CS 124 Programming Assignment 2 Write Up

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Introduction

Conventional matrix multiplication over two matrices $A, B \in \mathbb{R}^{n \times n}$ calculates entry i, j of the product AB as follows:

$$AB[i][j] = \sum_{k=0}^{n-1} A[i][k]B[k][j].$$

Assuming common operations like addition, multiplication, array access, etc are $\Theta(1)$, we see that this algorithm takes $\Theta(n)$ over n^2 entries, yielding a time complexity of $\Theta(n^3)$.

However, Strassen came up with a very clever algorithm to improve this $\Theta(n^3)$ time complexity to $\Theta(n^{\log_2(7)})$. The basic idea is to perform 7 matrix multiplications on $\frac{n}{2} \times \frac{n}{2}$ submatrices and add and subtract them in clever ways. Assuming the relevant (constant in n) number of matrix additions between the submatrix products take time $\Theta(n^2)$, Strassen's algorithm has recursive runtime of

$$T(n) = 7T(n/2) + \Theta(n^2).$$

The master method then yields our claimed time complexity of $\Theta(n^{\log_2(7)})$.

However, the asymptotic notation above may be hiding some very large constant factors in Strassen's algorithm; for small matrices, conventional matrix multiplication may indeed be faster. Traditional Strassen's recurses to a base case of 1×1 matrices, but recursing down to an experimental cutoff point at which point we switch to traditional matrix multiplication will yield performance benefits.

The purpose of our code will be to experimentally determine what this cutoff point is using randomly generated integer matrices with entries between 0 and 4. We will also use our Strassen's implementation to calculate the number of triangles in randomly generated graphs of size 1024, which can be done via matrix multiplication of the adjacency matrix (the theory of which is described later).

Analytically Estimating Cutoff

We first analytically estimate the crossover point between Strassen's and traditional matrix multiplication.

The problem statement seeks to find the value of n_0 for which Strassen's runs with minimum time. We estimate this by finding the cutoff n_0 at which Strassen's first runs faster than traditional matrix multiplication. (These are not neccessarily the same, but they should be close. Further discussion is given in the experiments section.)

Let $T(n)$ denote the time for traditional matrix multiplication. We then see that traditional multiplication runs n multiplications over n^2 terms, yielding total time n^3 from multiplication. Moreover, each n^2 term runs through $n-1$ additions, yielding $n^2(n-1)$ time through addition. We thus find that

$$T(n) \approx n^3 + n^2(n-1) = 2n^3 - n^2.$$

Let $S(n)$ now denote the time for Strassen's matrix multiplication with a threshold at $n_0 - 1$. We next assume that our Strassen's is implemented such that all calculations are done in-place (as we do in our code) to simplify time considerations from memory copying.
(small explanation: "in-place" means we don't use extra space memory to copy results for our sub-problems; instead, we always work only within space of initial two n-sized matrices)
In this case, the total time is again the number of multiplications and additions performed. Note again the recursion

$$S(n) = 7S(n/2) + f(n)$$

for $n \geq n_0$ and for $n < n_0$.

$$S(n) = 2n^3 - n^2.$$

We seek to estimate the functional form of $f(n) \in \Theta(n^2)$ to exactly solve this recursion.

Note that Strassen's first calculates 10 intermediate sums, each of which does $n^2/4$ additions. We then perform the recursive step, which is already baked-into the recursion. It next computes 4 intermediate sums. Two of these perform $3(n^2/4)$ additions, while two perform $n^2/4$ additions. This yields

$$f(n) = 10(n^2/4) + 2 \cdot 3(n^2/4) + 2(n^2/4) = 9/2n^2.$$

We seek to find the value of n_0 such that $S(n_0) < T(n_0)$, so that Strassen's is faster than traditional multiplication for all input greater than the threshold. Note that $\frac{n_0}{2} \leq n_0 - 1$ for $n_0 \geq 2$, which we assume. Thus

$$S(n_0) = 7S(n_0/2) + 9/2(n_0)^2 = 7(2(n_0/2)^3 - (n_0/2)^2) + \frac{9}{2}n_0^2.$$

Mathematica solves

$$7(2(n_0/2)^3 - (n_0/2)^2) + \frac{9}{2}n_0^2 < 2n_0^3 - n_0^2$$

to yield $n_0 > 15$. Thus the first point at which Strassen's outperforms traditional multiplication is at a threshold of 16. This is our **theoretical bound**.

