1) $-(-v) = v \quad (\forall)\ v \in V$

$$\left.\begin{array}{l} v + (-v) = 0 \quad \forall\ v \in V \text{ (additive inverse)} \\ (-v) + (-(-v)) = 0 \quad \forall\ v \in V \text{ (additive inverse)} \\ \text{The additive inverse of } (-v) \text{ is unique} \end{array}\right\} \Rightarrow$$

$\Rightarrow v = (-(-v))$

$\square$

2) $\underbrace{a \in \mathbb{F},\ v \in V,\ av = 0 \Rightarrow a = 0 \text{ or } v = 0.}$

Suppose $a \neq 0$. $a \in F \Rightarrow \exists\ a^{-1} \in F$ s.t. $aa^{-1} = 1$
(multiplicative inverse)

$av = 0 \quad | \cdot a^{-1}$

$(av)a^{-1} = 0 \cdot a^{-1}$

$(aa^{-1})v = 0$

$1 \cdot v = 0$

$\underline{v = 0}$

So if $a \neq 0 \Rightarrow v = 0$

$\square$

3) $v, w \in V$

$\exists$ unique $x \in V$ s.t. $v + 3x = w$

Existence: $v + 3x = w$

$$3x = w - v$$

$$x = \frac{1}{3}(w - v)$$

$w - v$ exists in V (vector addition + additive inverse)

$\frac{1}{3}(w - v)$ exists in V (scalar multiplication)

$\Rightarrow$ x exists in V.

Uniqueness: Suppose there $\exists$ $x_1, x_2 \in V$, $x_1 \neq x_2$ s.t. $\begin{cases} v + 3x_1 = w \\ v + 3x_2 = w \end{cases}$

$\begin{cases} v + 3x_1 = w \\ v + 3x_2 = w \end{cases}$ substract $\Leftrightarrow 3x_1 - 3x_2 = 0 \Leftrightarrow 3(x_1 - x_2) = 0$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$\Rightarrow$ x is unique

$\square$

4) additive identity

The $\emptyset$ doesn't contain any element hence it cannot contain an element 0 s.t. $v + 0 = v \ \forall v \in V$.

5) $0v = 0 \quad \forall v \in V$

scalar ↗ ↳ additive inverse of v

We'll show that if this condition holds along all other vector space axioms hold (except the additive inverse one), then additive inverses exist for all $v \in V$.

Consider $v \in V$, $(-1) \in F$, the additive inverse of $1 \in F$.

$$
\begin{aligned}
v + (-1)v &= 1v + (-1)v \\
&= (1 + (-1))v \\
&= 0v \\
&= 0
\end{aligned}
$$

$v + (-1)v = 0$, so $(-1)v$ is the additive inverse of v.

So if $0v = 0$ holds, then additive inverses exist in V.

6) $\mathbb{R} \cup \{\infty, -\infty\}$ vector space over $\mathbb{R}$?

Consider any $t \in \mathbb{R}$.

$$\left.\begin{aligned}(\infty + (-\infty)) + t &= 0 + t = t. \\ \infty + ((-\infty) + \infty) &= \infty + -\infty = 0.\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (\infty + (-\infty)) + t \neq \infty + ((-\infty) + \infty)$$

So addition over $\mathbb{R} \cup \{\infty, -\infty\}$ is not associative, thus $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space over $\mathbb{R}$.

7) For $f, g \in V^S$, $f + g \in V^S$ is: - ADDITION

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$

For $f \in V^S$, $\lambda \in V$, $\lambda f \in V^S$ is: - SCALAR MULTIPLICATION

$$(\lambda f)(x) = \lambda f(x) \quad \forall x \in S$$

1. Closure under addition

Holds - from our definition of addition $f + g \in V^S$, $\forall f, g \in V^S$. $\square$

2. Commutativity – $\forall f, g \in V^S$

$$
\begin{aligned}
f+g &= (f+g)(x) \\
&= f(x) + g(x) \\
&= g(x) + f(x) \\
&= (g+f)(x) \\
&= g+f.
\end{aligned}
$$

