

COURSE 4

2.3. Hermite interpolation

Example 1 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

<i>Time</i>	0	3	5	8	13
<i>Distance</i>	0	225	383	623	993
<i>Speed</i>	75	77	80	74	72

Let $x_k \in [a, b]$, $k = 0, 1, \dots, m$ be such that $x_i \neq x_j$, for $i \neq j$ and let $r_k \in \mathbb{N}$, $k = 0, 1, \dots, m$. Consider $f : [a, b] \rightarrow \mathbb{R}$ such that there exist $f^{(j)}(x_k)$, $k = 0, 1, \dots, m$; $j = 0, 1, \dots, r_k$ and $n = m + r_0 + \dots + r_m$.

The Hermite interpolation problem (HIP) consists in determining the polynomial P of the smallest degree for which

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j = 0, \dots, r_k.$$

Definition 2 A solution of (HIP) is called **Hermite interpolation polynomial**, denoted by $H_n f$.

Hermite interpolation polynomial, $H_n f$, satisfies the interpolation conditions:

$$(H_n f)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j = 0, \dots, r_k.$$

Hermite interpolation polynomial is given by

$$(H_n f)(x) = \sum_{k=0}^m \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \in \mathbb{P}_n, \quad (1)$$

where $h_{kj}(x)$ denote **the Hermite fundamental interpolation polynomials**. They fulfill the relations:

$$h_{kj}^{(p)}(x_\nu) = 0, \quad \nu \neq k, \quad p = 0, 1, \dots, r_\nu$$

$$h_{kj}^{(p)}(x_k) = \delta_{jp}, \quad p = 0, 1, \dots, r_k, \quad \text{for } j = 0, 1, \dots, r_k \text{ and } \nu, k = 0, 1, \dots, m,$$

$$\text{with } \delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$$

We denote by

$$u(x) = \prod_{k=0}^m (x - x_k)^{r_k+1} \quad \text{and} \quad u_k(x) = \frac{u(x)}{(x - x_k)^{r_k+1}}.$$

We have

$$h_{kj}(x) = \frac{(x - x_k)^j}{j!} u_k(x) \sum_{\nu=0}^{r_k-j} \frac{(x - x_k)^\nu}{\nu!} \left[\frac{1}{u_k(x)} \right]_{x=x_k}^{(\nu)}. \quad (2)$$

Example 3 Find the Hermite interpolation polynomial for a function f for which we know $f(0) = 1, f'(0) = 2$ and $f(1) = -3$ (equivalent with $x_0 = 0$ multiple node of order 2 or double node, $x_1 = 1$ simple node).

Sol. We have $x_0 = 0, x_1 = 1, m = 1, r_0 = 1, r_1 = 0, n = m + r_0 + r_1 = 2$

$$\begin{aligned} (H_2 f)(x) &= \sum_{k=0}^1 \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \\ &= h_{00}(x) f(0) + h_{01}(x) f'(0) + h_{10}(x) f(1). \end{aligned}$$

We have h_{00}, h_{01}, h_{10} . These fulfill relations:

$$h_{kj}^{(p)}(x_\nu) = 0, \quad \nu \neq k, \quad p = 0, 1, \dots, r_\nu$$

$$h_{kj}^{(p)}(x_k) = \delta_{jp}, \quad p = 0, 1, \dots, r_k, \quad \text{for } j = 0, 1, \dots, r_k \text{ and } \nu, k = 0, 1, \dots, m.$$

We have $h_{00}(x) = a_1x^2 + b_1x + c_1 \in \mathbb{P}_2$, with $a_1, b_1, c_1 \in \mathbb{R}$, and the system

$$\begin{cases} h_{00}(x_0) = 1 \\ h'_{00}(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{00}(0) = 1 \\ h'_{00}(0) = 0 \\ h_{00}(1) = 0 \end{cases}$$

that becomes

$$\begin{cases} c_1 = 1 \\ b_1 = 0 \\ a_1 + b_1 + c_1 = 0. \end{cases}$$

Solution is: $a_1 = -1, b_1 = 0, c_1 = 1$ so $h_{00}(x) = -x^2 + 1$.

