V real vector space

$v_1 \dots v_n$ basis of V.

$\overline{\qquad\qquad\qquad\qquad}$ +

$v_1, \dots, v_n$ basis of $V_{\mathbb{C}} = \{x+iy : x, y \in V\}$

$\overline{\qquad\qquad\qquad\qquad}$ #

Linear independence: Consider $a_1, \dots, a_n \in \mathbb{C}$

Let $a_j = \alpha_j + i\beta_j \ \forall j \in \{1..n\}, \ \alpha, \beta \in \mathbb{R}$

$$\sum_{j=1}^{n} a_j v_j = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} (\alpha_j + i\beta_j) v_j = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} \alpha v_j + i\beta_j v_j = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} \alpha_j v_j + i \sum_{j=1}^{n} \beta_j v_j = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} \alpha_j v_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \beta_j v_j = 0$$

Since $\{v_j\}$ is a basis in V, this is true $\Leftrightarrow$

$(\Leftarrow) \ \{\alpha_j\} = \{\beta_j\} = \{0\} \Rightarrow v_1 \ldots v_n$ lin indep. in $V_{\mathbb{C}}$. (1)

Spanning: Let $v \in V_{\mathbb{C}}$: $v = x + iy$, $x, y \in V$

$$\left.\begin{array}{l} \{v_j\} \text{ basis of } V \Rightarrow \exists \{a_j\} \in \mathbb{R} : \sum_{j=1}^{n} a_j v_j = x \\ \qquad\qquad\qquad\qquad \exists \{b_j\} \in \mathbb{R} : \sum_{j=1}^{n} b_j v_j = y \end{array}\right\} \Rightarrow$$

$$\Rightarrow v = x + iy = \sum_{j=1}^{n} a_j v_j + i \left(\sum_{j=1}^{n} b_j v_j \right)$$

$$(\Leftarrow) \ v = \sum_{j=1}^{n} \underbrace{(a_j + i b_j)}_{\in \mathbb{C}} v_j$$

$\Rightarrow \operatorname{span}(v_1 \ldots v_n) = V_{\mathbb{C}}$. (2)

(1), (2) $\Rightarrow v_1 \ldots v_n$ basis of $V_{\mathbb{C}}$.