$U = \{ p \in \mathcal{P}_4(F) : p(2) = p(5) = p(6) \}$

(a) Basis of U

$\forall p \in U$: Consider $r(x) = p(x) - p(2)$

$$\left.\begin{aligned} r(2) &= p(2) - p(2) = 0 \\ r(5) &= p(5) - p(2) = p(2) - p(2) = 0 \\ r(6) &= p(6) - p(2) = p(2) - p(2) = 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow r(x)$ has roots $2, 5, 6 \Rightarrow$

$\Rightarrow \exists\, g(x) = a + bx \in \mathcal{P}_1(F) : r(x) = (x-2)(x-5)(x-6)\, g(x)$

$r(x) = (x-2)(x-5)(x-6)(a+bx) \Rightarrow$

$$\left.\begin{aligned} &\Rightarrow p(x) = (x-2)(x-5)(x-6)(a+bx) + p(2) \\ &\quad p(2) \text{ is constant (i.e., deg 0)} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow p(x) = c \cdot 1 + a((x-2)(x-5)(x-6)) + b(x(x-2)(x-5)(x-6))$

$\Rightarrow \operatorname{span}(1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6)) = U.$

Since these polynomials have distinct degrees $(0, 3, 4)$ the list is also linearly independent.

$\Rightarrow B_U = (1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6))$ is a basis of U

(b) Extend B_U to be a basis of $\mathcal{P}_4(F)$

$\left.\begin{array}{l} E = (1, x, x^2, x^3, x^4) \text{ is a basis of } \mathcal{P}_4(F) \\ \dim \mathcal{P}_4(F) = 5, \ \dim U = 3 \end{array}\right\} \Rightarrow$

$\Rightarrow$ Extending B_U to be a basis of $\mathcal{P}_4(F)$ we must add 2 elements from E which $\notin \operatorname{span} B_U$.

$\left.\begin{array}{l} x \notin \operatorname{span} B_U \text{ (degree 1)} \\ x^2 \notin \operatorname{span} B_U \text{ (degree 2)} \end{array}\right\} \Rightarrow$

$\Rightarrow B = (1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6), x, x^2)$ is a basis of $\mathcal{P}_4(F)$.

(c) W subspace of $\mathcal{P}_4(F)$: $\mathcal{P}_4(F) = U \oplus W$

Let $W = \operatorname{span}(x, x^2)$

- $U + W = \mathcal{P}_4(F)$

$\left.\begin{array}{l} \text{Since } B \text{ is a basis of } \mathcal{P}_4(F) \Rightarrow \operatorname{span}(B) = \mathcal{P}_4(F). \\ \{1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6)\} \subset U \\ \{x, x^2\} \subset W \end{array}\right\} \Rightarrow$

$\Rightarrow \operatorname{span}(B) = \operatorname{span}(U \cup W) = U + W = \mathcal{P}_4(F)$

$\Rightarrow U + W = \mathcal{P}_4(F)$ (1)

- $U \cap W = \{0\}$

$$\left.\begin{array}{l} \dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) \\ \dim(U+W) = \dim \mathcal{P}_4(F) = 5 \\ \dim U = 3, \quad \dim W = 2 \end{array}\right\} \Rightarrow$$

$\Rightarrow \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W$

$\Rightarrow \dim(U \cap W) = 0$

$\Rightarrow U \cap W = \{0\}$ (2)

(1), (2) $\Rightarrow U \oplus W = \mathcal{P}_4(F)$