If we take into account how we implement Strassen's algorithm, however, we'll get different bound: our implementation, in order to stay in-place, ends up undoing the 10 $n^2/4$ additions on the intermediate matrices (as we must undo some of the arithmetic operations performed in-place to preserve the appropriate summands for final calculation, we indicated these adjustments with (*) in our code). This yields an additional $10(n^2/4)$ additions, giving $f(n) = 7n^2$. Also, in order to use results for previous recursive sub-problems, namely values $p_1, ..., p_7$ (they are called $prod_1, ..., prod_7$ in code) of products of sub-matricies, we have to **save** them into some local variables, implying copying of 7 matricies of size $n/2$, which adds $7(n^2/4)$ more operations, giving $f(n) = (35/4)n^2$. This yields the recursion $S(n) = 7S(n/2) + (35/4)n^2$, which yields the inequality

$$7(2(n_0/2)^3 - (n_0/2)^2) + (35/4)n_0^2 \leq 2n_0^3 - n_0^2,$$

which results in $n_0 \geq 32$. Since our implementation only depends on powers of 2 and recurses always to a base-case power of 2, our personal theoretical cutoff should be the next greatest power of 2: 32. (Our

implementation pads on extra 0's for non powers of 2, so we always recurse down to a base-case that is a power of 2.)
Although this doesn't change our bound, it's important consideration, because in case split point = 32 doesn't work (this might happen, perhaps, because of some other randomness, while the difference in RT might be very small in favor of trivial implementation), then the next candidate for split point is 64 – and we exactly observe these patterns in the experiment!

Implementations and Experimental Crossover Point Result

Our implementation of trivial matrix multiplication is done in the standard manner.

We implement Strassen's using the standard algorithm described in the lecture notes. We optimize to prevent needless memory copying by implementing the algorithm in-place (except for new memory allocation for our result matrix); we also allow for variable base case by setting a local variable *split_point* which determines when we switch to standard matrix multiplication. For non powers of 2, we simply pad extra 0's onto the matrices appropriately (as discussed earlier).

We present two notions of crossover point.
One is the n_0 at which Strassen's runtime is minimized. For this crossover n_0 we introduced crude theoretical bound in the previos section: if switching from current version n_0 to $n_0/2$ results in longer RT $\iff$ trivial implementation at this stage is faster than Strassen procedures + trivial at smaller stage.

To find it, we implement a function $find\_optimal\_split$ which takes in a local variable *dimension*. It then initializes two random matrices of the appropriate dimension with random values between 0 and 4. It then iterates throguh all possible *split_points* at which we can switch to traditional matrix multiplication (i.e., powers of 2 from 1 to *dimension*; non powers of 2 are equivalent to cutoffs at the next lower power of 2) and returns the split that takes the minimum amount of time to execute. This is then the second experimental crossover point.

Results are shown below.

Dimension	Cutoff
4	4
8	8
16	16
32	32
64	64
128	64
256	64
512	32
1024	32

It seems that, on our system, Strassen's is optimized at around a cutoff of 32×32 matrices – this is remarkably consistent with our theoretical result of 32!

There are things to take into account, though.
First, we assumed that multiplication is as expensive as addition on our architecture when analyzing the theoretical performance. This is not neccessarily the case, but the assumption was consistent with

theoretical results. Indeed, running a test comparing the runtime of multiplication and addition on our architecture yields practically identical results.

Then, cache concerns/low-level explanations could also explain discrepancy at lower values of n. This is also why we rely on the largest result of $n = 512, 1024$ as our decided cutoff (where we indeed observe result of 32).

At the end of the day, we also believe that the different cutoff values for some small matrices (for example the $n = 256$ case) is likely due to vanishingly small time differences at such small matrices regardless of optimization, so that randomness can lead to one cutoff being faster than another – this is especially true given how much our theoretical bound of 32 is tight, i.e. we have exactly the inequality $split \geq 32$.

