V_C - complex vector space. V - real vector space.

For any element $x \in V_C$: $x = u + iv$ with $u, v \in V$

1. Commutativity

Suppose $x, y \in V_C$.

$$
\begin{aligned}
x + y &= (x_u + i x_v) + (y_u + i y_v) \\
&= (x_u + y_u) + i(x_v + y_v) \\
&= (y_u + x_u) + i(y_v + x_v) \\
&= (y_u + i y_v) + (x_u + i x_v) \\
&= y + x
\end{aligned}
$$

$\square$

2. Associativity (addition)

Suppose $x, y, z \in V_C$.

$$
\begin{aligned}
x + (y + z) &= (x_u + i x_v) + ((y_u + z_u) + i(y_v + z_v)) \\
&= (x_u + y_u + z_u) + i(x_v + y_v + z_v) \\
&= ((x_u + y_u) + i(x_v + y_v)) + (z_u + i z_v) \\
&= (x + y) + z.
\end{aligned}
$$

$\square$

3. Associativity (multiplication)

Suppose $\alpha, \beta \in \mathbb{C}$, $x \in V_C$. $\alpha = a + bi$, $\beta = c + di$

$$(\alpha\beta)x = ((a+bi)(c+di))x$$
$$= ((\underbrace{ac-bd}_{t}) + i(\underbrace{ad+bc}_{p}))(x_u + ix_v)$$
$$= (t + ip)(x_u + ix_v)$$
$$= (tx_u - px_v) + i(tx_v + px_u)$$
$$= ((ac-bd)x_u - (ad+bc)x_v) + i((ac-bd)x_v + (ad+bc)x_u)$$
$$= (acx_u - bdx_u - adx_v - bcx_v) + i(acx_v - bdx_v + adx_u + bcx_u) \quad (1)$$

$$\alpha(\beta x) = (a+bi)((c+di)(x_u + ix_v))$$
$$= (a+bi)((\underbrace{cx_u - dx_v}_{m}) + i(\underbrace{cx_v + dx_u}_{n}))$$
$$= (a+bi)(m + in)$$
$$= (am - bn) + i(an + bm)$$
$$= (a(cx_u - dx_v) - b(cx_v + dx_u)) + i(a(cx_v + dx_u) + b(cx_u - dx_v))$$
$$= (acx_u - adx_v - bcx_v - bdx_u) + i(acx_v + adx_u + bcx_u - bdx_v) \quad (2)$$

$(1), (2) \implies (\alpha\beta)x = \alpha(\beta x)$

4) Additive identity

Suppose $0 \in V_C$; $0 = 0u + 0iv$ with $u, v \in V$
$\hookrightarrow \in \mathbb{R}$

$$x + 0 = (x_u + ix_v) + (0u + 0(iv))$$
$$= (x_u + 0) + i(x_v + 0)$$
$$= x_u + ix_v$$
$$= x.$$

So, $\exists$ element $0 \in V_c$ s.t. $x + 0 = x \quad \forall\, x \in V_c$.

5) Additive inverse

Suppose $-x \in V_c$ s.t. $-x = (-x_u) + i(-x_v)$

$-x_u, -x_v \in V$ and are the additive inverse of x_u, x_v

$$x + (-x) = (x_u + ix_v) + ((-x_u) + i(-x_v))$$
$$= (x_u + (-x_u)) + i(x_v + (-x_v))$$
$$= 0 + i0$$
$$= 0.$$

So, $\exists$ element $-x \in V_c$ s.t. $x + (-x) = 0 \quad \forall\, x \in V_c$.

6) Multiplicative identity

Consider the scalar $1 = 1 + 0i \in \mathbb{C}$

$$1x = (1 + 0i)(x_u + ix_v)$$