$\mathcal{U} = \{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0 \}$

(a) Basis of $\mathcal{U}$

Let $p(x) = a + bx + cx^2 + dx^3 + ex^4 \in \mathcal{P}_4(\mathbb{R})$

$\int_{-1}^{1} p(x)\,dx = 0 \iff$

$\iff \int_{-1}^{1} a + bx + cx^2 + dx^3 + ex^4 \; dx = 0$

$\iff \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4} + \frac{ex^5}{5} \right]_{-1}^{1} = 0$

$\iff \left(a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} \right) - \left(-a + \frac{b}{2} - \frac{c}{3} + \frac{d}{4} - \frac{e}{5} \right) = 0$

$\iff 2a + \frac{2c}{3} + \frac{2e}{5} = 0$

$\iff a + \frac{c}{3} + \frac{e}{5} = 0$

$\iff a = -\frac{c}{3} - \frac{e}{5}$

$$\left.\begin{aligned} (\forall)\, p(x) \in \mathcal{U} : p(x) &= -\frac{c}{3} - \frac{e}{5} + bx + cx^2 + dx^3 + ex^4 \\ &= bx + c\left(-\frac{1}{3} + x^2\right) + dx^3 + e\left(-\frac{1}{5} + x^4\right) \end{aligned}\right\} \Rightarrow$$

Let $B_u = \left(x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5} \right)$

$\Rightarrow \operatorname{Span}(B_u) = \mathcal{U}$

Since every element has a different deg, B_u is lin. independent. $\Rightarrow$

$\Rightarrow$ B_u is a basis of U.

(b) Extend B_u to be a basis of $\mathcal{P}_4(F)$

Let $E = (1, x, x^2, x^3, x^4)$, the standard basis of $\mathcal{P}_4(\mathbb{R})$

Since $\dim \mathcal{P}_4(F) = 5$ and $\dim U = 4$, we can extend B_u to be a basis of $\mathcal{P}_4(F)$ by adding one element from E that $\notin$ span B_u.

$1 \notin$ span B_u, since it's a deg 0 polynomial $\Rightarrow$

$\Rightarrow B = (x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}, 1)$ is a basis of $\mathcal{P}_4(F)$.

(c) W subspace of $\mathcal{P}_4(\mathbb{R})$: $\mathcal{P}_4(\mathbb{R}) = U \oplus W$

Let $W = \text{span}(1)$.

- $U + W = \mathcal{P}_4(F)$

Since B is a basis of $\mathcal{P}_4(F) \Rightarrow$ span $B = \mathcal{P}_4(F)$

$\{x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}\} \subset U$. $\Bigg\} \Rightarrow$

$\{1\} \subset W$

$\Rightarrow \mathcal{P}_4(F) = \operatorname{span} B = \operatorname{span}(U \cup W) = U + W$

$\Rightarrow U + W = \mathcal{P}_4(F) \quad (1)$

- $U \cap W = \{0\}$

$$\left.\begin{aligned} &\dim(U+W) = \dim U + \dim W - \dim(U \cap W) \\ &\dim(U+W) = \dim(\mathcal{P}_4(F)) = 5 \\ &\dim U = 4 \\ &\dim W = 1 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W$

$\Rightarrow \dim(U \cap W) = 0$

$\Rightarrow U \cap W = \{0\} \quad (2)$

$(1), (2) \Rightarrow U \oplus W = \mathcal{P}_4(F)$