$\square$

3. Associativity

– addition: $\forall f, g, h \in V^S$

$$
\begin{aligned}
(f+g)+h &= ((f+g)+h)(x) \\
&= (f+g)(x) + h(x) \\
&= (f(x) + g(x)) + h(x) \\
&= f(x) + (g(x) + h(x)) \\
&= f(x) + (g+h)(x) \\
&= (f+(g+h))(x) \\
&= f+(g+h)
\end{aligned}
$$

$\square$

– scalar multiplication

$\forall f \in V^S, \ \alpha, \beta \in V$

$$
\begin{aligned}
(\alpha\beta)f &= ((\alpha\beta)f)(x) \\
&= (\alpha\beta)f(x) \\
&= \alpha(\beta f(x)) \\
&= \alpha(\beta(f))
\end{aligned}
$$

$\square$

4. Additive identity

$\exists\ 0 : S \to V$ s.t. $0(x) = 0$. $\forall\ x \in S$

$$f+0 = (f+\underbrace{0}_{\in V^S})(x) = f(x) + 0(x) = f(x) + \underbrace{0}_{\in V} = f(x) = f.$$

So there exists $0 \in V^S$ s.t. $f+0 = f \quad \forall\ f \in V^S$.

$\square$

5. Additive inverse

$\forall\ f \in V^S$, $\exists\ -f : S \to V$ s.t. $(-f)(x) = -f(x)$, $\forall x \in S$.

$$
\begin{aligned}
(f+(-f))(x) &= f(x) + (-f)(x) \\
&= f(x) + (-f(x)) \\
&= 0
\end{aligned}
$$

So there exists $-f \in V^S$ s.t. $f+(-f) = 0$, $\forall f \in V^S$

$\square$

6. Multiplicative identity

Assume 1 is multiplicative identity in V.

$1f = (1f)(x) = 1f(x) = f(x)$.

So there exists $1 \in V$ s.t. $1f = f \quad \forall f \in V^S$.

7. Distributivity

Suppose $\alpha, \beta \in V$, $f, g \in V^S$.

$$\begin{cases} \alpha(f+g) = \alpha((f+g)(x)) \\ \qquad = \alpha(f(x)+g(x)) \\ \qquad = \alpha f(x) + \alpha g(x) \\ \qquad = \alpha f + \alpha g \\ \text{So } \alpha(f+g) = \alpha f + \alpha g \quad \forall \alpha \in V, f, g \in V^S. \end{cases}$$

$\square$

$$\begin{cases} (\alpha+\beta)f = ((\alpha+\beta)f)(x) \\ \qquad = (\alpha+\beta)f(x) \\ \qquad = \alpha f(x) + \beta f(x) \\ \qquad = \alpha f + \beta f \\ \text{So } (\alpha+\beta)f = \alpha f + \beta f \quad \forall \alpha, \beta \in V, f \in V^S. \end{cases}$$

$\square$

$1 \dots 7 \Rightarrow$ given our prior definitions of addition / scalar mult. V^S is a vector space.

8) $V_{\mathbb{C}}$ - complex vector space. V - real vector space.

For any element $x \in V_{\mathbb{C}}$: $x = u + iv$ with $u, v \in V$

1. Commutativity

Suppose $x, y \in V_{\mathbb{C}}$.

$$
\begin{aligned}
x + y &= (x_u + ix_v) + (y_u + iy_v) \\
&= (x_u + y_u) + i(x_v + y_v) \\
&= (y_u + x_u) + i(y_v + x_v) \\
&= (y_u + iy_v) + (x_u + ix_v) \\
&= y + x
\end{aligned}
$$

$\square$

2. Associativity (addition)

Suppose $x, y, z \in V_{\mathbb{C}}$.

$$
\begin{aligned}
x + (y + z) &= (x_u + ix_v) + ((y_u + z_u) + i(y_v + z_v)) \\
&= (x_u + y_u + z_u) + i(x_v + y_v + z_v) \\
&= ((x_u + y_u) + i(x_v + y_v)) + (z_u + iz_v) \\
&= (x + y) + z.
\end{aligned}
$$

