} f \subseteq V_2$ sspv.

Prop $f: V_1 \to V_2$ apl. liniară

1) f inj $\Leftrightarrow$ $\operatorname{Ker} f = \{0_{V_1}\}$

2) f surj $\Leftrightarrow$ $\dim \operatorname{Im} f = \dim V_2$.

Dem

1) "$\Rightarrow$" Ip: f inj

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{V_2}$.

$f: (V_1,+) \to (V_2,+)$ morf gr $\Rightarrow f(0_{V_1}) = 0_{V_2}$

$\Rightarrow f(0_{V_1}) = f(x) \overset{f \text{ inj}}{\Rightarrow} x = 0_{V_1}$.

$\Rightarrow \operatorname{Ker} f = \{0_{V_1}\}$

"$\Leftarrow$" Ip: $\operatorname{Ker} f = \{0_{V_1}\}$

Fie $x_1, x_2 \in V_1$ ai $f(x_1) = f(x_2) \Rightarrow f(x_1 - x_2) = 0_{V_2}$

$\Rightarrow x_1 - x_2 \in \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow x_1 - x_2 = 0_{V_1} \Rightarrow x_1 = x_2 \Rightarrow f$ inj

2) "$\Rightarrow$" Ip: f surj $\Rightarrow \operatorname{Im} f = V_2 \Rightarrow \dim \operatorname{Im} f = \dim V_2$

"$\Leftarrow$" $\left.\begin{array}{l} \dim \operatorname{Im} f = \dim V_2 \\ \operatorname{Im} f \subseteq V_2 \text{ sspv} \end{array}\right\} \Rightarrow \operatorname{Im} f = V_2 \Rightarrow f$ surj.

Teorema dimensiunii

$f: V_1 \longrightarrow V_2$ apl. liniară

$\Rightarrow \dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f.$

Dem

$\operatorname{Ker} f \subseteq V_1$ subsp. vect.

Fie $R_0 = \{e_1, .., e_k\}$ reper în $\operatorname{Ker} f$.

Extindem la un reper $\{e_1, .., e_k, e_{k+1}, .., e_n\}$ reper în V_1.

Dem că $R = \{f(e_{k+1}), .., f(e_n)\}$ reper în $\operatorname{Im} f$ $(k \leq n)$

- SLI

Fie $a_j \in \mathbb{K}$, $j = \overline{k+1, n}$ aî $\sum_{j=k+1}^{n} a_j f(e_j) = 0_{V_2} \Rightarrow f\left(\sum_{j=k+1}^{n} a_j e_j\right) = 0_{V_2}.$

$\Rightarrow \sum_{j=k+1}^{n} a_j e_j \in \operatorname{Ker} f = \langle R_0 \rangle \Rightarrow \exists\, a_i \in \mathbb{K}, i = \overline{1,k}$ aî

$$\sum_{j=k+1}^{n} a_j e_j = \sum_{i=1}^{k} a_i e_i \Rightarrow \sum_{i=1}^{k} a_i e_i - \sum_{j=k+1}^{n} a_j e_j = 0_{V_1} \xrightarrow[\text{SLI}]{\text{reper în } V_1}$$

$a_i = 0, \forall i = \overline{1,k}$

$a_j = 0, \forall j = \overline{k+1, n} \Rightarrow R$ este SLI

- SG Dem că $\operatorname{Im} f = \langle R \rangle$.

$\forall y \in \operatorname{Im} f \Rightarrow \exists\, x \in V_1 = \langle \{e_1, .., e_k, e_{k+1}, ..., e_n\} \rangle$ aî $f(x) = y$

$$y = f\left(\sum_{i=1}^{k} a_i e_i + \sum_{j=k+1}^{n} a_j e_j\right) = \underbrace{f\Big(\overbrace{\sum_{i=1}^{k} a_i e_i}^{\in \operatorname{Ker} f}\Big)}_{= 0_{V_2}} + \underline{f\left(\sum_{j=k+1}^{n} a_j e_j\right)}$$

$\Rightarrow y = \sum_{j=k+1}^{n} a_j f(e_j) \Rightarrow R$ este SG pt $\operatorname{Im} f$.

$$\dim V_1 = n = \underbrace{k}_{\dim \operatorname{Ker} f} + \underbrace{n - k}_{\dim \operatorname{Im} f}.$$

Prop $f: V_1 \to V_2$ liniară

a) f inj $\iff \dim V_1 = \dim \operatorname{Im} f$.

b) f surj $\iff \dim V_1 = \dim \operatorname{Ker} f + \dim V_2$.

c) f ~~surj~~ bij $\iff \dim V_1 = \dim V_2$.

