COURSE 5

2.4. Birkhoff interpolation

Let $x_k \in [a,b]$, $k = 0,1,...,m$, $x_i \neq x_j$ for $i \neq j, r_k \in \mathbb{N}$ and $I_k \subset \{0,1,...,r_k\}$, $k = 0,1,...,m$, $f : [a,b] \to \mathbb{R}$ s.t. $\exists f^{(j)}(x_k)$, $k = 0,...,m$, $j \in I_k$, and denote $n = |I_0| + ... + |I_m| - 1$, where $|I_k|$ is the cardinal of the set I_k.

The Birkhoff interpolation problem (BIP) consists in determining the polynomial P of the smallest degree such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0,...,m;\ j \in I_k.$$

Remark 1 *If* $I_k = \{0,1,...,r_k\}$, $k = 0,...,m$, *then (BIP) reduces to a (HIP). Birkhoff interpolation is also called* lacunary Hermite interpolation.

In order to check if (BIP) has an unique solution, we consider the polynomial $P(x) = a_n x^n + ... + a_0$ and the $(n+1) \times (n+1)$ linear system

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m;\ j \in I_k, \tag{1}$$

having as unknowns the coefficients of the polynomial. If the determinant of the system (1) is nonzero than (BIP) has an unique solution.

Definition 2 *A solution of (BIP), if exists, is called* **Birkhoff interpolation polynomial***, denoted by $B_n f$.*

Birkhoff interpolation polynomial is given by

$$(B_n f)(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k), \tag{2}$$

where $b_{kj}(x)$ denote the Birkhoff fundamental interpolation polynomials. They fulfill relations:

$$\begin{aligned} &b_{kj}^{(p)}(x_\nu) = 0,\ \nu \neq k, \quad p \in I_\nu \\ &b_{kj}^{(p)}(x_k) = \delta_{jp},\ p \in I_k, \quad \text{for } j \in I_k \text{ and } \nu, k = 0, 1, ..., m, \end{aligned} \tag{3}$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

Remark 3 *Because of the gaps of the interpolation conditions, it is hard to find an explicit expression for* b_{kj}, $k = 0, ..., m;\ j \in I_k$. *They are found using relations (3).*

Birkhoff interpolation formula is

$$f = B_n f + R_n f,$$

where $R_n f$ denotes the remainder term.

Example 4 *Let* $f \in C^2[0,1]$, *the nodes* $x_0 = 0$, $x_1 = 1$ *and we suppose that we know* $f(0) = 1$ *and* $f'(1) = \frac{1}{2}$. *Find the corresponding interpolation formula.*

Sol. *We have* $m = 1,\ I_0 = \{0\},\ I_1 = \{1\}$, *so* $n = 1 + 1 - 1 = 1$.

We check if there exists a solution of the problem.

Consider $P(x) = a_1 x + a_0 \in \mathbb{P}_1$ *and the system*

$$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \Longleftrightarrow \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases} .$$

The determinat of the system is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

so the problem has an unique solution.

The Birkhoff polynomial is

$$(B_1 f)(x) = b_{00}(x) f(0) + b_{11}(x) f'(1) \in \mathbb{P}_1.$$

We have $b_{00}(x) = ax + b \in \mathbb{P}_1$ *and*

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \Longleftrightarrow \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases} ,$$

whence

$$b_{00}(x) = 1.$$

For $b_{11}(x) = cx + d \in \mathbb{P}_1$ *we have*

$$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_1) = 1 \end{cases} \Longleftrightarrow \begin{cases} b_{11}(0) = 0 \\ b'_{11}(1) = 1 \end{cases} \Leftrightarrow \begin{cases} d = 0 \\ c = 1 \end{cases}$$

whence

$$b_{11}(x) = x.$$

So,

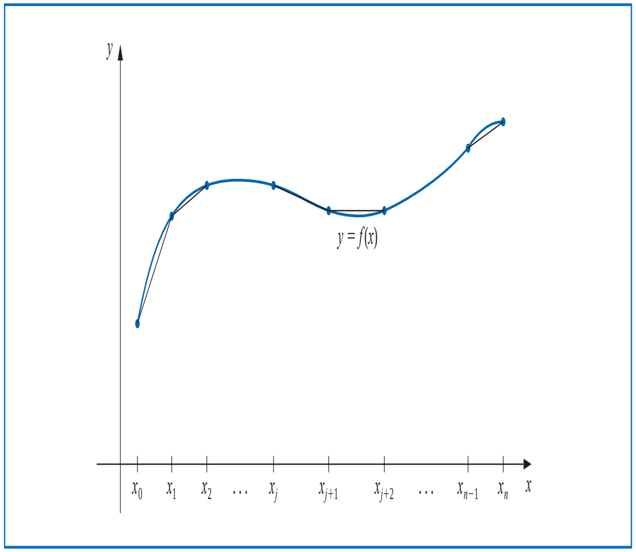
$$(B_1 f)(x) = f(0) + x f'(1) = 1 + \frac{1}{2}x.$$

Example 5 *Considering* $f'(0) = 1$, $f(1) = 2$ *and* $f'(2) = 1$. *Find the approximative value of* $f(\frac{1}{2})$.

