(a) $(1,0,\dots,0), (0,1,0\dots,0), \dots, (0,\dots,0,1)$ is a basis of F^n

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Let $e_i = (0,0,\dots,\underbrace{1}_{i},0,0)$: i^{th} coordinate is 1.

Spanning :

Consider any $v = (v_1, \dots, v_n) \in F^n$

$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n$

$\iff v = (v_1, 0, \dots 0) + (0, v_2, \dots, 0) + \dots + (0, 0, \dots, v_n)$

$\iff v = v.$

So, $\forall v \in F^n$, v can be defined as a linear combination of $(e_1, \dots, e_n) \implies \text{span}(e_1, \dots e_n) = F^n$ (1)

Linear independence:

Let $a_1, \dots, a_n \in F$

Consider $a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$

$\iff (a_1, 0, \dots, 0) + (0, a_2, \dots, 0) + \dots + (0, 0, \dots, a_n) = 0$

$\iff (a_1, a_2, \dots, a_n) = 0$

$\iff a_1 = a_2 = \dots = a_n = 0$

$\Rightarrow e_1, \dots e_n$ is linearly independent (2)

(1), (2) $\Rightarrow e_1, \dots e_n$ is a basis of F^n

(b) $(1,2), (3,5)$ basis of F^2

Spanning: Consider $v = (v_1, v_2) \in F^2$ $\quad a_1, a_2 \in F$

$$v = a_1(1,2) + a_2(3,5)$$

$$\iff v = (a_1 + 3a_2,\ 2a_1 + 5a_2)$$

$$\iff \begin{cases} a_1 + 3a_2 = v_1 \\ 2a_1 + 5a_2 = v_2 \end{cases} \iff \begin{cases} 2a_1 + 6a_2 = 2v_1 \\ 2a_1 + 5a_2 = v_2 \end{cases} \iff$$

$$\iff \begin{cases} a_2 = 2v_1 - v_2 \\ 2a_1 + 5a_2 = v_2 \end{cases} \iff \begin{cases} a_2 = 2v_1 - v_2 \\ a_1 = v_1 - 6v_1 + 3v_2 = -5v_1 + 3v_2 \end{cases}$$

$\forall\, v = (v_1, v_2) \in F^2$

$$v = (-5v_1 + 3v_2)(1,2) + (2v_1 - v_2)(3,5)$$

$\Rightarrow (1,2), (3,5)$ span F^2 $\quad$ (1)

Linear independence: Consider $a_1, a_2 \in F$

$$a_1(1,2) + a_2(3,5) = 0$$

$$\Leftrightarrow \begin{cases} a_1 + 3a_2 = 0 \\ 2a_1 + 5a_2 = 0 \end{cases} \overset{L_2 = 2L_1 - L_2}{\Leftrightarrow} \begin{cases} a_1 + 3a_2 = 0 \\ a_2 = 0 \end{cases} \Leftrightarrow a_1 = a_2 = 0$$

$\Rightarrow (1,2), (3,5)$ are linearly independent (2)

(1), (2) $\Rightarrow (1,2), (3,5)$ basis of F^2.

(c) $(1,2,-4), (7,-5,6)$ lin. indep., but not a basis in F^3

$(1,2,-4), (7,-5,6)$ does not span F^3

Consider $(1,0,0) \in F^3$, $a_1, a_2 \in F$.

$$(1,0,0) = a_1(1,2,-4) + a_2(7,-5,6)$$

$$\begin{cases} a_1 + 7a_2 = 1 \\ 2a_1 - 5a_2 = 0 \\ -4a_1 + 6a_2 = 0 \end{cases} \overset{L_2 = L_2 - 2L_1}{\Longleftrightarrow} \begin{cases} a_1 + 7a_2 = 1 \\ -19a_2 = -2 \\ -4a_1 + 6a_2 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 + 7a_2 = 1 \\ a_2 = \frac{2}{19} \\ -4a_1 + 6a_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_1 = 1 - \frac{14}{19} \\ a_2 = \frac{2}{19} \\ -4a_1 = -\frac{12}{19} \end{cases} \Leftrightarrow \begin{cases} a_1 = \frac{5}{19} \\ a_2 = \frac{2}{19} \\ a_1 = \frac{12}{76} = \frac{3}{19} \end{cases} \Rightarrow \text{Contradiction}$$

So, for $(1,0,0) \in F^3$, there is no $a_1, a_2 \in F$

s.t. $a_1(1,2,-4) + a_2(7,-5,6) = (1,0,0) \Rightarrow$

$\Rightarrow \text{span}((1,2,-4),(7,-5,6)) \neq F^3$.

