

$$\underline{U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}}$$

(a) Basis of U

$$\begin{aligned} (z_1, z_2, z_3, z_4, z_5) &= (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) \\ &= z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + \\ &\quad + z_5(0, 0, -3, 0, 1) \end{aligned}$$

So, $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$ spans U . These are also lin. independent \Rightarrow
 $\Rightarrow B = \{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)\}$
 is a basis of U .

(b) Extend basis B to \mathbb{C}^5

We add $e_2 = (0, 1, 0, 0, 0), e_3 = (0, 0, 1, 0, 0)$ to B .
 $e_2, e_3 \notin \text{span}(B)$

$$B = \{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$$

These vectors are lin. independent and span $\mathbb{C}^5 \Rightarrow$
 basis of \mathbb{C}^5

(c) Subspace $W \subseteq \mathbb{C}^5 : \mathbb{C}^5 = U \oplus W$

$$U = \text{span}((1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1))$$

$$\text{Let } W = \text{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$$

$$\underline{U+W = \mathbb{C}^5} : \text{Let } v \in \mathbb{C}^5$$

$$\text{span}(B) = \mathbb{C}^5 \Rightarrow \exists a_1, a_2, a_3, a_4, a_5 \in F :$$

$$v = \underbrace{a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, -3, 0, 1)}_u + \underbrace{a_4(0, 1, 0, 0, 0) + a_5(0, 0, 1, 0, 0)}_w$$

$$u \in U, w \in W \Rightarrow U+W = \mathbb{C}^5. \quad (1)$$

$$\underline{U \cap W = \{0\}} : \text{Let } v \in U \cap W \Rightarrow$$

$$\Rightarrow \exists a_1, a_2, a_3, a_4, a_5 \in F :$$

$$\left. \begin{aligned} v &= a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, -3, 0, 1) \\ v &= a_4(0, 1, 0, 0, 0) + a_5(0, 0, 1, 0, 0) \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow 0 = a_1(1, 6, 0, 0, 0) + a_2(0, 0, -2, 1, 0) + a_3(0, 0, -3, 0, 1) - a_4(0, 1, 0, 0, 0) - a_5(0, 0, 1, 0, 0)$$

Since these vectors are lin. independent \Rightarrow

$$\Rightarrow a_1 = \dots = a_5 = 0 \Rightarrow v = 0 \Rightarrow U \cap W = \{0\} \quad (2)$$

$$(1), (2) \Rightarrow \text{For } W = \text{span}((0, 1, 0, 0, 0), (0, 0, 1, 0, 0))$$

$$U \oplus W = \mathbb{C}^5$$