Dem a) f inj $\iff \operatorname{Ker} f = \{0_{V_1}\}$.

T. dim. $\dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$ $\Big| \Rightarrow$

f inj $\iff \dim V_1 = \dim \operatorname{Im} f$.

b) f surj $\iff \dim \operatorname{Im} f = \dim V_2$.

f surj $\overset{T.dim}{\iff} \dim V_1 = \dim \operatorname{Ker} f + \dim V_2$.

c) f bij $\iff \dim V_1 = \dim V_2$.

Teoremă $V_1 \simeq V_2$ (sp. vect. izomorfe i.e $\exists f: V_1 \to V_2$ izomorfism de sp. vect.)

$\iff \dim V_1 = \dim V_2$

Dem

„$\Rightarrow$" Ip. $\exists f: V_1 \to V_2$ izom. sp. vect $\begin{cases} \text{apl. lin} \\ \text{bij.} \end{cases}$

Cf Prop. preced c) $\dim V_1 = \dim V_2$

„$\Leftarrow$" $\dim V_1 = \dim V_2 = n$.

Construim $f: V_1 \to V_2$ izom de sp vect. aî $f(e_i) = e'_i$, $i = \overline{1,n}$

Fie $R_1 = \{e_1, .., e_n\}$ reper în V_1

$R_2 = \{e'_1, .., e'_n\}$ — \|\| — V_2.

Extindem f prin liniaritate

$x \in V_1$, $f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{= e'_i} = \sum_{i=1}^{n} x_i e'_i = x'$

f bij $\iff \forall \underbrace{x'}_{= \sum_{i=1}^{n} x_i e'_i} \in V_2$, $\exists! \underbrace{x}_{= \sum_{i=1}^{n} x_i e_i} \in V_1$ aî $f(x) = x'$ $\Big| \Rightarrow f$ izom sp vect $V_1 \simeq V_2$.

Prop $f: V_1 \to V_2$ apl. liniară

a) f inj $\Leftrightarrow$ f transformă $\forall$ SLI din V_1 într-un SLI din V_2.

b) f surj $\Leftrightarrow$ — " — SG — " — SG — " —

c) f bij $\Leftrightarrow$ — " — reper — " — reper — " —

Dem

a) "$\Rightarrow$" Ip f inj. Fie $S = \{v_1, \dots, v_n\}$ SLI în V_1. Dem că: $f(S) = \{f(v_1), \dots, f(v_n)\}$ SLI în V_2,

Fie $a_1, \dots, a_n \in \mathbb{K}$ ai $\sum_{i=1}^{n} a_i f(v_i) = 0_{V_2} \Rightarrow f\left(\sum_{i=1}^{n} a_i v_i\right) = 0_{V_2}$ $\quad \mathrm{Ker} f = \{0_{V_1}\}$

$\Rightarrow \sum_{i=1}^{n} a_i v_i = 0_{V_1} \overset{S}{\underset{\text{SLI}}{\Rightarrow}} a_i = 0, \forall i = \overline{1,n}$

Deci $f(S)$ e SLI în V_2

"$\Leftarrow$" f transf $\forall$ SLI din V_1 în SLI în V_2.

Fie $x \in \mathrm{Ker} f$. Ip abs $x \neq 0_{V_1} \Rightarrow \{x\}$ SLI în V_1 $\overset{ip}{\Rightarrow}$

$\Rightarrow \{\underset{0_{V_2}}{\underset{\parallel}{f(x)}}\}$ SLI în V_2 Contrad. Ip. este falsă și $x = 0_{V_1}$

$\Rightarrow f$ inj

b) "$\Rightarrow$" Ip f surj. $\Leftrightarrow$ $\mathrm{Im} f = V_2$. ($\Leftrightarrow \dim \mathrm{Im} f = \dim V_2$).

