V infinite-dimensional $\Longleftrightarrow \exists\, v_1, v_2, \ldots : v_1, \ldots, v_m$ is linearly independent $\forall\, m > 0$

$(\Rightarrow)$ Assume V is infinite-dimensional.

Suppose, for the sake of contradiction, $\nexists\, v_1, v_2, \ldots$ $: v_1, \ldots, v_m$ is linearly independent $\forall\, m > 0$.

We want to prove $\exists$ finite list S s.t. $\mathrm{span}(S) = V$.

Assume we follow a multi-step construction of a list S.

Step 1: Start with $S = \{\}$. This is linearly independent.

Step k: if $\mathrm{span}(S) = V$ we are done.

- if not, pick some $x \in V \setminus \mathrm{span}(S)$.
- Set $S = S \cup \{x\}$. This S is also linearly independent

because if S is already lin. independent and $x \notin \mathrm{span}(S)$, then $S \cup \{x\}$ is linearly independent.

This means that $\exists$ a finite list S s.t. $\mathrm{span}(S) = V$

This is a contradiction, since V is infinite-dimensional. $\Rightarrow$)

$\Rightarrow$ Our assumption cannot be true.

$\Rightarrow$ If V is infinite-dimensional $\Rightarrow \exists\, v_1, v_2 \dots$:

$v_1 \dots v_m$ is lin. independent $\forall\, m > 0$. (1)

($\Leftarrow$) Assume $\exists\, v_1, v_2, \dots : v_1, \dots, v_m$ is lin. independent

$\forall\, m > 0$.

if such a list exists then it's infinite.

if there exists an infinite list of linearly independent vectors $\Rightarrow \nexists$ a finite list which spans V, because the length of each linearly independent list of a vector space is $\leq$ length of any Spanning list.

if $\nexists$ a finite list which spans V $\Rightarrow$

$\Rightarrow$ V is infinite-dimensional. (2)

(1), (2) $\Rightarrow$ V infinite-dimensional $\Leftrightarrow \exists\, v_1, v_2, \dots : v_1, \dots, v_m$ is linearly independent $\forall\, m > 0$