$$p_k(x) = x^k (1-x)^{m-k}$$

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1$$

$p_0, \dots, p_m$ basis of $\mathcal{P}_m(F)$

$$(1-x)^{m-k} = \sum_{i=0}^{m-k} \binom{m-k}{i} 1^{m-k-i} (-x)^i$$

$$= \binom{m-k}{0} 1^{m-k} (-x)^0 + \sum_{i=1}^{m-k} \binom{m-k}{i} (-x)^i$$

$$= 1 + \sum_{i=1}^{m-k} \binom{m-k}{i} (-1)^i (x)^i$$

$$p_k(x) = x^k (1-x)^{m-k} = x^k \left(1 + \sum_{i=1}^{m-k} \binom{m-k}{i} (-1)^i (x)^i\right)$$

$$= x^k + \sum_{i=1}^{m-k} \binom{m-k}{i} (-1)^i (x)^{k+i}$$

Linear independence: Consider $a_0, \dots, a_m \in F$:

$$\sum_{k=0}^{m} a_k p_k(x) = 0$$

Suppose for the sake of contradiction $\exists\, a_k \neq 0$.

Let r be the smallest index : $a_r \neq 0$.

$$p_r(x) = x^r + \underbrace{\sum_{i=1}^{m-r} \binom{m-r}{i} (-1)^i (x)^{r+i}}_{\deg \geq r+1}$$

For every $k > r$, $\deg p_k \geq r+1$, so it contains no x^r term.

Hence, the coefficient of x^r is exactly a_r.

Since the sum is the 0 polynomial $\Rightarrow a_r = 0$, contradicting our assumption $\Rightarrow$

$\Rightarrow a_0 = a_1 = \dots = a_m = 0 \Rightarrow p_0, \dots, p_m$ are linearly independent.

Since $p_0, \dots, p_m$ are a lin. independent list of $m+1$ polynomials and $\dim \mathcal{P}_m(F) = m+1 \Rightarrow$

$\Rightarrow p_0, \dots, p_m$ is a basis of $\mathcal{P}_m(F)$.