CCCC Seminar săpt. 6

SSSS

1. $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x_1,x_2,x_3)=(2x_1+2x_2,\ x_1+x_3,\ x_1+3x_2-2x_3)$

a) f nu este izomorfism de sp. vec.

b) $f/_{V'}: V'\to V''$ izomorfism, unde:

$V'=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_2-x_3=0\}$

$V''=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid 3x_1-4x_2-2x_3=0\}$

c) Să se afle $f(V'\cap V'')$

d) $\mathbb{R}^3=V'\oplus W$. Dați un ex. de W

Fie $p:\mathbb{R}^3\to\mathbb{R}^3$ proiecția pe V'

$s:\mathbb{R}^3\to\mathbb{R}^3$ simetria față de V'

Să se calculeze $p(1,3,6)$, $s(1,3,6)$

Sol.:

a) $\mathcal{R}_0=\{e_1,e_2,e_3\}$ reperul canonic în $\mathbb{R}^3$

$[f]_{\mathcal{R}_0,\mathcal{R}_0}=A$

Metoda 1:

$f(x)=y \Leftrightarrow \underbrace{Y=AX}_{\Rightarrow f \text{ liniară}} \Leftrightarrow Y=\begin{pmatrix}2x_1+2x_2\\ x_1+x_3\\ x_1+3x_2-2x_3\end{pmatrix}=$

$=\underbrace{\begin{pmatrix}2&2&0\\1&0&1\\1&3&-2\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

Metoda 2:

$f(e_1)=f(1,0,0)=(2,1,1)=\boxed{2}\cdot e_1+\boxed{1}e_2+\boxed{1}e_3$

$f(e_2)=f(0,1,0)=(2,0,3)=2e_1+0\cdot e_2+3e_3$

$f(e_3)=f(0,0,1)=(0,1,-2)=0\cdot e_1+1\cdot e_2-2e_3$

$\det A=\begin{vmatrix}2&2&0\\1&0&1\\1&3&-2\end{vmatrix}=\begin{vmatrix}2&0&0\\1&-1&1\\1&2&2\end{vmatrix}=0 \Rightarrow A$ nu e inversabilă $\Rightarrow$

$\Rightarrow f$ nu este bij.

Obs! $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0\} = S(A)$

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\dim \ker f = 3 - rg\, A = 3 - 2 = 1 \Rightarrow f$ nu este injectivă

Teorema dimensiunii: $\dim_{\mathbb{R}} \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow$

$\Rightarrow \dim \operatorname{Im} f \Rightarrow f$ nu e surjectivă

b) $V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\} = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$ (scoatem pe x_3)

$\dim V' = 3 - 1 = 2$ $\qquad x_1(1,0,1) + x_2(0,1,1)$

$\left.\begin{array}{l} \mathcal{R}' = \{(1,0,1), (0,1,1)\} \text{ SG pt } V' \\ \dim V' = \dim \mathcal{R}' \end{array}\right| \Rightarrow \mathcal{R}'$ reper

$\mathcal{R}'' = f|_{V'}(\mathcal{R}') = \{f(1,0,1), f(0,1,1)\} = \{(2,2,-1), (2,1,1)\}$ ($u = (2,2,-1)$, $v = (2,1,1)$)

$\left.\begin{array}{l} u \in V'': \ 3 \cdot 2 - 4 \cdot 2 - 2 \cdot (-1) = 6 - 8 + 2 = 0 \\ v \in V'': \ 3 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 = 6 - 4 - 2 = 0 \end{array}\right| \Rightarrow \mathcal{R}'' \subset V''$

$$\left.\begin{array}{l} \dim V'' = 3 - 1 = 2 = \operatorname{card}(\mathcal{R}'') \\ rg \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max \Rightarrow \mathcal{R}'' \text{ SLI} \end{array}\right\} \Rightarrow \mathcal{R}'' \text{ reper în } V''$$

