$U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$

(a) Basis of U

Let $a + bx + cx^2 + dx^3 + ex^4 \in \mathcal{P}_4(\mathbb{R}) : a, b, c, d, e \in \mathbb{R}$

$p''(x) = 2c + 6dx + 12ex^2 \Rightarrow$

$\Rightarrow p''(6) = 2c + 36d + 432e$

$p''(6) = 0 \Leftrightarrow 2c + 36d + 432e = 0$

$\Leftrightarrow c = -18d - 216e$

$$\begin{aligned} p(x) &= a + bx + (-18d - 216e)x^2 + dx^3 + ex^4 \\ &= a + bx - 18dx^2 - 216ex^2 + dx^3 + ex^4 \\ &= a + bx + d(-18x^2 + x^3) + e(-216x^2 + x^4) \\ &= a + bx + d(x^3 - 18x^2) + e(x^4 - 216x^2) \end{aligned}$$

$\Rightarrow \operatorname{span}(1, x, x^3 - 18x^2, x^4 - 216x^2) = U$

$(1, x, x^3 - 18x^2, x^4 - 216x^2)$ are lin. independent since these polynomials have different deg. $\Rightarrow$

$\Rightarrow (1, x, x^3 - 18x^2, x^4 - 216x^2)$ is a basis of U

(b) extend $B_u = (1, x, x^3 - 18x^2, x^4 - 216x^2)$ to basis of $\mathcal{P}_4(\mathbb{R})$

$E = (1, x, x^2, x^3, x^4)$ is a basis of $\mathcal{P}_4(\mathbb{R})$. $\left.\right\} \Rightarrow$

$\dim \mathcal{P}_4(\mathbb{R}) = 5$

$\Rightarrow$ Extending B_u to be a basis of $\mathcal{P}_4(\mathbb{R})$ - we need to add one element from E which $\notin \text{span}(B_u)$.

$x^2 \notin \text{span}(B_u)$ since no element in B_u has $\deg = 2$.

$\Rightarrow B = (1, x, x^3 - 18x^2, x^4 - 216x^2, x^2)$ is a basis of $\mathcal{P}_4(\mathbb{R})$

(c) W subspace of $\mathcal{P}_4(\mathbb{R})$: $\mathcal{P}_4(\mathbb{R}) = U \oplus W$

Let $W = \text{span}(x^2) = \{\alpha x^2 : \alpha \in \mathbb{R}\}$

- $U + W = \mathcal{P}_4(\mathbb{F})$

$$\left.\begin{array}{l}\text{Since } B \text{ is a basis of } \mathcal{P}_4(\mathbb{R}) \Rightarrow \text{span } B = \mathcal{P}_4(\mathbb{R}) \\ \{1, x, x^3 - 18x^2, x^4 - 216x^2\} \subset U \\ \{x^2\} \subset W\end{array}\right\} \Rightarrow$$

$\Rightarrow \text{span } B = \text{span}(U \cup W) = U + W = \mathcal{P}_4(\mathbb{R})$

$\Rightarrow U + W = \mathcal{P}_4(\mathbb{R})$ (1)

- $U \cap W = \{0\}$

$$\left.\begin{aligned} &\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) \\ &\dim(U+W) = \dim(\mathcal{P}_4(\mathbb{R})) = 5 \\ &\dim U = 4, \quad \dim W = 1 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \dim(U) + \dim(W) - \dim(U \cap W) = \dim(U) + \dim(W) \Rightarrow$

$\Rightarrow \dim(U \cap W) = 0$

$\Rightarrow U \cap W = \{0\}$ (2)

(1), (2) $\Rightarrow U \oplus W = \mathcal{P}_4(\mathbb{R})$