For $f, g \in V^S$, $f+g \in V^S$ is:

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$ - ADDITION

For $f \in V^S$, $\lambda \in V$, $\lambda f \in V^S$ is: - SCALAR MULTIPLICATION

$$(\lambda f)(x) = \lambda f(x) \quad \forall x \in S$$

1. Closure under addition

Holds - from our definition of addition

$f+g \in V^S$, $\forall f, g \in V^S$. $\square$

2. Commutativity - $\forall f, g \in V^S$

$$\begin{aligned} f+g &= (f+g)(x) \\ &= f(x) + g(x) \\ &= g(x) + f(x) \\ &= (g+f)(x) \\ &= g+f. \end{aligned}$$

$\square$

3. Associativity

- addition: $\forall f, g, h \in V^S$

$$
\begin{aligned}
(f+g)+h &= ((f+g)+h)(x) \\
&= (f+g)(x)+h(x) \\
&= (f(x)+g(x))+h(x) \\
&= f(x)+(g(x)+h(x)) \\
&= f(x)+(g+h)(x) \\
&= (f+(g+h))(x) \\
&= f+(g+h)
\end{aligned}
$$

$\square$

- scalar multiplication

$\forall f \in V^S, \alpha, \beta \in V$

$$
\begin{aligned}
(\alpha\beta)f &= ((\alpha\beta)f)(x) \\
&= (\alpha\beta)f(x) \\
&= \alpha(\beta f(x)) \\
&= \alpha(\beta(f))
\end{aligned}
$$

$\square$

4. Additive identity

$\exists\ 0: S \to V$ s.t. $0(x) = 0. \quad \forall x \in S$

$$f+0 = (f+\underbrace{0}_{\in V^S})(x) = f(x)+0(x) = f(x)+\underbrace{0}_{\in V} = f(x) = f.$$

So there exists $0 \in V^S$ s.t. $f+0=f$ $\forall f \in V^S$.

$\square$

5. Additive inverse

$\forall f \in V^S$, $\exists -f : S \to V$ s.t. $(-f)(x) = -f(x)$, $\forall x \in S$.

$$\begin{aligned}(f+(-f))(x) &= f(x) + (-f)(x) \\ &= f(x) + (-f(x)) \\ &= 0\end{aligned}$$

So there exists $-f \in V^S$ s.t. $f+(-f) = 0$, $\forall f \in V^S$

$\square$

6. Multiplicative identity

Assume 1 is multiplicative identity in V.

$1f = (1f)(x) = 1f(x) = f(x)$.

So there exists $1 \in V$ s.t. $1f = f$ $\forall f \in V^S$.

7. Distributivity

Suppose $\alpha, \beta \in V$, $f, g \in V^S$.

$$\begin{cases} \alpha(f+g) = \alpha((f+g)(x)) \\ \qquad = \alpha(f(x)+g(x)) \\ \qquad = \alpha f(x) + \alpha g(x) \\ \qquad = \alpha f + \alpha g \\ \text{So } \alpha(f+g) = \alpha f + \alpha g \quad \forall \alpha \in V, f, g \in V^S. \end{cases} \quad \square$$

$$\begin{cases} (\alpha+\beta) f = ((\alpha+\beta)f)(x) \\ \qquad = (\alpha+\beta) f(x) \\ \qquad = \alpha f(x) + \beta f(x) \\ \qquad = \alpha f + \beta f \\ \text{So } (\alpha+\beta) f = \alpha f + \beta f \quad \forall \alpha, \beta \in V, f \in V^S. \end{cases} \quad \square$$

1...7 $\Rightarrow$ given our prior definitions of addition / scalar mult. V^S is a vector space.