We have $h_{01}(x) = a_2x^2 + b_2x + c_2 \in \mathbb{P}_2$, with $a_2, b_2, c_2 \in \mathbb{R}$. The system

is

$$\begin{cases} h_{01}(x_0) = 0 \\ h'_{01}(x_0) = 1 \\ h_{01}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{01}(0) = 0 \\ h'_{01}(0) = 1 \\ h_{01}(1) = 0 \end{cases}$$

and we get $h_{01}(x) = -x^2 + x$.

We have $h_{10}(x) = a_3x^2 + b_3x + c_3 \in \mathbb{P}_2$, with $a_3, b_3, c_3 \in \mathbb{R}$. The system is

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \end{cases} \Leftrightarrow \begin{cases} h_{10}(0) = 0 \\ h'_{10}(0) = 0 \\ h_{10}(1) = 1 \end{cases}$$

and we get $h_{10}(x) = x^2$.

The Hermite polynomial is

$$(H_2f)(x) = -x^2 + 1 - 2x^2 + 2x - 3x^2 = -6x^2 + 2x + 1.$$

The Hermite interpolation formula is

$$f = H_nf + R_nf,$$

where R_nf denotes the remainder term (the error).

Theorem 4 *If $f \in C^n[\alpha, \beta]$ and $f^{(n)}$ is derivable on (α, β) , with $\alpha = \min\{x, x_0, \dots, x_m\}$ and $\beta = \max\{x, x_0, \dots, x_m\}$, then there exists $\xi \in (\alpha, \beta)$ such that*

$$(R_nf)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \quad (3)$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_nf)(z) \\ u(x) & (R_nf)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β) .

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \quad k = 0, \dots, m; \quad j = 0, \dots, r_k;$$

because

$$u(x) = \prod_{k=0}^m (x - x_k)^{r_k+1} \Rightarrow u^{(j)}(x_k) = 0, \quad j = 0, \dots, r_k$$

and

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_n f)^{(j)}(x_k) = f^{(j)}(x_k) - f^{(j)}(x_k) = 0.$$

So, F and its derivatives have $n + 2$ distinct zeros in (α, β) . Applying successively Rolle's theorem it follows that F' has at least $n + 1$ zeros in $(\alpha, \beta) \Rightarrow \dots \Rightarrow F^{(n+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(n+1)}(\xi) = 0$.

We have

$$F^{(n+1)}(z) = \begin{vmatrix} u^{(n+1)}(z) & (R_n f)^{(n+1)}(z) \\ u(x) & (R_n f)(x) \end{vmatrix},$$

with $u(z) = \prod_{k=0}^m (z - z_k)^{r_k+1} \in \mathbb{P}_{n+1} \Rightarrow u^{(n+1)}(z) = (n + 1)!$, and $(R_n f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_n f)^{(n+1)}(z) = f^{(n+1)}(z)$ (as, $H_n f \in$

\mathbb{P}_n). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (3). ■

Corollary 5 *If $f \in C^{n+1}[a, b]$ then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \|f^{(n+1)}\|_\infty, \quad x \in [a, b]$$

where $\|\cdot\|_\infty$ denotes the uniform norm ($\|f\|_\infty = \max_{x \in [a, b]} |f(x)|$).

Remark 6 *In case of $m = 0$, i.e., $n = r_0$, (HIP) becomes **Taylor interpolation problem**. Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^n \frac{(x - x_0)^j}{j!} f^{(j)}(x_0).$$

Example 7 *Find the Hermite interpolation formula for the function $f(x) = xe^x$ for which we know $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$,*

$f(1) = 2.7183$, (equivalent with $x_0 = -1$ simple, $x_1 = 0$ multiple of order 2 and $x_2 = 1$ simple). Which is the limit of the error for approximating $f(\frac{1}{2})$?

Hermite interpolation with double nodes

Example 8 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

<i>Time</i>	0	3	5	8	13
<i>Distance</i>	0	225	383	623	993
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Consider $f : [a, b] \rightarrow \mathbb{R}$, $x_0, x_1, \dots, x_m \in [a, b]$

and the values $f(x_0), f(x_1), \dots, f(x_m), f'(x_0), f'(x_1), \dots, f'(x_m)$.

