$V = U \oplus W$

$u_1 \dots u_m$ basis of U ; $w_1 \dots w_n$ basis of W

$u_1 \dots u_m, w_1 \dots w_n$ basis of V

$$U \oplus W = V \Rightarrow \begin{cases} U \cap W = \{0\} \\ U + W = V \end{cases}$$

Spanning: Let $v \in V$.

$$\left.\begin{array}{l} \text{Since } U + W = V \Rightarrow \exists\, u \in U,\ w \in W : v = u + w \\ \text{Since } u_1 \dots u_m \text{ basis of } U \Rightarrow u = \sum_{i=1}^{m} a_i u_i \\ \text{Since } w_1 \dots w_n \text{ basis of } W \Rightarrow w = \sum_{i=1}^{n} b_i w_i \end{array}\right\} \Rightarrow$$

$$\Rightarrow v = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i \Rightarrow \operatorname{span}(u_1 \dots u_m, w_1 \dots w_n) = V \ (1)$$

Linear independence: Consider $\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0$

Let $u' = \sum_{i=1}^{m} a_i u_i \in U$, $w' = \sum_{i=1}^{n} b_i w_i \in W$

$$u' + w' = 0 \Rightarrow \underbrace{u'}_{\in U} = \underbrace{-w'}_{\in W} \Rightarrow u' \in U \cap W$$

But $U \cap W = \{0\} \Rightarrow u' = w' = 0$.

$$\left.\begin{aligned} u' &= \sum_{1}^{m} a_i u_i = 0 \text{, } \{u_i\} \text{ is a basis} \Rightarrow a_1 \dots a_m = 0 \\ w' &= \sum_{1}^{n} b_i w_i = 0 \text{, } \{w_i\} \text{ is a basis} \Rightarrow b_1 \dots b_n = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = 0 \iff a_1 = \dots = a_m = b_1 = \dots = b_n = 0 \Rightarrow$$

$\Rightarrow u_1, \dots, u_m, w_1, \dots, w_n$ are lin. independent (2)

(1), (2) $\Rightarrow u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for V.