Question 1: $f(n) = 3n^2 + 10n + 729$

1.1 Prove that $f(n) = O(n^2)$

Proof. To prove $f(n) = O(n^2)$ we must show that there exists a value, n_0, and a constant, c, such that:

$$3n^2 + 10n + 729 \leq cn^2, \forall n > n_0$$

If we can show that such an n_0 and c exist, that is the definition of Big O Notation. Let $n_0 = 27$ and $c = 15$. Since $n > 0$ we can upper bound $10n$ by $10n^2$:

$$3n^2 + 10n + 729 < 3n^2 + 10n^2 + 729 = 13n^2 + 729$$

Also since $n > n_0 = 27$ this gives us $729 < n^2$:

$$13n^2 + 729 < 15n^2$$

So, for $n > 27$, $f(n)$ was upper bounded by a constant multiplied by n^2. So by the definition of Big O Notation, the statement is shown.

(NOTE: Methodology for this proof comes from the "How to write excellent proofs" handout) □

1.2 Prove that $f(n) = O(n^3)$

Proof. To prove $f(n) = O(n^2)$ we must show that there exists a value, n_0, and a constant, c, such that:

$$3n^2 + 10n + 729 \leq cn^3, \forall n > n_0$$

If we can show that such an n_0 and c exist, that is the definition of Big O Notation. Let $n_0 = 9$ and $c = 15$. Since $n > 0$ we can upper bound $10n$ by $10n^3$:

$$3n^2 + 10n + 729 < 3n^3 + 10n^3 + 729 = 13n^3 + 729$$

Also since $n > n_0 = 9$ this gives us $729 < n^3$:

$$13n^2 + 729 < 15n^3$$

So, for $n > 9$, $f(n)$ was upper bounded by a constant multiplied by n^2. So by the definition of Big O Notation, the statement is shown.

(NOTE: As with the previous Proof, the methodology for this one also comes from the "How to write excellent proofs" handout)

$\square$

1.3 Prove that $f(n) = \Omega(n)$

Proof. To prove $f(n) = \Omega(n)$ we must show that there exists a **positive** value, n_0, and a positive constant, c, such that:

$$3n^2 + 10n + 729 \geq cn, \forall n > n_0$$

If we can show that such an n_0 and c exist, that is the definition of Ω Notation.

Let $n_0 = 0$ and $c = 1$. We know:

$$\begin{gathered} 10n \geq n \\ 10n + 729 \geq n \\ 3n^2 + 10n + 729 \geq n \end{gathered}$$

We know that $f(n) \geq cn$, $\forall n \geq 0$. Thus $f(n) = \Omega(n)$. $\square$

1.4 Prove that $f(n) = \Omega(n^2)$

Proof. To prove $f(n) = \Omega(n^2)$ we must show that there exists a **positive** value, n_0, and a positive constant, c, such that:

$$3n^2 + 10n + 729 \geq cn, \forall n > n_0$$

If we can show that such an n_0 and c exist, that is the definition of Ω Notation.

Let $n_0 = 0$ and $c = 1$. We know:

$$3n^2 \geq n^2, \forall n$$
$$3n^2 + 10n \geq n^2, \forall n \geq 0 \text{ and}$$
$$3n^2 + 10n + 729 \geq n^2$$

So, when $n \geq 0$

$$3n^2 + 10n + 729 \geq 3n^2 + 10n \geq n^2$$

Thus we know,

$$3n^2 + 10n + 729 \geq n^2$$

$f(n) \geq cn^2, \forall n \geq n_0 = 0$. Thus $f(n) = \Omega(n^2)$. $\square$

1.5 Best Bounds for Each

For both the Big O and Ω, I believe the best bounds are the ones that bound around the function the tightest. So while $f(n) = O(n^3)$ this is not as optimal as $f(n) = O(n^2)$ since the latter forms a tighter bound on $f(n)$. Similarly I believe $\Omega(n)$ is better than $\Omega(n^2)$ for the same reason.