Exercises 1A

1) $\alpha+\beta = \beta+\alpha \quad \forall\ \alpha, \beta \in \mathbb{C}$

Suppose $\alpha = a+bi,\ \beta = b+di \quad a,b,c,d \in \mathbb{R}$

$$\begin{aligned}
\alpha+\beta &= (a+bi) + (b+di) \\
&= (a+b) + (b+d)i \\
&= (b+a) + (d+b)i \\
&= \beta+\alpha
\end{aligned}$$

$\square$

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2) $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda), \ \forall\ \alpha,\beta,\lambda \in \mathbb{C}$

Suppose $\alpha = a+bi$, $\beta = c+di$, $\lambda = e+fi$ where $a,b,c,d,e,f \in \mathbb{R}$

$$\begin{aligned}
(\alpha+\beta)+\lambda &= ((a+bi)+(b+di)) + (e+fi) \\
&= ((a+b)+(b+d)i) + (e+fi) \\
&= (a+b+e) + (b+d+f)i \quad (1)
\end{aligned}$$

$$\begin{aligned}
\alpha+(\beta+\lambda) &= (a+bi) + ((b+di)+(e+fi)) \\
&= (a+bi) + ((b+e)+(d+f)i) \\
&= (a+b+e) + (b+d+f)i \ (2)
\end{aligned}$$

$(1), (2) \Rightarrow (\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$

$\square$

3) $(\alpha\beta)\lambda = \alpha(\beta\lambda) \quad \forall \; \alpha,\beta,\lambda \in \mathbb{C}$

$\alpha = a+bi \; , \; \beta = c+di \; , \; \lambda = e+fi \quad a,b,c,d,e,f \in \mathbb{R}$

$$(\alpha\beta)\lambda = ((a+bi)(c+di))(e+fi)$$

$$= ((\underbrace{ac-bd}_{x}) + (\underbrace{ad+bc}_{y})i)(e+fi)$$

$$= (x+yi)(e+fi)$$

$$= (xe-yf) + (xf+ye)i$$

$$= ((ac-bd)e - (ad+bc)f) + ((ac-bd)f + (ad+bc)e)i$$

$$= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)i \quad (1)$$

$$\alpha(\beta\lambda) = (a+bi)((c+di)(e+fi))$$

$$= (a+bi)((\underbrace{ce-df}_{g}) + (\underbrace{cf+de}_{h})i)$$

$$= (a+bi)(g+hi)$$

$$= (ag-bh) + (ah+bg)i$$

$$= (a(ce-df) - b(cf+de)) + (a(cf+de) + b(ce-df))i$$

$$= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i \quad (2)$$

$(1),(2) \Rightarrow (\alpha\beta)\lambda = \alpha(\beta\lambda)$ $\square$

4) $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta \quad (\forall)\ \lambda, \alpha, \beta \in \mathbb{C}$

Suppose $\alpha = a+bi$, $\beta = c+di$, $\lambda = e+fi$ where $a,b,c,d,e,f \in \mathbb{R}$

$$\lambda(\alpha+\beta) = (e+fi)(\underbrace{(a+c)}_{x} + \underbrace{(b+d)}_{y}i)$$

$$= (e+fi)(x+yi)$$

$$= (ex-fy) + (ey+fx)i$$

$$= (e(a+c) - f(b+d)) + (e(b+d) + f(a+c))i$$

$$= (ea+ec-fb-fd) + (eb+ed+fa+fc)i \quad (1)$$

$$\lambda\alpha + \lambda\beta = (e+fi)(a+bi) + (e+fi)(c+di)$$

$$= ((ea-fb)+(eb+fa)i) + ((ec-fd)+(ed+fc))i$$

$$= (ea-fb+ec-fd) + (eb+fa+ed+fc)i \quad (2)$$

$(1),(2) \Rightarrow \lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$

$\square$

5) $\forall\ \alpha \in \mathbb{C}\ \ \exists$ unique $\beta \in \mathbb{C}$ s.t. $\alpha + \beta = 0$

Suppose $\alpha = a+bi$ where $a, b \in \mathbb{R}$

Define $\beta = -a-bi$

$$\alpha + \beta = (a+bi) + (-a-bi) = (a-a) + (b-b)i$$

$$= 0 + 0i = 0. \quad \text{(Existence)}$$

Suppose $\exists\, \lambda \in \mathbb{C}$, $\lambda \neq \beta$ s.t. $\alpha + \lambda = 0$.