$\left.\begin{array}{l} \mathcal{R}' \text{ și } \mathcal{R}'' \text{ repere} \Rightarrow f|_{V'} \text{ bijecție} \\ f \text{ liniară} \end{array}\right| \Rightarrow f|_{V'}$ liniară $\Rightarrow$ izomorf

c) $f(V' \cap V'')$

$$V' \cap V'' = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases}\right\}$$

$$A' = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -4 & -2 \end{pmatrix} \qquad \dim(V' \cap V'') = 3 - rg\, A' = 1$$

$$x_3 = \alpha \Rightarrow \begin{cases} x_1 + x_2 = \alpha \\ 3x_1 - 4x_2 = 2\alpha \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{6\alpha}{7} \\ x_2 = \frac{\alpha}{7} \end{matrix}$$

$$V' \cap V'' = \{(\tfrac{6\alpha}{7}, \tfrac{\alpha}{7}, \alpha) \mid \alpha \in \mathbb{R}\} = \{\tfrac{\alpha}{7}(6,1,7) \mid -''-\}$$

$$= \langle \underbrace{\{(6,1,7)\}}_{\text{reper}} \rangle$$

$f|_{V' \cap V''} : V' \cap V'' \to \mathbb{R}^3$

$\dim(V' \cap V'') = \dim \ker f|_{V' \cap V''} + \dim f(V' \cap V'') \Rightarrow$

$\Rightarrow \dim f(V' \cap V'') \leq 1$

$f(6,1,7) = (14, 13, -5)$

$f(V' \cap V'') = \langle \{(14, 13, -5)\} \rangle \Rightarrow \dim f(V' \cap V'') = 1$

d) $\mathbb{R}^3 = V' \oplus W$

$u = v' + w$

$s, p : \mathbb{R}^3 \to \mathbb{R}^3$, $p =$ pr. pe V', $s =$ simetria față de V'

$p(u) = v'$

$s(u) = 2p(u) - u = 2v' - (v' + w) = v' - w$

$\mathcal{R}' = \{(1,0,1), (0,1,1)\}$ reper în V'. Extindem la un reper în $\mathbb{R}^3$

$$\mathrm{rg}\begin{pmatrix} 1 & 0 & \boxed{0} \\ 0 & 1 & \boxed{1} \\ 1 & 1 & \boxed{0} \end{pmatrix} = 3 = \max \Rightarrow W = \langle \{e_2\} \rangle$$

$$(1,3,6) = \overbrace{a(1,0,1) + b(0,1,1)}^{v'} + \overbrace{c(0,1,0)}^{w}$$

$$= (a, b+c, a+b) \Rightarrow a = 1,\ b = 5,\ c = -2$$

$(1, 5, -2)$ coord lui $u = (1,3,6)$

$v' = (1,5,6)$, $w = (0,-2,0)$

$p(u) = v' = (1,5,6)$

$s(u) = v' - w = (1,7,6)$

$$E_{11} = \begin{array}{cc} \text{col } 1 & \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{linia } 1 \end{array}$$

1 pe l1 col 1

4. $f: \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2^s(\mathbb{R}),\ f(A) = A + A^T$

a) $[f]_{R_0, R_0'}$ $\quad R_0 = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ reper în $\mathcal{M}_2(\mathbb{R})$

$R_0' = \{E_{11}, E_{12}+E_{21}, E_{22}\}$ reper în $\mathcal{M}_2^s(\mathbb{R})$

b) $\ker f$, $\operatorname{Im} f$

c) $f(V) = ?$, $V = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, c, d \in \mathbb{R} \right\}$

Sol.: a) $f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2E_{11} + 0(E_{12}+E_{21}) + 0 \cdot E_{22}$

$f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \ldots 1 \ldots 0$

$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \ldots 1 \ldots 0$

$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0 \ldots 0 \ldots 2$

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

b) $\ker f = \{A \in \mathcal{M}_2(\mathbb{R}) \mid f(A) = O_2\} = \mathcal{M}_2^a(\mathbb{R}) =$

$= \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\} = \left\langle \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right\rangle \Rightarrow$

$\Rightarrow \dim \ker f = 1$

$\dim \mathcal{M}_2(\mathbb{R}) = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3 = \dim \mathcal{M}_2^s(\mathbb{R}) \Rightarrow$

$\Rightarrow f$ surj. $\Rightarrow \operatorname{Im} f = \mathcal{M}_2^s(\mathbb{R})$

c) $f(V) = \{ f\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R} \}$

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 2d \end{pmatrix}$$

$$f(V) = \langle \{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \} \rangle \Rightarrow \dim f(V) = 2$$

