V_1, V_2 subspaces of V

$V_1 \cup V_2$ subspace of $V \iff V_1 \subseteq V_2$ or $V_2 \subseteq V_1$

Assume $V_1 \cup V_2$ subspace of V (1)

Suppose $V_1 \not\subseteq V_2$ and $V_2 \not\subseteq V_1$. $\Rightarrow \exists\, x \in V_1 \setminus V_2$
$y \in V_2 \setminus V_1$

Let $x \in V_1 \setminus V_2$ and $y \in V_2 \setminus V_1$.

$x + y \in V_1 \cup V_2$ (subspace of V)

- if $x+y \in V_1 \Rightarrow y \in V_1$ (closure under addition)
- if $x+y \in V_2 \Rightarrow x \in V_2$ (closure under addition)

Both cases contradict our assumption $\Rightarrow$

$\Rightarrow V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Assume $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$: (2)

if $V_1 \subseteq V_2$, $V_1 \cup V_2 = V_2$ which is a subspace of V

if $V_2 \subseteq V_1$, $V_1 \cup V_2 = V_1$ which is a subspace of V

$\Rightarrow V_1 \cup V_2$ subspace of V.

(1), (2) $\Rightarrow V_1 \cup V_2$ subspace of $V \iff V_1 \subseteq V_2$ or
$V_2 \subseteq V_1$.