$V = U + W$

$\exists$ basis of $V \subseteq U \cup W$

V finite dimensional $\Rightarrow$ U, W finite dimensional

Let $B_{U\cap W} = \{v_1, \dots, v_k\}$ be a basis for $U \cap W$.

$B_{U\cap W}$ basis $\Rightarrow$ $(v_1, \dots, v_k)$ lin. independent in U

We can extend it to form $B_U = \{v_1, \dots v_k, u_1, \dots, u_m\}$ basis of U.

Analogous, we can extend $B_{U\cap W}$ to form $B_W = \{v_1, \dots, v_k, w_1, \dots w_n\}$ basis of W.

Consider $B = B_{U\cap W} \cup B_U \cup B_W = \{v_1 \dots v_k, u_1 \dots u_m, w_1 \dots w_n\}$

$B \subset U \cup W$.

We want to show B is a basis of V.

Spanning: Let $x \in V$.

$V = U + W \Rightarrow x = u + w$ for some $u \in U$, $w \in W$.

Since B_u is a basis for U, $u \in U \Rightarrow$

$$u = \sum_{i=1}^{k+m} a_i (b_u)_i \ : \ a_i \in F, \ b_{u_i} \in B_u$$

Since B_w is a basis for W, $w \in W \Rightarrow$

$$w = \sum_{i=1}^{k+n} a_i (b_w)_i \ : \ a_i \in F, \ b_{w_i} \in B_w.$$

This means any $x = u + w$ can be expressed as a linear combination of vectors from $B_u \cup B_w = B$.
So $\operatorname{span}(B) = V$. (1)

Linear independence: Consider linear combination over B.

$$\sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{m} b_i u_i + \sum_{i=1}^{n} c_i w_i = 0$$

$$\Leftrightarrow \underbrace{\sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{m} b_i u_i}_{\in U} = -\underbrace{\sum_{i=1}^{n} c_i w_i}_{\in W}$$

Let $z = \sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{m} b_i u_i$

$z \in U$, $z \in W \Rightarrow z \in U \cap W$

Since $z \in U \cap W$ it can be written as a lin. combination of vectors from $B_{U \cap W}$:

$$z = \sum_{i=1}^{k} d_i v_i \Rightarrow$$

$$\sum_{i=1}^{k} d_i v_i - \sum_{i=1}^{m} c_i w_i = 0$$

Every vector in this equation $\in B_W$. B_W is a basis $\Rightarrow$ lin. independent $\Rightarrow$ $d_1 \dots d_k = 0$ and $c_1 \dots c_m = 0$

Substituting back: $\sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{m} b_i u_i + 0 = 0$

Every vector in this equation $\in B_U$. B_U is a basis $\Rightarrow$ lin independent $\Rightarrow$ $a_1 \dots a_k = 0$ and $b_1 \dots b_m = 0$

Since all coefficients must be 0 $\Rightarrow$ the vectors in B are lin. independent. (2)

(1), (2) $\Rightarrow$ $B \subset U \cup W$ is a basis