$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \ , x_3 = 7x_4\}$

(a) Find a basis of U

$(x_1, x_2, x_3, x_4, x_5) = (3x_2, x_2, 7x_4, x_4, x_5)$

$= x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1)$

So, $(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$

spans U and is also linearly independent $\Rightarrow$ a basis in U

(b) Extend B to be a basis of $\mathbb{R}^5$.

$B = \{(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)\}$

- We add $(1,0,0,0,0)$ which $\notin$ span(B).

Now $B = \{(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0)\}$

- We add $(0,0,1,0,0)$ which $\notin$ span(B).

Now $B = \{(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0)$
$(0,0,1,0,0)\}$.

Now, we extended B to a basis of $\mathbb{R}^5$.

The list of 5 vectors are lin. independent $\Rightarrow$ span $\mathbb{R}^5$.

(c) Subspace $W \subset \mathbb{R}^5$: $\mathbb{R}^5 = U \oplus W$

$U = \text{span}((3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1))$

Let $W = \text{span}((1,0,0,0,0), (0,0,1,0,0))$

$U + W = \mathbb{R}^5$: Let $v \in \mathbb{R}^5$

We know $(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0)$ spans $\mathbb{R}^5 \Rightarrow$

$\exists\ a_1, a_2, a_3, a_4, a_5 \in F$:

$\underbrace{a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1)}_{u} +$

$\underbrace{a_4(1,0,0,0,0) + a_5(0,0,1,0,0)}_{w} = v$

$u \in U,\ w \in W \Rightarrow U + W = \mathbb{R}^5$ (1)

$U \cap W = \{0\}$: Let $v \in U \cap W \Rightarrow$

$\Rightarrow \exists\ a_1, a_2, a_3, a_4, a_5 \in F$:

$\left.\begin{aligned} v &= a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) \\ v &= a_4(1,0,0,0,0) + a_5(0,0,1,0,0) \end{aligned}\right\} \Rightarrow$

$$\Rightarrow 0 = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) - a_4(1,0,0,0,0) - a_5(0,0,1,0,0)$$

Since, $(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0)$ are linearly independent

$\Rightarrow a_1 = \dots = a_5 = 0 \Rightarrow v = 0 \Rightarrow U \cap W = \{0\}$ (2)

(1), (2) $\Rightarrow$ For $W = \text{span}((1,0,0,0,0), (0,0,1,0,0))$

$U \oplus W = \mathbb{R}^5$.