COURSE 9

3.6. Quadrature formulas of Gauss type

All the previous rules can be written in the form

$$\int_a^b f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + R_m(f), \tag{1}$$

where the coefficients A_k, $k = 1, ..., m$, do not depend on the function f. We have picked the nodes x_k, $k = 1, ..., m$ equispaced and have then calculated the coefficients A_k, $k = 1, ..., m$. This guarantees that the rule is exact for polynomials of degree $\leq m$.

It is possible to make such a rule exact for polynomials of degree $\leq 2m - 1$, by choosing also the nodes appropriately. This is the basic idea of the gaussian rules.

Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $w : [a, b] \to \mathbb{R}_+$ a weight function, integrable on $[a, b]$.

Definition 1 *A formula of the following form*

$$\int_a^b w(x)f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + R_m(f) \tag{2}$$

is called **a quadrature formula of Gauss type** *or* **with maximum degree of exactness** *if the coefficients A_k and the nodes x_k, $k = 1, ..., m$ are determined such that the formula has the maximum degree of exactness.*

Remark 2 *The coefficients and the nodes are determined such that to minimize the error, to produce exact results for the largest class of polynomials.*

A_k and x_k, $k = 1, ..., m$ from (2) are $2m$ unknown parameters $\Rightarrow$ $2m$ equations obtained such that the formula (2) is exact for any polynomial degree at most $2m - 1$.

It is often possible to rewrite the integral $\int_a^b g(x)dx$ as $\int_a^b w(x)f(x)dx$, where $w(x)$ is a nonnegative integrable function, and $f(x) = \frac{g(x)}{w(x)}$ is smooth, or it is possible to consider the simple choice $w(x) = 1$.

For the general case, consider the elementary polynomials $e_k(x) = x^k$; $k = 0, ..., 2m-1$ and obtain the system s.t. $R_m(e_k) = 0$:

$$\begin{cases} \sum\limits_{k=1}^{m} A_k e_0(x_k) = \int_a^b w(x) e_0\,(x)\,dx \\ \sum\limits_{k=1}^{m} A_k e_1(x_k) = \int_a^b w(x) e_1\,(x)\,dx \\ ... \\ \sum\limits_{k=1}^{m} A_k e_{2m-1}(x_k) = \int_a^b w(x) e_{2m-1}\,(x)\,dx \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} A_1 + A_2 + ... + A_m = \mu_0 \\ A_1 x_1 + A_2 x_2 + ... + A_m x_m = \mu_1 \\ ... \\ A_1 x_1^{2m-1} + A_2 x_2^{2m-1} + ... + A_m x_m^{2m-1} = \mu_{2m-1} \end{cases} \tag{3}$$

with

$$\mu_k = \int_a^b w(x) x^k dx.$$

As, the system (3) is difficult to solve, there have been found other ways to find the unknown parameters.

If $w(x) = 1$ (this case was studied by Gauss), then the nodes are the roots of Legendre orthogonal polynomial

$$u(x) = \frac{m!}{(2m)!}\,[(x-a)^m(x-b)^m]^{(m)}$$

and for finding the coefficients we use the first m equations from the system (3).

For example, for $m = 2$, the interval $[-1, 1]$ and $w(x) = 1$, we get the system

$$\begin{cases} A_1 + A_2 = \int_{-1}^{1} dx = 2 \\ A_1x_1 + A_2x_2 = \int_{-1}^{1} xdx = 0 \\ A_1x_1^2 + A_2x_2^2 = \int_{-1}^{1} x^2dx = \frac{2}{3} \\ A_1x_1^3 + A_2x_2^3 = \int_{-1}^{1} x^3dx = 0 \end{cases} \tag{4}$$

with solution $A_1 = A_2 = 1$ and $x_1 = -\frac{\sqrt{3}}{3}, x_2 = \frac{\sqrt{3}}{3}$, which gives the fomula

$$\int_{-1}^{1} f(x)dx \simeq f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}).$$

This formula has degree of precision 3, i.e., it gives exact result for every polynomial of the 3−rd degree or less.

Remark 3 *The resulting rules look more complicated than the interpolatory rules. Both nodes and weights for gaussian rules are, in general, irrational numbers. But, on a computer, it usually makes no difference whether one evaluates a function at* $x = 3$ *or at* $x = 1/\sqrt{3}$*. Once the nodes and weights of such a rule are stored, these rules are as easily used as the trapezium rule or Simpson's rule. At the same time, these gaussian rules are usually much more accurate when compared with the last ones on the basis of number of function values used.*

Example 4 *Consider* $m = 1$ *and obtain the following Gauss type quadrature formula*

$$\int_a^b f(x)dx = A_1 f(x_1) + R_1(f).$$

The system (3) becomes

$$\begin{cases} A_1 = \int_a^b dx = b - a \\ A_1 x_1 = \int_a^b x dx = \frac{b^2 - a^2}{2}. \end{cases}$$

The unique solution of this system is $A_1 = b - a$, $x_1 = \frac{a+b}{2}$.

The same result is obtained considering x_1 *the root of the Legendre polynomial of the first degree,*

$$u(x) = \frac{1}{2}[(x-a)(x-b)]' = x - \frac{a+b}{2}.$$

The Gauss type quadrature formula with one node is

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + R_1(f),$$

with

$$R_1(f) = \frac{(b-a)^3}{24} f''(\xi), \qquad \xi \in [a,b]$$

which is called **the rectangle quadrature rule** (also called **the mid-point rule).**

The repeated rectangle (midpoint) quadrature formula is

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) + R_n(f),$$

$$R_n(f) = \frac{(b-a)^3}{24n^2} f''(\xi), \qquad \xi \in [a,b]$$

with $x_1 = a + \frac{b-a}{2n}$, $x_i = x_1 + (i-1)\frac{b-a}{n}$, $i = 2, ..., n.$

We have

$$|R_n(f)| \ \leq \frac{(b-a)^3}{24n^2} M_2 f, \quad \text{with } M_2 f = \max_{x \in [a,b]} |f''(x)|.$$

Remark 5 *Another* **rectangle rule** *is the following*:

$$\int_a^b f(x)dx = (b-a)f(a) + R(f),$$

with

$$R(f) = \frac{(b-a)^2}{2} f'(\xi), \qquad \xi \in [a,b].$$

Romberg's algorithm for the rectangle (midpoint) quadrature formula. Applying successively the rectangle formula on $[a, b]$, we get

$$Q_{D_0}(f) = (b-a)f(x_1), \quad x_1 = \frac{a+b}{2}$$
$$Q_{D_1}(f) = \frac{1}{3}Q_{D_0}(f) + \frac{b-a}{3}\left[f(x_2) + f(x_3)\right],$$
$$x_2 = a + \frac{b-a}{6},\ x_3 = b - \frac{b-a}{6}.$$

Continuing in an analogous manner, we obtain the sequence

$$Q_{D_0}(f),\ Q_{D_1}(f), ..., Q_{D_k}(f), ... \tag{5}$$

which converges to the value I of the integral $\int_a^b f(x)dx$.

If we want to approximate the integral I with error less than ε, we compute successively the elements of (5) until the first index for which

$$\left|Q_{D_m}(f) - Q_{D_{m-1}}(f)\right| \leq \varepsilon,$$

$Q_{D_m}(f)$ being the required value.

Example 6 *Approximate* $\ln 2 = \int_1^2 \frac{1}{x} dx$, *with* $\varepsilon = 10^{-2}$, *using the repeated rectangle (midpoint) method.*