$\lambda = -\alpha \implies \lambda = -a - bi \implies \lambda = \beta$ (Uniqueness)

$\square$

6) $\forall\, \alpha \in \mathbb{C}$, $\alpha \neq 0$ $\exists$ unique $\beta \in \mathbb{C}$ s.t. $\alpha\beta = 1$

Suppose $\alpha = a + bi$, $a, b \in \mathbb{R}$

Define $\beta = \dfrac{a - bi}{a^2 + b^2}$

$$\alpha\beta = (a+bi)\left(\frac{a-bi}{a^2+b^2}\right)$$

$$= \frac{(a+bi)(a-bi)}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

So $\exists\, \beta \in \mathbb{C}$ s.t. $\alpha\beta = 1$.

Suppose $\exists\, \lambda_1, \lambda_2 \in \mathbb{C}$ s.t. $\alpha\lambda_1 = 1$ and $\alpha\lambda_2 = 1$.

$\implies \alpha\lambda_1 = \alpha\lambda_2 \implies \alpha(\lambda_1 - \lambda_2) = 0$

Since $\alpha \neq 0$, $\alpha(\lambda_1 - \lambda_2) = 0 \implies \lambda_1 - \lambda_2 = 0 \implies$

$\underline{\lambda_1 = \lambda_2}$

Hence the inverse is unique.

$\square$

7) $a = \dfrac{-1+\sqrt{3}i}{2}$ cube root of 1.

Show $a^3 = 1$

$$a^3 = a^2 a = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 \left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$a^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{(-1+\sqrt{3}i)^2}{4} = \frac{1-2\sqrt{3}i+3i^2}{4} = \frac{-2\sqrt{3}i-2}{4} = \frac{-1-\sqrt{3}i}{2}$$

$$a^2 a = \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$= \frac{(-1-\sqrt{3}i)(-1+\sqrt{3}i)}{4}$$

$$= \frac{(-1)^2-(\sqrt{3}i)^2}{4} = \frac{1+3}{4} = \underline{\underline{1}}$$

$\square$

8) Find two distinct square roots of i

Trying to find $z \in \mathbb{C}$ s.t. $z^2 = i$. Let $z = a+bi$ where $a, b \in \mathbb{R}$

$$(a+bi)^2 = i = 0+1i$$

$$a^2-b^2+2abi = 0+1i \iff \begin{cases} a^2-b^2=0 \\ 2ab=1 \end{cases} \iff \begin{cases} a^2=b^2 \\ ab=\frac{1}{2} \end{cases} \iff \begin{cases} a=\pm b \\ ab=\frac{1}{2} \end{cases}$$

1. $\begin{cases} a=b \\ ab=\frac{1}{2} \end{cases} \iff \begin{cases} a=b \\ b^2=\frac{1}{2} \end{cases} \iff \begin{cases} a=b \\ b=\pm\frac{1}{\sqrt{2}} \end{cases} \implies \underline{\begin{cases} z_1=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i \\ z_2=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i \end{cases}}$

2. $\begin{cases} a=-b \\ ab=\frac{1}{2} \end{cases} \iff \begin{cases} a=-b \\ -b^2=\frac{1}{2} \end{cases} \to$ Not real

9) $x \in \mathbb{R}^4$ s.t. $(4,-3,1,7)+2x=(5,9,-6,8)$

$$\left.\begin{array}{l} 4+2x_1=5 \Rightarrow x_1=\frac{1}{2} \\ -3+2x_2=9 \Rightarrow x_2=6 \\ 1+2x_3=-6 \Rightarrow x_3=-\frac{5}{2} \\ 7+2x_4=8 \Rightarrow x_4=\frac{1}{2} \end{array}\right\} \Rightarrow x=\left(\frac{1}{2},6,-\frac{5}{2},\frac{1}{2}\right)$$

10) $\nexists \lambda \in \mathbb{C}$ s.t. $\lambda(2-3i, 5+4i, -6+7i)=(12-5i, 7+22i, -32-9i)$