Fie $S = \{v_1, \dots, v_n\}$ SG pt V_1 i.e. $V_1 = \langle S \rangle$.

Dem că $f(S) = \{f(v_1), \dots, f(v_n)\}$ SG pt V_2 i.e. $V_2 = \langle f(S) \rangle$.

$\forall y \in V_2, \exists x \in \underset{\langle S \rangle}{\underset{\parallel}{V_1}}$ ai $y = f(x) = f\left(\sum_{i=1}^{n} a_i v_i\right) \overset{f \text{ lin}}{=}$

$= \sum_{i=1}^{n} a_i f(v_i) \Rightarrow V_2 = \langle f(S) \rangle$

"$\Leftarrow$" Ip: $\langle S \rangle = V_1 \Rightarrow \langle f(S) \rangle = V_2$. Dem $\mathrm{Im} f = V_2$. "$\subseteq$"

Dem $V_2 \subseteq \mathrm{Im} f$.

Fie $y \in V_2 \Rightarrow y = \sum_{i=1}^{n} a_i f(v_i) \overset{f \text{ lin}}{=} f\left(\sum_{i=1}^{n} a_i v_i\right) \in f(V_1)$ $\quad V_1 = \langle S \rangle$

$\Rightarrow V_2 \subset \mathrm{Im} f$.

Matricea asociată unei aplicații liniare

$f: V_1 \longrightarrow V_2$ apl. liniară, $\dim V_1 = n$, $\dim V_2 = m$.

$R_1 = \{e_1, .., e_n\}$ reper în V_1 și $R_2 = \{e'_1, .., e'_m\}$ reper în V_2

$$f(e_i) = \sum_{j=1}^{m} a_{ji} e'_j, \ \forall i = \overline{1,n}$$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ji} e'_j =$$

$$\begin{cases} f(x) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i\right) e'_j \\ f(x) = y = \sum_{j=1}^{m} y_j e'_j \end{cases} \Rightarrow y_j = \sum_{i=1}^{n} a_{ji} x_i, \forall j = \overline{1,m}$$

$$\boxed{Y = AX}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$ Notăm $[f]_{R_1, R_2} = A$

Teorema de caract. a apl. liniare

$f: V_1 \longrightarrow V_2$ apl.

f e liniară $\Leftrightarrow \exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ a.î. coordonatele lui x în raport cu reperul $R_1 = \{e_1, .., e_n\}$ din V_1 și coordonatele lui $y = f(x)$ în raport cu reperul $R_2 = \{e'_1, .., e'_m\}$ din V_2 verifică $Y = AX$

$$\begin{pmatrix} y_1 \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x = \sum_{i=1}^{n} x_i e_i$$

$$y = \sum_{j=1}^{m} y_j e'_j$$

$\boxed{\text{OBS}}$ $A = [f]_{R_1, R_2}$

$$\begin{array}{ccc} R_1 = \{e_1, .., e_n\} \text{ în } V_1 & \longrightarrow & R_2 = \{e'_1, .., e'_m\} \text{ în } V_2 \\ GL(n, \mathbb{K}) \ni C \downarrow & A' = [f]_{\overline{R_1}, \overline{R_2}} & \downarrow D \in GL(m, \mathbb{K}) \\ \overline{R_1} = \{\overline{e_1}, .., \overline{e_n}\} \text{ în } V_1 & \longrightarrow & \overline{R_2} = \{\overline{e}'_1, .., \overline{e}'_m\} \text{ în } V_2 \end{array}$$

$\boxed{A' = D^{-1} A C}$ $\quad \operatorname{rg} A = \operatorname{rg} \overline{A}$ (invariant)

OBS $f \in End(V)$

$$\begin{array}{ccc} \mathcal{R}_1 = \{e_1, .., e_n\} & \longrightarrow & \mathcal{R}_1 = \{e_1, .., e_n\} \\ C \downarrow & & \downarrow D = C \\ \overline{\mathcal{R}}_1 = \{\overline{e}_1, .., \overline{e}_n\} & \longrightarrow & \overline{\mathcal{R}}_1 = \{\overline{e}_1, .., \overline{e}_n\} \end{array}$$