2.5. Cubic spline interpolation

Lagrange, Hermite, Birkhoff interpolants of large degrees could oscillate widely; a minor fluctuation over a small portion of the interval can induce large fluctuations over the entire interval.

An alternative: to divide the interval into a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. This is called **piecewise-polynomial approximation.**



Let $f : [a, b] \to \mathbb{R}$ be the approximating function. Examples of piecewise-polynomial interpolation:

- piecewise-linear interpolation: consists of joining a set of data points $\{(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))\}$ by a series of straight lines

  Disadvantage: there is likely no differentiability at the endpoints of the subintervals, (the interpolating function is not "smooth"). Often, from physical conditions, that smoothness is required.

- Hermite interpolation when values of f and f' are known at the points $x_0 < x_1 < ... < x_n$;

  Disadvantage: we need to know f' and this is frequently unavailable.

- spline interpolation: piecewise polynomials that require no specific derivative information, except perhaps at the endpoints of the interval.

Definition 6 *The piecewise-polynomial approximation that uses cubic spline polynomials between each successive pair of nodes is called* **cubic spline interpolation.**

(The word "spline" was used to refer to a long flexible strip, generally of metal, that could be used to draw continuous smooth curves by forcing the strip to pass through specified points and tracing along the curve.)

Definition 7 *Let* $f : [a, b] \to \mathbb{R}$ *and the nodes* $a = x_0 < x_1 < ... < x_n = b$, *a* **cubic spline interpolant** S *for* f *is the function that satisfies the following conditions:*

(a) $S(x)$ *is a cubic polynomial, denoted* $S_j(x)$ *on the subinterval* $[x_j, x_{j+1}]$, $\forall j = 0, 1, ..., n-1$, *i.e.,*

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \\ ... & \\ S_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

(b) $S_j(x_j) = f(x_j)$ *and* $S_j(x_{j+1}) = f(x_{j+1})$, $\forall j = 0, 1, ..., n-1$;

(c) $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$, $\forall j = 0, 1, ..., n-2$;

(d) $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1})$, $\forall j = 0, 1, ..., n-2$;

(e) $S_j''(x_{j+1}) = S_{j+1}''(x_{j+1})$, $\forall j = 0, 1, ..., n-2$;

(f) *One of the following boundary conditions is satisfied:*

(i) $S''(x_0) = S''(x_n) = 0$ ($\Longleftrightarrow S''_0(x_0) = S''_{n-1}(x_n) = 0$ *natural (or free) boundary)* **natural spline***;*

(ii) $S^{'}(x_0) = f'(x_0)$ *and* $S'(x_n) = f'(x_n)$ ($\Longleftrightarrow S^{'}_0(x_0) = f'(x_0)$ *and* $S'_{n-1}(x_n) = f'(x_n)$ *clamped boundary)* **clamped spline***;*

(iii) $S_1(x) = S_2(x)$ *and* $S_{n-2} = S_{n-1}$ *(***de Boor spline***).*

Remark 8 *A cubic spline function defined on an interval divided into* n *subintervals will require determining* $4n$ *constants.*

We have the following expression of a cubic spline:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad \forall j = 0, 1, ..., n-1. \tag{4}$$

Theorem 9 *If* f *is defined at* $a = x_0 < x_1 < ... < x_n = b$, *then* f *has a unique natural spline interpolant* S *on the nodes* x_0, $x_1, ..., x_n$; *that satisfies the natural boundary conditions* $S''(a) = 0$ *and* $S''(b) = 0$.

Theorem 10 *If f is defined at $a = x_0 < x_1 < ... < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, ..., x_n$; that satisfies the clamped boundary conditions $S'(a) = f'(a)$ și $S'(b) = f'(b)$.*