The second notion of crossover is the n_0 at which Strassen's **first** outperforms traditional matrix multiplication for given size n. This crossover is different from that done in theoretical analysis (and thus the previous crossover): in limit ($n \to \infty$) this crossover becomes equal to 1 because, as we know, Strassen is $\Theta(n^{\log_2(7)})$, while trivial is $\Theta(n^3)$ implying that even if we don't optimize for Strassen for small cases, we still will do faster.
However, as we demonstrate in our experiment, for relatively small values of $n \leq 1024$ this crossover becomes 16 or 32, which again justifies decision to switch from Strassen to trivial at small cases.
To analyze this, we programmed a helper $find\_first\_cross$ which found the cutoff n_0 when Strassen's was first faster than traditional matrix multiplication. This helper binary searches through all cutoff recursive thresholds to find the smallest one that is faster than traditional matrix multiplication. The binary search is justified by the theoretical analysis above (as we should be faster than traditional matrix multiplication past a certain point, and slower for all thresholds below), and it ends up holding in some experiments we did as well (which timed Strassen's faster past a threshold, and slower below a given threshold). Our results are shown below.

Dimension	Cutoff
4	4
8	8
16	16
32	32
64	34
128	16
256	32
512	32
1024	16

Interestingly, results here also somewhat align with previous bounds of 32.

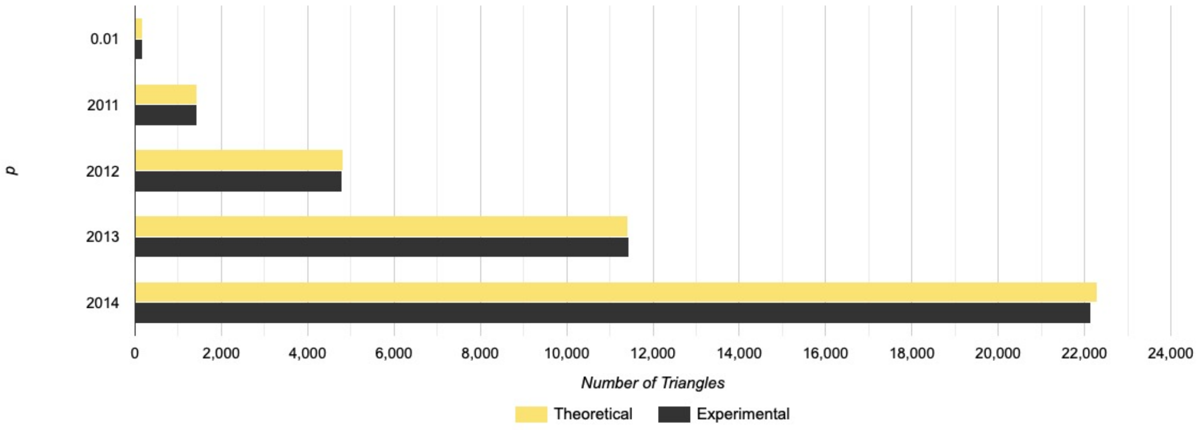
On a given graph G with adjacency matrix A, the number of triangles is given by $\text{tr}(A^3)/6$, where $\text{tr}(A^3)$ denotes the sum of diagonal entries of A^3. We thus initialize such adjacency matrices by including a given edge with probability p.

We thus initialize such a random adjacency matrix by including a given edge with probability p. We calculate $\text{tr}(A^3)/6$, repeat 5 times, and average. We then compare our number of triangles to the theoretical average, which is easily calculated via linearity of expectation to be $\binom{1024}{3}p^3$.

A table of our experimentally determined calculations against theoretical calculations is given below.

p	Experimental	(Approximate) Theoretical
0.01	174.2	178.433
0.02	1430.2	1427.46
0.03	4795.2	4817.69
0.04	11435.2	11419.7
0.05	22145.4	22304.1

The bar chart below summarizes our results.



Concluding Remarks

Our experimental crossover of 32 relies on the fact that we are multiplying integer matrices; speed of multiplication vs addition over floating point numbers likely has a different ratio and could result in a different optimal crossover.

Strassen's algorithm is indeed faster than naive multiplication using the base-case optimizations above, even for n that are arguably small (e.g. $n = 256$). This indicates that Strassen's is NOT practically useless, as was implied by the analysis quoted in the progset. We are curious whether such practical adjustments can be made for matrix algorithms that are asymptotically better than even Strassen's.

Strassen's on non-square matrices would require a totally different analysis and likely have cutoffs dependent on appropriate matrices.