(a) Let $v \in V$

$\underline{\overline{v \text{ is linearly independent} \iff v \neq 0}}$

($\Rightarrow$)

Assume v is linearly independent $\Rightarrow av = 0 \iff a = 0$.

Assume, for the sake of contradiction, that $v = 0$.

Then $av = 0$ for $\forall a \in F : a \neq 0 \Rightarrow$

$\Rightarrow v$ is NOT linearly independent $\Rightarrow$

$\Rightarrow v \neq 0$. (1)

($\Leftarrow$) Assume $v \neq 0$.

Let av, $a \in F$ be a linear combination of v

Since $v \neq 0$, $av = 0 \iff a = 0 \Rightarrow$

v is linearly independent. (2)

(1), (2) $\Rightarrow v$ is linearly independent $\iff v \neq 0$.

(b) Let $v_1, v_2 \in V$

$$v_1, v_2 \text{ linearly independent} \iff \nexists\, \alpha \in F : v_1 = \alpha v_2$$

($\Rightarrow$) Assume v_1, v_2 are linearly independent. $\Rightarrow$

$\Rightarrow a_1 v_1 + a_2 v_2 = 0 \iff a_1 = a_2 = 0$

Suppose, for the sake of contradiction, $\exists\, \alpha \in F : v_1 = \alpha v_2$.

$$\left.\begin{array}{l} \text{Let } a_1 \neq 0,\ a_2 = -a_1\alpha \\ a_1 v_1 + a_2 v_2 = a_1\alpha v_2 - a_1\alpha v_2 = 0. \end{array}\right\} \Rightarrow$$

$\Rightarrow v_1, v_2$ are linearly dependent : CONTRADICTION

$\Rightarrow$ our assumption is false $\Rightarrow \nexists\, \alpha \in F : v_1 = \alpha v_2$.

$\Rightarrow v_1, v_2$ linearly independent $\Rightarrow \nexists\, \alpha \in F : v_1 = \alpha v_2$.

($\Leftarrow$) Assume, $\nexists\, \alpha \in F : v_1 = \alpha v_2$.

We want to prove v_1, v_2 linearly independent

i.e. $a_1 v_1 + a_2 v_2 = 0 \iff a_1 = a_2 = 0$.

Suppose, for the sake of contradiction,

$\exists\, a_1, a_2 \in F : a_1, a_2 \neq 0,\ a_1 v_1 + a_2 v_2 = 0.$

$a_1 v_1 + a_2 v_2 = 0 \iff$

$\iff a_1 v_1 = -a_2 v_2$

$\iff v_1 = -\dfrac{a_2}{a_1} v_2$: Contradiction.

$\Rightarrow \nexists\, \lambda \in F : v_1 = \lambda v_2 \Rightarrow v_1, v_2$ are linearly independent.