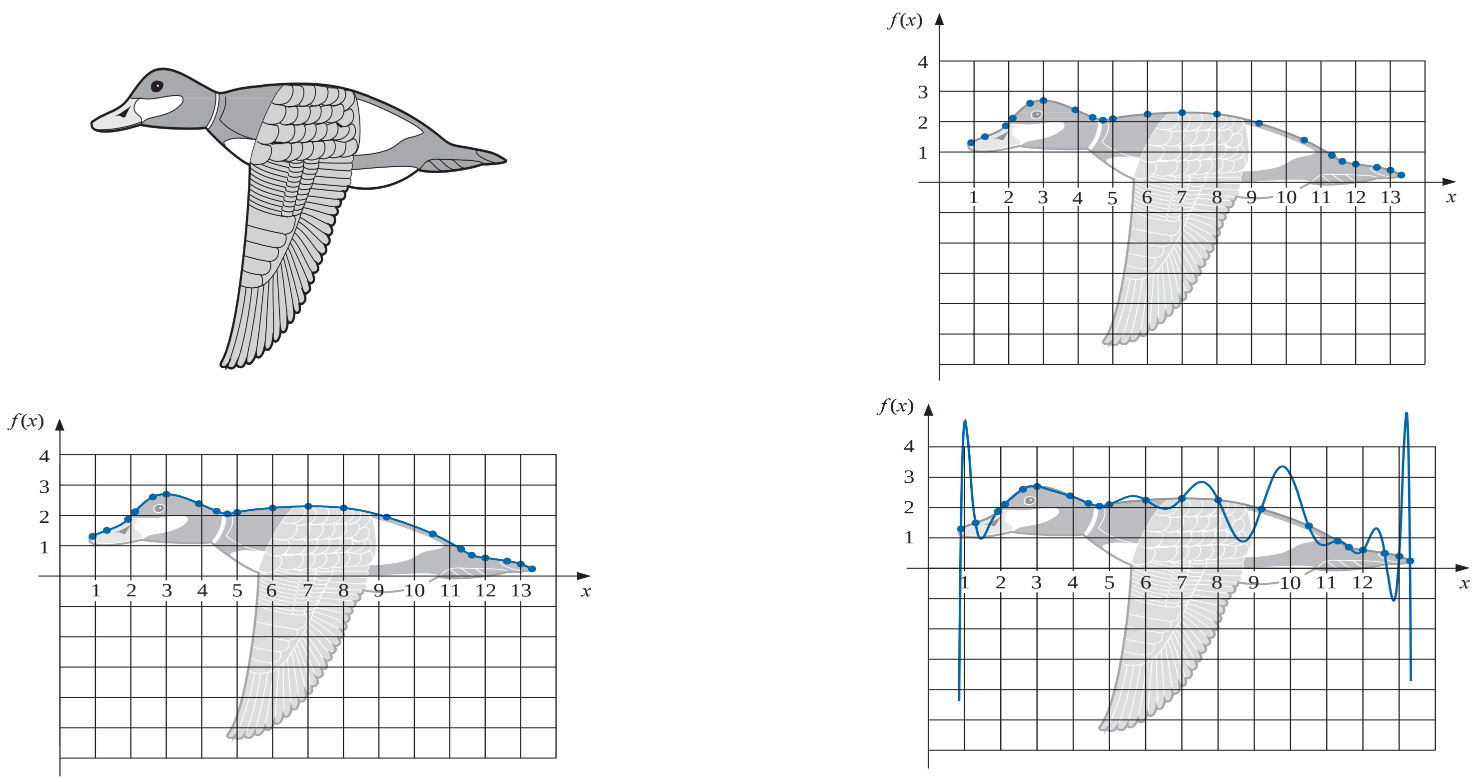
Theorem 11 *Let $f \in C^4[a, b]$ with $\max\limits_{a \le x \le b} |f^{(4)}(x)| = M$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes $a = x_0 < x_1 < \cdots < x_n = b$, then for all x in $[a, b]$,*

$$|f(x) - S(x)| \le \frac{5M}{384} \max_{0 \le j \le n-1} (x_{j+1} - x_j)^4.$$

Remark 12 *A fourth-order error-bound result also holds in the case of natural boundary conditions, but it is more difficult to express.*

Remark 13 *The natural boundary conditions will generally give less accurate results than the clamped conditions near the ends of the interval $[x_0, x_n]$ unless the function f happens to nearly satisfy $f''(x_0) = f''(x_n) = 0$.*

Illustration. To approximate the top profile of a duck, we have chosen 21 points along the curve through which we want the approximating curves to pass.



1) The duck in flight. 2) The points. 3) The natural cubic spline. 4) The Lagrange interpolation polynomial.

Example 14 *Construct* **a natural cubic spline** *that passes through the points* $(1, 2)$, $(2, 3)$ *and* $(3, 5)$.

Sol. (Sketch of the solution) *We follow Definition 7:*

Here $S(x)$ *consists of two cubic splines,* $S_j(x)$ *on the subinterval* $[x_j, x_{j+1}]$, $\forall j = 0, 1$, *i.e.,*

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \end{cases}$$

given by (4),

$$S_0(x) = a_0 + b_0(x - 1) + c_0(x - 1)^2 + d_0(x - 1)^3,$$
$$S_1(x) = a_1 + b_1(x - 2) + c_1(x - 2)^2 + d_1(x - 2)^3.$$

There are 8 constants $(a_i, b_i, c_i, d_i,\ i = 0, 1)$ *to be determined, which requires 8 conditions, that come from (b),(c),(d),(e),(i).*

Example 15 *Construct* **a clamped spline** S *that passes through the points* $(1, 2)$, $(2, 3)$ *and* $(3, 5)$ *and that has* $S'(1) = 2$ *and* $S'(3) = 1$.