The Hermite interpolation polynomial with double nodes, H_{2m+1} , satisfies the interpolation properties:

$$\begin{aligned} H_{2m+1}(x_i) &= f(x_i), \quad i = \overline{0, m}, \\ H'_{2m+1}(x_i) &= f'(x_i), \quad i = \overline{0, m}. \end{aligned}$$

It is a polynomial of $n = 2m + 1$ degree.

For computation: use Lagrange polynomial written in Newton form, with divided differences table having each node x_i written twice.

Consider $z_0 = x_0, z_1 = x_0, z_2 = x_1, z_3 = x_1, \dots, z_{2m} = x_m, z_{2m+1} = x_m$.

Form divided differences table: each node appear twice, in the first column write the values of f for each node twice; in the second column, at the odd positions put the values of the derivatives of f ; the other elements are computed using the rule from divided differences.

We obtain the following table:

z_0	$f(z_0)$	$(\mathcal{D}^1 f)(z_0) = f'(x_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m} f)(z_0)$	$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D}^1 f)(z_1)$	\vdots		$(\mathcal{D}^{2m} f)(z_1)$	
z_2	$f(z_2)$	$(\mathcal{D}^1 f)(z_2) = f'(x_1)$				
z_3	$f(z_3)$	\vdots				
\vdots	\vdots	$(\mathcal{D}^1 f)(z_{2m-1})$	$(\mathcal{D}^2 f)(z_{2m-1})$	\ddots		
z_{2m}	$f(z_{2m})$	$(\mathcal{D}^1 f)(z_{2m}) = f'(x_m)$		\dots		
z_{2m+1}	$f(z_{2m+1})$			\dots		

Newton interpolation polynomial for the nodes x_0, \dots, x_n is

$$(N_n f)(x) = f(x_0) + \sum_{i=1}^n (x - x_0) \dots (x - x_{i-1}) (\mathcal{D}^i f)(x_0),$$

and similarly, Hermite interpolation polynomial is

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0) \dots (x - z_{i-1}) (\mathcal{D}^i f)(z_0),$$

where $(\mathcal{D}^i f)(z_0)$, $i = 1, \dots, 2m + 1$ are the elements from the first line and columns $2, \dots, 2m + 1$.

Example 9 Consider the double nodes $x_0 = -1$ and $x_1 = 1$, and $f(-1) = -3$, $f'(-1) = 10$, $f(1) = 1$, $f'(1) = 2$. Find the Hermite interpolation polynomial, that approximates the function f , in both forms, using the classical formula and using divided differences.

Sol. We present here the method with divided differences. We have $m = 1, r_0 = r_1 = 1 \Rightarrow n = 3$

$z_0 = -1$	$f(-1) = -3$	$f'(-1) = 10$	$\frac{\frac{f(1)-f(-1)}{2} - f'(-1)}{z_2 - z_0} = -4$	$\frac{0 - (-4)}{z_3 - z_0} = 2$
$z_1 = -1$	$f(-1) = -3$	$\frac{f(1)-f(-1)}{z_2 - z_1} = 2$	$\frac{f'(1) - \frac{f(1)-f(-1)}{2}}{z_3 - z_1} = 0$	
$z_2 = 1$	$f(1) = 1$	$f'(1) = 2$		
$z_3 = 1$	$f(1) = 1$			

The Hermite interpolation polynomial is

$$\begin{aligned}
 (H_3 f)(x) &= f(z_0) + \sum_{i=1}^3 (x - z_0) \dots (x - z_{i-1}) (\mathcal{D}^i f)(z_0) \\
 &= f(z_0) + (x - z_0) (\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1) (\mathcal{D}^2 f)(z_0) \\
 &\quad + (x - z_0)(x - z_1)(x - z_2) (\mathcal{D}^3 f)(z_0)
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 (H_3 f)(x) &= f(-1) + (x + 1)f'(-1) + (x + 1)^2 \frac{f(1) - f(-1) - 2f'(-1)}{4} \\
 &\quad + (x + 1)^2 (x - 1) \frac{2f'(1) - f(1) + f(-1)}{4} \\
 &= -3 + 10(x + 1) - 4(x + 1)^2 + 2(x + 1)^2(x - 1) \\
 &= 2x^3 - 2x^2 + 1.
 \end{aligned}$$

Example 10 *Considering the the following data*

x	0	2	3
$f(x)$	0	10	12
$f'(x)$	5	3	7

find the corresponding Hermite interpolation polynomial.