$\Rightarrow (1,2,-4),(7,-5,6)$ not a basis of F^3.

(d) $(1,2),(3,5),(4,13)$ not a basis in F^2

$(1,0),(0,1)$ is a basis in $F^2 \Rightarrow \text{span}((1,0),(0,1)) = F^2$.

The length of every spanning list in $F^2 \geq$ the length of every linearly independent list $\Rightarrow$

$\Rightarrow (1,2),(3,5),(4,13)$ is not linearly independent in F^2

$\Rightarrow$ not a basis.

(e) $(1,1,0),(0,0,1)$ basis of $V = \{(x,x,y) \in F^3 : x,y \in F\}$

Spanning: Let $v = (v_1, v_1, v_2) \in V$, $v_1, v_2 \in F$

$v = v_1(1,1,0) + v_2(0,0,1)$

$\Rightarrow \forall v \in V$ we can write it as a linear comb. of $(1,1,0),(0,0,1) \Rightarrow \text{span}((1,1,0),(0,0,1)) = V$. (1)

Linear independence: Let $a_1, a_2 \in F$

Consider $a_1(1,1,0) + a_2(0,0,1) = 0$

$\iff (a_1, a_1, a_2) = 0$

$\iff a_1 = a_2 = 0$

$\Rightarrow (1,1,0), (0,0,1)$ are linear independent (2)

(1), (2) $\Rightarrow (1,1,0), (0,0,1)$ basis of V.

(f) $(1,-1,0), (1,0,-1)$ basis of $V = \{(x,y,z) \in F^3 : x+y+z=0\}$

$V = \{(-y-z, y, z) : x+y+z=0\}$

Spanning: Let $v = (-y-z, y, z) \in V$, $y, z \in F$.

$v = -y(1,-1,0) + -z(1,0,-1)$

$\iff v = (-y-z, y, z)$

$\Rightarrow$ We can define $\forall v \in V$ as a linear combination of $(1,-1,0), (1,0,-1)$.

$\Rightarrow \operatorname{Span}((1,-1,0), (1,0,-1)) = V$ (1)

Linear independence: Let $a_1, a_2 \in F$.

Consider $a_1(1,-1,0) + a_2(1,0,-1) = 0$

$\Leftrightarrow (a_1 + a_2, -a_1, -a_2) = 0$

$\Leftrightarrow a_1 = a_2 = 0$

$\Rightarrow (1,-1,0), (1,0,-1)$ linear independent (2)

(1),(2) $\Rightarrow (1,-1,0), (1,0,-1)$ basis of V.

(g) $1, z, \dots, z^m$ basis of $\mathcal{P}_m(F)$

Spanning: Consider any $p(z) \in \mathcal{P}_m(F)$

$p(z) = a_0 \cdot 1 + a_1 z + a_2 z^2 + \dots + a_m z^m \Rightarrow$

$\Rightarrow p$ linear combination of $1, z, \dots, z^m \Rightarrow$

$\Rightarrow \operatorname{span}(1, z, \dots, z^m) = \mathcal{P}_m(F)$ (1)

Linear independence: Consider $a_0, \dots, a_m \in F$:

$a_0 \cdot 1 + a_1 z + \dots + a_m z^m = 0$

Assume, for the sake of contradiction, that not all

$a_0 = \dots = a_m = 0 \Rightarrow$

$\Rightarrow \exists\, a_k \neq 0$. Let k be the largest index s.t. $a_k \neq 0$. Our lhs. polynomial then has deg k.

The rhs. zero-polynomial has deg $-\infty$.
This is a contradiction, since two polynomials are equal $\Leftrightarrow$ all corresponding coefficients are equal.

$\Rightarrow a_0 = a_1 \dots = a_m = 0$.
$\Rightarrow 1, z, \dots, z^m$ linearly independent. (2)

(1), (2) $\Rightarrow 1, z, \dots, z^m$ basis of $\mathcal{P}_m(F)$.