$w_k = v_1 + \dots + v_k$

$v_1, \dots, v_m$ basis $\Leftrightarrow$ $w_1, \dots, w_m$ basis

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($\Rightarrow$) Assume $v_1 \dots v_m$ basis. Prove $w_1, \dots, w_k$ basis

Linear independence: Let $a_1, a_2 \dots a_m \in F$. Consider

$$a_1 w_1 + a_2 w_2 + \dots + a_m w_m = 0$$

$$\Leftrightarrow a_1 v_1 + a_2(v_1 + v_2) + \dots + a_m(v_1 + \dots + v_m)$$

$$\Leftrightarrow a_1 v_1 + a_2 v_1 + a_2 v_2 + \dots + a_m v_1 + \dots + a_m v_m = 0$$

$$\Leftrightarrow (a_1 + a_2 + \dots + a_m) v_1 + (a_2 + \dots + a_m) v_2 + \dots + a_m v_m = 0$$

Since $v_1, \dots v_m$ is a basis, so lin. independent, this is true $\Leftrightarrow$ $a_1 = \dots = a_m = 0$. $\Rightarrow$ $w_1, \dots w_m$ lin. independent. (1)

Spanning: Let $v \in V$. Consider whether $\exists\, a_1, \dots, a_m \in F$ s.t.:

$$v = a_1 w_1 + \dots + a_m w_m$$

$$\Leftrightarrow v = (a_1 + a_2 + \dots + a_m) v_1 + (a_2 + \dots + a_m) v_2 + \dots + a_m v_m$$

Since $v_1 \dots v_m$ is a basis, so it spans V $\Rightarrow$ such $a_1 \dots a_m \in F$ exists. $\Rightarrow$ We can define an arbitrary $v \in V$ as a lin combination of $w_1, \dots, w_m$ $\Rightarrow$ $\operatorname{span}(w_1, \dots, w_m) = V$. (2)

$(1),(2) \Rightarrow v_1, \dots, v_m$ basis $\Rightarrow w_1, \dots, w_m$ basis. $(*)$

$(\Leftarrow)$ Assume $w_1, \dots w_m$ basis. Prove $v_1, \dots, v_m$ basis.

$$\begin{cases} v_1 = w_1 \\ v_k = w_k - w_{k-1} \end{cases}$$

Linear independence: Let $a_1, \dots, a_m \in F$. Consider

$a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0$

$\Leftrightarrow a_1 w_1 + a_2 (w_2 - w_1) + \dots + a_m (w_m - w_{m-1}) = 0$

$\Leftrightarrow a_1 w_1 + a_2 w_2 - a_2 w_1 + \dots + a_m w_m - a_m w_{m-1} = 0$

$\Leftrightarrow (a_1 - a_2) w_1 + (a_2 - a_3) w_2 + \dots + (a_{m-1} - a_m) w_m + a_m w_m = 0$

Since $w_1 \dots, w_m$ is a basis, so lin. independent, this is true $\Leftrightarrow a_1 = \dots = a_m = 0$. $\Rightarrow v_1, \dots v_m$ lin. independent. (1)

Spanning: Let $v \in V$. Consider whether $\exists\, a_1 \dots, a_m \in F$ s.t.:

$v = a_1 v_1 + \dots + a_m v_m$

$\Leftrightarrow v = (a_1 - a_2) w_1 + (a_2 - a_3) + \dots + (a_{m-1} - a_m) w_m + a_m w_m$

Since $w_1, \dots, w_m$ spans V, $\exists\, a_1 \dots a_m$ with this cond.

$\Rightarrow$ we can write any $v \in V$ as a lin. combination of $v_1 \dots v_m \Rightarrow \operatorname{span}(v_1, \dots, v_m) = V$. (2)

$(1),(2) \Rightarrow w_1 \dots w_m$ basis $\Rightarrow v_1 \dots v_m$ basis $(**)$

$(*),(**) \Rightarrow v_1, \dots, v_m$ basis $\Leftrightarrow w_1 \dots w_m$ basis.