$v_1 \dots v_m$ is linearly independent in V and $w \in V$

$v_1 \dots v_m, w$ linearly independent $\Longleftrightarrow w \notin \operatorname{span}(v_1 \dots v_m)$

($\Longrightarrow$) Assume $v_1, \dots v_m, w$ linearly independent, prove $w \notin \operatorname{span}(v_1 \dots v_m)$

Suppose, for the sake of contradiction, $w \in \operatorname{span}(v_1, \dots, v_m)$

$w \in \operatorname{span}(v_1, \dots, v_m) \Rightarrow \exists a_1, \dots, a_m : w = \sum_{i=1}^{m} a_i v_i$

$\sum_{i=1}^{m} a_i v_i + (-1) w = 0$ is a non-trivial relation between $v_1, \dots, v_m, w$.

This contradicts linearly independent $v_1, \dots v_m, w$

$\Rightarrow$ Our assumption is false and $w \notin \operatorname{span}(v_1, \dots, v_m)$ (1)

($\Longleftarrow$) Assume $w \notin \operatorname{span}(v_1 \dots v_m)$

Consider $\sum_{i=1}^{m} a_i v_i + \alpha w = 0$.

Suppose, for the sake of contradiction $\alpha \neq 0 \Rightarrow$

$$\Rightarrow w = -\frac{1}{\alpha} \sum_{i=1}^{m} a_i v_i \in \operatorname{span}(v_1 \dots v_m)$$

This contradicts $w \notin \operatorname{span}(v_1 \dots v_m)$, so $\alpha = 0$.

$$\left.\begin{array}{l} \text{This contradicts } w \notin \operatorname{span}(v_1 \dots v_m), \text{ so } \alpha = 0. \\ \sum_{i=1}^{m} a_i v_i = 0 \Leftrightarrow a_1 = \dots = a_m = 0 \text{ (linearly independent)} \end{array}\right\} \Rightarrow$$

$\Rightarrow v_1 \dots v_m, w$ are linearly independent (2)

(1), (2) $\Rightarrow$ $v_1 \dots v_m, w$ linearly independent $\Leftrightarrow w \notin \operatorname{span}(v_1 \dots v_m)$