$\square$

3. Associativity (multiplication)

Suppose $\alpha, \beta \in \mathbb{C}$, $x \in V_c$. $\quad \alpha = a+bi$, $\beta = c+di$

$$(\alpha\beta)x = ((a+bi)(c+di))x$$
$$= ((\underbrace{ac-bd}_{t}) + i(\underbrace{ad+bc}_{p}))(x_u + i x_v)$$
$$= (t + ip)(x_u + i x_v)$$
$$= (t x_u - p x_v) + i(t x_v + p x_u)$$
$$= ((ac-bd)x_u - (ad+bc)x_v) + i((ac-bd)x_v + (ad+bc)x_u)$$
$$= (acx_u - bdx_u - adx_v - bcx_v) + i(acx_v - bdx_v + adx_u + bcx_u) \quad (1)$$

$$\alpha(\beta x) = (a+bi)((c+di)(x_u + i x_v))$$
$$= (a+bi)((\underbrace{cx_u - dx_v}_{m}) + i(\underbrace{cx_v + dx_u}_{n}))$$
$$= (a+bi)(m + in)$$
$$= (am - bn) + i(an + bm)$$
$$= (a(cx_u - dx_v) - b(cx_v + dx_u)) + i(a(cx_v + dx_u) + b(cx_u - dx_v))$$
$$= (acx_u - adx_v - bcx_v - bdx_u) + i(acx_v + adx_u + bcx_u - bdx_v) \quad (2)$$

$(1),(2) \Rightarrow (\alpha\beta)x = \alpha(\beta x)$

4) Additive identity

Suppose $0 \in V_c$; $0 = 0u + 0iv$ with $u, v \in V$
$\hookrightarrow \in \mathbb{R}$.

$$x + 0 = (x_u + ix_v) + (0u + 0(iv))$$

$$= (x_u + 0) + i(x_v + 0)$$

$$= x_u + ix_v$$

$$= x.$$

So, $\exists$ element $0 \in V_c$ s.t. $x + 0 = x \quad \forall\, x \in V_c$.

5) Additive inverse

Suppose $-x \in V_c$ s.t. $-x = (-x_u) + i(-x_v)$

$-x_u, -x_v \in V$ and are the additive inverse of x_u, x_v

$$x + (-x) = (x_u + ix_v) + ((-x_u) + i(-x_v))$$

$$= (x_u + (-x_u)) + i(x_v + (-x_v))$$

$$= 0 + i0$$

$$= 0.$$

So, $\exists$ element $-x \in V_c$ s.t. $x + (-x) = 0 \quad \forall\, x \in V_c$.

6) Multiplicative identity

Consider the scalar $1 = 1 + 0i \in \mathbb{C}$

$$1x = (1+0i)(x_u + ix_v)$$
$$= (1x_u - 0x_v) + i(1x_v + 0x_u)$$
$$= x_u + ix_v$$
$$= x.$$

7) Distributivity

Consider $x, y \in V_{\mathbb{C}}$. $\alpha, \beta \in \mathbb{C}$.

$$\alpha(x+y) = (a+bi)((x_u + y_u) + i(x_v + y_v))$$
$$= (a(x_u + y_u) - b(x_v + y_v)) + i(a(x_v + y_v) + b(x_u + y_u))$$
$$= (ax_u + ay_u - bx_v - by_v) + i(ax_v + ay_v - bx_u + by_u)$$
$$= ((ax_u - bx_v) + i(ax_v + bx_u)) + ((ay_u - by_v) + i(ay_v + by_u))$$
$$= (a+bi)(x_u + ix_v) + (a+bi)(y_u + iy_v)$$
$$= \alpha x + \alpha y$$

$$(\alpha+\beta)x = ((a+bi)+(c+di))(x_u+ix_v)$$

$$=((a+c)+i(b+d))(x_u+ix_v)$$

$$=((a+c)x_u-(b+d)x_v)+i((a+c)x_v+(b+d)x_u)$$

$$=(ax_u+cx_u-bx_v-dx_v)+i(ax_v+cx_v+bx_u+dx_u)$$

$$=((ax_u-bx_v)+i(ax_v+bx_u))+((cx_u-dx_v)+i(cx_v+dx_u))$$

$$=(a+bi)(x_u+ix_v) + (c+di)(x_u+ix_v)$$

$$=\alpha x + \beta x$$