(a) $b \in F$

$\{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\} = U$

U Subspace of $F^4 \iff b = 0$

U is a subspace of $F^4 \iff 0 = (0,0,0,0) \in U$

$\iff 0 = 5 \cdot 0 + b$

$\iff \underline{b = 0}.$

$\square$

(b) $C([0,1], \mathbb{R})$ is a subspace of $\mathbb{R}^{[0,1]}$.

· Additive identity

$f : [0,1] \to \mathbb{R} : f(x) = 0$ is Continuous

so $0 \in C[0,1]$. $\square$

· Closed under addition

Let $f, g \in C[0,1]$

$f+g = (f+g)(x) = f(x) + g(x)$, $x \in [0,1]$

$f(x) + g(x)$ is Continuous if f, g are Continuous

$\Rightarrow f+g \in C[0,1]$. $\square$

- Closed under scalar multiplication

Let $f \in C[0,1]$, $a \in \mathbb{R}$.

The product of a scalar and a continuous function is a continuous function. $\Rightarrow$

$\Rightarrow af \in C[0,1]$. $\square$

$C([0,1], \mathbb{R})$ is a subspace of $\mathbb{R}^{[0,1]}$.

(c) $D = \{ f \in \mathbb{R}^{\mathbb{R}} : f$ is differentiable $\}$

D is a subspace of $\mathbb{R}^{\mathbb{R}}$

- Additive identity:

Let $f : \mathbb{R} \to \mathbb{R} : f(x) = 0$

f is differentiable on $\mathbb{R}$ with $f'(x) = 0 \Rightarrow f \in D$. $\square$

- Closure under addition.

Let $f, g \in D$

f, g are differentiable on $\mathbb{R}$.

$f+g = (f+g)(x) = f(x) + g(x)$ is differentiable

(Sum of differentiable functions is differentiable)

$(f+g)' = f' + g' \Rightarrow (f+g) \in D.$ $\square$

- Closure under scalar multiplication

Let $f \in D$, $a \in \mathbb{R}$.

af is differentiable on $\mathbb{R}$.

$(af)' = af' \in D.$ $\square$

So D is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(d) $D = \{ f : (0,3) \to \mathbb{R} : f \text{ differentiable}, f'(2) = b \}$

D is subspace of $\mathbb{R}^{(0,3)} \iff b = 0$

Suppose D is a subspace of $\mathbb{R}^{(0,3)}$.

$\Rightarrow f : (0,3) \to \mathbb{R} : f(x) = 0 \in D$ (additive identity)

For this, $f'(x) = 0 \quad \forall x \in \mathbb{R}$

$f'(2) = 0 \Rightarrow b = 0.$

So, the zero function (additive identity) is in D

$\iff b = 0$

D is a subspace of $\mathbb{R}^{(0,3)} \iff b=0$. (1)

Suppose $b=0$. We prove D is a subspace of $\mathbb{R}^{(0,3)}$

$D = \{ f:(0,3) \to \mathbb{R} : f'(2) = 0 \}$

· Additive identity

$f:(0,3) \to \mathbb{R}$, $f(x) = 0 \in D$ $(f'(x) = 0 = f'(2))$.

$\square$

· Closed under addition

Let $f, g \in D$, so $f'(2) = 0$, $g'(2) = 0$

$f+g = f(x) + g(x)$

$(f+g)' = f'(x) + g'(x)$

$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0. \Rightarrow$

$\Rightarrow f+g \in D.$ $\square$

· Closed under scalar multiplication

Let $f \in D$, $a \in \mathbb{R}$ so $f'(2) = 0$.

$af = af(x)$

$(af)' = af'(x)$

$(af)'(2) = af'(2) = a \cdot 0 = 0. \Rightarrow$

$\Rightarrow af \in D.$ $\square$

So if $b=0$, D is a subspace. (2)

(1),(2) $\Rightarrow$ D is a subspace of $\mathbb{R}^{(0,3)}$ $\Leftrightarrow$ $b=0$.

(e) $V = \{f : \mathbb{N} \to \mathbb{C} : \lim_{n\to\infty} f(n) = 0\}$

$\boxed{V \text{ is a subspace of } \mathbb{C}^\infty}$

- Additive identity

Let $f : \mathbb{N} \to \mathbb{C}$ s.t. $f(n) = 0$ (zero func)

$\lim_{n\to\infty} f(n) = 0.$

$\Rightarrow f(n) = 0 \in V.$

- Closure under addition

Let $f, g \in V$, so $\lim_{n\to\infty} f(n) = 0$, $\lim_{n\to\infty} g(n) = 0.$

$f+g = f(n) + g(n)$

$$\lim_{n\to\infty}(f+g) = \lim_{n\to\infty}(f(n)+g(n)) =$$

$$= \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n)$$

$$= 0 + 0$$

$$= 0 \implies f+g \in V.$$

· Closure under scalar multiplication

Let $f \in V$, $a \in \mathbb{C}$ so $\lim_{n\to\infty} f(n) = 0$

$$af = (af)(n) = af(n)$$

$$\lim_{n\to\infty} af(n) = a \lim_{n\to\infty} f(n) = a \cdot 0 = 0 \implies$$

$$\implies af \in V$$

So, V is a subspace of C^∞.