$A = [f]_{\mathcal{R}_1, \mathcal{R}_1}$

$A' = [f]_{\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_1}$

$$\boxed{A' = C^{-1} A C}$$

EX $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (x_1 + x_2, 2x_2)$

$\mathcal{R}_0 = \{e_1 = (1,0), e_2 = (0,1)\}$ reperul canonic în $\mathbb{R}^2$

$\mathcal{R}' = \{e_1' = e_1 - 2e_2, \ e_2' = e_1 + e_2\}$

a) $[f]_{\mathcal{R}_0, \mathcal{R}_0}$; b) $[f]_{\mathcal{R}', \mathcal{R}'}$

SOL M1

a) $f(x) = y \Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} x_1 + x_2 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$X = x_1 e_1 + x_2 e_2$.

$A = [f]_{\mathcal{R}_0, \mathcal{R}_0}$

M2 $f(e_1) = f(1,0) = (1,0) = \boxed{1} e_1 + \boxed{0} e_2$.

$f(e_2) = f(0,1) = (1,2) = (1,0) + (0,2) = \boxed{1} e_1 + \boxed{2} e_2$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

b) M1 $f(e_1') = f((1,-2)) = (1-2, 2(-2)) = (-1,-4) = a e_1' + b e_2'$

$(-1,-4) = a(1,-2) + b(1,1) = (a+b, -2a+b)$

$A' = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\begin{cases} a + b = -1 \\ -2a + b = -4 \end{cases} \ominus$$

$3a \ / = 3$

$a = 1$

$b = -2$

$A' = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

$f(e_2') = f(1,1) = (2,2) = c e_1' + d e_2' = (c+d, -2c+d)$

$$\begin{cases} c + d = 2 \\ -2c + d = 2 \end{cases} \ominus$$

$3c \ / = 0$

$c = 0$

$d = 2$

M2 $\mathcal{R}_0 = \{e_1, e_2\} \xrightarrow{C} \mathcal{R}' = \{e_1' = e_1 - 2e_2, \ e_2' = e_1 + e_2\}$

$C = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

$A' = C^{-1} A C$

Prop $f: V_1 \to V_2$ liniară $\quad A = [f]_{R_1, R_2}$ $\forall R_1$ reper în V_1, $\forall R_2$ reper în V_2

a) f inj $\iff \dim V_1 = rg\, A$

b) f surj $\iff \dim V_2 = rg\, A$

c) f bij $\iff A \in GL(n, \mathbb{K})$ $\quad (\dim V_1 = \dim V_2 = n)$

Dem

a) f inj $\iff \underset{\{x \in V_1 \mid AX = \left(\begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix}\right)\} = S(A)}{\underset{\|}{\operatorname{Ker} f}} = \{0_{V_1}\}$ $\qquad f(x) = y \iff Y = AX$

$\iff 0 = \dim V_1 - rg\, A \iff \dim V_1 = rg\, A$

b) f surj $\iff \dim \operatorname{Im} f = \dim V_2$

T dim $\quad \cancel{\dim V_1} = \underset{\cancel{\dim V_1} - rg\, A}{\underset{\|}{\dim \operatorname{Ker} f}} + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = rg\, A$

$\Rightarrow f$ surj $\iff \dim V_2 = rg\, A$

c) Consec pt a), b).

OBS a) $U \xrightarrow{f} V \xrightarrow{g} W$, $h = g \circ f$, f, g liniare

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V \arrow[r, "g"] & W \end{tikzcd}$$

R_1 reper în U

$x \longrightarrow y = f(x) \to z = g(y) = g(f(x))$ $\quad R_2$ reper în V, R_3 reper în W.

$$\left.\begin{array}{l} Z = A_h \cdot X \\ Y = A_f \cdot X, \quad Z = A_g Y = A_g \cdot A_f \cdot X \end{array}\right\} \Rightarrow A_h = A_g \cdot A_f$$

$$\boxed{A_{g \circ f} = A_g \cdot A_f}$$

b) $\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "id_U"'] & V \arrow[r, "f^{-1}"] & U \end{tikzcd}$ $\quad I_n = A_f \cdot A_{f^{-1}}$

$\begin{tikzcd} V \arrow[r, "f^{-1}"] \arrow[rr, bend right, "id_V"'] & U \arrow[r, "f"] & V \end{tikzcd}$. $\quad f$ izom sp vect. $\quad I_n = A_{f^{-1}} \cdot A_f$

$\Rightarrow (A_f)^{-1} = A_{f^{-1}}$.

c) $GL(V) = Aut(V) = \{f \in End(V) \mid f \text{ bij}\}$.

