$p_0, p_1, \dots, p_m \in \mathcal{P}_m(F) : p_k(2) = 0 \ \forall k \in \{0, \dots, m\}$

$p_0, \dots, p_m$ linearly dependent in $\mathcal{P}_m(F)$

$p_k(2) = 0 \iff x - 2 \mid p_k(x)$

Hence, $\forall k \in \{0, \dots, m\}, \exists g_k \in \mathcal{P}_{m-1}(F)$ s.t.

$$p_k(x) = (x-2) g_k(x)$$

$\mathcal{P}_{m-1}(F) = \text{span}(1, x, \dots, x^{m-1})$

We know any list of linearly independent vectors $\leq$ a spanning list.

So an arbitrary list of len $m+1$, $g_0, \dots, g_m \in \mathcal{P}_{m-1}(F)$ is linearly dependent since $1, x, \dots, x^{m-1}$ of len m spans $\mathcal{P}_{m-1}(F)$. $\implies$

$\exists a_0, \dots, a_m \in F$, not all zero: $\sum_{i=0}^{m} a_0 g_i(x) = 0$

Multiplying by $x-2$: $(x-2) \sum_{i=0}^{m} a_0 g_i(x) = 0$

$$\Longleftrightarrow \sum_{i=0}^{m} a_0 \left((x-2)\, g_i(x)\right) = 0$$

$$\Longleftrightarrow \sum_{i=0}^{m} a_0\, p_i(x) = 0$$

$\Longrightarrow$ $p_0, \dots, p_m$ are linearly dependent.