V_1, V_2, V_3 subspaces of V

$V_1 \cup V_2 \cup V_3$ subspace of $V \iff$ one contains the other

$(\Rightarrow)$

Assume $V_1 \cup V_2 \cup V_3$ subspace of V.

Assume, for contradiction, no subspace contains the other two $\Rightarrow$

$$\Rightarrow \left| \begin{array}{l} 1) V_1 \not\subseteq V_2 \text{ and } V_1 \not\subseteq V_3 \Rightarrow \exists\, x \in V_1 : x \notin V_2, x \notin V_3 \\ 2) V_2 \not\subseteq V_1 \text{ and } V_2 \not\subseteq V_3 \Rightarrow \exists\, y \in V_2 : y \notin V_1, y \notin V_3 \\ 3) V_3 \not\subseteq V_1 \text{ and } V_3 \not\subseteq V_2 \Rightarrow \exists\, z \in V_3 : z \notin V_1, z \notin V_2 \end{array} \right.$$

Consider $x + \lambda y : \lambda \in F, \lambda \neq 0$.

Since $V_1 \cup V_2 \cup V_3$ is a subspace of V,

$x + \lambda y \in V_1 \cup V_2 \cup V_3$ closed under addition + closed under scalar multiplication

Case 1: Assume $x + \lambda y \in V_1$

$x \in V_1 \Rightarrow -x \in V_1$ (V_1 closed under scalar multiplication)

$$\left.\begin{array}{l} x + \lambda y \in V_1 \\ -x \in V_1 \end{array}\right\} \Rightarrow (x + \lambda y) - x \in V_1 \Rightarrow \lambda y \in V_1.$$

$$\Rightarrow y \in V_1$$

Contradicts 2).

Case 2: Assume $x + \lambda y \in V_2$.

$\lambda y \in V_2 \Rightarrow -\lambda y \in V_2$ (V_2 closed under scalar mult.)

$$\left.\begin{array}{l} x+\lambda y \in V_2 \\ -\lambda y \in V_2 \end{array}\right\} \Rightarrow (x+\lambda y) - \lambda y \in V_2 \Rightarrow x \in V_2.$$

Contradicts 1).

Case 3. Assume $x + \lambda y \in V_3$

Assume $|F| > 2$!!

Consider $\mu \in F$, $\lambda \neq \mu \neq 0$.

$x + \mu y \in V_3$ (Analogous proof)

$$\left.\begin{array}{l} x+\lambda y \in V_3 \\ x+\mu y \in V_3 \end{array}\right\} \Rightarrow (x+\lambda y) - (x+\mu y) \in V_3 \Rightarrow$$

(closed under addition, scalar mult.)

$\Rightarrow \lambda y - \mu y \in V_3$

$$\left.\begin{array}{l} \Rightarrow (\lambda - \mu) y \in V_3 \\ \lambda - \mu \neq 0 \end{array}\right\} \Rightarrow y \in V_3$$

Contradicts 2).

So, our assumption is false $\Rightarrow$ one subspace must contain the other two

if $V_1 \cup V_2 \cup V_3$ subspace $\Rightarrow$ one subspace must contain the other two. (1)

($\Leftarrow$) If one subspace contains the other two, say $V_1 \supseteq V_2, V_3 \Rightarrow V_1 \cup V_2 \cup V_3 = V_1$, which is a subspace. (2)

(1), (2) $\Rightarrow V_1 \cup V_2 \cup V_3$ subspace $\Leftrightarrow$ one contains the other two.