Ex 2/3. $f: \mathbb{R}^3 \to \mathbb{R}_2[X]$ liniară

$f(1,1,-1) = x^2 + 3x + 3$, $f(-1,1,0) = -x + 1$, $f(0,1,-1) = 2x^2$

a) $f = ?$

b) $[f]_{R_0, R_0'} = A = ?$ $\quad R_0 = \{e_1, e_2, e_3\}$ repere canonice în $\mathbb{R}^3$, $\quad R_0' = \{1, x, x^2\}$ repere canonice în $\mathbb{R}_2[X]$

Sol.:

b) $\begin{cases} f(e_1) + f(e_2) - f(e_3) = x^2 + 3x - 3 \\ -f(e_1) + f(e_2) = -x + 1 \\ f(e_2) - f(e_3) = 2x^2 \quad (1-3) \end{cases}$

$$f(e_1) = -x^2 + 3x + 3 = 3 \cdot 1 + 3 \cdot x - 1 \cdot x^2$$
$$f(e_2) = -x^2 + 2x + 4 = 4 \cdot 1 + 2 \cdot x - 1 \cdot x^2$$
$$f(e_3) = -3x^2 + 2x + 4 = 4 \cdot 1 + 2 \cdot x - 3 \cdot x^2$$

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 3 & 2 & 2 \\ 1 & -1 & -3 \end{pmatrix}$$

a) $f(a,b,c) = a f(e_1) + b f(e_2) + c f(e_3)$

$= a(3 + 3x - x^2) + b(4 + 2x - x^2) + c(4 + 2x - 3x^2)$

$= 3a + 4b + 4c + x(3a + 2b + 2c) + x^2(-a - b - 3c)$

Ex3/3. $f: \mathbb{R}_1[X] \to \mathcal{M}_{2,1}(\mathbb{R})$

$$[f]_{\mathcal{R}_1,\mathcal{R}_2} = A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad \mathcal{R}_1 = \{1+x, -x\} \quad \mathcal{R}_2 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}$$

$f = ?$

<u>Sol.:</u> $f(1+x) = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$$f(-x) = 1\begin{pmatrix}1\\1\end{pmatrix} - 1\cdot\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$f(a+bx) = af(1) + bf(x)$$

$$\left.\begin{aligned} \begin{pmatrix}1\\1\end{pmatrix} &= f(1) + f(x) \\ \begin{pmatrix}2\\1\end{pmatrix} &= \cancel{f(1)} - f(x) \end{aligned}\right| \Rightarrow f(1) = \begin{pmatrix}3\\2\end{pmatrix}$$

$$f(x) = \begin{pmatrix}-2\\-1\end{pmatrix}$$

$$f(a+bx) = a\cdot\begin{pmatrix}3\\2\end{pmatrix} + b\cdot\begin{pmatrix}-2\\-1\end{pmatrix}$$

Ex4/3 Sol: $f^2 = f + id_{\mathbb{R}^n} \approx A^2 = A + I_n$

$$A^2 - A = I_n$$

$$A(A - I_n) = I_n$$

$$\overset{\shortparallel}{(A - I_n)A} \Rightarrow A \text{ este inversabilă} \Rightarrow f \text{ automorfism}$$

$$f^2 - f = id$$

$$f\underbrace{(f - id)}_{f^{-1}} = id$$

Ex 1/
/ 1. $f:\mathbb{R}_2[X]\to\mathbb{R}_2[X],\ f(p)=p+p'+p'',\ p=\tilde{p}$

a) Este f apl. bij?

b) $[f]_{R_0,R_0}=?$ $R_0=\{1,x,x^2\}$

Sol.: $f(1)=1=1\cdot 1+0\cdot x+0\cdot x^2$

$f(x)=1+x=1\cdot 1+1\cdot x+0\cdot x^2$

$f(x^2)=x^2+2x+2=2\cdot 1+2\cdot x+1\cdot x^2$

$$A=\begin{pmatrix}1&1&2\\0&1&2\\0&0&1\end{pmatrix},\ \det A=1\Rightarrow f \text{ bij}$$

$$\left.\begin{array}{l}\det A=1\Rightarrow f\text{ bij}\\ \begin{cases}f(p+q)=f(p)+f(q)\\ f(\alpha p)=\alpha f(p)\end{cases}\end{array}\right|\Rightarrow f\in Aut(\mathbb{R}_2[X])$$