$(GL(V), \circ) \xrightarrow{\varphi} (GL(n, \mathbb{K}), \cdot)$, $\varphi(f) = A_f$

izomorfism de grupuri

$\varphi(f \circ g) = \varphi(f) \cdot \varphi(g)$ și φ bij.

__Def__ (spațiul dual)

$(V, +, \cdot)/\mathbb{K}$ sp. vectorial

$(V^* = \{f : V \to \mathbb{K} \mid f \text{ liniară}\}, +, \cdot)/\mathbb{K}$ __sp. vect. dual__

$+ : V^* \times V^* \longrightarrow V^*$ $\quad (f+g)(x) := f(x) + g(x)$

$\cdot : \mathbb{K} \times V^* \longrightarrow V^*$ $\quad (\alpha f)(x) := \alpha \cdot f(x), \forall x \in V$
$\forall \alpha \in \mathbb{K}$.

__Teoremă__ $V \simeq V^*$ (sp. vect izomorfe)

__Dem__

Fie $\mathcal{R} = \{e_1, .., e_n\}$ reper în V

Considerăm $\mathcal{R}^* = \{e_1^*, .., e_n^*\}$, $e_i^* : V \to \mathbb{K}$ liniare.

ai $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

$e_i^*(x) = e_i^*\left(\sum_{j=1}^n x_j e_j\right) = \sum_{j=1}^n x_j e_i^*(e_j) =$

(extindem prin liniaritate

$= x_1 \underbrace{e_i^*(e_1)}_{= 0} + \ldots + x_i \underbrace{e_i^*(e_i)}_{= 1} + \ldots + x_n \underbrace{e_i^*(e_n)}_{= 0} = x_i, \forall i = \overline{1,n}$

Dem că $\mathcal{R}^* = \{e_1^*, .., e_n^*\}$ reper în V^*

1) SLI $\quad \sum_{i=1}^n a_i e_i^* = 0 \quad \mid (e_j) \quad \forall j = \overline{1,n}$

$\sum_{i=1}^n a_i e_i^*(e_j) = 0 \Rightarrow a_j = 0 \Rightarrow$ SLI

2) SG $\quad V^* = \langle R^* \rangle$

$\forall f \in V^* \implies f = \sum_{i=1}^{n} f_i e_i^*, \; f_i \in \mathbb{K}.$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} \underbrace{x_i}_{e_i^*(x)} \underbrace{f(e_i)}_{f_i \in \mathbb{K}} = \sum_{i=1}^{n} f_i e_i^*(x), \; \forall x \in V$$

$f = \sum_{i=1}^{n} f_i e_i^* \implies R^* = \{e_1^*, \dots, e_n^*\}$ reper în V^*

$\dim V = \dim V^* \implies V \simeq V^*$

<u>Exemple de endomorfisme</u> (<u>proiecții</u> și <u>simetrii</u>)

<u>Def</u> $\quad V = V_1 \oplus V_2, \; p \in End(V)$

$p(v) = p(v_1 + v_2) = v_1 \in V_1$

$p =$ proiecție pe V_1, de-a lungul lui V_2.

<u>Prop</u> $\; p \in End(V)$

$p =$ proiecție $\iff p \circ p = p$.

<u>Def</u> $\; s \in End(V)$

s s.n. <u>simetrie</u> $\iff s \circ s = id_V$

<u>Prop</u> $\; p : V \longrightarrow V, \; V = V_1 \oplus V_2$ proiecția pe V_1, de-a lungul lui V_2

$\iff s = 2p - id_V$ este simetrie față de V_1

$(\mathrm{char}\, \mathbb{K} \neq 2)$