$b, c \in \mathbb{R}$

$T: \mathbb{R}^3 \to \mathbb{R}^2 \quad T(x,y,z) = (2x - 4y + 3z + b,\ 6x + cxyz)$

T linear $\iff b = c = 0$

($\Rightarrow$) Assume T is linear, prove $b = c = 0$

$\left.\begin{array}{l} T \text{ is linear} \Rightarrow T(0) = 0 \\ T(0,0,0) = (b, 0) \end{array}\right\} \Rightarrow b = 0.$

T is linear $\Rightarrow T(\lambda x, \lambda y, \lambda z) = \lambda T(x,y,z) \ \forall x, y, z, \lambda \in \mathbb{R}$.

Considering only the second component

$6\lambda x + c(\lambda x)(\lambda y)(\lambda z) = \lambda(6x + cxyz)$

$\iff 6\lambda x + c\lambda^3 xyz = 6\lambda x + c\lambda xyz$

$\iff c\lambda^3 xyz - c\lambda xyz = 0$

$\iff cxyz(\lambda^3 - \lambda) = 0$

$\iff c = 0.$

$\Rightarrow T$ is linear $\Rightarrow b = c = 0$.

($\Leftarrow$) Assume $b = c = 0$, prove T is linear

$b = c = 0 \Rightarrow T(x,y,z) = (2x - 4y + 3z,\ 6x)$

Additivity: $T(u+v) = Tu + Tv \quad \forall u, v \in \mathbb{R}^3$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$T(u+v) = (2(x_1+x_2) - 4(y_1+y_2) + 3(z_1+z_2),\ 6(x_1+x_2))$

$= (2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2,\ 6x_1 + 6x_2)$

$= (2x_1 - 4y_1 + 3z_1,\ 6x_1) + (2x_2 - 4y_2 + 3z_2,\ 6x_2)$

$= Tu + Tv$

Homogeneity: $T(\lambda v) = \lambda Tv \quad \forall v \in \mathbb{R}^3, \lambda \in \mathbb{R}$

Let $v = (x, y, z)$

$T(\lambda(x,y,z)) = T(\lambda x, \lambda y, \lambda z)$

$= (2\lambda x - 4\lambda y + 3\lambda z,\ 6\lambda x)$

$= \lambda(2x - 4y + 3z,\ 6x)$

$= \lambda Tv$

So if $b = c = 0 \Rightarrow T$ is linear (2)

(1), (2) $\Rightarrow T$ is linear $\iff b = c = 0$.