Gauss elimination

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$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_i \\
\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n
\end{cases}
\qquad
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

A nonsingular/invertible ($det\ A \neq 0$), $A\,x = b \implies x = A^{-1}b$.

Naïve Gauss elimination:

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{23} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & a_{i2} & a_{i3} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
\quad \cdots \quad
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & \cdots & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & \cdots & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & & & \vdots \\
0 & 0 & 0 & \cdots\ a_{kk} & \cdots\ a_{kj} & \cdots & a_{kn} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots\ a_{ik} & \cdots\ a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots\ a_{nk} & \cdots\ a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
\quad \cdots
$$

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots \qquad \cdots + a_{1n}x_n = b_1 \\
\qquad a_{22}x_2 + a_{23}x_3 + \cdots \qquad \cdots + a_{2n}x_n = b_2 \\
\qquad\qquad a_{33}x_3 + \cdots \qquad + a_{3n}x_n = b_3 \\
\qquad\qquad\qquad \ddots \qquad\qquad \vdots \qquad \vdots \\
\qquad\qquad\qquad a_{ii}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n = b_i \\
\qquad\qquad\qquad\qquad \ddots \qquad \vdots \qquad \vdots \\
\qquad\qquad\qquad\qquad a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\
\qquad\qquad\qquad\qquad\qquad a_{nn}x_n = b_n
\end{cases}
$$

Example (when the naïve Gauss elimination fails):

$$
\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}
\qquad
\begin{cases} \varepsilon x_1 + \qquad x_2 = 1 \\ (1-\varepsilon^{-1})x_2 = 2 - \varepsilon^{-1} \end{cases}
\qquad
x_2 = \frac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \approx 1, \qquad x_1 = \varepsilon^{-1}(1-x_2) \approx 0 \qquad \text{(wrong)}
$$

$$
x_1 = \frac{1}{1-\varepsilon} \approx 1, \qquad x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1 \qquad \text{(correct)}
$$

$$
\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases}
\qquad
\begin{cases} x_1 + \qquad x_2 = 2 \\ (1-\varepsilon)x_2 = 1 - 2\varepsilon \end{cases}
\qquad
x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1, \qquad x_1 = 2 - x_2 \approx 1
$$