COURSE 2

2.2. Lagrange interpolation

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

The Lagrange interpolation problem (LIP) consists in determining the polynomial P of the smallest degree for which

$$P(x_i) = f(x_i),\ i = 0, 1, ..., m \tag{1}$$

i.e., the polynomial of the smallest degree which passes through the distinct points $(x_i, f(x_i))$, $i = 0, 1, ..., m$.

Since in (1) there are $m + 1$ conditions to be satisfied, we need $m + 1$ degrees of freedom. Consider the m-th degree polynomial

$$P(x) = a_0 + a_1 x + ... + a_{m-1} x^{m-1} + a_m x^m. \tag{2}$$

The $m+1$ coefficients $\{a_i\}$ have to be determined in such way that (1) are satisfied. This leads to the linear system of equations:

$$\begin{cases} a_0 + a_1 x_0 + ... + a_{m-1} x_0^{m-1} + a_m x_0^m = f(x_0) \\ a_0 + a_1 x_1 + ... + a_{m-1} x_1^{m-1} + a_m x_1^m = f(x_1) \\ \\ a_0 + a_1 x_m + ... + a_{m-1} x_m^{m-1} + a_m x_m^m = f(x_m). \end{cases}$$

Written in the matrix form, the system is

$$\underbrace{\begin{pmatrix} 1 & x_0 & ... & x_0^{m-1} & x_0^m \\ 1 & x_1 & ... & x_1^{m-1} & x_1^m \\ \vdots & & & & \\ 1 & x_m & ... & x_m^{m-1} & x_m^m \end{pmatrix}}_{V} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m). \end{pmatrix}.$$

The matrix V with the special structure containing the powers of the nodes is called a Vandermonde matrix.

Remark 1 *For $m+1$ distinct nodes the Vandermonde matrix is non-singular and there exists a unique interpolating polynomial P of degree less or equal to m with $P(x_i) = f(x_i),\ i = 0, 1, ..., m$.*

Remark 2 *Because the Vandermonde matrix is ill conditioned this method is not recomended for computing the Lagrange polynomial.*

Definition 3 *A solution of (LIP) is called* **Lagrange interpolation polynomial**, *denoted by $L_m f$.*

Remark 4 *We have $(L_m f)(x_i) = f(x_i),\ i = 0, 1, ..., m.$*

$L_m f \in \mathbb{P}_m$ ($\mathbb{P}_m$ is the space of polynomials of at most m-th degree).

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} \ell_i(x) f(x_i), \tag{3}$$

where by $\ell_i(x)$ denote **the Lagrange fundamental interpolation polynomials.**

We have

$$u(x) = \prod_{j=0}^{m} (x - x_j),$$

$$u_i(x) = \frac{u(x)}{x - x_i} = (x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m) = \prod_{\substack{j=0 \\ j \neq i}}^{m} (x - x_j)$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{(x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m)}{(x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_m)} = \prod_{\substack{j=0 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j}, \tag{4}$$

for $i = 0, 1, ..., m$.

How do we know that the interpolation polynomial expanded in powers of x as in (2) and the polynomial constructed as in (3) represent the same polynomial?

Assume we have computed two interpolating polynomials $Q(x)$ and $P(x)$ each of degree m such that

$$Q(x_j) = f(x_j) = P(x_j), \quad j = 0, ..., m.$$

Then we can form the difference

$$d(x) = Q(x) - P(x),$$

that is a polynomial of degree less or equal to m.

Because of the interpolation property of P and Q, we have

$$d(x_j) = Q(x_j) - P(x_j) = 0, \quad j = 0, ..., m.$$

A non-zero polynomial of degree less than or equal to m cannot have more than m zeros. But d has $m+1$ distint zeros, hence it must be indentically zero, so $Q(x) = P(x)$.

Proposition 5 *We also have*

$$\ell_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \ i = 0, 1, ..., m. \tag{5}$$

Proof. We have $u_i(x) = \frac{u(x)}{x-x_i}$, so $u(x) = u_i(x)(x - x_i)$. We get $u'(x) = u_i(x) + (x - x_i)u_i'(x)$, whence it follows $u'(x_i) = u_i(x_i)$. So, as

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

we get

$$\ell_i(x) = \frac{u_i(x)}{u'(x_i)} = \frac{u(x)}{(x - x_i)u'(x_i)}, \;\; i = 0, 1, ..., m. \tag{6}$$

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Theorem 6 *The operator L_m is linear.*

Proof.

$$\begin{aligned} L_m(\alpha f + \beta g)(x) &= \sum_{i=0}^{m} \ell_i(x)(\alpha f + \beta g)(x_i) = \sum_{i=0}^{m} [\ell_i(x)\alpha f(x_i) + \ell_i(x)\beta g(x_i)] \\ &= \alpha(L_m f)(x) + \beta(L_m g)(x), \end{aligned}$$

so

$$L_m(\alpha f + \beta g) = \alpha L_m f + \beta L_m g, \qquad \forall f, g : [a, b] \to \mathbb{R} \text{ and } \alpha, \beta \in \mathbb{R}.$$

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Example 7 *1) Consider the nodes x_0, x_1 and a function f to be interpolated.*

We have $m = 1$,

$$\begin{aligned} u(x) &= (x - x_0)(x - x_1) \\ u_0(x) &= x - x_1 \\ u_1(x) &= x - x_0 \end{aligned}$$

$$\begin{aligned} (L_1 f)(x) &= l_0(x) f(x_0) + l_1(x) f(x_1) \\ &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1), \end{aligned}$$

which is the line passing through the given points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

Example 8 *Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.*

x	-1	0	3
$f(x)$	8	-2	4

Sol. *We have* $m = 2$. *The Lagrange polynomial is*

$$(L_2 f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2).$$

$u(x) = (x+1)(x-0)(x-3)$ *and it follows*

$$l_0(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4}x(x-3)$$
$$l_1(x) = \frac{(x+1)(x-3)}{(0+1)(0-3)} = -\frac{1}{3}(x+1)(x-3)$$
$$l_2(x) = \frac{(x+1)(x-0)}{(3+1)(3-0)} = \frac{1}{12}x(x+1),$$

The polynomial is

$$(L_2 f)(x) = 2x(x-3) + \frac{2}{3}(x+1)(x-3) + \frac{1}{3}x(x+1).$$

and $(L_2 f)(-0.5) = 2.25$.

Remark 9 *Disadvantages of the form (3) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum\limits_{i=0}^{m} l_i(x) f(x_i)}{\sum\limits_{i=0}^{m} l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^{m} (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod\limits_{j=0, j \neq i}^{m} (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \tag{7}$$

called **the barycentric form** *of Lagrange interpolation polynomial*.

Remark 10 *Formula (7) needs half of the number of arithmetic operations needed for (3) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 11 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{8}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \;\; F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^{m} (z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned} (R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z) \end{aligned}$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi)$, whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$. ■

Corolar 12 *If* $f \in C^{m+1}[a,b]$ *then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm, and* $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.