$U = \{ p \in \mathcal{P}_4(F) : p(2) = p(5) \}$

Find basis of U

$\forall p \in U$: Consider $r(x) = p(x) - p(2)$

$$\left.\begin{aligned} r(2) &= p(2) - p(2) = 0 \\ r(5) &= p(5) - p(2) = 0. \end{aligned}\right\} \Rightarrow$$

$\Rightarrow r(x)$ has roots $2, 5$ and $r(x) \in \mathcal{P}_4(F)$

$$\left.\begin{aligned} &\exists\, g(x) \in \mathcal{P}_2(F) : r(x) = (x-2)(x-5)\, g(x) \\ &g(x) = a + bx + cx^2 : a, b, c \in F \end{aligned}\right\} \Rightarrow$$

$\Rightarrow r(x) = (x-2)(x-5)(a + bx + cx^2)$

$\Rightarrow p(x) = (x-2)(x-5)(a + bx + cx^2) + p(2)$

$\Rightarrow p(x) = d \cdot 1 + a(x-2)(x-5) + b(x(x-2)(x-5)) + c(x^2(x-2)(x-5))$

$\Rightarrow \operatorname{span}(1, (x-2)(x-5), x(x-2)(x-5), x^2(x-2)(x-5)) = U$

Since these polynomials have distinct degs, the list is also linearly independent.

$\Rightarrow B_U = 1, (x-2)(x-5), x(x-2)(x-5), x^2(x-2)(x-5)$ basis of U.

(b) Extend B_u to basis of $\mathcal{P}_4(F)$

$E = (1, x, x^2, x^3, x^4)$ is a basis of $\mathcal{P}_4(F)$. $\Big\}$ $\Rightarrow$
$\dim \mathcal{P}_4(F) = 5$

$\Rightarrow$ Extending B_u to be a basis of $\mathcal{P}_4(F)$ - we need to add one element from E which $\notin \operatorname{span}(B_u)$.

$x \notin \operatorname{span}(B_u)$ since no element in B_u has deg 1.
or $x(2) = 2 \neq 5 = x(5)$.

$\Rightarrow B = (1, (x-2)(x-5), x(x-2)(x-5), x^2(x-2)(x-5), x)$ is a basis of $\mathcal{P}_4(F)$.

(c) W subspace of $\mathcal{P}_4(F)$: $\mathcal{P}_4(F) = U \oplus W$

Let $W = \operatorname{span}(x) = \{\beta x : \beta \in F\}$

We show $U + W = \mathcal{P}_4(F)$: $\forall p \in \mathcal{P}_4(F)$: $p(x) = u(x) + \beta x$

$\Leftrightarrow u(x) = p(x) - \beta x$

$$\Leftrightarrow \begin{cases} u(2) = p(2) - 2\beta \\ u(5) = p(5) - 5\beta \\ u(2) = u(5) \end{cases} \Leftrightarrow 3\beta = p(5) - p(2) \Rightarrow \beta = \frac{p(5) - p(2)}{3}$$

$$\Rightarrow \text{for } \beta = \frac{p(1)-p(2)}{3}, \; u(x) \in U$$

$$p(x) = \underbrace{u(x)}_{\in U} + \underbrace{\frac{p(1)-p(2)}{3}x}_{\in W}$$

$$\Rightarrow U + W = \mathcal{P}_4(F) \; (1)$$

- We show that $U \cap W = \{0\}$

$$\left.\begin{aligned} &\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) \\ &\dim(U+W) = \dim \mathcal{P}_4(\mathbb{R}) = 5 \\ &\dim(U) = 4 \\ &\dim(W) = 1 \end{aligned}\right\} \Rightarrow$$

$$\dim(U) + \dim(W) - \dim(U \cap W) = \dim(U) + \dim(W)$$

$$\Rightarrow \dim(U \cap W) = 0$$

$$\Rightarrow U \cap W = \{0\} \; (2)$$

$$(1),(2) \Rightarrow U \oplus W = \mathcal{P}_4(F)$$