$\lambda = a+bi$, $a,b \in \mathbb{R}$

$(a+bi)(2-3i)=12-5i \Leftrightarrow (2a-3b)+(-3a+2b)i=12-5i$

$$\Leftrightarrow \begin{cases} 2a-3b=12 \\ -3a+2b=-5 \end{cases}$$

$(a+bi)(5+4i)=7+22i \Leftrightarrow (5a-4b)+(4a+5b)i=7+22i$

$$\Leftrightarrow \begin{cases} 5a-4b=7 \\ 4a+5b=22 \end{cases}$$

$(a+bi)(-6+7i)=-32-9i \Leftrightarrow (-6a-7b)+(7a-6b)i=-32-9i$

$$\Leftrightarrow \begin{cases} -6a-7b=-32 \\ 7a-6b=-9 \end{cases}$$

$$A=\begin{bmatrix} 2 & -3 \\ -3 & 2 \\ 5 & -4 \\ 4 & 5 \\ -6 & -7 \\ 7 & -6 \end{bmatrix}$$

$(2,-3)$, $(-3,2)$ are linearly independent $\Rightarrow \operatorname{rank}(A)=2$

$$[A|b] = \left[\begin{array}{cc|c} 2 & -3 & 12 \\ -3 & 2 & -5 \\ 5 & -4 & 7 \\ 4 & 5 & 22 \\ -6 & -7 & -32 \\ 7 & -6 & -9 \end{array}\right] \qquad \begin{vmatrix} 2 & -3 & 12 \\ -3 & 2 & -5 \\ 5 & -4 & 7 \end{vmatrix} = 24 \neq 0$$

$$\Rightarrow \operatorname{rank}([A|b]) = 3.$$

$\operatorname{rank}(A) \neq \operatorname{rank}([A|b])$ so by Rouché Capelli the system is inconsistent.

$\square$

11) $(x+y)+z = x+(y+z)$ $(\forall)\ x,y,z \in F^n$

$(x+y)+z = ((x_1,\dots,x_n)+(y_1,\dots,y_n))+(z_1,\dots,z_n)$

$= (x_1+y_1,\dots,x_n+y_n)+(z_1,\dots,z_n)$

$= (x_1+y_1+z_1,\dots,x_n+y_n+z_n) \quad (1)$

$x+(y+z) = (x_1,\dots x_n)+((y_1,\dots,y_n)+(z_1,\dots,z_n))$

$= (x_1,\dots,x_n)+(y_1+z_1,\dots,y_n+z_n)$

$= (x_1+y_1+z_1,\dots,x_n+y_n+z_n) \quad (2)$

$(1),(2) \Rightarrow (x+y)+z = x+(y+z)$ $(\forall)\ x,y,z \in F^n$

$\square$

12) $(ab)x = a(bx) \quad \forall\, x \in F^n, \; a, b \in F$

$$(ab)x = (ab)(x_1, \dots, x_n)$$
$$= (abx_1, \dots, abx_n)$$
$$= (a(bx_1), \dots, a(bx_n))$$
$$= a(bx)$$

$\square$

13) $1x = x \quad \forall\, x \in \mathbb{F}^n$

$$1x = (1x_1, \dots, 1x_n) = (x_1, \dots, x_n) = x$$

↪ 1 multiplicative identity in F

$\square$

14) $\lambda(x+y) = \lambda x + \lambda y \quad \forall\, \lambda \in \mathbb{F}, \; x, y \in \mathbb{F}^n$

$$\lambda(x+y) = \lambda(x_1+y_1, \dots, x_n+y_n)$$
$$= (\lambda(x_1+y_1), \dots, \lambda(x_n+y_n))$$
$$= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n)$$
$$= \lambda x + \lambda y$$

$\square$

15) $(a+b)x = ax + bx \quad \forall\, a,b \in \mathbb{F},\ x \in \mathbb{F}^n$

$$
\begin{aligned}
(a+b)x &= (a+b)(x_1, \dots, x_n) \\
&= ((a+b)x_1, \dots, (a+b)x_n) \\
&= (ax_1 + bx_1, \dots, ax_n + bx_n) \\
&= ax + bx
\end{aligned}
$$

$\square$