1) Consider $V = \{0\}$ (zero space)

Let $B = ()$ (empty list / $\emptyset$)

$\left.\begin{array}{l} \text{span}(B) = \{0\} = V \\ B \text{ is also linearly independent} \end{array}\right\} \Rightarrow B$ is a basis of V

Uniqueness:

Consider any $W \in V \setminus B : \{0\} \subseteq W \Rightarrow W$ is not linearly independent.

$\Rightarrow$ For the zero-space there is exactly one basis.

2) Let $V = \{(x_1, x_2 \ldots, x_n) \in F^n : n \geq 2)$

Let $B = (v_1, v_2, \ldots v_n)$ be a basis of V. $n \geq 2$.

Consider $B' = (v_1 + v_2, v_2, \ldots v_n)$:

$\left.\begin{array}{l} v_1 = (v_1 + v_2) - v_2 \Rightarrow \text{span}(B) \subseteq \text{span}(B') \\ v_1 + v_2 \in \text{span}(B) \Rightarrow \text{span}(B') \subseteq \text{span}(B) \end{array}\right\} \Rightarrow$

$\Rightarrow \text{span}(B') = \text{span}(B) = V$

Let $a_1, a_2, \dots a_n \in F$

$a_1(v_1+v_2)+a_2v_2+\dots+a_nv_n=0$

$\Leftrightarrow a_1v_1+a_1v_2+a_2v_2+\dots+a_nv_n=0$

$\Leftrightarrow a_1v_1+(a_1+a_2)v_2+\dots+a_nv_n=0$

True only for $a_1=a_2=\dots=a_n=0$, because $B=(v_1,v_2,\dots,v_n)$ is a basis, hence linearly independent.

$\Rightarrow B'$ - linearly independent

So, for $V=\{(x_1,x_2\dots,x_n)\in F^n : n\geq 2\}$, a basis cannot be unique.

3) Let $V=\{x\in F^1\}$ - one dimensional space.

Let $B=(v)$ a basis of V.

$\forall\, \alpha\in F,\ \alpha\notin\{0,1\},\ B'=(\alpha v)$ is linearly independent and $\mathrm{span}(B)=\mathrm{span}(B') \Rightarrow B'$ is also a basis.

$\Rightarrow B$ is not unique.

$\Rightarrow$ For each one-dimensional space (i.e. where we have one coordinate basis) we can create a different basis by rescaling iff we have other scalars than $\{0,1\}$ in F.

Let $V = \{ x \in F_2^1 \}$ - one dimensional space over $F_2 = \{0, 1\}$ with the usual ops mod 2.

if $B = (v)$ is a basis of $V \Rightarrow$

$\Rightarrow \forall w \in V, w \neq 0 : w \in \operatorname{span}(B)$

$\Rightarrow w = \alpha v, \alpha \in F_2$ $\qquad |F^\times| = 1$

Because $w \neq 0 \Rightarrow \alpha \neq 0 \Rightarrow \alpha = 1$

So, $w = v$

This means, B is an unique basis for V.

So, the only vector spaces which have one unique basis are $\{0\}$ - the zero space and a one-dim. vector space over F_2, since $|F^\times| = 1$. Otherwise, we can create different basis by either rescaling or combining elements of another basis.