(a) $U = \{p \in \mathcal{P}_4(F) : p(6) = 0\}$.

Basis of U?

$p(6) = 0 \iff \forall p \in U : p(x) = (x-6)\, g(x)$ where $g \in \mathcal{P}_3(F)$.

$(1, x, x^2, x^3)$ is a basis for g. $\Rightarrow$

$\Rightarrow g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, \dots, a_3 \in F$

$\Rightarrow p(x) = (x-6)(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$\iff p(x) = a_0(x-6) + a_1(x(x-6)) + a_2(x^2(x-6)) + a_3(x^3(x-6))$

$\Rightarrow \operatorname{span}((x-6), x(x-6), x^2(x-6), x^3(x-6)) = U$. (1)

Consider $B_u = a_0(x-6) + a_1(x(x-6)) + a_2(x^2(x-6)) + a_3(x^3(x-6)) = 0$

For some $a_0, \dots, a_3 \in F$.

$\iff (x-6)(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = 0$

$\iff a_0 + a_1 x + a_2 x^2 + a_3 x^3 \equiv 0 \iff a_0 = a_1 = a_2 = a_3 = 0$

$\Rightarrow B_u$ is linearly independent (2)

(1), (2) $\Rightarrow B_u = x-6, x(x-6), x^2(x-6), x^3(x-6)$ basis of U

(b) Extend the basis to a basis of $\mathcal{P}_4(F)$

$1, x, x^2, x^3, x^4$ is a basis of $\mathcal{P}_4(F)$

Let $B = x-6, x(x-6), x^2(x-6), x^3(x-6), 1, x, x^2, x^3, x^4$

This is a spanning list of $\mathcal{P}_4(F)$.

We'll reduce this to be a basis of $\mathcal{P}_4(F)$

Every element of B_u is linearly independent $\Rightarrow$

$\Rightarrow \forall\ v_k \in B_u : v_k \notin \operatorname{span}(v_1 \dots v_{k-1})$

We start from 1:

$\forall\ p \in \operatorname{span}(B_u) : p(6) = 0$. $\quad 1(6) = 1 \neq 0 \Rightarrow$

$\Rightarrow 1 \notin \operatorname{span}(B_u) \Rightarrow$ we keep 1.

Since $B = (x-6, x(x-6), x^2(x-6), x^3(x-6), 1)$ is linearly independent and has len $= \dim \mathcal{P}_4(F) = 5 \Rightarrow$

$\Rightarrow B$ is already a basis of $\mathcal{P}_4(F)$ and we discard the rest of the procedure.

(c) W subspace of $\mathcal{P}_4(F) : \mathcal{P}_4(F) = U \oplus W$

Let $W = \operatorname{span}(1) = \{a : a \in F\}$

- $U + W = \mathcal{P}_4(F)$

Let $u(x) = p(x) - p(6)$

Since for $u(6) = p(6) - p(6) = 0 \Rightarrow u \in U$

Let $w(x) = p(6)$, $w \in W$

$p(x) = p(x) - p(6) + p(6) = u(x) + w(x) \Rightarrow$

$\Rightarrow \mathcal{P}_4(F) = U + W$

$$\left.\begin{array}{l} \dim(U) = 4 \\ \dim(W) = 1 \\ \dim(\mathcal{P}_4(F)) = 5 \end{array}\right\} \Rightarrow 5 = \dim \mathcal{P}_4(F) = \dim(U+W) = \\ = \dim U + \dim W = 4 + 1.$$

$\dim(U+W) = \dim U + \dim W \Rightarrow U + W$ direct sum

$\Rightarrow \mathcal{P}_4(F) = U \oplus W$