$b, c \in \mathbb{R}$

$T: \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2 : Tp = \left(3p(4) + 5p'(6) + bp(1)p(2),\ \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\right)$

T linear $\Leftrightarrow$ $b = c = 0$

($\Rightarrow$) Assume T is linear. Prove $b = c = 0$.

T is linear $\Rightarrow T(\lambda p) = \lambda Tp$

Consider the first component of T:

$T(\lambda p) = 3(\lambda p)(4) + 5(\lambda p)'(6) + b(\lambda p)(1)(\lambda p)(2)$

$= 3\lambda p(4) + 5\lambda p'(6) + b\lambda^2 p(1)p(2)$

$\lambda Tp = \lambda(3p(4) + 5p'(6) + bp(1)p(2))$

$= 3\lambda p(4) + 5\lambda p'(6) + b\lambda p(1)p(2)$

$\Rightarrow b\lambda^2 p(1)p(2) = b\lambda p(1)p(2) \Leftrightarrow b\lambda p(1)p(2)(\lambda - 1) = 0$

Let $p(x) = 1,\ \lambda = 2$

$b \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 0 \Rightarrow 2b = 0 \Leftrightarrow \underline{b = 0}$

Since this must hold for $\forall$ p and $\lambda \Rightarrow \underline{b = 0}$

Consider the second component of T:

$T(\lambda p) = \int_{-1}^{2} x^3 (\lambda p)(x) + c\sin((\lambda p)(0))$

$= \int_{-1}^{2} \lambda x^3 p(x) + c\sin(\lambda p(0))$

$= \lambda \int_{-1}^{2} x^3 p(x) + c\sin(\lambda p(0))$

$\lambda Tp = \lambda\left(\int_{-1}^{2} x^3 p(x) + c\sin(p(0))\right)$

$= \lambda \int_{-1}^{2} x^3 p(x) + \lambda c\sin(p(0))$

$\Rightarrow c\sin(\lambda p(0)) = \lambda c\sin(p(0))$

$\Leftrightarrow c(\sin(\lambda p(0)) - \lambda\sin(p(0))) = 0$

Let $p(x) = 1,\ \lambda = \pi$

$\Rightarrow c(\sin(\pi) - \pi\sin(1)) = 0$

$\Leftrightarrow c(0 - \pi\sin(1)) = 0$

Since $\sin(1) \neq 0 \Rightarrow \underline{c = 0}$

So, if T is linear $\Rightarrow b = c = 0$.

($\Leftarrow$) Assume $b = c = 0$. Prove T is linear

$b = c = 0 \Rightarrow Tp = \left(3p(4) + 5p'(6),\ \int_{-1}^{2} x^3 p(x)\right)$

Additivity: $T(p+g) = Tp + Tg$

$T(p+g) = \left(3(p+g)(4) + 5(p+g)'(6),\ \int_{-1}^{2} x^3 (p+g)(x)\right)$

$= \left(3p(4) + 3g(4) + 5p'(6) + 5g'(6),\ \int_{-1}^{2} x^3 p(x) + x^3 g(x)\right)$

$= \left(3p(4) + 5p'(6) + 3g(4) + 5g'(6),\ \int_{-1}^{2} x^3 p(x) + \int_{-1}^{2} x^3 g(x)\right)$

$= Tp + Tg$ (1)

Homogeneity: $T(\lambda p) = \lambda Tp$

$T(\lambda p) = \left(3(\lambda p)(4) + 5(\lambda p)'(6),\ \int_{-1}^{2} x^3 (\lambda p)(x)\right)$

$= \left(\lambda 3p(4) + \lambda 5p'(6),\ \int_{-1}^{2} \lambda x^3 p(x)\right)$

$= \left(\lambda(3p(4) + 5p'(6)),\ \lambda\int_{-1}^{2} x^3 p(x)\right)$

$= \lambda\left(3p(4) + 5p'(6),\ \int_{-1}^{2} x^3 p(x)\right)$

$= \lambda Tp$ (2)

(1),(2) $\Rightarrow$ if $b = c = 0 \Rightarrow$ T is linear

So, T is linear $\Leftrightarrow b = c = 0$.